ELECTRONIC RESEARCH ARCHIVE
Volume **29**, Number **5**, November **2021**

doi:10.3934/era.2021050
pp. **3509–3533**

GLOBAL EXISTENCE FOR A TWO-SPECIES CHEMOTAXIS-NAVIER-STOKES SYSTEM WITH p-LAPLACIAN

Jiayi Han and Changchun Liu*

Department of Mathematics, Jilin University
Changchun 130012, China

(Communicated by Elisabetta Rocca)

Abstract. We consider a two-species chemotaxis-Navier-Stokes system with p-Laplacian in three-dimensional smooth bounded domains. It is proved that for any $p \geq 2$, the problem admits a global weak solution.

1. **Introduction.** In this paper, we are concerned with the following two-species chemotaxis-Navier-Stokes system:

$$\begin{cases} (n_1)_t + u \cdot \nabla n_1 = \nabla \cdot (|\nabla n_1|^{p-2} \nabla n_1) - \chi_1 \nabla \cdot (n_1 \nabla c) \\ \qquad\qquad +\mu_1 n_1 (1 - n_1 - a_1 n_2), \\ (n_2)_t + u \cdot \nabla n_2 = \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) + \mu_2 n_2 (1 - a_2 n_1 - n_2), \\ c_t + u \cdot \nabla c = \Delta c - (\alpha n_1 + \beta n_2) c, \\ u_t + (u \cdot \nabla) u = \Delta u + \nabla P + (n_1 + n_2) \nabla \Phi, \quad \nabla \cdot u = 0, \end{cases} \tag{1.1}$$

in $Q = \Omega \times [0, \infty)$, where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary, $\mu_1, \mu_2, \alpha, \beta, \chi_1, \chi_2$, are positive constants, and $a_1, a_2 \geq 0$ are constants. This system describes the evolution of two kinds of aerobic bacteria, that compete according to Iotka-Volterra competitive kinetics in a liquid surrounding environment. Here n_1 and n_2 represent the population densities of two species respectively, c stands for the concentration of oxygen, u shows the fluid velocity field, and P represents the pressure of the fluid. The given function Φ represents the gravitational potential.

The problem (1.1) is a generalized system to the chemotaxis-fluid system, which is proposed by Tuval et al. in [21]. The chemotaxis-Navier-Stokes system models have been widely studied by many researchers ([22, 23, 25]).

What's more, the investigation of the problems involving chemotaxis-Navier-Stokes system models with p-Laplacian has been addressed by several authors. Tao

2020 *Mathematics Subject Classification.* Primary: 92C17, 35K51; Secondary: 35K65, 35K92.

Key words and phrases. Chemotaxis-Navier-Stokes system, p-Laplacian diffusion, global existence.

This work is supported by the Jilin Scientific and Technological Development Program.

* Corresponding author: Changchun Liu.

and Li [19] discussed the following chemotaxis-Navier-Stokes system:

$$\begin{cases} n_t + u \cdot \nabla n = \nabla \cdot (|\nabla n|^{p-2} \nabla n) - \nabla \cdot (n\chi(c)\nabla c)\,, \\ c_t + u \cdot \nabla c = \Delta c - nf(c), \\ u_t + (u \cdot \nabla)u = \Delta u + \nabla P + n\nabla\phi, \\ \nabla \cdot u = 0. \end{cases}$$

They got that if $p > \frac{32}{15}$, under appropriate assumptions on f and χ, for all sufficiently smooth initial data (n_0, c_0, u_0), the system owns at least one global weak solution in three dimensional spaces. Furthermore, Tao and Li [20] proved that global bounded weak solutions of the chemotaxis-Stokes system exist whenever $p > \frac{23}{11}$. Liu [10] investigated the following problem:

$$\begin{cases} \rho_t + u \cdot \nabla \rho = \nabla \cdot (|\nabla \rho|^{p-2} \nabla \rho) - \nabla \cdot (\rho \nabla c) - \rho m, & x \in \Omega, t > 0, \\ c_t - u \cdot \nabla c = \Delta c - c + m, & x \in \Omega, t > 0, \\ m_t + u \cdot \nabla m = \Delta m - \rho m, & x \in \Omega, t > 0, \\ u_t + \kappa (u \cdot \nabla)u = \Delta u + \nabla P + (\rho + m)\nabla\phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary. It is proved that if either $p > 2$ for $\kappa \in \mathbb{R}$, $N = 2$ or $p > \frac{94}{45}$ for $\kappa = 0$, $N = 3$ is satisfied, then for each properly chosen initial data and associated initial-boundary problem admits a global weak solution which is bounded. The relevant equations have also been studied in [11, 12].

On the other hand, two-species competitive chemotaxis systems have been studied by many authors [1, 15] recently, mainly about the global existence and asymptotic stability of solution. Cao, Kurima and Mizukami [3] considered the following two-species chemotaxis-Stokes system:

$$\begin{cases} (n_1)_t + u \cdot \nabla n_1 = \Delta n_1 - \chi_1 \nabla \cdot (n_1 \nabla c) \\ \qquad\qquad +\mu_1 n_1 (1 - n_1 - a_1 n_2), & x \in \Omega, t > 0, \\ (n_2)_t + u \cdot \nabla n_2 = \Delta n_2 - \chi_2 \nabla \cdot (n_2 \nabla c) \\ \qquad\qquad +\mu_2 n_2 (1 - a_2 n_1 - n_2), & x \in \Omega, t > 0, \\ c_t + u \cdot \nabla c = \Delta c - c(\alpha n_1 + \beta n_2), & x \in \Omega, t > 0, \\ u_t + \kappa (u \cdot \nabla)u = \Delta u + \nabla P + (\gamma n_1 + \delta n_2)\nabla\phi, & x \in \Omega, t > 0, \\ \nabla \cdot u = 0, & x \in \Omega, t > 0. \end{cases} \tag{1.2}$$

They proved the global existence, boundedness and stabilization of solutions to the above system in the 3-dimensional case. Hirata et al [6] gave more complete stabilization of solutions for (1.2) in the 2-dimensional case. Moreover Liu and Li [13] proved that the system (1.2) admits a time periodic solution under some conditions. The relevant equations have also been studied in [8].

As mentioned above, two-species chemotaxis-Stokes system and one species chemotaxis-Stokes system with p-Laplacian were studied by many authors. However the combination of these two kinds of problems has not been studied. Thus, we are inspired to investigate the case that the two species have different diffusion law, namely one according to the p-Laplacian diffusion and the other according to standard Laplacian diffusion. From a physical point of view, in the same liquid surrounding environment, one species is influenced by ions and molecules and thus its mobility is described by a nonlinear function of the cells, but the other species is not affected by ions or molecules thus diffuse by linear Laplacian diffusion.

Obviously, Cao, Kurima and Mizukami solved the problem (1.1) when $p = 2$. If $p > 2$, from a mathematical point of view, p-Laplacian diffusion term $\nabla \cdot (|\nabla n_1|^{p-2}\nabla n_1)$ lead to a lot of difficulties in our proof of main result because of its nonlinear character. To be specific, a main difficulty arises in the estimate of n_1 in contrast to the case of Laplacian, and the property of the Neumann heat semigroup becomes useless and so on. In order to overcome the difficulties bring by the p-Laplacian diffusion term, we consider a regularized problems of (1.1) and establish an energy-type inequality in Section 3 as a starting point to discuss the problem.

In this paper, we shall consider (1.1) along with the boundary conditions

$$|\nabla n_1|^{p-2}\frac{\partial n_1}{\partial \nu} = \frac{\partial n_2}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0 \text{ and } u = 0, \text{ on } Q_\Gamma, \tag{1.3}$$

where $Q_\Gamma = \partial\Omega \times [0, \infty)$, and the initial conditions

$$n_i(x,0) = n_{i,0}(x), \quad c(x,0) = c_0(x), \quad u(x,0) = u_0(x), \quad x \in \Omega, \quad i = 1,2. \tag{1.4}$$

Assume that $n_{1,0}$, $n_{2,0}$, c_0, u_0 and Φ are given functions satisfying

$$\Phi \in W^{1,\infty}(\Omega), \tag{1.5}$$

and

$$\begin{cases} n_{1,0} \in L^2(\Omega), \text{ and } n_{1,0} > 0, \\ n_{2,0} \in L\log L(\Omega), \text{ and } n_{2,0} > 0, \\ c_0 \in L^\infty(\Omega), \ c_0 > 0, \text{ and } \sqrt{c_0} \in W^{1,2}(\Omega), \\ u_0 \in L^2_\sigma(\Omega), \end{cases} \tag{1.6}$$

where $L\log L(\Omega)$ is the standard Orlicz space associated with the Young function $(0,\infty) \ni z \mapsto z\ln(1+z)$ and $L^2_\sigma(\Omega) := \{\varphi \in (L^2(\Omega))^3 | \nabla \cdot \varphi = 0\}$.

Let us first give the definition of weak solution.

Definition 1.1. We call (n_1, n_2, c, u) a global weak solution of (1.1), (1.3) and (1.4) if

$$n_1 \in L^1_{loc}([0,\infty); L^1(\Omega)), \quad n_2 \in L^1_{loc}([0,\infty)); W^{1,1}(\Omega)),$$

$$c \in L^1_{loc}([0,\infty); W^{1,1}(\Omega)), \quad u \in (L^1_{loc}([0,\infty); W^{1,1}_0(\Omega)))^3,$$

such that $n_1 \geq 0, n_2 \geq 0$ and $c \geq 0$ $a.e$ in Q and that

$$\mu_1 n_1(1 - n_1 - a_1 n_2), \ \mu_2 n_2(1 - a_2 n_1 - n_2), \ (\alpha n_1 + \beta n_2)c \in L^1_{loc}([0,\infty); L^1(\Omega)),$$

$$|\nabla n_1|^{p-2}\nabla n_1, \ n_1\nabla c, \ n_2\nabla c, \ n_1 u, \ n_2 u, \ cu \in (L^1_{loc}([0,\infty); L^1(\Omega)))^3,$$

$$u \otimes u \in (L^1_{loc}([0,\infty); L^1(\Omega)))^{3\times 3},$$

and that

$$\int_0^\infty\int_\Omega (n_1)_t\phi_1 - \int_0^\infty\int_\Omega n_1 u\cdot\nabla\phi_1 = -\int_0^\infty\int_\Omega |\nabla n_1|^{p-2}\nabla n_1\cdot\nabla\phi_1 + \int_0^\infty\int_\Omega \chi_1 n_1\nabla c\cdot\nabla\phi_1 + \int_0^\infty\int_\Omega \mu_1 n_1\left(1-n_1-a_1n_2\right)\phi_1,$$

$$\int_0^\infty\int_\Omega (n_2)_t\phi_2 - \int_0^\infty\int_\Omega n_2 u\cdot\nabla\phi_2 = -\int_0^\infty\int_\Omega \nabla n_2\cdot\nabla\phi_2 + \int_0^\infty\int_\Omega \chi_2 n_2\nabla c\cdot\nabla\phi_2 + \int_0^\infty\int_\Omega \mu_2 n_2(1-a_2n_1-n_2)\phi_2,$$

$$\int_0^\infty\int_\Omega c_t\phi_3 - \int_0^\infty\int_\Omega cu\cdot\nabla\phi_3 = -\int_0^\infty\int_\Omega \nabla c\cdot\nabla\phi_3 - \int_0^\infty\int_\Omega(\alpha n_1+\beta n_2)c\phi_3,$$

as well as

$$\int_0^\infty\int_\Omega u_t\phi_4 - \int_0^\infty\int_\Omega u\otimes u\nabla\phi_4 = -\int_0^\infty\int_\Omega \nabla u\cdot\nabla\phi_4 + \int_0^\infty\int_\Omega (n_1+n_2)\nabla\Phi\cdot\phi_4,$$

hold for all $\phi_1,\phi_2,\phi_3 \in C_0^\infty(\bar\Omega\times[0,\infty))$ and $\phi_4 \in (C_0^\infty(\Omega\times[0,\infty)))^3$ satisfying $\nabla\cdot\phi_4 = 0$.

The plan of this paper is as follows. In Section 2, we list some lemmas, which will be used throughout this paper. In Section 3, we consider a family of regularized problems and show the global existence of the regularized problems, by establishing an energy-type inequality and using the Moser-Alikakos iteration procedure. Finally, in Section 4, we show that the problem (1.1), (1.3) and (1.4) admits a global-in-time weak solution.

2. Preliminaries.

In this section, we recall some lemmas, which will be used throughout the paper. Before going further, we first list the Gagliardo-Nirenberg interpolation inequality [16] for the convenience of application.

Lemma 2.1. *For functions $u:\Omega\to\mathbb{R}$ defined on a bounded Lipschitz domain $\Omega\subset\mathbb{R}^N$, we have*

$$\|D^j u\|_{L^p} \le C\|D^m u\|_{L^r}^b\|u\|_{L^q}^{1-b} + C\|u\|_{L^s},$$

where $1\le q,r\le\infty$, $\frac{j}{m}\le b\le 1$,

$$\frac{1}{p} = \frac{j}{N} + (\frac{1}{r}-\frac{m}{N})b + \frac{1-b}{q},$$

and $s>0$ is arbitrary.

Next, we list the following Lemma 2.2 [18].

Lemma 2.2. *Let $T>0, \tau\in(0,T), a>0, b>0$, and suppose that $y:[0,T)\to[0,\infty)$ is absolutely continuous such that*

$$y'(t) + ay(t) \le h(t), \ \textit{for } t\in[0,T),$$

where $h\ge 0, h(t)\in L^1_{loc}([0,T))$ and

$$\int_{t-\tau}^t h(s)ds \le b, \ \textit{for all } t\in[\tau,T).$$

Then

$$y(t) \le \max\{y(0)+b, \frac{b}{a\tau}+2b\} \ \textit{for all } t\in[0,T).$$

Finally, we also give a generalized lemma of Lemma 2.2 [7].

Lemma 2.3. *Let $T > 0, \tau \in (0,T), \sigma \geq 0, a > 0, b \geq 0$, and suppose that $f : [0,T) \to [0,\infty)$ is absolutely continuous and satisfies*

$$f'(t) + af^{1+\sigma}(t) \leq h(t),\ t \in \mathbb{R},$$

where $h \geq 0,\ h(t) \in L^1_{loc}([0,T))$ and

$$\int_{t-\tau}^{t} h(s)ds \leq b,\ \textit{for all}\ t \in [\tau, T).$$

Then

$$\begin{aligned}&\sup_{t\in(0,T)} f(t) + a \sup_{t\in(\tau,T)} \int_{t-\tau}^{t} f^{1+\sigma}(s)ds\\ \leq& b + 2\max\{f(0) + b + a\tau, \frac{b}{a\tau} + 1 + 2b + 2a\tau\}.\end{aligned}$$

3. **Regularized problem.** Inspired by the idea from [22], in order to construct a global weak solution of (1.1), (1.3) and (1.4), we first consider the following appropriately regularized problem:

$$\begin{cases}\partial_t n_{1\varepsilon} + u_\varepsilon \cdot \nabla n_{1\varepsilon} = \nabla \cdot \left(\left(|\nabla n_{1\varepsilon}|^2 + \varepsilon\right)^{\frac{p-2}{2}} \nabla n_{1\varepsilon}\right) \\ \qquad -\chi_1 \nabla \cdot (n_{1\varepsilon} F'_\varepsilon(n_{1\varepsilon})\nabla c_\varepsilon) + \mu_1 n_{1\varepsilon}(1 - n_{1\varepsilon} - a_1 n_{2\varepsilon}), (x,t) \in Q, \\ \partial_t n_{2\varepsilon} + u_\varepsilon \cdot \nabla n_{2\varepsilon} = \Delta n_{2\varepsilon} - \chi_2 \nabla \cdot (n_{2\varepsilon} F'_\varepsilon(n_{2\varepsilon})\nabla c_\varepsilon) \\ \qquad +\mu_2 n_{2\varepsilon}(1 - a_2 n_{1\varepsilon} - n_{2\varepsilon}), & (x,t) \in Q, \\ \partial_t c_\varepsilon + u_\varepsilon \cdot \nabla c_\varepsilon = \Delta c_\varepsilon - (\alpha F_\varepsilon(n_{1\varepsilon}) + \beta F_\varepsilon(n_{2\varepsilon}))\, c_\varepsilon, & (x,t) \in Q, \\ \partial_t u_\varepsilon + (Y_\varepsilon u_\varepsilon \cdot \nabla) u_\varepsilon = \Delta u_\varepsilon + \nabla P_\varepsilon + (n_{1\varepsilon} + n_{2\varepsilon})\nabla \Phi, & (x,t) \in Q, \\ \nabla \cdot u_\varepsilon = 0, & (x,t) \in Q, \\ \frac{\partial n_{1\varepsilon}}{\partial \nu} = \frac{\partial n_{2\varepsilon}}{\partial \nu} = \frac{\partial c_\varepsilon}{\partial \nu} = 0, \quad u_\varepsilon = 0, & (x,t) \in Q_\Gamma, \\ n_{i,\varepsilon}(x,0) = n_{0i\varepsilon}(x),\ c(x,0) = c_{0\varepsilon}(x),\ u(x,0) = u_{0\varepsilon}(x),\ x \in \Omega,\ i = 1,2, \end{cases} \tag{3.1}$$

for $\varepsilon \in (0,1)$. We take $F_\varepsilon(s) := \frac{1}{\varepsilon}\ln(1+\varepsilon s)$, for $s \geq 0$ and utilize the standard Yosida approximation Y_ε [14, 17], which was defined by

$$Y_\varepsilon v := (1 + \varepsilon A)^{-1} v, \ \text{ for all }\ v \in L^2_\sigma(\Omega).$$

By $\mathbf{P}$ we mean the Helmholtz projection in $L^2(\Omega)$, then we represent A as the realization of Stokes operator $-\mathbf{P}\Delta$ in $L^2_\sigma(\Omega)$, with domain $D(A) = W^{2,2}(\Omega) \cap W^{1,2}_{0,\sigma}(\Omega)$, where $W^{1,2}_{0,\sigma}(\Omega) = W^{1,2}_0(\Omega) \cap L^2_\sigma(\Omega) = \overline{C^\infty_{0,\sigma}(\Omega)}^{\|\cdot\|_{W^{1,2}(\Omega)}}$, with $C^\infty_{0,\sigma}(\Omega) = C^\infty_0(\Omega) \cap L^2_\sigma(\Omega)$. Thereafter, it is obvious to see that our choice of F_ε ensures that

$$0 \leq F_\varepsilon(s) \leq s, \text{ for all } s \geq 0, \tag{3.2}$$

$$0 \leq F'_\varepsilon(s) = \frac{1}{1+\varepsilon s} \leq 1 \text{ and } 0 \leq sF'_\varepsilon(s) = \frac{s}{1+\varepsilon s} \leq \frac{1}{\varepsilon}, \text{ for all } s \geq 0, \tag{3.3}$$

and

$$F_\varepsilon(s) \nearrow s \text{ and } F'_\varepsilon(s) \nearrow 1, \text{ as } \varepsilon \searrow 0, \text{ for all } s \geq 0. \tag{3.4}$$

The families of approximate initial date $n_{01\varepsilon}(x)>0$, $n_{02\varepsilon}(x)>0$, $c_{0\varepsilon}(x)>0$ and $u_{0\varepsilon}$ satisfy that

$$\begin{cases} n_{01\varepsilon}\in C_0^\infty(\Omega),\ \int_\Omega n_{01\varepsilon}=\int_\Omega n_{1,0} \text{ for all } \varepsilon\in(0,1),\\ n_{01\varepsilon}\to n_{1,0} \text{ in } L^2(\Omega) \text{ as } \varepsilon\searrow 0,\end{cases} \tag{3.5}$$

$$\begin{cases} n_{02\varepsilon}\in C_0^\infty(\Omega),\ \int_\Omega n_{02\varepsilon}=\int_\Omega n_{2,0} \text{ for all } \varepsilon\in(0,1),\\ n_{02\varepsilon}\to n_{2,0} \text{ in } LlogL(\Omega) \text{ as } \varepsilon\searrow 0,\end{cases} \tag{3.6}$$

$$\begin{cases} \sqrt{c_{0\varepsilon}}\in C_0^\infty(\Omega),\ \|c_{0\varepsilon}\|_{L^\infty(\Omega)}\le\|c_0\|_{L^\infty(\Omega)} \text{ for all } \varepsilon\in(0,1),\\ \sqrt{c_{0\varepsilon}}\to\sqrt{c_0} \text{ a.e. in } \Omega \text{ and } W^{1,2}(\Omega) \text{ as } \varepsilon\searrow 0,\end{cases} \tag{3.7}$$

and

$$u_{0\varepsilon}\in C_{0,\sigma}^\infty(\Omega), \text{ with } \|u_{0\varepsilon}\|_{L^2(\Omega)}=\|u_0\|_{L^2(\Omega)} \text{ for all } \varepsilon\in(0,1). \tag{3.8}$$

Firstly, we give the local smooth solutions existence result of the above approximate problem as follows.

Lemma 3.1. *Taking $p\ge 2$, then for each $\varepsilon\in(0,1)$, there exist $T_{max,\varepsilon}\in(0,\infty]$ and uniquely determined functions*

$$\begin{aligned}&n_{1\varepsilon}\in C^0(\bar\Omega\times[0,T_{\max,\varepsilon}))\cap C^{2,1}(\bar\Omega\times(0,T_{\max,\varepsilon})),\\ &n_{2\varepsilon}\in C^0(\bar\Omega\times[0,T_{\max,\varepsilon}))\cap C^{2,1}(\bar\Omega\times(0,T_{\max,\varepsilon})),\\ &c_\varepsilon\in C^0(\bar\Omega\times[0,T_{\max,\varepsilon}))\cap C^{2,1}(\bar\Omega\times(0,T_{\max,\varepsilon})) \text{ and}\\ &u_\varepsilon\in C^0(\bar\Omega\times[0,T_{\max,\varepsilon});\mathbb{R}^3)\cap C^{2,1}(\bar\Omega\times(0,T_{\max,\varepsilon});\mathbb{R}^3),\end{aligned}$$

such that $(n_{1\varepsilon},n_{2\varepsilon},c_\varepsilon,u_\varepsilon)$ is a classical solution of (3.1) in $\Omega\times(0,T_{\max,\varepsilon})$ with some $P_\varepsilon\in C^{1,0}(\Omega\times(0,T_{\max,\varepsilon}))$. Moreover, we have $n_{1\varepsilon}>0$, $n_{2\varepsilon}>0$, and $c_\varepsilon>0$ in $\bar\Omega\times(0,T_{\max,\varepsilon})$ and if $T_{\max,\varepsilon}<\infty$, then

$$\|n_{1\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)}+\|n_{2\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)}+\|c_\varepsilon(\cdot,t)\|_{W^{1,q}(\Omega)}+\|A^\gamma u_\varepsilon(\cdot,t)\|_{L^2(\Omega)}\to\infty,$$

as $t\nearrow T_{\max,\varepsilon}$, for all $q>3$ and $\gamma>\frac{3}{4}$.

Proof. Combination of arguments Lemma 2.1 in [23] and Lemma 2.1 in [19], which is based on a standard Schauder fixed point argument and a parabolic regularity theory, entails the existence of classical solution. Since $(n_{1\varepsilon},n_{2\varepsilon},c_\varepsilon,u_\varepsilon)$ is a smooth solution of (3.1), the nonnegativity of $n_{1\varepsilon},n_{2\varepsilon}$ and c_ε follows from the maximum principle [23, 5, 2]. □

We are now ready to construct some basic estimates of $(n_{1\varepsilon},n_{2\varepsilon},c_\varepsilon,u_\varepsilon)$. In what follows, without special explanation, we take $\tau=\min\left\{1,\frac{T_{\max,\varepsilon}}{2}\right\}$, what's more, all constants C denote some different constants from line to line, which are only depend on the given coefficients, initial date and Ω. All these estimates of this form $\int_{t-\tau}^t\cdots ds$ in this paper can be replaced by $\int_0^{T_{\max,\varepsilon}}\cdots ds$ if $\tau<1$, so these estimates do not dependent on τ. The following estimates of $n_{1\varepsilon}$, $n_{2\varepsilon}$ and c_ε are obvious but important in the proof of our result.

Lemma 3.2. *For each $\varepsilon\in(0,1)$, the solution of (3.1) satisfies*

$$\sup_{t\in(0,T_{max,\varepsilon})}\|n_{1\varepsilon}(\cdot,t)\|_{L^1(\Omega)}+\sup_{t\in(\tau,T_{max,\varepsilon})}\int_{t-\tau}^t\int_\Omega n_{1\varepsilon}^2dxds\le C, \tag{3.9}$$

$$\sup_{t\in(0,T_{max,\varepsilon})}\|n_{2\varepsilon}(\cdot,t)\|_{L^1(\Omega)}+\sup_{t\in(\tau,T_{max,\varepsilon})}\int_{t-\tau}^t\int_\Omega n_{2\varepsilon}^2dxds\le C, \tag{3.10}$$

and

$$\|c_\varepsilon(\cdot,t)\|_{L^\infty(\Omega)} \leq \|c_0\|_{L^\infty(\Omega)} = s_0 \quad \textit{for all} \quad t \in (0, T_{max,\varepsilon}). \tag{3.11}$$

Proof. Integrating the first equation in (3.1) to see

$$\frac{d}{dt}\int_\Omega n_{1\varepsilon} + \mu_1 \int_\Omega (n_{1\varepsilon}^2 + a_1 n_{1\varepsilon} n_{2\varepsilon}) = \mu_1 \int_\Omega n_{1\varepsilon},$$

in both cases $a_1 = 0$ and $a_1 > 0$, we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega n_{1\varepsilon} + \mu_1 \int_\Omega n_{1\varepsilon}^2 &\leq \mu_1 \int_\Omega n_{1\varepsilon} \\ &\leq \frac{\mu_1}{2}\int_\Omega n_{1\varepsilon}^2 + C(\Omega), \end{aligned}$$

with some $C(\Omega) > 0$. We derive from the Hölder inequality that

$$\frac{(\int_\Omega n_{1\varepsilon})^2}{|\Omega|} \leq \int_\Omega n_{1\varepsilon}^2,$$

hence we further have

$$\frac{d}{dt}\int_\Omega n_{1\varepsilon} + \frac{\mu_1}{2|\Omega|}\left(\int_\Omega n_{1\varepsilon}\right)^2 \leq C(\Omega).$$

By Lemma 2.3, we can obtain (3.9). Then completely similar to the proof of (3.10). Finally, an application of the maximum principle to the third equation in (3.1) gives (3.11). ☐

Now we are in the position to derive an energy-type inequality which will be used in the reduction of further estimates. To this end, we first list an inequality which is crucial in the proof of the energy-type inequality. More details of the proof please refer to [4].

Lemma 3.3. *Suppose that $h \in C^2(\mathbb{R})$. Then for all $\varphi \in C^2(\bar{\Omega})$ fulfilling $\frac{\partial \varphi}{\partial \nu} = 0$ on $\partial\Omega$ we have*

$$\begin{aligned} \int_\Omega h'(\varphi)|\nabla\varphi|^2\Delta\varphi = &-\frac{2}{3}\int_\Omega h(\varphi)|\Delta\varphi|^2 + \frac{2}{3}\int_\Omega h(\varphi)|D^2\varphi|^2 - \frac{1}{3}\int_\Omega h''(\varphi)|\nabla\varphi|^4 \\ &- \frac{1}{3}\int_{\partial\Omega} h(\varphi)\frac{\partial}{\partial\nu}|\nabla\varphi|^2. \end{aligned}$$

Then we derive the decisive energy-type inequality from the first three equations in (3.1). The main idea of the proof is similar to the strategy introduced in [22] (see also [19]).

Lemma 3.4. *Assume that $p \geq 2$. There exist constants $C > 0$, $K > 0$ such that for any $\varepsilon \in (0,1)$, the solution of (3.1) satisfies*

$$\begin{aligned} &\frac{d}{dt}\left(\bar{A}\int_\Omega (n_{1\varepsilon}\ln n_{1\varepsilon} - n_{1\varepsilon}) + \bar{B}\int_\Omega (n_{2\varepsilon}\ln n_{2\varepsilon} - n_{2\varepsilon}) + \frac{1}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}\right) \\ &+ \frac{1}{K}\cdot\left\{\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega (|\nabla n_{1\varepsilon}|^2 + \varepsilon)^{\frac{p-2}{2}}\frac{|\nabla n_{1\varepsilon}|^2}{n_{1\varepsilon}} + \int_\Omega \frac{|\nabla n_{2\varepsilon}|^2}{n_{2\varepsilon}}\right\} \\ \leq &K\int_\Omega |\nabla u|^2 + C, \end{aligned} \tag{3.12}$$

where $\bar{A} = \frac{\alpha}{\chi_1}$, $\bar{B} = \frac{\beta}{\chi_2}$.

Proof. Testing the first equation of (3.1) by $\ln n_{1\varepsilon}$ and integrating by parts to compute

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (n_{1\varepsilon}\ln n_{1\varepsilon}-n_{1\varepsilon})+\int_\Omega (|\nabla n_{1\varepsilon}|^2+\varepsilon)^{\frac{p-2}{2}}\frac{|\nabla n_{1\varepsilon}|^2}{n_{1\varepsilon}}\\ =&\chi_1\int_\Omega \nabla F_\varepsilon(n_{1\varepsilon})\cdot\nabla c_\varepsilon+\mu_1\int_\Omega n_{1\varepsilon}(1-n_{1\varepsilon}-a_1n_{2\varepsilon})\ln n_{1\varepsilon}\\ =&\chi_1\int_\Omega \nabla F_\varepsilon(n_{1\varepsilon})\cdot\nabla c_\varepsilon+\mu_1\int_\Omega n_{1\varepsilon}(1-n_{1\varepsilon})\ln n_{1\varepsilon}-\mu_1a_1\int_\Omega n_{1\varepsilon}n_{2\varepsilon}\ln n_{1\varepsilon},\end{aligned}$$

for all $t\in(0,T_{\max,\varepsilon})$. Here we use that $s\ln s\geq -\frac{1}{e}$, for all $s\geq 0$ to see that

$$-\mu_1a_1\int_\Omega n_{1\varepsilon}n_{2\varepsilon}\ln n_{1\varepsilon}\leq\frac{\mu_1a_1}{e}\int_\Omega n_{2\varepsilon},$$

for all $t\in(0,T_{\max,\varepsilon})$. Moreover, due to the fact that $(1-n_{1\varepsilon})\ln n_{1\varepsilon}\leq 0$ for all $(x,t)\in\Omega\times(0,T_{\max,\varepsilon})$, we have

$$\mu_1\int_\Omega n_{1\varepsilon}(1-n_{1\varepsilon})\ln n_{1\varepsilon}\leq 0,\qquad \text{for all } t\in(0,T_{\max,\varepsilon}).$$

Furthermore, we see that

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (n_{1\varepsilon}\ln n_{1\varepsilon}-n_{1\varepsilon})+\int_\Omega (|\nabla n_{1\varepsilon}|^2+\varepsilon)^{\frac{p-2}{2}}\frac{|\nabla n_{1\varepsilon}|^2}{n_{1\varepsilon}}\\ \leq&\chi_1\int_\Omega \nabla F_\varepsilon(n_{1\varepsilon})\cdot\nabla c_\varepsilon+\frac{\mu_1a_1}{e}\int_\Omega n_{2\varepsilon}.\end{aligned}\tag{3.13}$$

Proceeding similarly, we obtain

$$\begin{aligned}&\frac{d}{dt}\int_\Omega (n_{2\varepsilon}\ln n_{2\varepsilon}-n_{2\varepsilon})+\int_\Omega \frac{|\nabla n_{2\varepsilon}|^2}{n_{2\varepsilon}}\\ =&\chi_2\int_\Omega \nabla F_\varepsilon(n_{2\varepsilon})\cdot\nabla c_\varepsilon+\mu_2\int_\Omega n_{2\varepsilon}(1-a_2n_{1\varepsilon}-n_{2\varepsilon})\ln n_{2\varepsilon}\\ \leq&\chi_2\int_\Omega \nabla F_\varepsilon(n_{2\varepsilon})\cdot\nabla c_\varepsilon+\frac{\mu_2a_2}{e}\int_\Omega n_{1\varepsilon}.\end{aligned}\tag{3.14}$$

Finally, we have the following inequality

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}+\frac{1}{2(2+\sqrt{3})}\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}\\ \leq&-\int_\Omega \nabla c_\varepsilon\cdot\nabla\left(\alpha F_\varepsilon(n_{1\varepsilon})+\beta F_\varepsilon(n_{2\varepsilon})\right)+\frac{2+\sqrt{3}}{2}s_0\int_\Omega|\nabla u_\varepsilon|^2,\end{aligned}\tag{3.15}$$

for all $t\in(0,T_{\max,\varepsilon})$. Indeed, by a straightforward calculation and integration by parts we see

$$\frac{1}{2}\frac{d}{dt}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}=\frac{1}{2}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}c_{\varepsilon t}-\int_\Omega\frac{\Delta c_\varepsilon}{c_\varepsilon}\cdot c_{\varepsilon t},$$

for all $t\in(0,T_{\max,\varepsilon})$. In view of the third equation of (3.1) and integration by parts, we further have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}\\ =&\frac{1}{2}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}\Delta c_\varepsilon-\int_\Omega\frac{|\Delta c_\varepsilon|^2}{c_\varepsilon}-\frac{1}{2}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}\left(\alpha F_\varepsilon(n_{1\varepsilon})+\beta F_\varepsilon(n_{2\varepsilon})\right)\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \Delta c_\varepsilon\cdot(\alpha F_\varepsilon(n_{1\varepsilon})+\beta F_\varepsilon(n_{2\varepsilon}))-\frac{1}{2}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}u_\varepsilon\cdot\nabla c_\varepsilon+\int_\Omega\frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon\cdot\nabla c_\varepsilon)\\
=&\frac{1}{2}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}\Delta c_\varepsilon-\int_\Omega\frac{|\Delta c_\varepsilon|^2}{c_\varepsilon}-\frac{1}{2}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}\left(\alpha F_\varepsilon(n_{1\varepsilon})+\beta F_\varepsilon(n_{2\varepsilon})\right)\\
&-\int_\Omega\nabla c_\varepsilon\cdot\nabla\left(\alpha F_\varepsilon(n_{1\varepsilon})+\beta F_\varepsilon(n_{2\varepsilon})\right)-\frac{1}{2}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)+\int_\Omega\frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon\cdot\nabla c_\varepsilon),
\end{aligned}
$$

for all $t\in(0,T_{\max,\varepsilon})$. On the other hand, considering the Lemma 3.3 with $\varphi=c_\varepsilon$ and $h(\varphi)=-\frac{1}{\varphi}$ to observe that

$$
-\int_\Omega\frac{|\Delta c_\varepsilon|^2}{c_\varepsilon}=-\int_\Omega\frac{|D^2c_\varepsilon|^2}{c_\varepsilon}-\frac{3}{2}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}\Delta c_\varepsilon+\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\frac{1}{2}\int_{\partial\Omega}\frac{1}{c_\varepsilon}\frac{\partial}{\partial\nu}|\nabla c_\varepsilon|^2.
$$

What's more, we observe that

$$
\begin{aligned}
\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2&=\int_\Omega\frac{1}{c_\varepsilon}|D^2c_\varepsilon|^2-2\int_\Omega\frac{1}{c_\varepsilon^2}(D^2c_\varepsilon\cdot\nabla c_\varepsilon)\cdot\nabla c_\varepsilon+\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_{\varepsilon^3}}\\
&=\int_\Omega\frac{1}{c_\varepsilon}|D^2c_\varepsilon|^2-\int_\Omega\frac{1}{c_\varepsilon^2}\nabla|\nabla c_\varepsilon|^2\cdot\nabla c_\varepsilon+\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_{\varepsilon^3}}\\
&=\int_\Omega\frac{1}{c_\varepsilon}|D^2c_\varepsilon|^2+\int_\Omega\frac{1}{c_\varepsilon^2}|\nabla c_\varepsilon|^2\Delta c_\varepsilon-\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_{\varepsilon^3}}.
\end{aligned}
$$

Combining the above three equations, we further have

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}=&-\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2+\frac{1}{2}\int_{\partial\Omega}\frac{1}{c_\varepsilon}\frac{\partial}{\partial\nu}|\nabla c_\varepsilon|^2\\
&-\frac{1}{2}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon}\left(\alpha F_\varepsilon(n_{1\varepsilon})+\beta F_\varepsilon(n_{2\varepsilon})\right)+\int_\Omega\frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon\cdot\nabla c_\varepsilon)\\
&-\int_\Omega\nabla c_\varepsilon\cdot\nabla\left(\alpha F_\varepsilon(n_{1\varepsilon})+\beta F_\varepsilon(n_{2\varepsilon})\right)-\frac{1}{2}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon).
\end{aligned}
$$

Then applying Lemma 3.3 in [23], the inequality

$$
\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}\le(2+\sqrt{3})\int_\Omega c_\varepsilon|D^2\ln c_\varepsilon|^2
$$

holds for all $t\in(0,T_{\max,\varepsilon})$. Since the convexity of Ω and $\frac{\partial c_\varepsilon}{\partial\nu}=0$ on $\partial\Omega$, it follows that $\frac{\partial}{\partial\nu}|\nabla c_\varepsilon|^2\le0$ on $\partial\Omega$([4]). Moreover because of $\nabla\cdot u_\varepsilon=0$, integration by parts and the Young's inequality we find

$$
\begin{aligned}
&\int_\Omega\frac{\Delta c_\varepsilon}{c_\varepsilon}(u_\varepsilon\cdot\nabla c_\varepsilon)\\
=&\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)-\int_\Omega\frac{\nabla c_\varepsilon}{c_\varepsilon}\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon)-\int_\Omega\frac{u_\varepsilon}{c_\varepsilon}\cdot(D^2c_\varepsilon\cdot\nabla c_\varepsilon)\\
=&\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)-\int_\Omega\frac{\nabla c_\varepsilon}{c_\varepsilon}\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon)-\frac{1}{2}\int_\Omega\frac{u_\varepsilon}{c_\varepsilon}\cdot\nabla|\nabla c_\varepsilon|^2\\
=&\frac{1}{2}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)-\int_\Omega\frac{\nabla c_\varepsilon}{c_\varepsilon}\cdot(\nabla u_\varepsilon\cdot\nabla c_\varepsilon)\\
\le&\frac{1}{2}\int_\Omega\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon)+\frac{1}{2(2+\sqrt{3})}\int_\Omega\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3}+\frac{2+\sqrt{3}}{2}\int_\Omega c_\varepsilon|\nabla u_\varepsilon|^2
\end{aligned}
$$

$$\leq \frac{1}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon^2}(u_\varepsilon\cdot\nabla c_\varepsilon) + \frac{1}{2(2+\sqrt{3})}\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{2+\sqrt{3}}{2}s_0\int_\Omega |\nabla u_\varepsilon|^2.$$

Combined the above discussion with (3.2), we arrive at (3.15).

Finally, considering the consequence of Lemma 3.2 and combining (3.13)-(3.15) with $C = \frac{\bar{A}\mu_1 a_1}{e}\int_\Omega n_{2\varepsilon} + \frac{\bar{B}\mu_2 a_2}{e}\int_\Omega n_{1\varepsilon}$, and $K = \max\left\{\frac{2+\sqrt{3}}{2}s_0, 2(2+\sqrt{3}), \frac{1}{\bar{A}}, \frac{1}{\bar{B}}\right\}$, we arrive at (3.12). □

Lemma 3.5. *Assume that $p \geq 2$. $\bar{A}, \bar{B}$ and K be given in Lemma 3.4. Then for any $\varepsilon \in (0,1)$ there exists $K_0 > 0$ large enough such that*

$$\begin{aligned}
&\frac{d}{dt}\left(\bar{A}\int_\Omega (n_{1\varepsilon}\ln n_{1\varepsilon} - n_{1\varepsilon}) + \bar{B}\int_\Omega (n_{2\varepsilon}\ln n_{2\varepsilon} - n_{2\varepsilon}) + \frac{1}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + K\int_\Omega |u_\varepsilon|^2\right)\\
&+\frac{1}{K_0}\cdot\left\{\int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_\Omega \frac{|\nabla n_{1\varepsilon}|^p}{n_{1\varepsilon}} + \int_\Omega \frac{|\nabla n_{2\varepsilon}|^2}{n_{2\varepsilon}} + \int_\Omega |\nabla u_\varepsilon|^2\right\}\\
&\leq K_0, \qquad (3.16)
\end{aligned}$$

for all $t \in (0, T_{max,\varepsilon})$.

Proof. Testing the fourth equation in (3.1) with u_ε and considering Hölder's inequality, the sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^6(\Omega)$ as well as Minkowski inequality show that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega |u_\varepsilon|^2 + \int_\Omega |\nabla u_\varepsilon|^2\\
=&\int_\Omega (n_{1\varepsilon} + n_{2\varepsilon})\nabla\Phi\cdot u_\varepsilon\\
\leq&\|\nabla\Phi\|_{L^\infty(\Omega)}\|n_{1\varepsilon}+n_{2\varepsilon}\|_{L^{\frac{6}{5}}(\Omega)}\|u_\varepsilon\|_{L^6(\Omega)}\\
\leq&\frac{1}{4}\|\nabla u_\varepsilon\|^2_{L^2(\Omega)} + \|\nabla\Phi\|^2_{L^\infty(\Omega)}\left(\|n_{1\varepsilon}\|^2_{L^{\frac{6}{5}}(\Omega)} + \|n_{2\varepsilon}\|^2_{L^{\frac{6}{5}}(\Omega)}\right), \qquad (3.17)
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$. Here we have used that

$$\begin{aligned}
&\int_\Omega (Y_\varepsilon u_\varepsilon\cdot\nabla)u_\varepsilon\cdot u_\varepsilon = -\int_\Omega (\nabla\cdot Y_\varepsilon u_\varepsilon)|u_\varepsilon|^2 - \frac{1}{2}\int_\Omega Y_\varepsilon u_\varepsilon\cdot\nabla|u_\varepsilon|^2\\
=&-\frac{1}{2}\int_\Omega (\nabla\cdot Y_\varepsilon u_\varepsilon)|u_\varepsilon|^2 = 0,
\end{aligned}$$

since $\nabla\cdot u_\varepsilon = 0$ and $\nabla\cdot(1+\varepsilon A)^{-1}u_\varepsilon = 0$.

Let $\theta := \frac{p-1}{4(2p-3)} \in (0,1)$, then θ satisfies $\frac{5(p-1)}{6p} = \theta(\frac{1}{p} - \frac{1}{3}) + (1-\theta)\frac{p-1}{p}$. An application of the Gagliardo-Nirenberg inequality shows that

$$\begin{aligned}
\|n_{1\varepsilon}\|^2_{L^{\frac{6}{5}}(\Omega)} &= \left\|n_{1\varepsilon}^{\frac{p-1}{p}}\right\|^{\frac{2p}{p-1}}_{L^{\frac{6p}{5(p-1)}}(\Omega)}\\
&\leq C_1\left\|\nabla n_{1\varepsilon}^{\frac{p-1}{p}}\right\|^{\frac{2p}{p-1}\theta}_{L^p(\Omega)}\left\|n_{1\varepsilon}^{\frac{p-1}{p}}\right\|^{\frac{2p}{p-1}(1-\theta)}_{L^{\frac{p}{p-1}}(\Omega)} + C_1\left\|n_{1\varepsilon}^{\frac{p-1}{p}}\right\|^{\frac{2p}{p-1}}_{L^{\frac{p}{p-1}}(\Omega)}\\
&= C_1\left\|\nabla n_{1\varepsilon}^{\frac{p-1}{p}}\right\|^{\frac{2p}{p-1}\theta}_{L^p(\Omega)}\|n_{1\varepsilon}\|^{2(1-\theta)}_{L^1(\Omega)} + C_1\|n_{1\varepsilon}\|^2_{L^1(\Omega)},
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$, with some $C_1 > 0$. Due to our assumption $p \geq 2 > \frac{7}{4}$, we have $\frac{2p}{p-1}\theta < p$. Using the Young's inequality together with (3.9) to estimate

$$\|n_{1\varepsilon}\|_{\frac{6}{5}}^2 \leq \delta \left\|\nabla n_{1\varepsilon}^{\frac{p-1}{p}}\right\|_p^p + C_2 C_\delta, \tag{3.18}$$

for all $t \in (0, T_{\max,\varepsilon})$, with some $C_2 > 0$.

Let $\eta := \frac{1}{4} \in (0,1)$, then η satisfies $\frac{5}{12} = \eta\left(\frac{1}{2} - \frac{1}{3}\right) + \frac{1}{2}(1-\eta)$. Once again we use the Gagliardo-Nirenberg inequality to see that

$$\begin{aligned}
\|n_{2\varepsilon}\|_{L^{\frac{6}{5}}(\Omega)}^2 &= \left\|n_{2\varepsilon}^{\frac{1}{2}}\right\|_{L^{\frac{12}{5}}(\Omega)}^4 \\
&\leq C_3 \left\|\nabla n_{2\varepsilon}^{\frac{1}{2}}\right\|_{L^2(\Omega)}^{4\eta} \left\|n_{2\varepsilon}^{\frac{1}{2}}\right\|_{L^2(\Omega)}^{4(1-\eta)} + C_3 \left\|n_{2\varepsilon}^{\frac{1}{2}}\right\|_{L^2(\Omega)}^4 \\
&= C_3 \left\|\nabla n_{2\varepsilon}^{\frac{1}{2}}\right\|_{L^2(\Omega)}^{4\eta} \|n_{2\varepsilon}\|_{L^1(\Omega)}^{2(1-\eta)} + C_3 \|n_{2\varepsilon}\|_{L^1(\Omega)}^2,
\end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$, with some $C_3 > 0$. Since $4\eta < 2$, we use Young's inequality together with (3.10) to estimate

$$\|n_{2\varepsilon}\|_{L^{\frac{6}{5}}(\Omega)}^2 \leq \vartheta \|\nabla n_{2\varepsilon}^{\frac{1}{2}}\|_{L^2(\Omega)}^2 + C_4 C_\vartheta, \tag{3.19}$$

for all $t \in (0, T_{\max,\varepsilon})$, with some $C_4 > 0$.

Noticing that

$$\int_\Omega (|\nabla n_{1\varepsilon}|^2 + \varepsilon)^{\frac{p-2}{2}} \frac{|\nabla n_{1\varepsilon}|^2}{n_{1\varepsilon}} \geq \int_\Omega \frac{|\nabla n_{1\varepsilon}|^p}{n_{1\varepsilon}}.$$

Then considering the consequence of Lemma 3.4 and combining (3.17)-(3.19) with $\delta = \frac{p^p}{4K\|\nabla\Phi\|_\infty^2 (p-1)^p}$ and $\vartheta = \frac{1}{K\|\nabla\Phi\|_\infty^2}$, we arrive at (3.16). □

We can thereby establish the following consequences.

Lemma 3.6. *Assume that $p \geq 2$. $\bar{A}, \bar{B}$ and K be given in Lemma 3.4. Then for any $\varepsilon \in (0,1)$, we have*

$$\begin{aligned}
&\bar{A}\int_\Omega (n_{1\varepsilon}\ln n_{1\varepsilon} - n_{1\varepsilon}) + \bar{B}\int_\Omega (n_{2\varepsilon}\ln n_{2\varepsilon} - n_{2\varepsilon}) \\
&\quad + \frac{1}{2}\int_\Omega \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + K\int_\Omega |u_\varepsilon|^2 \leq C,
\end{aligned} \tag{3.20}$$

for all $t \in (0, T_{max,\varepsilon})$, and

$$\begin{aligned}
&\int_0^T \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \int_0^T \int_\Omega \left|\nabla n_{1\varepsilon}^{\frac{p-1}{p}}\right|^p + \int_0^T \int_\Omega \left|\nabla n_{2\varepsilon}^{\frac{1}{2}}\right|^2 + \int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \\
&\leq C(T+1), \quad \textit{for any fixed } T \in (0, T_{max,\varepsilon}).
\end{aligned} \tag{3.21}$$

Proof. Set

$$y_\varepsilon(t) := \int_\Omega \left\{\bar{A}(n_{1\varepsilon}\ln n_{1\varepsilon} - n_{1\varepsilon}) + \bar{B}(n_{2\varepsilon}\ln n_{2\varepsilon} - n_{2\varepsilon}) + \frac{1}{2}\frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} + K|u_\varepsilon|^2\right\}(\cdot, t),$$

for all $t \in (0, T_{\max,\varepsilon})$, and

$$h_\varepsilon(t) := \int_\Omega \left\{\frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \left|\nabla n_{1\varepsilon}^{\frac{p-1}{p}}\right|^p + \left|\nabla n_{2\varepsilon}^{\frac{1}{2}}\right|^2 + |\nabla u_\varepsilon|^2\right\}(\cdot, t),$$

for all $t \in (0, T_{\max,\varepsilon})$. Obviously, Lemma 3.5 shows that

$$y_\varepsilon'(t) + \frac{1}{K_0} h_\varepsilon(t) \le K_0 \quad \text{for all} \quad t \in (0, T_{\max,\varepsilon}).$$

Now, we are going to show that $y_\varepsilon(t)$ is dominated by $h_\varepsilon(t)$. Firstly, employing the standard Poincaré inequality, there exists $C_1 > 0$ such that

$$K \int_\Omega |u_\varepsilon|^2 \le C_1 \int_\Omega |\nabla u_\varepsilon|^2 \quad \text{for all } t \in (0, T_{\max,\varepsilon}). \tag{3.22}$$

Obviously, using that $s \ln s \le 2s^{\frac{6}{5}}$, for all $s > 0$ and combining the Young's inequality together with (3.18), we see that

$$\begin{aligned}&\int_\Omega \bar{A}(n_{1\varepsilon} \ln n_{1\varepsilon} - n_{1\varepsilon}) \le \bar{A} \int_\Omega n_{1\varepsilon} \ln n_{1\varepsilon} \\ \le& 2\bar{A} \int_\Omega n_{1\varepsilon}^{\frac{6}{5}} \le C_2 \left\| \nabla n_{1\varepsilon}^{\frac{p-1}{p}} \right\|_{L^p(\Omega)}^p + C_2,\end{aligned} \tag{3.23}$$

with some $C_2 > 0$. Similar to the proof of (3.23), due to the Young's inequality and (3.19), we can also obtain

$$\bar{B} \int_\Omega (n_{2\varepsilon} \ln n_{2\varepsilon} - n_{2\varepsilon}) \le C_3 \left\| \nabla n_{2\varepsilon}^{\frac{1}{2}} \right\|_{L^2(\Omega)}^2 + C_3, \tag{3.24}$$

with some $C_3 > 0$. Finally, according to the Young's inequality and (3.11), there exists C_4 such that

$$\frac{1}{2} \frac{|\nabla c_\varepsilon|^2}{c_\varepsilon} \le \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + \frac{1}{16} \int_\Omega c_\varepsilon \le \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} + C_4. \tag{3.25}$$

(3.22)-(3.25) ensure that $y_\varepsilon \le C_5 h_\varepsilon(t) + C_5$, with some $C_5 > 0$. And then

$$y_\varepsilon'(t) + \frac{1}{2C_5K_0} y_\varepsilon + \frac{1}{2K_0} h_\varepsilon(t) \le K_0 + \frac{1}{2K_0}, \text{ for all } t \in (0, T_{\max,\varepsilon}).$$

By Lemma 2.2, it is easy to see that (3.20) and (3.21) hold. The proof is complete. □

Lemma 3.7. *Assume that $p \ge 2$, $\xi \in (1, 11 - \frac{12}{p}]$, $\zeta \in (1, 5]$. For any fixed $T \in (0, T_{max,\varepsilon})$, there exists $C > 0$ such that the solution of (3.1) satisfies*

$$\int_\Omega n_{1\varepsilon}^\xi + \int_0^T \int_\Omega n_{1\varepsilon}^{\xi+1} + \int_0^T \int_\Omega n_{1\varepsilon}^{\xi-2} |\nabla n_{1\varepsilon}|^p \le C(T+1), \tag{3.26}$$

and

$$\int_\Omega n_{2\varepsilon}^\zeta + \int_0^T \int_\Omega n_{1\varepsilon}^{\zeta+1} + \int_0^T \int_\Omega n_{2\varepsilon}^{\zeta-2} |\nabla n_{2\varepsilon}|^2 \le C(T+1). \tag{3.27}$$

Proof. By Lemma 3.6, we have

$$\int_0^T \int_\Omega |\nabla c_\varepsilon|^4 = \int_0^T \int_\Omega \frac{|\nabla c_\varepsilon|^4}{c_\varepsilon^3} \cdot c_\varepsilon^3 \le s_0^3 C(T+1). \tag{3.28}$$

When multiplying the first equation of (3.1) by $\xi n_{1\varepsilon}^{\xi-1}$, and integrating by parts, due to Young's inequality and (3.3) we can estimate

$$\begin{aligned}&\frac{d}{dt} \int_\Omega n_{1\varepsilon}^\xi + \xi(\xi-1) \int_\Omega n_{1\varepsilon}^{\xi-2} |\nabla n_{1\varepsilon}|^p + \xi\mu_1 \int_\Omega n_{1\varepsilon}^{\xi+1} \\ \le& \frac{d}{dt} \int_\Omega n_{1\varepsilon}^\xi + \xi(\xi-1) \int_\Omega n_{1\varepsilon}^{\xi-2} \left(|\nabla n_{1\varepsilon}|^2 + \varepsilon\right)^{\frac{p-2}{2}} |\nabla n_{1\varepsilon}|^2 + \xi\mu_1 \int_\Omega n_{1\varepsilon}^{\xi+1}\end{aligned}$$

$$
\begin{aligned}
=&\xi(\xi-1)\chi_1\int_\Omega n_{1\varepsilon}F'_\varepsilon(n_{1\varepsilon})n_{1\varepsilon}^{\xi-2}\nabla n_{1\varepsilon}\cdot\nabla c_\varepsilon+\xi\mu_1\int_\Omega n_{1\varepsilon}^\xi-\xi\mu_1 a_1\int_\Omega n_{1\varepsilon}^\xi n_{2\varepsilon}\\
\le&\frac{\xi(\xi-1)\chi_1}{\varepsilon}\int_\Omega n_{1\varepsilon}^{\xi-2}\nabla n_{1\varepsilon}\cdot\nabla c_\varepsilon+\xi\mu_1\int_\Omega n_{1\varepsilon}^\xi\\
\le&\frac{1}{2}\xi(\xi-1)\int_\Omega n_{1\varepsilon}^{\xi-2}|\nabla n_{1\varepsilon}|^p+C_2\int_\Omega n_{1\varepsilon}^{(\xi-2)}|\nabla c_\varepsilon|^{\frac{p}{p-1}}+\xi\mu_1\int_\Omega n_{1\varepsilon}^\xi,
\end{aligned}
$$

with $C_2=2^{\frac{1}{p-1}}(\frac{\chi_1}{\varepsilon})^{\frac{p}{p-1}}\xi(\xi-1)$. Using the Young's inequality again, we further have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega n_{1\varepsilon}^\xi+\frac{1}{2}\xi(\xi-1)\int_\Omega n_{1\varepsilon}^{\xi-2}|\nabla n_{1\varepsilon}|^p+\xi\mu_1\int_\Omega n_{1\varepsilon}^{\xi+1}\\
\le&\frac{\xi\mu_1}{2}\int_\Omega n_{1\varepsilon}^{(\xi-2)\frac{4(p-1)}{3p-4}}+C_3\int_\Omega|\nabla c_\varepsilon|^4+\frac{\xi\mu_1}{4}\int_\Omega n_{1\varepsilon}^{\zeta+1}+C_4|\Omega|,
\end{aligned}
$$

where $C_3=(\frac{2C_2}{\xi\mu_1})^{\frac{3p-4}{p}}$, $C_4=4^\xi\xi\mu_1$. If $\xi\in(1,11-\frac{12}{p}]$, we have $(\xi-2)\frac{4(p-1)}{3p-4}\le\xi+1$, this implies that

$$
\frac{d}{dt}\int_\Omega n_{1\varepsilon}^\xi+\frac{1}{2}\xi(\xi-1)\int_\Omega n_{1\varepsilon}^{\xi-2}|\nabla n_{1\varepsilon}|^p+\frac{1}{4}\xi\mu_1\int_\Omega n_{1\varepsilon}^{\xi+1}\le C_3\int_\Omega|\nabla c_\varepsilon|^4+C_5|\Omega|,
$$

where $C_5=\frac{\xi\mu_1}{2}+C_4$. Considering Lemma 2.3 and (3.28), then there exists $C_6>0$ such that $\int_\Omega n_{1\varepsilon}^\xi\le C_6$, for all $t\in(0,T_{\max,\varepsilon})$ and $\xi\in(1,11-\frac{12}{p}]$. Since $p\ge2$, ensuring that $11-\frac{12}{p}\ge5$, we obtain $\int_\Omega n_{1\varepsilon}^5\le C_6$, for all $t\in(0,T_{\max,\varepsilon})$.

Next we test the second equation from (3.1) by $\zeta n_{2\varepsilon}^{\zeta-1}$ with $\zeta\in(1,5]$, to see by neglecting several non positive contributions and employing (3.3) as well as Young's inequality that

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega n_{2\varepsilon}^\zeta+\zeta(\zeta-1)\int_\Omega n_{2\varepsilon}^{\zeta-2}|\nabla n_{2\varepsilon}|^2+\zeta\mu_2\int_\Omega n_{2\varepsilon}^{\zeta+1}\\
\le&\zeta(\zeta-1)\chi_2\int_\Omega n_{2\varepsilon}F'_\varepsilon(n_{2\varepsilon})n_{2\varepsilon}^{\zeta-2}\nabla n_{2\varepsilon}\cdot\nabla c_\varepsilon+\zeta\mu_2\int_\Omega n_{2\varepsilon}^\zeta\\
\le&\frac{\chi_2}{\varepsilon}\zeta(\zeta-1)\int_\Omega n_{2\varepsilon}^{\zeta-2}\nabla n_{2\varepsilon}\cdot\nabla c_\varepsilon+\zeta\mu_2\int_\Omega n_{2\varepsilon}^\zeta\\
\le&\frac{1}{2}\zeta(\zeta-1)\int_\Omega n_{2\varepsilon}^{\zeta-2}|\nabla n_{2\varepsilon}|^2+\frac{\chi_2^2}{2\varepsilon^2}\zeta(\zeta-1)\int_\Omega n_{2\varepsilon}^{\zeta-2}|\nabla c_\varepsilon|^2+\zeta\mu_2\int_\Omega n_{2\varepsilon}^\zeta.
\end{aligned}
$$

Considering the fact that $2(\zeta-2)\le\zeta+1$, since $\zeta\in(1,5]$, and combining the Young's inequality, we see that

$$
\begin{aligned}
&\frac{\chi_2^2}{2\varepsilon^2}\zeta(\zeta-1)\int_\Omega n_{2\varepsilon}^{\zeta-2}|\nabla c_\varepsilon|^2\\
\le&\frac{1}{2}\zeta\mu_2\int_\Omega n_{2\varepsilon}^{2(\zeta-2)}+C_7\int_\Omega|\nabla c_\varepsilon|^4\\
\le&\frac{1}{2}\zeta\mu_2\int_\Omega n_{2\varepsilon}^{\zeta+1}+C_7\int_\Omega|\nabla c_\varepsilon|^4+\frac{\zeta\mu_2}{2}|\Omega|,
\end{aligned}
$$

and that

$$
\zeta\mu_2\int_\Omega n_{2\varepsilon}^\zeta\le\frac{1}{4}\zeta\mu_2\int_\Omega n_{2\varepsilon}^{\zeta+1}+4^\zeta\zeta\mu_2|\Omega|,
$$

where $C_7 = \frac{8\zeta(\zeta-1)^2\chi_2^4}{\mu_2\varepsilon^4}$. Letting $C_8 = \frac{\zeta\mu_2}{2} + 4^\zeta\zeta\mu_2$, then we have

$$\frac{d}{dt}\int_\Omega n_{2\varepsilon}^\zeta + \frac{1}{2}\zeta(\zeta-1)\int_\Omega n_{2\varepsilon}^{\zeta-2}|\nabla n_{2\varepsilon}|^2 + \frac{1}{4}\zeta\mu_2\int_\Omega n_{2\varepsilon}^{\zeta+1} \le C_7\int_\Omega |\nabla c_\varepsilon|^4 + C_8|\Omega|.$$

Finally, by Lemma 2.3 and (3.28), we also have that there exists $C_{10} > 0$ such that $\int_\Omega n_{2\varepsilon}^5 \le C_{10}$, for all $t \in (0, T_{\max,\varepsilon})$. □

Lemma 3.8. *Assume that $p \ge 2$. For all $\varepsilon \in (0,1)$, the solution of (3.1) is global in time; that is, we have $T_{max,\varepsilon} = \infty$.*

Proof. Assume that $T_{\max,\varepsilon} < \infty$, for some $\varepsilon \in (0,1)$. By Lemma 3.6, it is easy to see that the following inequality holds

$$\int_0^{T_{\max,\varepsilon}}\int_\Omega |\nabla c_\varepsilon(\cdot,t)|^4 \le C_1 \quad \text{and} \quad \int_\Omega |u_\varepsilon(\cdot,t)|^2 \le C_2, \tag{3.29}$$

for all $t \in (0, T_{\max,\varepsilon})$, with some $C_1, C_2 > 0$. Combining (3.26), (3.27) with (3.29), we obtain

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le C_3 \quad \text{for all} \quad t \in (0, T_{\max,\varepsilon}), \tag{3.30}$$

and

$$\|\nabla c_\varepsilon\|_{L^4(\Omega)} \le C_4 \quad \text{for all} \quad t \in (0, T_{\max,\varepsilon}), \tag{3.31}$$

with some $C_3 > 0,\ C_4 > 0$. Because the proof of (3.30) and (3.31) is similar to [22] Lemma 3.9, we refer the reader to it for more details. Noticing the third equation of (3.1) and using the variation-of-constant formula, we can obtain

$$\begin{aligned} c_\varepsilon(t) &= e^{-t}e^{t\Delta}c_{0\varepsilon} \\ &\quad + \int_0^t e^{-(t-s)}e^{(t-s)\Delta}(c_\varepsilon - \alpha F_\varepsilon(n_{1\varepsilon})c_\varepsilon - \beta F_\varepsilon(n_{2\varepsilon})c_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon)(s)ds, \end{aligned}$$

for all $t \in (0, T_{\max,\varepsilon})$, where the $\{e^{t\Delta}\}_{t\ge 0}$ is the Neumann heat semigroup in Ω. For more details of Neumann heat semigroup, please refer to [24]. In conjunction with (3.2), (3.7), (3.26), (3.27), (3.30) and (3.31), we further have

$$\begin{aligned} &\|\nabla c_(\cdot,t)\|_{L^\infty(\Omega)} \\ \le& \left\|e^{-t}\nabla e^{t\Delta}c_{0\varepsilon}\right\|_{L^\infty(\Omega)} \\ &+ \left\|\int_0^t e^{-(t-s)}\nabla e^{(t-s)\Delta}\left(c_\varepsilon - \alpha F_\varepsilon(n_{1\varepsilon})c_\varepsilon - \beta F_\varepsilon(n_{2\varepsilon})c_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon\right)(s)ds\right\|_{L^\infty(\Omega)} \\ \le& C_5 t^{-\frac{1}{2}}e^{-t}\|c_{0\varepsilon}\|_{L^\infty(\Omega)} \\ &+ C_5\int_0^t e^{-(t-s)}(t-s)^{-\frac{1}{2}-\frac{3}{2}\cdot\frac{1}{4}}\|c_\varepsilon - \alpha F_\varepsilon(n_{1\varepsilon})c_\varepsilon - \beta F_\varepsilon(n_2)c_\varepsilon - u_\varepsilon\cdot\nabla c_\varepsilon\|_{L^4(\Omega)}ds \\ \le& C_5\tau^{-\frac{1}{2}}e^{-t}\|c_{0\varepsilon}\|_{L^\infty(\Omega)} \\ &+ C_6\int_0^t e^{-(t-s)}(t-s)^{-\frac{1}{2}-\frac{3}{2}\cdot\frac{1}{4}}\left(1 + \|n_{1\varepsilon}\|_{L^4(\Omega)} + \|n_{2\varepsilon}\|_{L^4(\Omega)} + \|\nabla c_\varepsilon\|_{L^4(\Omega)}\right)ds \\ \le& C_7, \end{aligned} \tag{3.32}$$

for all $t \in (\tau, T_{\max,\varepsilon})$ and any fixed $\tau \in (0, T_{\max,\varepsilon})$, with some $C_5,\ C_6,\ C_7 > 0$ since $1 - \frac{1}{2} - \frac{3}{2}\cdot\frac{1}{4} > 0$.

In what follows, we are in the position to estimate $\|n_{1\varepsilon}\|_{L^\infty(\Omega)}$ and $\|n_{2\varepsilon}\|_{L^\infty(\Omega)}$. We first combine the estimates for Neumann heat semigroup and Young's inequality to obtain that

$$\begin{aligned}
&\|n_{2\varepsilon}(\cdot,t)\|_{L^\infty(\Omega)}\\
=&\left\|e^{(t-\tau)\Delta}n_{2\varepsilon}(\cdot,\tau)-\int_\tau^t \nabla e^{(t-s)\Delta}\left(\chi_2 n_{2\varepsilon}F'_\varepsilon(n_{2\varepsilon})\nabla c_\varepsilon+n_{2\varepsilon}u_\varepsilon\right)ds\right.\\
&\left.+\int_\tau^t e^{(t-s)\Delta}\mu_2 n_{2\varepsilon}(1-a_2n_{1\varepsilon}-n_{2\varepsilon})ds\right\|_{L^\infty(\Omega)}\\
\le& C_8(t-\tau)^{-\frac{3}{2}}\|n_{2\varepsilon}(\cdot,\tau)\|_{L^1(\Omega)}+C_8\int_\tau^t (t-s)^{-\frac{1}{2}-\frac{3}{2}\cdot\frac{1}{4}}(1+\|n_{2\varepsilon}\|_{L^4(\Omega)})ds\\
&+C_8\int_\tau^t(t-s)^{-\frac{3}{2}\cdot\frac{1}{2}}\|\mu_2n_{2\varepsilon}(1-a_2n_{1\varepsilon}-n_{2\varepsilon})\|_{L^2(\Omega)}\\
\le& C_8(t-\tau)^{-\frac{3}{2}}\|n_{2\varepsilon}(\cdot,\tau)\|_{L^1(\Omega)}+C_8\int_\tau^t (t-s)^{-\frac{1}{2}-\frac{3}{2}\cdot\frac{1}{4}}(1+\|n_{2\varepsilon}\|_{L^4(\Omega)})ds\\
&+C_9\int_\tau^t(t-s)^{-\frac{3}{2}\cdot\frac{1}{2}}\left(\|n_{2\varepsilon}\|_{L^2(\Omega)}+\|n_{1\varepsilon}\|^2_{L^4(\Omega)}+\|n_{2\varepsilon}\|^2_{L^4(\Omega)}\right)ds\\
\le& C_{10},
\end{aligned}$$

for all $t\in(2\tau,T_{\max,\varepsilon})$ and any fixed $\tau\in(0,T_{\max,\varepsilon})$, with some $C_8,\ C_9,\ C_{10}>0$, since (3.10), (3.26), (3.27), (3.30) and (3.32) hold.

Once again, we multiply the first equation in (3.1) by $\xi n_{1\varepsilon}^{\xi-1}$, $\xi>1$ to see upon integrating by parts that again by Young's inequality, (3.3) and (3.32)

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega n_{1\varepsilon}^\xi+\xi(\xi-1)\int_\Omega n_{1\varepsilon}^{\xi-2}|\nabla n_{1\varepsilon}|^p+\xi\mu_1\int_\Omega n_{1\varepsilon}^{\xi+1}\\
\le&\frac{d}{dt}\int_\Omega n_{1\varepsilon}^\xi+\xi(\xi-1)\int_\Omega n_{1\varepsilon}^{\xi-2}\left(|\nabla n_{1\varepsilon}|^2+\varepsilon\right)^{\frac{p-2}{2}}|\nabla n_{1\varepsilon}|^2+\xi\mu_1\int_\Omega n_{1\varepsilon}^{\xi+1}\\
=&\xi(\xi-1)\chi_1\int_\Omega n_{1\varepsilon}F'_\varepsilon(n_{1\varepsilon})n_{1\varepsilon}^{\xi-2}\nabla n_{1\varepsilon}\cdot\nabla c_\varepsilon+\xi\mu_1\int_\Omega n_{1\varepsilon}^\xi-\xi\mu_1a_1\int_\Omega n_{1\varepsilon}^\xi n_{2\varepsilon}\\
\le& C_{11}\int_\Omega n_{1\varepsilon}^{\xi-2}|\nabla n_{1\varepsilon}|+\xi\mu_1\int_\Omega n_{1\varepsilon}^\xi\\
\le&\int_\Omega n_{1\varepsilon}^{\xi-2}\left(\xi(\xi-1)|\nabla n_{1\varepsilon}|^p+C_{12}\right)+\frac{1}{2}\xi\mu_1\int_\Omega n_{1\varepsilon}^{\xi+1}+C_{13}\\
\le&\xi(\xi-1)\int_\Omega n_{1\varepsilon}^{\xi-2}|\nabla n_{1\varepsilon}|^p+\frac{3}{4}\xi\mu_1\int_\Omega n_{1\varepsilon}^{\xi+1}+C_{14},
\end{aligned}$$

for all $t\in(\tau,T_{\max,\varepsilon})$. Then by Lemma 2.3, we have $\int_\Omega n_{1\varepsilon}^\xi\le C_{15}$, for all $t\in(\tau,T_{\max,\varepsilon})$, with any $\xi>1$ and some $C_{15}>0$.

Finally, based on a Moser-type iteration method, we achieve L^∞ bounds for $n_{1\varepsilon}$. Taking $r_k=2r_{k-1}+2-p$, $k=\{1,2,3\cdots\cdots\}$ and $r_0>p$ is a positive constant. It is not hard to see that $\{r_k\}_{k\in N}$ is a nonnegative strictly increasing sequence, $r_k>p$, for all k and $r_k\nearrow\infty$, as $k\to\infty$. What's more, there exist $c_1,c_2>0$, which are independent of k such that

$$c_12^k\le r_k\le c_22^k,\ \text{for all } k\in N. \tag{3.33}$$

Letting

$$M_k := \sup_{t\in(\tau,T_{\max,\varepsilon})} \int_\Omega \widehat{n_{1\varepsilon}}^{r_k}(x,t)dx,\ k\in N,\ \text{where } \widehat{n_{1\varepsilon}} := \max\{n_{1\varepsilon}(x,t),1\},$$

for all $x \in \bar{\Omega}$ and $t \in (\tau, T_{\max,\varepsilon})$.

We multiply the first equation of (3.1) by $r_k n_{1\varepsilon}^{r_k-1}$ and invoke the Young's inequality along with (3.3) and (3.32) to see that

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega n_{1\varepsilon}^{r_k} + r_k(r_k-1)\int_\Omega n_{1\varepsilon}^{r_k-2}\left(|\nabla n_{1\varepsilon}|^2+\varepsilon\right)^{\frac{p-2}{2}}|\nabla n_{1\varepsilon}|^2 \\
&+ r_k\mu_1\int_\Omega n_{1\varepsilon}^{r_k+1} + \int_\Omega n_{1\varepsilon}^{r_k} \\
\le& r_k(r_k-1)\chi_1\int_\Omega n_{1\varepsilon}^{r_k-1}F'_\varepsilon(n_{1\varepsilon})\nabla c_\varepsilon\cdot\nabla n_{1\varepsilon} + (r_k\mu_1+1)\int_\Omega n_{1\varepsilon}^{r_k} \\
\le& C_{16}r_k(r_k-1)\int_\Omega n_{1\varepsilon}^{r_k-1}|\nabla n_{1\varepsilon}| + (r_k\mu_1+1)\int_\Omega n_{1\varepsilon}^{r_k} \\
\le& C_{16}r_k(r_k-1)\int_\Omega \left(\frac{1}{2C_{16}}n_{1\varepsilon}^{r_k-2}|\nabla n_{1\varepsilon}|^p\right. \\
&\left.+\frac{p-1}{p}\left(\frac{p}{2C_{16}}\right)^{-\frac{1}{p-1}}\cdot n_{1\varepsilon}^{(\frac{2-r_k}{p}+r_k-1)\frac{p}{p-1}}\right) + (r_k\mu_1+1)\int_\Omega n_{1\varepsilon}^{r_k} \\
\le& \frac{1}{2}r_k(r_k-1)\int_\Omega n_{1\varepsilon}^{r_k-2}|\nabla n_{1\varepsilon}|^p + C_{17}r_k(r_k-1)\int_\Omega n_{1\varepsilon}^{r_k+p'-2} \\
&+ (r_k\mu_1+1)\int_\Omega n_{1\varepsilon}^{r_k},
\end{aligned}$$

for all $t \in (\tau, T_{\max,\varepsilon})$. Taking $\theta_k := 2\cdot\frac{r_k+p-2}{r_k+p'-2} \ge 2$, $\lambda_k := \frac{2(r_k+p-2)}{r_k} \ge 2$ and considering the Hölder inequality, there must exist $C_{18} > 0$ fulfilling

$$\begin{aligned}
&\frac{d}{dt}\int_\Omega n_{1\varepsilon}^{r_k} + \frac{1}{2}r_k(r_k-1)\left(\frac{p}{r_k-2+p}\right)^p\int_\Omega\left|\nabla n_{1\varepsilon}^{\frac{r_k-2+p}{p}}\right|^p + \int_\Omega n_{1\varepsilon}^{r_k} \\
=&\frac{d}{dt}\int_\Omega n_{1\varepsilon}^{r_k} + \frac{1}{2}r_k(r_k-1)\int_\Omega n_{1\varepsilon}^{r_k-2}|\nabla n_{1\varepsilon}|^p + \int_\Omega n_{1\varepsilon}^{r_k} \\
\le& C_{18}r_k(r_k-1)\left(\int_\Omega n_{1\varepsilon}^{(r_k+p'-2)\theta_k}\right)^{\frac{1}{\theta_k}} + C_{18}r_k(r_k-1)\left(\int_\Omega n_{1\varepsilon}^{r_k\lambda_k}\right)^{\frac{1}{\lambda_k}},
\end{aligned} \tag{3.34}$$

for all $t \in (\tau, T_{\max,\varepsilon})$, since $r_k - 1 > 1$, for all k. Using the Gagliardo-Nirenberg inequality and the Young's inequality, there exists $C_{19} > 0$ such that

$$\begin{aligned}
&C_{18}\left(\int_\Omega n_{1\varepsilon}^{(r_k-2+p')\theta_k}\right)^{\frac{1}{\theta_k}} \\
=&C_{18}\left\|n_{1\varepsilon}^{\frac{r_k+p-2}{p}}\right\|_{L^{2p}(\Omega)}^{\frac{2p}{\theta_k}} \\
\le& C_{19}\left\|\nabla n_{1\varepsilon}^{\frac{r_k+p-2}{p}}\right\|_{L^p(\Omega)}^{\frac{2pa}{\theta_k}}\cdot\left\|n_{1\varepsilon}^{\frac{r_k+p-2}{p}}\right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{2p(1-a)}{\theta_k}} + C_{19}\left\|n_{1\varepsilon}^{\frac{r_k+p-2}{p}}\right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{2p}{\theta_k}} \\
=&C_{19}\left\|\nabla n_{1\varepsilon}^{\frac{r_k+p-2}{p}}\right\|_{L^p(\Omega)}^{\frac{2pa}{\theta_k}}\cdot\left\|n_{1\varepsilon}^{r_{k-1}}\right\|_{L^1(\Omega)}^{\frac{4(1-a)}{\theta_k}} + C_{19}\left\|n_{1\varepsilon}^{r_{k-1}}\right\|_{L^1(\Omega)}^{\frac{4}{\theta_k}}
\end{aligned}$$

$$
\begin{aligned}
&\le C_{19} M_{k-1}^{\frac{4(1-a)}{\theta_k}} \left(\int_\Omega \left| \nabla n_{1\varepsilon}^{\frac{r_k+p-2}{p}} \right|^p \right)^{\frac{2a}{\theta_k}} + C_{19} M_{k-1}^{\frac{4}{\theta_k}} \\
&\le C_{19} \eta \int_\Omega \left| \nabla n_{1\varepsilon}^{\frac{r_k+p-2}{p}} \right|^p + C_{19} C_\eta \left(M_{k-1}^{\frac{4(1-a)}{\theta_k}} \right)^{\frac{\theta_k}{\theta_k - 2a}} + C_{19} M_{k-1}^{\frac{4}{\theta_k}}, \qquad (3.35)
\end{aligned}
$$

for all $t \in (\tau, T_{\max,\varepsilon})$, with $a = \frac{9}{2p+6} \in (0,1)$ satisfying $\frac{1}{2p} = (\frac{1}{p} - \frac{1}{3})a + \frac{2}{p}(1-a)$.

Similarly, we also have

$$
\begin{aligned}
&C_{18} \left(\int_\Omega n_{1\varepsilon}^{r_k \lambda_k} \right)^{\frac{1}{\lambda_k}} \\
=&C_{18} \left\| n_{1\varepsilon}^{\frac{r_k+p-2}{p}} \right\|_{L^{2p}(\Omega)}^{\frac{2p}{\lambda_k}} \\
\le& C_{20} \left\| \nabla n_{1\varepsilon}^{\frac{r_k+p-2}{p}} \right\|_{L^p(\Omega)}^{\frac{2pa}{\lambda_k}} \cdot \left\| n_{1\varepsilon}^{\frac{r_k+p-2}{p}} \right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{2p(1-a)}{\lambda_k}} + C_{20} \left\| n_{1\varepsilon}^{\frac{r_k+p-2}{p}} \right\|_{L^{\frac{p}{2}}(\Omega)}^{\frac{2p}{\lambda_k}} \\
=&C_{20} \left\| \nabla n_{1\varepsilon}^{\frac{r_k+p-2}{p}} \right\|_{L^p(\Omega)}^{\frac{2pa}{\lambda_k}} \cdot \| n_{1\varepsilon}^{r_{k-1}} \|_{L^1(\Omega)}^{\frac{4(1-a)}{\lambda_k}} + C_{20} \| n_{1\varepsilon}^{r_{k-1}} \|_{L^1(\Omega)}^{\frac{4}{\lambda_k}} \\
\le& C_{20} M_{k-1}^{\frac{4(1-a)}{\lambda_k}} \left(\int_\Omega \left| \nabla n_{1\varepsilon}^{\frac{r_k+p-2}{p}} \right|^p \right)^{\frac{2a}{\lambda_k}} + C_{20} M_{k-1}^{\frac{4}{\lambda_k}} \\
\le& C_{20} \delta \int_\Omega \left| \nabla n_{1\varepsilon}^{\frac{r_k+p-2}{p}} \right|^p + C_{20} C_\delta \left(M_{k-1}^{\frac{4(1-a)}{\lambda_k}} \right)^{\frac{\lambda_k}{\lambda_k - 2a}} + C_{20} M_{k-1}^{\frac{4}{\lambda_k}}, \qquad (3.36)
\end{aligned}
$$

for all $t \in (\tau, T_{\max,\varepsilon})$, with some $C_{27} > 0$. Taking $\eta = \frac{1}{4C_{19}} (\frac{p}{r_k+p-2})^p, \delta = \frac{1}{4C_{20}} (\frac{p}{r_k+p-2})^p$ and substituting (3.35) and (3.36) into (3.34), we obtain

$$
\begin{aligned}
\frac{d}{dt} \int_\Omega n_{1\varepsilon}^{r_k} + \int_\Omega n_{1\varepsilon}^{r_k} \le& C_{19} C_\eta r_k (r_k - 1) M_{k-1}^{\frac{4(1-a)}{\theta_k - 2a}} + C_{19} r_k (r_k - 1) M_{k-1}^{\frac{4}{\theta_k}} \\
&+ C_{20} C_\delta r_k (r_k - 1) M_{k-1}^{\frac{4(1-a)}{\lambda_k - 2a}} + C_{20} r_k (r_k - 1) M_{k-1}^{\frac{4}{\lambda_k}},
\end{aligned}
$$

where

$$
C_\eta = \left(\eta \frac{\theta_k}{2a} \right)^{-\frac{1}{\frac{\theta_k}{2a} - 1}} \cdot \frac{\frac{\theta_k}{2a} - 1}{\frac{\theta_k}{2a}} = \frac{\theta_k - 2a}{\theta_k} \left(\frac{\theta_k}{2a} \eta \right)^{-\frac{2a}{\theta_k - 2a}}.
$$

Letting

$$
\tilde{b} = 2^{\frac{ap}{1-a}} > 1,
$$

then by (3.33) we have

$$
\begin{aligned}
C_\eta &\le \eta^{-\frac{2a}{\theta_k - 2a}} \\
&= (4C_{19})^{\frac{2a}{\theta_k - 2a}} \left(\frac{r_k + p - 2}{p} \right)^{\frac{2a}{\theta_k - 2a} p} \\
&\le (4C_{19})^{\frac{2a}{\theta_k - 2a}} \left(r_k^{\frac{2a}{\theta_k - 2a} p} + 1 \right) \\
&\le (4C_{19})^{\frac{a}{1-a}} 2 r_k^{\frac{a}{1-a} p} \\
&\le C_{21} \tilde{b}^k.
\end{aligned}
$$

A quite similar computation gives

$$C_\delta \le C_{22}\tilde{b}^k,$$

with some $C_{22} > 0$. Noticing that $\frac{4(1-a)}{\theta_k - 2a}, \frac{4}{\theta_k}, \frac{4(1-a)}{\lambda_k - 2a}$ and $\frac{4}{\lambda_k} < 2$, then we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega n_{1\varepsilon}^{r_k} + \int_\Omega n_{1\varepsilon}^{r_k} \le & C_{19}C_{21}r_k(r_k-1)\tilde{b}^k M_{k-1}^{\frac{4(1-a)}{\theta_k-2a}} + C_{19}r_k(r_k-1)M_{k-1}^{\frac{4}{\theta_k}} \\ & + C_{20}C_{22}r_k(r_k-1)\tilde{b}^k M_{k-1}^{\frac{4(1-a)}{\lambda_k-2a}} + C_{20}r_k(r_k-1)M_{k-1}^{\frac{4}{\lambda_k}} \\ \le & 2C_{19}C_{21}r_k(r_k-1)\tilde{b}^k M_{k-1}^2 + 2C_{20}C_{22}r_k(r_k-1)\tilde{b}^k M_{k-1}^2.\end{aligned}$$

An integration of this ODI shows that

$$M_k \le \max\left\{\int_\Omega n_{01\varepsilon}^{r_k}, \left(2C_{19}C_{21} + 2C_{20}C_{22}\right) r_k(r_k-1)\tilde{b}^k M_{k-1}^2\right\}.$$

On the one hand, if

$$\left(2C_{19}C_{21} + 2C_{20}C_{22}\right) r_k(r_k-1)\tilde{b}^k M_{k-1}^2 < \int_\Omega n_{01\varepsilon}^{r_k},$$

holding for infinitely many $k \ge 1$, we see

$$\sup_{t\in(0,\infty)} \left(\int_\Omega n_{1\varepsilon}^{r_{k-1}}\right)^{\frac{1}{r_{k-1}}} \le \left(\int_\Omega n_{01\varepsilon}^{r_k}\right)^{\frac{1}{2r_{k-1}}},$$

for all such k, and hence conclude that $\|n_{1\varepsilon}(t)\|_{L^\infty(\Omega)} \le \|n_{01\varepsilon}\|_{L^\infty(\Omega)}$, for all $t > 0$.

On the other hand, in the opposite case, upon enlarging C_{19} and C_{20} if necessary we have that $M_k \le \left(2C_{19}C_{21} + 2C_{20}C_{22}\right) r_k(r_k-1)\tilde{b}^k M_{k-1}^2$, for all $k \ge 1$. Let $C_{23} := \left(2C_{19}C_{21} + 2C_{20}C_{22}\right) c_2^2$, then from the definition of $\tilde{b}$ and (3.33), we see

$$\begin{aligned}M_k &\le \left(2C_{19}C_{21} + 2C_{20}C_{22}\right) r_k^2 \tilde{b}^k M_{k-1}^2 \\ &\le \left(2C_{19}C_{21} + 2C_{20}C_{22}\right) (c_2 2^k)^2 \tilde{b}^k M_{k-1}^2 \\ &\le C_{23}(4\tilde{b})^k M_{k-1}^2,\end{aligned}$$

for all $k \ge 1$. Furthermore, we have

$$M_k \le C_{23}^{\sum_{j=0}^{k-1} 2^j} (4\tilde{b})^{\sum_{j=0}^{k-1}(k-j)2^j} M_0^{2^k},$$

for all $k \ge 1$. A straight calculation shows that $\sum_{j=0}^{k-1} 2^j = 2^k - 1 \le 2^k$ and $\sum_{j=0}^{k-1}(k-j)2^j = 2^{k+1} - 2 - k \le 2^{k+1}$, then we have

$$M_k \le C_{23}^{2^k} (4\tilde{b})^{2^{k+1}} M_0^{2^k}.$$

Finally, combining the definition of M_k, we concludes that

$$\|n_{1\varepsilon}(t)\|_{L^\infty(\Omega)}^{c_1} \le C_{24} \cdot \|n_{01\varepsilon}\|_{L^\infty(\Omega)}, \quad \text{for all } \; t \in (\tau, T_{\max,\varepsilon}),$$

with $C_{24} = C_{23}(4\tilde{b})^2$, which confirms $\|n_{1\varepsilon}\|_{L^\infty(\Omega)}$ is bounded. By Lemma 3.1, $T_{\max,\varepsilon} = \infty$. □

Lemma 3.9. *There exists $C > 0$ such that for any $\varepsilon \in (0,1)$, the solution of (3.1) satisfies*

$$\int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} \le C(T+1), \tag{3.37}$$

for all $T > 0$. And

$$\int_0^T \int_\Omega n_{1\varepsilon}^r \leq C(T+1), \tag{3.38}$$

for all $T > 0$, where $r \in [1, \frac{4p}{3} + 3]$. And

$$\int_0^T \int_\Omega n_{2\varepsilon}^m \leq C(T+1), \tag{3.39}$$

for all $T > 0$, where $m \in [1, \frac{17}{3}]$.

Proof. Fixing $C_1 > 0$ and $C_2 > 0$ such that in accordance with Lemma 3.6, we have that

$$\int_\Omega |u_\varepsilon|^2 \leq C_1(T+1), \tag{3.40}$$

and that

$$\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 \leq C_2(T+1). \tag{3.41}$$

Then we combine the Gagliardo-Nirenberg inequality with Poincaré inequality to fix $C_3, C_4 > 0$ such that

$$\begin{aligned}
\|u_\varepsilon\|_{L^{\frac{10}{3}}((0,T)\times\Omega)}^{\frac{10}{3}} &= \int_0^T \|u_\varepsilon\|_{L^{\frac{10}{3}}(\Omega)}^{\frac{10}{3}} \\
&\leq C_3 \int_0^T \|\nabla u_\varepsilon\|_{L^2(\Omega)}^{\frac{10}{3}\cdot\frac{3}{5}} \|u_\varepsilon\|_{L^2(\Omega)}^{\frac{10}{3}\cdot\frac{2}{5}} \\
&\leq C_4 \int_0^T \|\nabla u_\varepsilon\|_{L^2(\Omega)}^2 \\
&\leq C_2 C_4 (T+1).
\end{aligned}$$

Recalling Lemma 3.7, particularly, when $\xi = 5$, we have

$$\int_0^T \int_\Omega \left|\nabla n_{1\varepsilon}^{\frac{3+p}{p}}\right|^p = \left(\frac{3+p}{p}\right)^p \int_0^T \int_\Omega n_{1\varepsilon}^3 |\nabla n_{1\varepsilon}|^p \leq C_5(T+1),$$

with some $C_5 > 0$. Taking $\tilde{a} = \frac{\frac{p+3}{pr} - \frac{(p+3)r}{pr}}{\frac{1}{p} - \frac{1}{3} - \frac{p+3}{p}} = \frac{3(p+3)(r-1)}{(4p+6)r}$, when $r \leq \frac{4p}{3} + 3$, we see that $\tilde{a} \in [0, 1)$ and $\frac{pr}{p+3} \cdot \tilde{a} \leq p$. Thus, invoking the Gagliardo-Nirenberg inequality along with (3.9), we obtain $C_6 > 0$ and $C_7 > 0$ such that

$$\begin{aligned}
\int_0^T \|n_{1\varepsilon}\|_{L^r(\Omega)}^r &= \int_0^T \left\| n_{1\varepsilon}^{\frac{p+3}{p}} \right\|_{L^{\frac{pr}{3+p}}(\Omega)}^{\frac{pr}{3+p}} \\
&\leq C_6 \int_0^T \left\| \nabla n_{1\varepsilon}^{\frac{p+3}{p}} \right\|_{L^p(\Omega)}^{\frac{pr}{p+3}\cdot\tilde{a}} \left\| n_{1\varepsilon}^{\frac{p+3}{p}} \right\|_{L^{\frac{p}{p+3}}(\Omega)}^{\frac{pr}{p+3}\cdot(1-\tilde{a})} + C_6 \int_0^T \left\| n_{1\varepsilon}^{\frac{p+3}{p}} \right\|_{L^{\frac{p}{p+3}}(\Omega)}^{\frac{pr}{p+3}} \\
&\leq C_6 \int_0^T \left\| \nabla n_{1\varepsilon}^{\frac{p+3}{p}} \right\|_{L^p(\Omega)}^p + C_6 \int_0^T \left(\|n_{1\varepsilon}\|_{L^1(\Omega)}^{r(1-\tilde{a})\sigma} + \|n_{1\varepsilon}\|_{L^1(\Omega)}^r \right) \\
&\leq C_7(T+1),
\end{aligned}$$

with $\sigma = \frac{p+3}{p+3-r\tilde{a}}$.

Again by Lemma 3.7 we also have

$$\int_0^T \int_\Omega \left|\nabla n_{2\varepsilon}^{\frac{5}{2}}\right|^2 = (\frac{5}{2})^2 \int_0^T n_{2\varepsilon}^3 |\nabla n_{2\varepsilon}|^2 \le C_8(T+1),$$

with some $C_8 > 0$. Upon another application of the Gagliardo-Nirenberg inequality in precisely the same way we see

$$\begin{aligned}\int_0^T \|n_{2\varepsilon}\|_{L^m(\Omega)}^m &= \int_0^T \left\|n_{2\varepsilon}^{\frac{5}{2}}\right\|_{L^{\frac{2m}{5}}(\Omega)}^{\frac{2m}{5}} \\ &\le C_9 \int_0^T \left\|\nabla n_{2\varepsilon}^{\frac{5}{2}}\right\|_{L^2(\Omega)}^{\frac{2m}{5}\cdot\hat{a}} \left\|n_{2\varepsilon}^{\frac{5}{2}}\right\|_{L^{\frac{2}{5}}(\Omega)}^{\frac{2m}{5}\cdot(1-\hat{a})} + C_9 \int_0^T \left\|n_{2\varepsilon}^{\frac{5}{2}}\right\|_{L^{\frac{2}{5}}(\Omega)}^{\frac{2m}{5}} \\ &\le C_9 \int_0^T \left\|\nabla n_{2\varepsilon}^{\frac{5}{2}}\right\|_{L^2(\Omega)}^{\frac{2m}{5}\cdot\hat{a}} \|n_{2\varepsilon}\|_{L^1(\Omega)}^{m\cdot(1-\hat{a})} + C_9 \int_0^T \|n_{2\varepsilon}\|_{L^1(\Omega)}^m ,\end{aligned}$$

where $\hat{a} = \frac{\frac{5}{2m} - \frac{5}{2}}{\frac{1}{2} - \frac{1}{3} - \frac{5}{2}} = \frac{15}{14}(1 - \frac{1}{m})$. Since $m \le \frac{17}{3}$, we have $\frac{2m}{5} \cdot \hat{a} \le 2$. Then employing (3.10) and Young's inequality, we have

$$\int_0^T \|n_{2\varepsilon}\|_{L^m(\Omega)}^m \le C_{10}(T+1).$$

□

Lemma 3.10. *There exists $C > 0$ such that for any $\varepsilon \in (0,1)$ and any $T > 0$, the solution of (3.1) satisfies*

$$\int_0^T \|\partial_t n_{1\varepsilon}\|_{(W^{1,p}(\Omega))^*}^{p'} \le C(T+1), \tag{3.42}$$

$$\int_0^T \|\partial_t n_{2\varepsilon}(\cdot,t)\|_{(W^{1,\frac{5}{2}}(\Omega))^*}^{\frac{5}{3}} \le C(T+1), \tag{3.43}$$

$$\int_0^T \|\partial_t c_\varepsilon(\cdot,t)\|_{(W^{1,\frac{10}{7}}(\Omega))^*}^{\frac{10}{3}} \le C(T+1), \tag{3.44}$$

$$\int_0^T \|\partial_t u_\varepsilon(\cdot,t)\|_{(W_{0,\sigma}^{1,5}(\Omega))^*}^{\frac{5}{4}} \le C(T+1). \tag{3.45}$$

Proof. Testing the first equation in (3.1) by any $\varphi \in C^\infty(\bar{\Omega})$, and integrating by parts, due to Hölder's inequality, and (3.3) we can estimate

$$\begin{aligned}&\left|\int_\Omega \partial_t n_{1\varepsilon}(\cdot,t)\varphi\right| \\ =&\left|-\int_\Omega \left(|\nabla n_{1\varepsilon}|^2+\varepsilon\right)^{\frac{p-2}{2}} \nabla n_{1\varepsilon}\cdot\nabla\varphi + \int_\Omega \chi_1 n_{1\varepsilon} F_\varepsilon'(n_{1\varepsilon})\nabla c_\varepsilon\cdot\nabla\varphi + \int_\Omega n_{1\varepsilon}u_\varepsilon\cdot\nabla\varphi\right. \\ &\left.+\int_\Omega \mu_1 n_{1\varepsilon}\left(1-n_{1\varepsilon}-a_1 n_{2\varepsilon}\right)\right| \\ \le& \int_\Omega \left(|\nabla n_{1\varepsilon}|^2+\varepsilon\right)^{\frac{p-2}{2}} |\nabla n_{1\varepsilon}\|\,\nabla\varphi| + C_1\|n_{1\varepsilon}\nabla c_\varepsilon\|_{L^{p'}(\Omega)}\|\nabla\varphi\|_{L^p(\Omega)} \\ &+\|n_{1\varepsilon}u_\varepsilon\|_{L^{p'}(\Omega)}\|\nabla\varphi\|_{L^p(\Omega)} + \|\mu_1 n_{1\varepsilon}(1-n_{1\varepsilon}-a_1 n_{2\varepsilon})\|_{L^{p'}(\Omega)}\|\varphi\|_{L^p(\Omega)} \\ \le& C_1\left(\|(|\nabla n_{1\varepsilon}|^2+\varepsilon)^{\frac{p-1}{2}}\|_{L^{p'}(\Omega)} + \|n_{1\varepsilon}\nabla c_\varepsilon\|_{L^{p'}(\Omega)} + \|n_{1\varepsilon}u_\varepsilon\|_{L^{p'}(\Omega)}\right.\end{aligned}$$

$$+\|\mu_1 n_{1\varepsilon}(1-n_{1\varepsilon}-a_1 n_{2\varepsilon})\|_{L^{p'}(\Omega)}\Big)\|\nabla\varphi\|_{L^p(\Omega)},$$

with some $C_1>0$. Then we use Young's inequality (3.26), (3.27), (3.28), (3.37) and the fact that $\frac{1}{5}+\frac{1}{4}<\frac{1}{5}+\frac{3}{10}<\frac{1}{p'}$ to see

$$\begin{aligned}
&\int_0^T \|\partial_t n_{1\varepsilon}(\cdot,t)\|^{p'}_{(W^{1,p}(\Omega))^*}\\
\le& C_2\left(\int_0^T\int_\Omega \left(|\nabla n_{1\varepsilon}|^2+1\right)^{\frac{(p-1)p'}{2}}+\int_0^T\int_\Omega |n_{1\varepsilon}\nabla c_\varepsilon|^{p'}+\int_0^T\int_\Omega |n_{1\varepsilon}u_\varepsilon|^{p'}\right.\\
&\left.+\int_0^T\int_\Omega (|n_{1\varepsilon}|^{p'}+|n_{1\varepsilon}^2|^{p'}+|n_{1\varepsilon}n_{2\varepsilon}|^{p'})\right)\\
\le& C_3\left(\int_0^T\int_\Omega (|\nabla n_{1\varepsilon}|^p+1)+\int_0^T\int_\Omega |n_{1\varepsilon}|^5+\int_0^T\int_\Omega |\nabla c_\varepsilon|^4\right.\\
&\left.+\int_0^T\int_\Omega |n_{1\varepsilon}|^5+\int_0^T\int_\Omega |u_\varepsilon|^{\frac{10}{3}}+\int_0^T\int_\Omega |n_{2\varepsilon}|^5+|\Omega|T\right)\le C(T+1).
\end{aligned}$$

By Lemma 3.6, we see there exists $C_4>0$ such that

$$\int_0^T\int_\Omega \frac{|\nabla n_{2\varepsilon}|^2}{n_{2\varepsilon}}\le C_4(T+1).$$

Upon the Hölder's inequality, we furthermore see that

$$\begin{aligned}
\int_0^T\int_\Omega |\nabla n_{2\varepsilon}|^{\frac{5}{3}}=&\int_0^T\int_\Omega \left|\left(\frac{|\nabla n_{2\varepsilon}|^2}{n_{2\varepsilon}}\right)^{\frac{5}{6}}\cdot n_{2\varepsilon}^{\frac{5}{6}}\right|\\
\le&\left(\int_0^T\int_\Omega \frac{|\nabla n_{2\varepsilon}|^2}{n_{2\varepsilon}}\right)^{\frac{5}{6}}\left(\int_0^T\int_\Omega n_{2\varepsilon}^5\right)^{\frac{1}{6}}\\
\le& C_5(T+1),
\end{aligned}\tag{3.46}$$

with some $C_5>0$. We next test the second equation in (3.1) by any $\varphi\in C^\infty(\bar\Omega)$, and integrate by parts, due to Hölder's inequality and (3.3) we can obtain

$$\begin{aligned}
&\left|\int_\Omega \partial_t n_{2\varepsilon}(\cdot,t)\varphi\right|\\
=&\left|-\int_\Omega \nabla n_{2\varepsilon}\cdot\nabla\varphi+\int_\Omega \chi_2 n_{2\varepsilon}F'_\varepsilon(n_{2\varepsilon})\nabla c_\varepsilon\cdot\nabla\varphi+\int_\Omega n_{2\varepsilon}u_\varepsilon\cdot\nabla\varphi\right.\\
&\left.+\mu_2\int_\Omega n_{2\varepsilon}\left(1-a_2n_{1\varepsilon}-n_{2\varepsilon}\right)\varphi\right|\\
\le& C_6\Big(\|\nabla n_{2\varepsilon}\|_{L^{\frac{5}{3}}(\Omega)}+\|n_{2\varepsilon}\nabla c_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}\\
&+\|n_{2\varepsilon}u_\varepsilon\|_{L^{\frac{5}{3}}(\Omega)}+\|n_{2\varepsilon}(1-a_2n_{1\varepsilon}-n_{2\varepsilon})\|_{L^{\frac{5}{3}}(\Omega)}\Big)\|\varphi\|_{W^{1,\frac{5}{2}}(\Omega)}.
\end{aligned}$$

We notice that $\frac{1}{5}+\frac{1}{4}<\frac{1}{5}+\frac{3}{10}<\frac{3}{5}$. By the Young's inequality, we can find $C_7>0$ such that

$$\int_0^T \|\partial_t n_{2\varepsilon}(\cdot,t)\|^{\frac{5}{3}}_{(W^{1,\frac{5}{2}}(\Omega))^*}$$

$$
\begin{aligned}
\le & C_6 \int_0^T \int_\Omega |\nabla n_{2\varepsilon}|^{\frac{5}{3}} + \int_0^T \int_\Omega |n_{2\varepsilon} \nabla c_\varepsilon|^{\frac{5}{3}} \\
& + \int_0^T \int_\Omega |n_{2\varepsilon} u_\varepsilon|^{\frac{5}{3}} + \int_0^T \int_\Omega |n_{2\varepsilon}(1 - a_2 n_{1\varepsilon} - n_{2\varepsilon})|^{\frac{5}{3}} \\
\le & C_7 \int_0^T \int_\Omega |\nabla n_{2\varepsilon}|^{\frac{5}{3}} + \int_0^T \int_\Omega |n_{2\varepsilon}|^5 + \int_0^T \int_\Omega |\nabla c_\varepsilon|^4 \\
& + \int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} + \int_0^T \int_\Omega |n_{1\varepsilon}|^5 + |\Omega| T.
\end{aligned}
$$

Recalling (3.26), (3.27), (3.28) with (3.37), we obtain (3.43).

Likewise, given any $\varphi \in C^\infty(\bar{\Omega})$, testing the third equation in (3.1) by φ to obtain that

$$
\begin{aligned}
& \left| \int_\Omega \partial_t c_\varepsilon(\cdot, t) \varphi \right| \\
= & \left| - \int_\Omega \nabla c_\varepsilon \cdot \nabla \varphi - \int_\Omega \left(\alpha F_\varepsilon(n_{1\varepsilon}) c_\varepsilon + \beta F_\varepsilon(n_{2,\varepsilon}) \right) c_\varepsilon \varphi + \int_\Omega c_\varepsilon u_\varepsilon \cdot \nabla \varphi \right| \\
\le & C_8 \left(\|\nabla c_\varepsilon\|_{L^{\frac{10}{3}}(\Omega)} + \|n_{1\varepsilon} c_\varepsilon\|_{L^{\frac{10}{3}}(\Omega)} + \|n_{2\varepsilon} c_\varepsilon\|_{L^{\frac{10}{3}}(\Omega)} + \|c_\varepsilon u_\varepsilon\|_{L^{\frac{10}{3}}(\Omega)} \right) \|\varphi\|_{W^{1,\frac{10}{7}}(\Omega)}.
\end{aligned}
$$

Thereafter we use (3.11) to see

$$
\begin{aligned}
& \int_0^T \|\partial_t c_\varepsilon(\cdot, t)\|_{(W^{1,\frac{10}{7}}(\Omega))^*}^{\frac{10}{3}} \\
\le & C_8 \int_0^T \int_\Omega |\nabla c_\varepsilon|^{\frac{10}{3}} + \int_0^T \int_\Omega |n_{1\varepsilon} c_\varepsilon|^{\frac{10}{3}} + \int_0^T \int_\Omega |n_{2\varepsilon} c_\varepsilon|^{\frac{10}{3}} + \int_0^T \int_\Omega |c_\varepsilon u_\varepsilon|^{\frac{10}{3}} \\
\le & C_9 \int_0^T \int_\Omega |\nabla c_\varepsilon|^4 + \int_0^T \int_\Omega |n_{1\varepsilon}|^5 + \int_0^T \int_\Omega |n_{2\varepsilon}|^5 + \int_0^T \int_\Omega |u_\varepsilon|^{\frac{10}{3}} + |\Omega| T.
\end{aligned}
$$

Combining (3.26), (3.27) (3.28) and (3.37) entail (3.44).

Finally, noticing that $\|Y_\varepsilon v\|_{L^2(\Omega)} \le \|v\|_{L^2(\Omega)}$ for all $v \in L^2_\sigma(\Omega)$ and the Young's inequality as well as that $\nabla \Phi \in L^\infty(\Omega)$, there exist constants $C_{10}, C_{11} > 0$ such that for any $\varphi \in (C^\infty_{0,\sigma}(\Omega); R^3)$

$$
\begin{aligned}
& \int_0^T \|\partial_t u_\varepsilon(\cdot, t)\|_{(W^{1,5}_{0,\sigma}(\Omega))^*}^{\frac{5}{4}} dt \\
\le & C_{10} \int_0^T \int_\Omega |\nabla u_\varepsilon|^{\frac{5}{4}} + \int_0^T \int_\Omega |Y_\varepsilon u_\varepsilon \otimes u_\varepsilon|^{\frac{5}{4}} + \int_0^T \int_\Omega |(n_{1\varepsilon} + n_{2\varepsilon}) \nabla \Phi|^{\frac{5}{4}} \\
\le & C_{11} (\int_0^T \int_\Omega |\nabla u_\varepsilon|^2 + \int_0^T \int_\Omega |Y_\varepsilon u_\varepsilon|^2 + \int_0^T \int_\Omega u_\varepsilon^{\frac{10}{3}} \\
& + \int_0^T \int_\Omega n_{1\varepsilon}^5 + \int_0^T \int_\Omega n_{2\varepsilon}^5 + |\Omega| T) \le C(T+1).
\end{aligned}
$$

The proof is complete. □

4. **The main result and its proof.** In this section, we are going to proof the existence of weak solution for the problem (1.1), (1.3) and (1.4). With the above compactness properties at hand, by means of a standard extraction procedure we can now derive the following theorem which actually is our main result.

Theorem 4.1. *Let $\Omega \subset R^3$ be a bounded convex domain with smooth boundary. Suppose that the assumptions (1.5) and (1.6) hold. Then for $p \geq 2$, there exists at least one global weak solution (in the sense of Definition 1.1) of (1.1),(1.3) and (1.4).*

Proof. If $(n_{1\varepsilon}, n_{2\varepsilon}, c_\varepsilon, u_\varepsilon)$ is the global solution of (3.1), considering Lemma 3.6-Lemma 3.10 to see that

$$\|n_{1\varepsilon}\|_{L^p_{\text{loc}}([0,\infty);W^{1,p}(\Omega))} \leq C(T+1), \tag{4.1}$$

$$\|(n_{1\varepsilon})_t\|_{L^{p'}_{\text{loc}}([0,\infty);(W^{1,p}(\Omega))^*)} \leq C(T+1), \tag{4.2}$$

$$\|n_{2\varepsilon}\|_{L^{\frac{5}{3}}_{\text{loc}}([0,\infty);W^{1,\frac{5}{3}}(\Omega))} \leq C(T+1), \tag{4.3}$$

$$\|(n_{2\varepsilon})_t\|_{L^{\frac{5}{3}}_{\text{loc}}([0,\infty);(W^{1,\frac{5}{2}}(\Omega))^*)} \leq C(T+1), \tag{4.4}$$

and that

$$\|c_\varepsilon\|_{L^4_{\text{loc}}([0,\infty);W^{1,4}(\Omega))} \leq C(T+1), \tag{4.5}$$

$$\|(c_\varepsilon)_t\|^{\frac{10}{3}}_{L^{\frac{10}{3}}_{\text{loc}}([0,T);(W^{1,\frac{10}{7}}(\Omega))^*)} \leq C(T+1), \tag{4.6}$$

$$\|u_\varepsilon\|_{L^2_{\text{loc}}([0,\infty);W^{1,2}(\Omega))} \leq C(T+1), \tag{4.7}$$

$$\|u_\varepsilon\|_{L^{\frac{5}{4}}_{\text{loc}}([0,\infty);(W^{1,5}_{0,\sigma}(\Omega))^*)} \leq C(T+1). \tag{4.8}$$

According to the Aubin-Lions lemma ([9]), there exists $(\varepsilon_j)_{j\in N} \in (0,1)$ such that $\varepsilon_j \searrow 0$ as $j \to \infty$, and such that as $\varepsilon = \varepsilon_j \searrow 0$, we have

$$n_{1\varepsilon} \to n_1, \qquad \text{in } L^p_{\text{loc}}(\Omega \times [0,\infty)) \text{ and a.e. in } \Omega \times [0,\infty), \tag{4.9}$$

$$n_{2\varepsilon} \to n_2, \qquad \text{in } L^{\frac{5}{3}}_{\text{loc}}(\Omega \times [0,\infty)) \text{ and a.e. in } \Omega \times [0,\infty), \tag{4.10}$$

$$c_\varepsilon \to c, \qquad \text{in } L^4_{\text{loc}}(\Omega \times [0,\infty)) \text{ and a.e. in } \Omega \times [0,\infty), \tag{4.11}$$

$$u_\varepsilon \to u, \qquad \text{in } L^2_{\text{loc}}(\Omega \times [0,\infty)) \text{ and a.e. in } \Omega \times [0,\infty). \tag{4.12}$$

Considering Lemma 3.7-Lemma 3.9, we also have

$$n_{1\varepsilon} \rightharpoonup n_1, \ \text{ in } L^r_{\text{loc}}(\Omega \times [0,\infty)), \ \text{ for } \ r \in [1, \frac{4p}{3}+3], \tag{4.13}$$

$$\nabla n_{1\varepsilon}^{\frac{\xi-2+p}{p}} \rightharpoonup \nabla n_1^{\frac{\xi-2+p}{p}}, \ \text{ in } L^p_{\text{loc}}(\Omega \times [0,\infty)), \ \text{ for } \ \xi \in (1, 11 - \frac{12}{p}], \tag{4.14}$$

$$n_{2\varepsilon} \rightharpoonup n_2, \ \text{ in } L^{\frac{17}{3}}_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.15}$$

$$\nabla n_{2\varepsilon}^{\frac{\zeta}{2}} \rightharpoonup \nabla n_2^{\frac{\zeta}{2}}, \ \text{ in } L^2_{\text{loc}}(\Omega \times [0,\infty)), \ \text{ for } \ \zeta \in (1,5], \tag{4.16}$$

as well as

$$c_\varepsilon \overset{*}{\rightharpoonup} c, \ \text{ in } L^\infty(\Omega \times [0,\infty)), \tag{4.17}$$

$$\nabla c_\varepsilon \rightharpoonup \nabla c, \ \text{ in } L^4_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.18}$$

$$u_\varepsilon \rightharpoonup u, \quad \text{in } L^{\frac{10}{3}}_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.19}$$

$$\nabla u_\varepsilon \rightharpoonup \nabla u, \quad \text{in } L^2_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.20}$$

as $\varepsilon \searrow 0$. Combining the Lemma 6.1 and Lemma 6.2 in [19], we further see

$$n_{1\varepsilon} \to n_1, \quad \text{in } L^5_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.21}$$

$$n_{2\varepsilon} \to n_2, \quad \text{in } L^5_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.22}$$

$$c_\varepsilon u_\varepsilon \to cu, \quad \text{in } L^1_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.23}$$

$$n_{1\varepsilon} u_\varepsilon \to n_1 u, \quad \text{in } L^1_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.24}$$

and

$$n_{2\varepsilon} u_\varepsilon \to n_2 u, \quad \text{in } L^1_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.25}$$

$$Y_\varepsilon u_\varepsilon \otimes u_\varepsilon \to u \otimes u, \quad \text{in } L^1_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.26}$$

$$n_{1\varepsilon} F'_\varepsilon(n_{1\varepsilon}) \nabla c_\varepsilon \to n_1 \nabla c, \quad \text{in } L^2_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.27}$$

$$n_{2\varepsilon} F'_\varepsilon(n_{2\varepsilon}) \nabla c_\varepsilon \to n_2 \nabla c, \quad \text{in } L^2_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.28}$$

as well as

$$(\alpha F_\varepsilon(n_{1\varepsilon}) + \beta F_\varepsilon(n_{2\varepsilon}))c_\varepsilon \to (\alpha n_1 + \beta n_2)c, \quad \text{in } L^1_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.29}$$

$$(|\nabla n_{1\varepsilon}|^2 + \varepsilon)^{\frac{p-2}{2}} \nabla n_{1\varepsilon} \to |\nabla n_1|^{p-2} \nabla n_1, \quad \text{in } L^{\frac{p}{p-1}}_{\text{loc}}(\Omega \times [0,\infty)), \tag{4.30}$$

as $\varepsilon = \varepsilon_j \searrow 0$. According to these convergence properties, by using the standard arguments and letting $\varepsilon = \varepsilon_j \searrow 0$ in each term of the natural weak formulation of (3.1) separately. Then we can verify that (n_1, n_2, c, u) is a weak solution of (1.1), (1.3) and (1.4). The proof is complete. □

Acknowledgment. The authors would like to express their deep thanks to the referee's valuable suggestions for the revision and improvement of the manuscript.

REFERENCES

[1] X. Bai and M. Winkler, Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics, *Indiana Univ. Math. J.*, **65** (2016), 553–583.

[2] N. Bellomo, A. Bellouquid, Y. Tao and M. Winkler, Toward a mathematical theory of Keller-Segel models of pattern formation in biological tissues, *Math. Models Methods Appl. Sci.*, **25** (2015), 1663–1763.

[3] X. Cao, S. Kurima and M. Mizukami, Global existence and asymptotic behavior of classical solutions for a 3D two-species chemotaxis-Stokes system with competitive kinetics, *Math. Meth. Appl. Sci.*, **41** (2018), 3138–3154.

[4] R. Dal Passo, H. Garcke and G. Grün, On a fourth-order degenerate parabolic equation: global entropy estimates, existence, and qualitative behavior of solutions, *SIAM J. Math. Anal.*, **29** (1998), 321–342.

[5] K. Fujie, A. Ito and T. Yokota, Existence and uniqueness of local classical solutions to modified tumor invasion models of Chaplain-Anderson type, *Adv. Math. Sci. Appl.*, **24** (2014), 67–84.

[6] M. Hirata, S. Kurima, M. Mizukami and T. Yokota, Boundedness and stabilization in a two-dimensional two-species chemotaxis-Navier-Stokes system with competitive kinetics, *J. Differential Equations*, **263** (2017), 470–490.

[7] C. Jin, Boundedness and global solvability to a chemotaxis-haptotaxis model with slow and fast diffusion, *Discrete Contin. Dyn. Syst. Ser. B*, **23** (2018), 1675–1688.

[8] H.-Y. Jin and T. Xiang, Convergence rates of solutions for a two-species chemotaxis-Navier-Stokes sytstem with competitive kinetics, *Discrete Contin. Dyn. Syst. Ser. B*, **24** (2019), 1919–1942.

[9] J.-L. Lions, *Quelques méthodes de résolution des problémes aux limites non linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[10] J. Liu, Boundedness in a chemotaxis-Navier-Stokes System modeling coral fertilization with slow p-Laplacian diffusion, *J. Math. Fluid Mech.*, **22** (2020), No. 10, 31 pp.

[11] C. Liu and P. Li, Global existence for a chemotaxis-haptotaxis model with p-Laplacian, *Commun. Pure Appl. Anal.*, **19** (2020), 1399–1419.

[12] C. Liu and P. Li, Boundedness and global solvability for a chemotaxis-haptotaxis model with p-Laplacian diffusion, *Electron. J. Differential Equations*, (2020), Paper No. 16, 16 pp.

[13] C. Liu and P. Li, Time periodic solutions for a two-species chemotaxis-Navier-Stokes system, *Discrete and Continuous Dynamical Systems Series B*, **26** (2021), 4567–4585.

[14] T. Miyakawa and H. Sohr, On energy inequality, smoothness and large time behaviour in L^2 for weak solutions of the Navier-Stokes equations in exterior domains, *Math. Z.*, **199** (1988), 455–478.

[15] M. Mizukami, Boundedness and asymptotic stability in a two-species chemotaxis-competition model with signal-dependent sensitivity, *Discrete Contin. Dyn. Syst. Ser. B*, **22** (2017), 2301–2319.

[16] L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **13** (1959), 115–162.

[17] H. Sohr, *The Navier-Stokes Equations. An Elementary Functional Analytic Approach*, Birkhäuser, Basel, 2001.

[18] C. Stinner, C. Surulescu and M. Winkler, Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion, *SIAM J. Math. Anal.*, **46** (2014), 1969–2007.

[19] W. Tao and Y. Li, Global weak solutions for the three-dimensional chemotaxis-Navier-Stokes system with slow p-Laplacian diffusion, *Nonlinear Anal. Real World Appl.*, **45** (2019), 26–52.

[20] W. Tao and Y. Li, Boundedness of weak solutions of a chemotaxis-Stokes system with slow p-Laplacian diffusion, *J. Differential Equations*, **268** (2020), 6872–6919.

[21] I. Tuval, L. Cisneros, C. Dombrowski, C. W. Wolgemuth, J. O. Kessler and R. E. Goldstein, Bacterial swimming and oxygen transport near contact lines, *Proc. Natl. Acad. Sci. USA*, **102** (2005), 2277–2282.

[22] M. Winkler, Global weak solutions in a three-dimensional chemotaxis-Navier-Stokes system, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **33** (2016), 1329–1352.

[23] M. Winkler, Global large-data solutions in a chemotaxis-(Navier-)Stokes system modeling cellular swimming in fluid drops, *Comm. Partial Differential Equations*, **37** (2012), 319–351.

[24] M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, *J. Differential Equations*, **248** (2010), 2889–2905.

[25] M. Winkler, How far do chemotaxis-driven forces influence regularity in the Navier-Stokes system? *Trans. Amer. Math. Soc.*, **369** (2017), 3067–3125.

Received February 2021; revised May 2021; early access July 2021.

E-mail address: 1834406995@qq.com
E-mail address: liucc@jlu.edu.cn