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ASYMPTOTIC BEHAVIOR OF WEAK AND STRONG SOLUTIONS OF THE MAGNETOHYDRODYNAMIC EQUATIONS

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Abstract. We prove some results on the stability of slow stationary solutions of the MHD equations in two- and three-dimensional bounded domains for external force fields that are asymptotically autonomous. Our results show that weak solutions are asymptotically stable in time in the L^2-norm. Further, assuming certain regularity hypotheses on the problem data, strong solutions are asymptotically stable in the H^1 and H^2-norms.

1. **Introduction.** In several situations the motion of incompressible electrical conducting fluid can be modeled by the magnetohydrodynamic (MHD) equations, which correspond to the Navier-Stokes equations coupled with the Maxwell equations. In the presence of a free motion of heavy ions, not directly due to the electrical field (see Schlüter [19] and Pikelner [15]), the MHD equations can be reduced to

$$\begin{aligned} \frac{\partial \boldsymbol{u}}{\partial t} - \frac{\eta}{\rho}\Delta \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u} - \frac{\mu}{\rho}\boldsymbol{h}\cdot\nabla\boldsymbol{h} &= \boldsymbol{f} - \frac{1}{\rho}\nabla\left(p^* + \frac{\mu}{2}\boldsymbol{h}^2\right), \\ \frac{\partial \boldsymbol{h}}{\partial t} - \frac{1}{\mu\sigma}\Delta \boldsymbol{h} + \boldsymbol{u}\cdot\nabla\boldsymbol{h} - \boldsymbol{h}\cdot\nabla\boldsymbol{u} &= -\operatorname{grad}\omega, \\ \operatorname{div}\boldsymbol{u} = \operatorname{div}\boldsymbol{h} = 0, \end{aligned} \tag{1}$$

together with the following boundary and initial conditions:

$$\begin{aligned} &\boldsymbol{u}(x,t) = 0,\ \boldsymbol{h}(x,t) = 0, \quad \text{on } \partial\Omega\times(0,T), \\ &\boldsymbol{u}(x,0) = \boldsymbol{u}_0(x),\ \boldsymbol{h}(x,0) = \boldsymbol{h}_0(x), \quad \text{in } \Omega. \end{aligned} \tag{2}$$

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In the previous expressions, $\boldsymbol{u}$ and $\boldsymbol{h}$ are respectively the unknown velocity and magnetic field; p^* is the unknown hydrostatic pressure; w is an unknown function related to the heavy ions (in such a way that the density of electric current, j_0, generated by this motion satisfies the relation $\mathrm{rot} j_0 = -\sigma \nabla \omega$), ρ is the density of mass of the fluid (assumed to be a positive constant); $\mu > 0$ is the constant magnetic permeability of the medium; $\sigma > 0$ is the constant electric conductivity; $\eta > 0$ is the constant viscosity of the fluid and $\boldsymbol{f}$ is a given external force field.

Due to its importance, the MHD system has been discussed in a broad variety of studies encompassing subjects such as the existence of weak solutions and strong solutions, uniqueness and regularity criteria. See e.g. [6], [13], [12], [14], [7], [17], [18] and the references therein.

In the present work we discuss the stability of stationary solutions of the MHD equations in two- and three-dimensional bounded domains with respect to both initial conditions and external forcing variations. Under certain regularity hypotheses on the problem data, we establish in Theorem 3.2 the aforementioned stability in the L^2-norm for weak slow flow stationary solutions. Additionally, in Theorem 4.4 and Theorem 5.2 we discuss respectively the H^1-stability and the H^2-stability for strong solutions. We note that, for a fixed given external force field, our results in particular imply the asymptotic stability of such stationary solutions.

The issue of stability of solutions is an important one, since solutions of any dynamical system are thought to be physically reasonable only if they are stable. There exists a number of ways in which stability can be examined. In past years, many efforts have been made to study the asymptotic behavior of classical Navier-Stokes equations. We refer the reader to Heywood and Rannacher [11], Beirão da Veiga [2], Qu and Wang [16], Zhang [21] and the references therein.

A few of the references mentioned above, e.g. [8], are closely related to the contents of this paper. In effect, it was shown in [8] that, under condition (52) stated in Section 5 below, the strong solution of the two dimensional Navier-Stokes equation is asymptotically stable in a bounded domain of $\mathbb{R}^2$. In this paper we establish the corresponding result for the magnetohydrodynamic equations, assuming instead condition (34) below, which is weaker than the used in [8], both in two dimensional and three dimensional domains. Further, under hypotheses (52) we will show that stability actually holds in the H^2-norm. Thus, our results improve the existing ones even for Navier-Stokes equations.

2. Notation and preliminaries.

We will consider the usual Sobolev spaces $W^{m,q}(\Omega) = \{f \in L^q(\Omega); \|\partial^\alpha f\|_{L^q(\Omega)} < +\infty, |\alpha| \le m\}$, for $m = 0, 1, 2, \ldots,$ $1 \le q \le +\infty$, with the usual norm. When $q = 2$, we write $H^m(\Omega) = W^{m,2}(\Omega)$ and set $H_0^m(\Omega) =$ closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. The L^q-norm is denoted by $\|\cdot\|_{L^q(\Omega)}$. When $q = 2$, the L^2-norm is denoted by $\|\cdot\|$ and the associated inner product in $L^2(\Omega)$ by $(\cdot,\cdot)$.

If X is a Banach space, we denote by $L^q(0,T;X)$ the Banach space of the X-valued functions defined in the interval $[0,T]$ that are L^q-integrable in the sense of Bochner. In addition, vector spaces will be denoted by boldface letters.

We also consider the following spaces of divergence free functions:

$$\begin{aligned} \boldsymbol{C}_{0,\sigma}^\infty(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{C}_0^\infty(\Omega) : \operatorname{div} \boldsymbol{v} = 0\}, \\ \boldsymbol{H} &= \text{closure of } \boldsymbol{C}_{0,\sigma}^\infty(\Omega) \text{ in } \boldsymbol{L}^2(\Omega), \\ \boldsymbol{V} &= \text{closure of } \boldsymbol{C}_{0,\sigma}^\infty(\Omega) \text{ in } \boldsymbol{H}^1(\Omega). \end{aligned}$$

Throughout the paper, the Helmholtz projection P is the orthogonal projection from $\boldsymbol{L}^2(\Omega)$ into $\boldsymbol{H}$ and $A = -P\Delta$ with $D(A) = \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)$ is the usual Stokes operator. We observe that, by the regularity of the Stokes operator, it is usually assumed that Ω is of class C^3 in order to apply Cattabriga's results [5]. However, we use the stronger results of Amrouche and Girault [1], which imply, in particular, that when $A\boldsymbol{u} \in \boldsymbol{L}^2(\Omega)$, then $\boldsymbol{u} \in \boldsymbol{H}^2(\Omega)$ and $\|\boldsymbol{u}\|_{\boldsymbol{H}^2(\Omega)}$ and $\|A\boldsymbol{u}\|$ are equivalent norms when Ω is of Class $C^{1,1}$.

For ease of reference, we also recall the following inequalities which are consequences of the Sobolev and Hölder inequalities:

Lemma 2.1. *Let $\Omega \subseteq \mathbb{R}^3$ be bounded. Then*
(a) There is a constant $C_L > 0$ such that for any $\boldsymbol{u} \in \boldsymbol{V}$

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)} \le C_L \|\nabla \boldsymbol{u}\|. \tag{3}$$

(b) If each integral makes sense, for $p, q, r > 0$ and $1/p + 1/q + 1/r = 1$, we have

$$|(\boldsymbol{u} \cdot \nabla \boldsymbol{v}, \boldsymbol{w})| \le 3^{\frac{1}{p}+\frac{1}{r}} \|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^q(\Omega)} \|\boldsymbol{w}\|_{\boldsymbol{L}^r(\Omega)}. \tag{4}$$

We also need the following regularity result for the Stokes problem (see Temam [20])

$$\begin{array}{lcll} -\mu\Delta \boldsymbol{v} + \nabla \eta & = & \boldsymbol{g}, & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} & = & 0, & \text{in } \Omega, \\ \boldsymbol{v} & = & 0, & \text{on } \partial\Omega. \end{array} \tag{5}$$

Proposition 1. *Let Ω be an open set in $\mathbb{R}^n$ of class C^r, where $n = 2$ or 3 and $r = \max(m+2, 2)$ for some integer $m \ge -1$, and let $\boldsymbol{g} \in \boldsymbol{W}^{m,q}(\Omega)$, $1 < q < \infty$. Then there exist unique functions $\boldsymbol{v}$ and η (to be precise, η is unique up to a constant) that are solutions of (5) and satisfy*

$$\boldsymbol{v} \in \boldsymbol{W}^{m+2,q}(\Omega), \quad \eta \in W^{m+1,q}(\Omega)$$

with

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{m+2,q}(\Omega)} + \|\eta\|_{W^{m+1,q}(\Omega)/\mathbb{R}} \le C\|\boldsymbol{g}\|_{\boldsymbol{W}^{m,q}(\Omega)}$$

where C is a constant depending on q, μ, m, Ω.

2.1. **Mathematical setting of the problem.** By applying the Helmholtz operator P to both sides of the first equation in problem (1), and by taking into account the previous considerations, one obtains the operational form of the problem:

$$\begin{cases} \alpha \boldsymbol{u}_t + \nu A\boldsymbol{u} + \alpha P(\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - P(\boldsymbol{h} \cdot \nabla \boldsymbol{h}) = \alpha P\boldsymbol{f}, \\ \boldsymbol{h}_t + \gamma A\boldsymbol{h} + P(\boldsymbol{u} \cdot \nabla \boldsymbol{h}) - P(\boldsymbol{h} \cdot \nabla \boldsymbol{u}) = 0, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \\ \boldsymbol{h}(0) = \boldsymbol{h}_0. \end{cases} \tag{6}$$

Here we have set

$$\alpha = \rho/\mu, \quad \nu = \eta/\mu \quad \text{and} \quad \gamma = 1/(\mu\sigma).$$

The associated variational formulation is the following: to find $(\boldsymbol{u}, \boldsymbol{h})$ in suitable functional spaces such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\boldsymbol{h}(0) = \boldsymbol{h}_0$ and, for every $(\boldsymbol{v}, \boldsymbol{b}) \in \boldsymbol{V} \times \boldsymbol{V}$, the following holds:

$$\begin{cases} \alpha(\boldsymbol{u}_t, \boldsymbol{v}) + \nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \alpha(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{v}) - (\boldsymbol{h} \cdot \nabla \boldsymbol{h}, \boldsymbol{v}) = (\alpha \boldsymbol{f}, \boldsymbol{v}) \\ (\boldsymbol{h}_t, \boldsymbol{b}) + \gamma(\nabla \boldsymbol{h}, \nabla \boldsymbol{b}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{h}, \boldsymbol{b}) - (\boldsymbol{h} \cdot \nabla \boldsymbol{u}, \boldsymbol{b}) = 0. \end{cases} \tag{7}$$

The corresponding stationary system in operational form is

$$\begin{cases} \nu A\boldsymbol{u}_\infty + \alpha P(\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty = \alpha P\boldsymbol{f}_\infty + P(\boldsymbol{h}_\infty \cdot \nabla \boldsymbol{h}_\infty), \\ \gamma A\boldsymbol{h}_\infty + P(\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{h}_\infty - P(\boldsymbol{h}_\infty \cdot \nabla)\boldsymbol{u}_\infty = 0. \end{cases} \tag{8}$$

In this last system we considered a time-independent external force field $\boldsymbol{f}_\infty$, possibly different from the previous $\boldsymbol{f}$, because we want to check also the stability associated to changes in the external force field.

This last problem, in its associated variational formulation becomes: find $(\boldsymbol{u}_\infty, \boldsymbol{h}_\infty) \in \boldsymbol{V} \times \boldsymbol{V}$ such that, for every $(\boldsymbol{v}, \boldsymbol{b}) \in \boldsymbol{V} \times \boldsymbol{V}$, the following holds:

$$\left\{ \begin{array}{l} \nu(\nabla \boldsymbol{u}_\infty, \nabla \boldsymbol{v}) + \alpha(\boldsymbol{u}_\infty \cdot \nabla \boldsymbol{u}_\infty, \boldsymbol{v}) - (\boldsymbol{h}_\infty \cdot \nabla \boldsymbol{h}_\infty, \boldsymbol{v}) = \alpha(\boldsymbol{f}_\infty, \boldsymbol{v}), \\ \gamma(\nabla \boldsymbol{h}_\infty, \nabla \boldsymbol{b}) + (\boldsymbol{u}_\infty \cdot \nabla \boldsymbol{h}_\infty, \boldsymbol{b}) - (\boldsymbol{h}_\infty \cdot \nabla \boldsymbol{u}_\infty, \boldsymbol{b}) = 0. \end{array} \right. \tag{9}$$

We call such pair $(\boldsymbol{u}_\infty, \boldsymbol{h}_\infty)$ a **weak solution** of the stationary problem (9) (or (8)).

By using the Galerkin method, it is possible to show the following result on existence of weak solutions of (9) (see Chizhonkov [6]):

Proposition 2. *Problem (9) admits at least one weak solution $(\boldsymbol{u}_\infty, \boldsymbol{h}_\infty) \in \boldsymbol{V} \times \boldsymbol{V}$. Further, it satisfies the estimate*

$$\frac{\nu}{2} \| \nabla \boldsymbol{u}_\infty \|^2 + \gamma \| \nabla \boldsymbol{h}_\infty \|^2 \leq \frac{\alpha^2}{\nu} \| \boldsymbol{f}_\infty \|^2_{\boldsymbol{V}^*}. \tag{10}$$

Under smallness conditions, we also have uniqueness of such solutions:

Proposition 3. *(Uniqueness) Any stationary weak solution satisfying the conditions*

$$\frac{\sqrt{3}\, C_L}{\nu} \left(\alpha \| \boldsymbol{u}_\infty \|_{\boldsymbol{L}^3(\Omega)} + \| \boldsymbol{h}_\infty \|_{\boldsymbol{L}^3(\Omega)} \right) < 1, \tag{11}$$

$$\frac{\sqrt{3}\, C_L}{\gamma} \left(\| \boldsymbol{u}_\infty \|_{\boldsymbol{L}^3(\Omega)} + \| \boldsymbol{h}_\infty \|_{\boldsymbol{L}^3(\Omega)} \right) < 1, \tag{12}$$

where $0 < C_L$ is the constant appearing in (3), is unique.

Proof. (of Proposition 3) Let $(\boldsymbol{u}^1_\infty, \boldsymbol{h}^1_\infty)$ be a slow-flow solution of (9), that is, a weak solution satisfying (11) and (12) , and let $(\boldsymbol{u}^2_\infty, \boldsymbol{h}^2_\infty)$ be another tentative weak solution of (9). By setting $\boldsymbol{u} = \boldsymbol{u}^1_\infty - \boldsymbol{u}^2_\infty$ and $\boldsymbol{h} = \boldsymbol{h}^1_\infty - \boldsymbol{h}^2_\infty$, we have

$$\nu\, (\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) + \alpha(\boldsymbol{u} \cdot \nabla \boldsymbol{u}^1_\infty, \boldsymbol{v}) + \alpha(\boldsymbol{u}^2_\infty \cdot \nabla \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{h} \cdot \nabla \boldsymbol{h}^1_\infty, \boldsymbol{v}) + (\boldsymbol{h}^2_\infty \cdot \nabla \boldsymbol{h}, \boldsymbol{v}),$$
$$\gamma\, (\nabla \boldsymbol{h}, \nabla \boldsymbol{b}) + (\boldsymbol{u} \cdot \nabla \boldsymbol{h}^1_\infty, \boldsymbol{b}) + (\boldsymbol{u}^2_\infty \cdot \nabla \boldsymbol{h}, \boldsymbol{b}) - (\boldsymbol{h} \cdot \nabla \boldsymbol{u}^1_\infty, \boldsymbol{b}) - (\boldsymbol{h}^2_\infty \cdot \nabla \boldsymbol{u}, \boldsymbol{b}) = 0\,.$$

We take $\boldsymbol{v} = \boldsymbol{u}$ and $\boldsymbol{b} = \boldsymbol{h}$ in the above equalities and obtain

$$\nu\, \|\nabla \boldsymbol{u}\|^2 = -\alpha(\boldsymbol{u} \cdot \nabla \boldsymbol{u}^1_\infty, \boldsymbol{u}) + (\boldsymbol{h} \cdot \nabla \boldsymbol{h}^1_\infty, \boldsymbol{u}) + (\boldsymbol{h}^2_\infty \cdot \nabla \boldsymbol{h}, \boldsymbol{u})\,, \tag{13}$$
$$\gamma\, \|\nabla \boldsymbol{h}\|^2 = (\boldsymbol{h} \cdot \nabla \boldsymbol{u}^1_\infty, \boldsymbol{h}) + (\boldsymbol{h}^2_\infty \cdot \nabla \boldsymbol{u}, \boldsymbol{h}) - (\boldsymbol{u} \cdot \nabla \boldsymbol{h}^1_\infty, \boldsymbol{h})\,. \tag{14}$$

By Lemma 2.1, we have

$$\begin{aligned} |\alpha(\boldsymbol{u} \cdot \nabla \boldsymbol{u}^1_\infty, \boldsymbol{u})| = \alpha|(\boldsymbol{u} \cdot \nabla \boldsymbol{u}, \boldsymbol{u}^1_\infty)| &\leq \alpha\sqrt{3}\, \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\, \|\nabla \boldsymbol{u}\|\, \|\boldsymbol{u}^1_\infty\|_{\boldsymbol{L}^3(\Omega)} \\ &\leq \alpha\sqrt{3}\, C_L\, \|\nabla \boldsymbol{u}\|^2\, \|\boldsymbol{u}^1_\infty\|_{\boldsymbol{L}^3(\Omega)}\,, \\ |(\boldsymbol{h} \cdot \nabla \boldsymbol{h}^1_\infty, \boldsymbol{u})| = |(\boldsymbol{h} \cdot \nabla \boldsymbol{u}, \boldsymbol{h}^1_\infty)| &\leq \sqrt{3} \|\boldsymbol{h}\|_{\boldsymbol{L}^6(\Omega)} \|\nabla \boldsymbol{u}\|\, \|\boldsymbol{h}^1_\infty\|_{\boldsymbol{L}^3(\Omega)} \\ &\leq \sqrt{3}\, C_L\, \|\nabla \boldsymbol{h}\|\, \|\nabla \boldsymbol{u}\|\, \|\boldsymbol{h}^1_\infty\|_{\boldsymbol{L}^3(\Omega)}\,, \end{aligned}$$

$$\begin{aligned} |(\boldsymbol{h} \cdot \nabla \boldsymbol{u}^1_\infty, \boldsymbol{h})| = |(\boldsymbol{h} \cdot \nabla \boldsymbol{h}, \boldsymbol{u}^1_\infty)| &\leq \sqrt{3}\, \|\boldsymbol{h}\|_{\boldsymbol{L}^6(\Omega)}\, \|\nabla \boldsymbol{h}\|\, \|\boldsymbol{u}^1_\infty\|_{\boldsymbol{L}^3(\Omega)} \\ &\leq \sqrt{3}\ C_L\, \|\nabla \boldsymbol{h}\|^2\, \|\boldsymbol{u}^1_\infty\|_{\boldsymbol{L}^3(\Omega)}\,, \\ |(\boldsymbol{u} \cdot \nabla \boldsymbol{h}^1_\infty, \boldsymbol{h})| = |(\boldsymbol{u} \cdot \nabla \boldsymbol{h}, \boldsymbol{h}^1_\infty)| &\leq \sqrt{3}\ \|\boldsymbol{u}\|_{\boldsymbol{L}^6(\Omega)}\, \|\nabla \boldsymbol{h}\|\, \|\boldsymbol{h}^1_\infty\|_{\boldsymbol{L}^3(\Omega)} \\ &\leq \sqrt{3}\, C_L\, \|\nabla \boldsymbol{u}\|\, \|\nabla \boldsymbol{h}\|\, \|\boldsymbol{h}^1_\infty\|_{\boldsymbol{L}^3(\Omega)}\,. \end{aligned}$$

By adding equalities (13) and (14), using Young's inequality, and the last estimates, we obtain

$$\nu \,\|\nabla \boldsymbol{u}\|^2 + \gamma \,\|\nabla \boldsymbol{h}\|^2 \leq \sqrt{3}\, C_L \left(\alpha \|\boldsymbol{u}^1_\infty\|_{\boldsymbol{L}^3(\Omega)} + \|\boldsymbol{h}^1_\infty\|_{\boldsymbol{L}^3(\Omega)}\right)\|\nabla \boldsymbol{u}\|^2 + \sqrt{3}\, C_L \left(\|\boldsymbol{u}^1_\infty\|_{\boldsymbol{L}^3(\Omega)} + \|\boldsymbol{h}^1_\infty\|_{\boldsymbol{L}^3(\Omega)}\right)\|\nabla \boldsymbol{h}\|^2,$$

which, together with hypothesis (11) and (12), implies that $\|\nabla \boldsymbol{u}\| = \boldsymbol{0}$ and $\|\nabla \boldsymbol{h}\| = \boldsymbol{0}$. Since $(\boldsymbol{u}, \boldsymbol{h}) \in \boldsymbol{V} \times \boldsymbol{V}$, we obtain that $\boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{h} = \boldsymbol{0}$, i.e., $(\boldsymbol{u}^1_\infty, \boldsymbol{h}^1_\infty) = (\boldsymbol{u}^2_\infty, \boldsymbol{h}^2_\infty)$ which completes the proof. □

Remark 1. Since, by (10), $\|\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)} \leq C\|\nabla \boldsymbol{u}\|$ and $\|\boldsymbol{h}\|_{\boldsymbol{L}^3(\Omega)} \leq C\|\nabla \boldsymbol{h}\|$, conditions (11) and (12) can be interpreted either as saying that ν, γ are suficiently large or that $\|\boldsymbol{f}_\infty\|_{\boldsymbol{V}^*}$ is sufficiently small. In these cases, we say that the associated $(\boldsymbol{u}_\infty, \boldsymbol{h}_\infty)$ is a stationary **slow flow solution**.

Next, we show that the regularity of the weak solutions of the boundary value problem (9) correlates with that of $\boldsymbol{f}_\infty$, i.e., the more regular $\boldsymbol{f}_\infty$ is, the more regular the indicated solutions will be. To this end, we note that, by putting the nonlinearities on the right-hand side of (9), the stationary problem is equivalent to the following two coupled Stokes problems. The first Stokes problem is:

$$\begin{cases} -\nu\Delta \boldsymbol{u}_\infty + \nabla p_\infty = \alpha \boldsymbol{f}_\infty - \alpha \boldsymbol{u}_\infty \cdot \nabla \boldsymbol{u}_\infty + \boldsymbol{h}_\infty \cdot \nabla \boldsymbol{h}_\infty \equiv \mathcal{F} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{u}_\infty = 0 & \text{in } \Omega, \\ \boldsymbol{u}_\infty = 0 & \text{on } \partial\Omega, \end{cases} \tag{15}$$

where $p_\infty = \left(p^*_\infty + \frac{\mu}{2}\boldsymbol{h}^2_\infty\right)$. The second Stokes problem is

$$\begin{cases} -\gamma\Delta \boldsymbol{h}_\infty + \operatorname{grad} \omega_\infty = -\boldsymbol{u}_\infty \cdot \nabla \boldsymbol{h}_\infty + \boldsymbol{h}_\infty \cdot \nabla \boldsymbol{u}_\infty \equiv \mathcal{G} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{h}_\infty = 0 & \text{in } \Omega, \\ \boldsymbol{h}_\infty = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

Proposition 4. *Under the assumptions of Proposition 2 and the condition $\boldsymbol{f}_\infty \in \boldsymbol{L}^2(\Omega)$, we have $\boldsymbol{u}_\infty, \boldsymbol{h}_\infty \in \boldsymbol{H}^2(\Omega) \cap \boldsymbol{V}$. Moreover, the following inequality holds:*

$$\|\boldsymbol{u}_\infty\|_{\boldsymbol{H}^2} + \|p_\infty\|_{H^1} + \|\boldsymbol{h}_\infty\|_{\boldsymbol{H}^2} + \|\omega_\infty\|_{H^1} \leq \Psi(\|\boldsymbol{f}_\infty\|), \tag{17}$$

where Ψ is a continuous and nondecreasing function of its argument such that $\Psi(0) = 0$.

Proof. We begin by discussing the first problem (15) in $(\boldsymbol{u}_\infty, p_\infty)$ with given $\boldsymbol{h}_\infty \in \boldsymbol{V}$. By using the L^q-regularity properties of the Stokes problem given in Proposition 1, we conclude that

$$\|\boldsymbol{u}_\infty\|_{\boldsymbol{W}^{2,q}(\Omega)} + \|p_\infty\|_{W^{1,q}(\Omega)} \leq C\|\mathcal{F}\|_{\boldsymbol{L}^q}, \quad q > 1. \tag{18}$$

Fix $q = 3/2$ and let us estimate the terms on the right-hand side $\mathcal{F}$ of (15); by using the embedding $H^1_0(\Omega) \hookrightarrow L^6(\Omega)$ and Holder's inequality, we have

$$\begin{aligned} \int_\Omega |(\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty|^{3/2} &\leq C\int_\Omega |\boldsymbol{u}_\infty|^{3/2}|\nabla \boldsymbol{u}_\infty|^{3/2} \\ &\leq C\Big(\int_\Omega |\boldsymbol{u}_\infty|^6\Big)^{1/4}\Big(\int_\Omega |\nabla \boldsymbol{u}_\infty|^2\Big)^{3/4} \\ &\leq C\Big(\int_\Omega |\nabla \boldsymbol{u}_\infty|^2\Big)^{3/4}\Big(\int_\Omega |\nabla \boldsymbol{u}_\infty|^2\Big)^{3/4}, \end{aligned}$$

hence

$$\|(\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty\|_{\boldsymbol{L}^{3/2}(\Omega)} \le C\|\nabla \boldsymbol{u}_\infty\|^2. \tag{19}$$

Similarly,

$$\|(\boldsymbol{h}_\infty \cdot \nabla)\boldsymbol{h}_\infty\|_{\boldsymbol{L}^{3/2}(\Omega)} \le C\|\nabla \boldsymbol{h}_\infty\|^2. \tag{20}$$

As $\boldsymbol{f}_\infty \in \boldsymbol{L}^2(\Omega)$, we have $\alpha \boldsymbol{f}_\infty - \alpha(\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty + (\boldsymbol{h}_\infty \cdot \nabla)\boldsymbol{h}_\infty \in \boldsymbol{L}^{3/2}(\Omega)$, so that $\mathcal{F} \in \boldsymbol{L}^{3/2}(\Omega)$. By (18), we conclude that $\boldsymbol{u}_\infty \in \boldsymbol{W}^{2,3/2}(\Omega)$ and $p_\infty \in W^{1,3/2}(\Omega)$.

Next, we consider the problem (16) and use the already known fact that $\boldsymbol{u}_\infty \in \boldsymbol{W}^{2,3/2}(\Omega)$. Again by the regularity properties of the Stokes problem, we have

$$\|\boldsymbol{h}_\infty\|_{\boldsymbol{W}^{2,q}(\Omega)} + \|\omega_\infty\|_{W^{1,q}(\Omega)} \le C\|\mathcal{G}\|_{\boldsymbol{L}^q(\Omega)}, \quad q > 1. \tag{21}$$

We must now estimate the terms on the right-hand side $\mathcal{G}$ of (16). We have:

$$\begin{aligned}
\int_\Omega |(\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{h}_\infty|^{3/2} &\le C\int_\Omega |\boldsymbol{u}_\infty|^{3/2}|\nabla \boldsymbol{h}_\infty|^{3/2} \\
&\le C\Big(\int_\Omega |\boldsymbol{u}_\infty|^6\Big)^{1/4}\Big(\int_\Omega |\nabla \boldsymbol{h}_\infty|^2\Big)^{3/4} \\
&\le C\Big(\int_\Omega |\nabla \boldsymbol{u}_\infty|^2\Big)^{3/4}\Big(\int_\Omega |\nabla \boldsymbol{h}_\infty|^2\Big)^{3/4},
\end{aligned}$$

hence

$$\|(\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{h}_\infty\|_{\boldsymbol{L}^{3/2}(\Omega)} \le C\|\nabla \boldsymbol{u}_\infty\|^{3/2}\|\nabla \boldsymbol{h}_\infty\|^{3/2}.$$

Similarly,

$$\|(\boldsymbol{h}_\infty \cdot \nabla)\boldsymbol{u}_\infty\|_{\boldsymbol{L}^{3/2}(\Omega)} \le C\|\nabla \boldsymbol{u}_\infty\|^{3/2}\|\nabla \boldsymbol{h}_\infty\|^{3/2}.$$

Consequently, $\mathcal{G} \in \boldsymbol{L}^{3/2}(\Omega)$, and so $\boldsymbol{h}_\infty \in \boldsymbol{W}^{2,3/2}(\Omega)$ and $\omega_\infty \in W^{1,3/2}(\Omega)$.

Now, since $W^{2,3/2}(\Omega) \hookrightarrow W^{1,3}(\Omega) \hookrightarrow L^6(\Omega)$, we obtain

$$\|(\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{u}_\infty\| \le C\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^6(\Omega)}\|\nabla \boldsymbol{u}_\infty\|_{\boldsymbol{L}^3(\Omega)} \le C\|\boldsymbol{u}_\infty\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{u}_\infty\|_{\boldsymbol{W}^{2,3/2}(\Omega)}$$

and

$$\|(\boldsymbol{h}_\infty \cdot \nabla)\boldsymbol{h}_\infty\| \le C\|\boldsymbol{h}_\infty\|_{\boldsymbol{L}^6(\Omega)}\|\nabla \boldsymbol{h}_\infty\|_{\boldsymbol{L}^3(\Omega)} \le C\|\boldsymbol{h}_\infty\|_{\boldsymbol{H}^1}\|\boldsymbol{h}_\infty\|_{\boldsymbol{W}^{2,3/2}(\Omega)}.$$

Using the above and noting that $\alpha \boldsymbol{f}_\infty \in \boldsymbol{L}^2(\Omega)$, we obtain $\mathcal{F} \in \boldsymbol{L}^2(\Omega)$. Thus, (18) with $q = 2$ yields $\boldsymbol{u}_\infty \in \boldsymbol{H}^2(\Omega)$ and $p_\infty \in H^1(\Omega)$.

We conclude that

$$\|(\boldsymbol{u}_\infty \cdot \nabla)\boldsymbol{h}_\infty\| \le C\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla \boldsymbol{h}_\infty\| \le C\|\boldsymbol{u}_\infty\|_{\boldsymbol{H}^2(\Omega)}\|\nabla \boldsymbol{h}_\infty\|$$

and

$$\|(\boldsymbol{h}_\infty \cdot \nabla)\boldsymbol{u}_\infty\| \le C\|\boldsymbol{h}_\infty\|_{\boldsymbol{L}^6(\Omega)}\|\nabla \boldsymbol{u}_\infty\|_{\boldsymbol{L}^3(\Omega)} \le C\|\boldsymbol{u}_\infty\|_{\boldsymbol{H}^2(\Omega)}\|\nabla \boldsymbol{h}_\infty\|$$

Consequently, $\mathcal{G} \in \boldsymbol{L}^2(\Omega)$. Thus, from (21) with $q = 2$, we obtain $\boldsymbol{h}_\infty \in \boldsymbol{H}^2(\Omega)$ and $\omega_\infty \in \boldsymbol{H}^1(\Omega)$.

By following the previous estimates and using Proposition 2, we obtain (17), which completes the proof. □

In the remainder of this work, $(\boldsymbol{u}_\infty, \boldsymbol{h}_\infty)$ will denote a stationary slow-flow solution of the type discussed in this Section, i.e., $(\boldsymbol{u}_\infty, \boldsymbol{h}_\infty)$ satisfies the conclusions of Proposition 2 and 3.

3. L^2-stability.

Definition 3.1. For any given $\boldsymbol{u}_0, \boldsymbol{h}_0 \in \boldsymbol{H}$ and $\boldsymbol{f} \in L^\infty(0,\infty,\boldsymbol{V}^*)$, we say that a pair $(\boldsymbol{u},\boldsymbol{h})$ is a weak solution of (1)-(2) if $\boldsymbol{u},\boldsymbol{h} \in L^\infty(0,\infty,\boldsymbol{H}) \cap L^2_{loc}(0,\infty,\boldsymbol{V})$ and (7) holds.

Proposition 5. *Let $\boldsymbol{f} \in L^\infty(0,\infty;\boldsymbol{V}^*)$, $\boldsymbol{f}_\infty \in \boldsymbol{V}^*$ and let $(\boldsymbol{u},\boldsymbol{h})$ be a weak solution of (1)-(2) such that*

$$\boldsymbol{u}_t, \boldsymbol{h}_t \in L^2(0,T;\boldsymbol{V}^*) \text{ for all } T > 0, \tag{22}$$

Further, we assume that $(\boldsymbol{u}_\infty, \boldsymbol{h}_\infty)$ is a weak slow- flow solution of (9), i.e., (11) and (12) hold. Then there exists a positive constant $\beta_0 > 0$ such that, for every $\beta \in (0,\beta_0]$, we have

$$\begin{aligned} &\alpha\|\boldsymbol{u}(t)-\boldsymbol{u}_\infty\|^2 + \|\boldsymbol{h}(t)-\boldsymbol{h}_\infty\|^2 \\ &\leq e^{-2\beta t}(\alpha\|\boldsymbol{u}_0-\boldsymbol{u}_\infty\|^2 + \|\boldsymbol{h}_0-\boldsymbol{h}_\infty\|^2) + 2\alpha^2 e^{-2\beta t}\int_0^t e^{2\beta s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds \end{aligned} \tag{23}$$

Proof. Let

$$\boldsymbol{w} = \boldsymbol{u} - \boldsymbol{u}_\infty, \quad \text{and} \quad \boldsymbol{z} = \boldsymbol{h} - \boldsymbol{h}_\infty. \tag{24}$$

Then

$$\begin{aligned} \alpha(\boldsymbol{w}_t,\boldsymbol{v}) + \nu(A\boldsymbol{w},\boldsymbol{v}) &= -\alpha(\boldsymbol{w}\cdot\nabla\boldsymbol{u},\boldsymbol{v}) - \alpha(\boldsymbol{u}_\infty\cdot\nabla\boldsymbol{w},\boldsymbol{v}) \\ &\quad +(\boldsymbol{z}\cdot\nabla\boldsymbol{h},\boldsymbol{v}) + (\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{z},\boldsymbol{v}) + (\alpha(\boldsymbol{f}-\boldsymbol{f}_\infty),\boldsymbol{v}), \quad (25) \\ (\boldsymbol{z}_t,\boldsymbol{b}) + \gamma(A\boldsymbol{z},\boldsymbol{b}) &= -(\boldsymbol{w}\cdot\nabla\boldsymbol{h},\boldsymbol{b}) - (\boldsymbol{u}_\infty\cdot\nabla\boldsymbol{z},\boldsymbol{b}) \\ &\quad +(\boldsymbol{z}\cdot\nabla\boldsymbol{u},\boldsymbol{b}) + (\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{w},\boldsymbol{b}). \quad (26) \end{aligned}$$

By taking $\boldsymbol{v} = \boldsymbol{w}$ in (25), we obtain

$$\frac{\alpha}{2}\frac{d}{dt}\|\boldsymbol{w}\|^2 + \nu\|\nabla\boldsymbol{w}\|^2 = -\alpha(\boldsymbol{w}\cdot\nabla\boldsymbol{u},\boldsymbol{w}) + (\boldsymbol{z}\cdot\nabla\boldsymbol{h},\boldsymbol{w}) + (\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{z},\boldsymbol{w}) + (\alpha(\boldsymbol{f}-\boldsymbol{f}_\infty),\boldsymbol{w}). \tag{27}$$

On the other hand, taking $\boldsymbol{b} = \boldsymbol{z}$ in (26), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{z}\|^2 + \gamma\|\nabla\boldsymbol{z}\|^2 = -(\boldsymbol{w}\cdot\nabla\boldsymbol{h},\boldsymbol{z}) + (\boldsymbol{z}\cdot\nabla\boldsymbol{u},\boldsymbol{z}) + (\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{w},\boldsymbol{z}). \tag{28}$$

To estimate the terms on the right-hand side of the above expressions, we first note that

$$(\boldsymbol{w}\cdot\nabla\boldsymbol{u},\boldsymbol{w}) = -(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{u}) = -(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{w}) - (\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{u}_\infty) = -(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{u}_\infty).$$

Similarly,

$$\begin{gathered} (\boldsymbol{z}\cdot\nabla\boldsymbol{h},\boldsymbol{w}) = -(\boldsymbol{z}\cdot\nabla\boldsymbol{w},\boldsymbol{h}) = -(\boldsymbol{z}\cdot\nabla\boldsymbol{w},\boldsymbol{z}) - (\boldsymbol{z}\cdot\nabla\boldsymbol{w},\boldsymbol{h}_\infty), \\ -(\boldsymbol{w}\cdot\nabla\boldsymbol{h},\boldsymbol{z}) = (\boldsymbol{w}\cdot\nabla\boldsymbol{z},\boldsymbol{h}) = (\boldsymbol{w}\cdot\nabla\boldsymbol{z},\boldsymbol{z}) + (\boldsymbol{w}\cdot\nabla\boldsymbol{z},\boldsymbol{h}_\infty) = (\boldsymbol{w}\cdot\nabla\boldsymbol{z},\boldsymbol{h}_\infty), \\ (\boldsymbol{z}\cdot\nabla\boldsymbol{u},\boldsymbol{z}) = -(\boldsymbol{z}\cdot\nabla\boldsymbol{z},\boldsymbol{u}) = -(\boldsymbol{z}\cdot\nabla\boldsymbol{z},\boldsymbol{w}) - (\boldsymbol{z}\cdot\nabla\boldsymbol{z},\boldsymbol{u}_\infty). \end{gathered}$$

The above equalities together with (27) and (28) imply the following differential identity

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\alpha\|\boldsymbol{w}\|^2 + \|\boldsymbol{z}\|^2) + \nu\|\nabla\boldsymbol{w}\|^2 + \gamma\|\nabla\boldsymbol{z}\|^2 &= \alpha(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{u}_\infty) - (\boldsymbol{z}\cdot\nabla\boldsymbol{w},\boldsymbol{h}_\infty) \\ &\quad +(\boldsymbol{w}\cdot\nabla\boldsymbol{z},\boldsymbol{h}_\infty) - (\boldsymbol{z}\cdot\nabla\boldsymbol{z},\boldsymbol{u}_\infty) + (\alpha(\boldsymbol{f}-\boldsymbol{f}_\infty),\boldsymbol{w}). \end{aligned} \tag{29}$$

The terms on the right-hand side of (29) can be estimated as follows. From Lemma 2.1 and the Young inequality, we have

$$|-(\boldsymbol{w}\cdot\nabla\boldsymbol{w},\boldsymbol{u}_\infty)| \leq \sqrt{3}\|\boldsymbol{w}\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{w}\|\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^3(\Omega)} \leq \sqrt{3}C_L\|\nabla\boldsymbol{w}\|^2\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^3(\Omega)}.$$

Similarly,

$$|-(\boldsymbol{z}\cdot\nabla\boldsymbol{z},\boldsymbol{u}_\infty)| \leq \sqrt{3}C_L\|\nabla\boldsymbol{z}\|^2\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^3(\Omega)},$$

$$|-(\boldsymbol{z}\cdot\nabla\boldsymbol{w},\boldsymbol{h}_\infty)| \leq \sqrt{3}C_L\|\nabla\boldsymbol{w}\|\|\nabla\boldsymbol{z}\|\|\boldsymbol{h}_\infty\|_{\boldsymbol{L}^3(\Omega)},$$

$$|(\boldsymbol{w}\cdot\nabla\boldsymbol{z},\boldsymbol{h}_\infty)| \leq \sqrt{3}C_L\|\nabla\boldsymbol{w}\|\|\nabla\boldsymbol{z}\|\|\boldsymbol{h}_\infty\|_{\boldsymbol{L}^3(\Omega)}.$$

Using the above in equality (29), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\alpha\|\boldsymbol{w}\|^2+\|\boldsymbol{z}\|^2)+\nu\|\nabla\boldsymbol{w}\|^2+\gamma\|\nabla\boldsymbol{z}\|^2\\
&\quad\leq \sqrt{3}C_L\alpha\|\nabla\boldsymbol{w}\|^2\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^3(\Omega)}+\sqrt{3}C_L\|\nabla\boldsymbol{z}\|^2\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^3(\Omega)}\\
&\quad+2\sqrt{3}C_L\|\nabla\boldsymbol{w}\|\|\nabla\boldsymbol{z}\|\|\boldsymbol{h}_\infty\|_{\boldsymbol{L}^3(\Omega)}+\alpha^2\|\boldsymbol{f}-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}\\
&\quad\leq \sqrt{3}C_L\{\alpha\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^3(\Omega)}+\|\boldsymbol{h}_\infty\|_{\boldsymbol{L}^3(\Omega)}\}\|\nabla\boldsymbol{w}\|^2\\
&\quad+\sqrt{3}C_L\{\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^3(\Omega)}+\|\boldsymbol{h}_\infty\|_{\boldsymbol{L}^3(\Omega)}\}\|\nabla\boldsymbol{z}\|^2+\alpha^2\|\boldsymbol{f}-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}.
\end{aligned} \tag{30}$$

Now we observe that (11) and (12) imply that

$$\begin{aligned}
\bar{\nu}=\nu-\sqrt{3}C_L\{\alpha\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^3(\Omega)}+\|\boldsymbol{h}_\infty\|_{\boldsymbol{L}^3(\Omega)}\} &> 0,\\
\bar{\gamma}=\gamma-\sqrt{3}C_L\{\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^3(\Omega)}+\|\boldsymbol{h}_\infty\|_{\boldsymbol{L}^3(\Omega)}\} &> 0.
\end{aligned}$$

Therefore, by (30), we have

$$\frac{1}{2}\frac{d}{dt}\{\alpha\|\boldsymbol{w}\|^2+\|\boldsymbol{z}\|^2\}+\frac{\bar{\nu}}{\alpha}\alpha\|\nabla\boldsymbol{w}\|^2+\bar{\gamma}\|\nabla\boldsymbol{z}\|^2 \leq \alpha^2\|\boldsymbol{f}-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}. \tag{31}$$

Now, using the embedding $\boldsymbol{H}^1_0(\Omega)\hookrightarrow\boldsymbol{L}^2(\Omega)$, we have

$$\frac{1}{2}\frac{d}{dt}\{\alpha\|\boldsymbol{w}\|^2+\|\boldsymbol{z}\|^2\}+\beta\{\alpha\|\boldsymbol{w}\|^2+\|\boldsymbol{z}\|^2\} \leq \alpha^2\|\boldsymbol{f}-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*},$$

for every $\beta\in(0,\beta_0]$, where

$$\beta_0=\min\{\bar{\nu}/\alpha,\bar{\gamma}\}C_{e1}, \tag{32}$$

with C_{e1} equal to the embedding constant of $\boldsymbol{H}^1_0(\Omega)\hookrightarrow\boldsymbol{L}^2(\Omega)$. By integrating this inequality, we obtain the desired decay property (23). □

In the next section we will also need the following estimate.

Proposition 6. *Let* $(\boldsymbol{u},\boldsymbol{h})$, $(\boldsymbol{u}_\infty,\boldsymbol{h}_\infty)$ *and* $\beta>0$ *be as in Proposition 5. Then,*

$$\begin{aligned}
&2\bar{\nu}e^{-\beta t}\int_0^t e^{\beta s}\|\nabla\boldsymbol{u}(s)-\nabla\boldsymbol{u}_\infty\|^2ds+2\bar{\gamma}e^{-\beta t}\int_0^t e^{\beta s}\|\nabla\boldsymbol{h}(s)-\nabla\boldsymbol{h}_\infty\|^2ds\\
&\quad\leq 2e^{-\beta t}(\alpha\|\boldsymbol{w}_0\|^2+\|\boldsymbol{z}_0\|^2)+4\alpha^2e^{-\beta t}\int_0^t e^{\beta s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds
\end{aligned} \tag{33}$$

Proof. Multiplying inequality (31) by $e^{\beta t}$ yields

$$\begin{aligned}
&\frac{d}{dt}e^{\beta t}\{\alpha\|\boldsymbol{w}\|^2+\|\boldsymbol{z}\|^2\}+2e^{\beta t}\bar{\nu}\|\nabla\boldsymbol{w}\|^2+2e^{\beta t}\bar{\gamma}\|\nabla\boldsymbol{z}\|^2\\
&\quad\leq \beta e^{\beta t}\{\alpha\|\boldsymbol{w}\|^2+\|\boldsymbol{z}\|^2\}+2\alpha^2e^{\beta t}\|\boldsymbol{f}-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}.
\end{aligned}$$

Now, integrating this last inequality from 0 to t and then multiplying by $e^{-\beta t}$, we obtain

$$\begin{aligned}
&2\bar{\nu}e^{-\beta t}\int_0^t e^{\beta s}\|\nabla \boldsymbol{w}(s)\|^2 ds + 2\bar{\gamma}e^{-\beta t}\int_0^t e^{\beta s}\|\nabla \boldsymbol{z}(s)\|^2 ds \\
&\le e^{-\beta t}(\alpha\|\boldsymbol{w}_0\|^2 + \|\boldsymbol{z}_0\|^2) + \beta e^{-\beta t}\int_0^t e^{\beta s}\{\alpha\|\boldsymbol{w}(s)\|^2 + \|\boldsymbol{z}(s)\|^2\}ds \\
&\quad +2\alpha^2 e^{-\beta t}\int_0^t e^{\beta s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds \\
&\le e^{-\beta t}(2-e^{-\beta t})(\alpha\|\boldsymbol{w}_0\|^2 + \|\boldsymbol{z}_0\|^2) \\
&\quad +2\alpha^2\beta e^{-\beta t}\int_0^t e^{-2\beta s}\int_0^s e^{2\beta s_1}\|\boldsymbol{f}(s_1)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds_1 ds \\
&\quad +2\alpha^2 e^{-\beta t}\int_0^t e^{\beta s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds \\
&\le e^{-\beta t}(2-e^{-\beta t})(\alpha\|\boldsymbol{w}_0\|^2 + \|\boldsymbol{z}_0\|^2) + 4\alpha^2 e^{-\beta t}\int_0^t e^{\beta s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds \\
&\quad -2\alpha^2 e^{-2\beta t}\int_0^t e^{2\beta s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds,
\end{aligned}$$

Using (23) and Fubini's theorem, the above yields

$$\begin{aligned}
&2\bar{\nu}e^{-\beta t}\int_0^t e^{\beta s}\|\nabla \boldsymbol{w}(s)\|^2 ds + 2\bar{\gamma}e^{-\beta t}\int_0^t e^{\beta s}\|\nabla \boldsymbol{z}(s)\|^2 ds \\
&\le e^{-\beta t}(\alpha\|\boldsymbol{w}_0\|^2 + \|\boldsymbol{z}_0\|^2) + \beta e^{-\beta t}\int_0^t e^{\beta s}\{\alpha\|\boldsymbol{w}(s)\|^2 + \|\boldsymbol{z}(s)\|^2\}ds \\
&\quad +2\alpha^2 e^{-\beta t}\textstyle\int_0^t e^{\beta s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds \\
&\le e^{-\beta t}(2-e^{-\beta t})(\alpha\|\boldsymbol{w}_0\|^2 + \|\boldsymbol{z}_0\|^2) \\
&\quad +2\alpha^2\beta e^{-\beta t}\int_0^t e^{-2\beta s}\int_0^s e^{2\beta s_1}\|\boldsymbol{f}(s_1)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds_1 ds \\
&\quad +2\alpha^2 e^{-\beta t}\int_0^t e^{\beta s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds \\
&\le e^{-\beta t}(2-e^{-\beta t})(\alpha\|\boldsymbol{w}_0\|^2 + \|\boldsymbol{z}_0\|^2) + 4\alpha^2 e^{-\beta t}\int_0^t e^{\beta s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds \\
&\quad -2\alpha^2 e^{-2\beta t}\int_0^t e^{2\beta s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds
\end{aligned}$$

whence estimate (33) follows. □

We can now prove the following stability result.

Theorem 3.2. *(*L^2**-stability***) Assume the hypotheses of Proposition 5 and also that* $\lim_{t\to\infty}\|\boldsymbol{f}(t)-\boldsymbol{f}_\infty\|_{\boldsymbol{V}^*}=0$. *Then* $\|\boldsymbol{u}(t)-\boldsymbol{u}_\infty\|\to 0$ *and* $\|\boldsymbol{h}(t)-\boldsymbol{h}_\infty\|\to 0$ *as* $t\to+\infty$.

Proof. We must show that $\lim_{t\to+\infty}\alpha\|\boldsymbol{u}(t)-\boldsymbol{u}_\infty\|^2+\|\boldsymbol{h}(t)-\boldsymbol{h}_\infty\|^2=0$, that is, given any $\epsilon>0$ (<1, without loss of generality), there exists a T_ϵ such that $\alpha\|\boldsymbol{u}(t)-\boldsymbol{u}_\infty\|^2+\|\boldsymbol{h}(t)-\boldsymbol{h}_\infty\|^2<\epsilon$ for $t>T_\epsilon$.

To find a T_ϵ, let us start by considering $\delta>0$ (which will be chosen later in function of ϵ). Since $\lim_{t\to\infty}\|\boldsymbol{f}(t)-\boldsymbol{f}_\infty\|_{\boldsymbol{V}^*}=0$, we can choose a T_δ such that $\|\boldsymbol{f}(t)-\boldsymbol{f}_\infty\|_{\boldsymbol{V}^*}<\delta$ for $t>T_\delta$. Also note that $\|\boldsymbol{f}(t)-\boldsymbol{f}_\infty\|_{\boldsymbol{V}^*}\le\|\boldsymbol{f}\|_{L^\infty(0,\infty;\boldsymbol{V}^*)}+\|\boldsymbol{f}_\infty\|_{\boldsymbol{V}^*}$ for all t.

Now, by (23) with a fixed $\beta \in (0, \beta_0]$, we have

$$\begin{aligned}
&\alpha\|\boldsymbol{u}(t)-\boldsymbol{u}_\infty\|^2+\|\boldsymbol{h}(t)-\boldsymbol{h}_\infty\|^2\\
&\le e^{-2\beta t}(\alpha\|\boldsymbol{u}_0-\boldsymbol{u}_\infty\|^2+\|\boldsymbol{h}_0-\boldsymbol{h}_\infty\|^2)+2\alpha^2e^{-2\beta t}\int_0^t e^{2\beta s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2_{\boldsymbol{V}^*}ds\\
&\le e^{-2\beta t}(\alpha\|\boldsymbol{u}_0-\boldsymbol{u}_\infty\|^2+\|\boldsymbol{h}_0-\boldsymbol{h}_\infty\|^2)\\
&+2\alpha^2e^{-2\beta t}\int_0^{T_\delta} e^{2\beta s}(\|\boldsymbol{f}\|_{L^\infty(0,\infty;\boldsymbol{V}^*)}+\|\boldsymbol{f}_\infty\|_{\boldsymbol{V}^*})^2ds+2\alpha^2e^{-2\beta t}\int_{T_\delta}^t e^{2\beta s}\delta^2(s)ds\\
&\le e^{-2\beta t}(\alpha\|\boldsymbol{u}_0-\boldsymbol{u}_\infty\|^2+\|\boldsymbol{h}_0-\boldsymbol{h}_\infty\|^2)\\
&+e^{-2\beta(t-T_\delta)}\alpha^2\frac{(\|\boldsymbol{f}\|_{L^\infty(0,\infty;\boldsymbol{V}^*)}+\|\boldsymbol{f}_\infty\|_{\boldsymbol{V}^*})^2}{\beta}+\frac{\delta^2}{\beta}\alpha^2
\end{aligned}$$

We now choose δ so that $\delta^2\alpha^2/\beta < \epsilon/3$, i.e., $\delta < (\beta\epsilon/(3\alpha^2))^{1/2}$, which yields the corresponding T_δ. Then, from the last estimate we see that it suffices to choose T_ϵ sufficiently large so that, for $t > T_\epsilon$, we have

$$e^{-2\beta t}(\alpha\|\boldsymbol{u}_0-\boldsymbol{u}_\infty\|^2+\|\boldsymbol{h}_0-\boldsymbol{h}_\infty\|^2) < \epsilon/3 \quad \text{and}$$

$$e^{-2\beta(t-T_\delta)}\alpha^2\frac{(\|\boldsymbol{f}\|_{L^\infty(0,\infty;\boldsymbol{V}^*)}+\|\boldsymbol{f}_\infty\|_{\boldsymbol{V}^*})^2}{\beta} < \epsilon/3.$$

These conditions are satisfied with

$$T_\epsilon > \max\{T_\delta, \frac{1}{2\beta}\ln\frac{3(\alpha\|\boldsymbol{u}_0-\boldsymbol{u}_\infty\|^2+\|\boldsymbol{h}_0-\boldsymbol{h}_\infty\|^2)}{\epsilon},$$

$$T_\delta+\frac{1}{2\beta}\ln\frac{3\alpha^2(\|\boldsymbol{f}\|_{L^\infty(0,\infty;\boldsymbol{V}^*)}+\|\boldsymbol{f}_\infty\|_{\boldsymbol{V}^*})^2}{\beta\epsilon}\}$$

Thus, $\alpha\|\boldsymbol{u}(t)-\boldsymbol{u}_\infty\|^2+\|\boldsymbol{h}(t)-\boldsymbol{h}_\infty\|^2 < \epsilon$, which completes the proof. □

Remark 2. When $n = 2$, there exists a unique global weak solution $(\boldsymbol{u}, \boldsymbol{h})$ of (7) satisfying the initial condition $(\boldsymbol{u}_0, \boldsymbol{h}_0) \in \boldsymbol{H} \times \boldsymbol{H}$. Moreover, it is not difficult to check that $\boldsymbol{u}_t, \boldsymbol{h}_t \in L^2(0, T; \boldsymbol{V}^*)$ (see [17]). Thus the estimates stated in Proposition 5 and therefore the conclusions of Theorem 3.2, hold. In particular, the above implies that any stationary slow-flow solution is weakly asymptotically stable.

When $n = 3$ and $\boldsymbol{u}, \boldsymbol{h} \in L^s(0, T; \boldsymbol{L}^r(\Omega))$ with $2/s + 3/r \le 1$ and $r > 3$, it can be shown that the solution satisfies $\boldsymbol{u}_t, \boldsymbol{h}_t \in L^2(0, T; \boldsymbol{V}^*)$ and is, furthermore, unique (see [7]). In this setting, the estimates stated in Proposition 5 and therefore the conclusions of Theorem 3.2 hold. Moreover, as before, any stationary slow-flow solution is weakly asymptotically stable.

The following is an immediate corollary of the theorem

Corollary 1. *If we set $\boldsymbol{f}(t) = \boldsymbol{f}_\infty$ in the Theorem 3.2, then the convergence rate there is exponential in the L^2-norm.*

4. $\boldsymbol{H}^1$-stability.

Definition 4.1. Let $\boldsymbol{u}_0, \boldsymbol{h}_0 \in \boldsymbol{V}$ and let $\boldsymbol{f} \in L^\infty([0, \infty); \boldsymbol{L}^2(\Omega))$. By a strong solution of problem (1) we mean a pair of vector-valued functions $\boldsymbol{u}, \boldsymbol{h}$ such that $\boldsymbol{u}, \boldsymbol{h} \in L^\infty(0, \infty; \boldsymbol{V}) \cap L^2_{loc}(0, \infty; \boldsymbol{H}^2(\Omega) \cap \boldsymbol{V})$ which satisfy (1).

The following conditions on the initial data will remain in force throughout this section:

$$\begin{cases} \boldsymbol{u}_0, \boldsymbol{h}_0 \in \boldsymbol{V} \\ \boldsymbol{f}_\infty \in \boldsymbol{L}^2(\Omega),\ \boldsymbol{f} \in L^\infty([0,\infty); \boldsymbol{L}^2(\Omega)) \\ \|\nabla \boldsymbol{u}_0\| + \|\nabla \boldsymbol{h}_0\| + \sup\limits_{t\geq 0} \|\boldsymbol{f}(t)\| \leq M_1. \end{cases} \tag{34}$$

Under assumptions (34), the existence and uniqueness of a local solution was established in [4], as follows:

Theorem 4.2. *The conditions* (34) *imply the existence of a positive constant* $T > 0$ *and functions* $\boldsymbol{u}, \boldsymbol{h} \in C([0,T); V)$ *and* $p, \omega \in L^2(0,T; H^1(\Omega)\backslash\mathbb{R})$ *which are the unique strong solution of* (1)-(2).

The following global existence theorem was proved in [18]:

Theorem 4.3. *Assume that* $n = 2$ *or that* $n = 3$ *and the constant* M_1 *in (34) is appropriately small. Then the solution in Theorem 4.2 exists for every* $t \geq 0$ *and it satisfies*

$$\sup_{t\geq 0}\{\|\nabla \boldsymbol{u}(t)\|, \|\nabla \boldsymbol{h}(t)\|\} < \infty.$$

As in the case of the standard Navier-Stokes equations, it is unknown whether or not the conclusion of Theorem 4.3 holds in general for large data in three dimensions. In what follows, we will work under the assumption that it does, i.e., we henceforth assume that there exist constants M_2 and T, where $0 < T \leq \infty$ is as in Theorem 4.2, such that

$$\sup_{t\geq 0}\{\|\nabla \boldsymbol{u}(t)\|, \|\nabla \boldsymbol{h}(t)\|\} = M_2 < \infty. \tag{35}$$

We note that it is also possible to carry out the following discussion without making the above assumption, namely by repeating the preceding smallness condition in three dimensions whenever needed. This approach, however, complicates the exposition and we avoid it.

Clearly, by Theorem 4.3, condition (35) holds without additional hypotheses in the two-dimensional case.

Remark 3. Assumption (35) was previously used by Heywood [9] and Heywood and Rannacher [11] in the study of convergence of Galerkin and finite element methods in the study of the classical Navier-Stokes equations, respectively.

Proposition 7. *Let* $(\boldsymbol{u}_\infty, \boldsymbol{h}_\infty)$ *and* $(\boldsymbol{u}, \boldsymbol{h})$ *be as in the last Section. Assume also that (34), (35) hold (thus* $(\boldsymbol{u}_\infty, \boldsymbol{h}_\infty)$ *and* $(\boldsymbol{u}, \boldsymbol{h})$ *are strong solutions). Then, there exists a positive constant* κ_0*, which depends only on* Ω *and on given parameters of the problem, such that, for every* $\kappa \in (0, \kappa_0]$ *we have:*

$$\begin{aligned} &\alpha\|\nabla \boldsymbol{u}(t) - \nabla \boldsymbol{u}_\infty\|^2 + \|\nabla \boldsymbol{h}(t) - \nabla \boldsymbol{h}_\infty\|^2 \\ &\leq C_1 e^{-\kappa t}(\alpha\|\nabla \boldsymbol{u}_0 - \nabla \boldsymbol{u}_\infty\|^2 + \|\nabla \boldsymbol{h}_0 - \nabla \boldsymbol{h}_\infty\|^2) \\ &+ C_2 e^{-\kappa t}\int_0^t e^{\kappa s}\|\boldsymbol{f}(s) - \boldsymbol{f}_\infty\|^2 ds \end{aligned} \tag{36}$$

Proof. By taking $\boldsymbol{v} = A\boldsymbol{w}$ in (25) and $\boldsymbol{b} = A\boldsymbol{z}$ in (26), we obtain

$$\begin{aligned}
&\frac{\alpha}{2}\frac{d}{dt}\|\nabla \boldsymbol{w}\|^2 + \nu\|A\boldsymbol{w}\|^2 \\
= \;& -\alpha(\boldsymbol{w}\cdot\nabla\boldsymbol{u}, A\boldsymbol{w}) - \alpha(\boldsymbol{u}_\infty\cdot\nabla\boldsymbol{w}, A\boldsymbol{w}) \\
&+(\boldsymbol{z}\cdot\nabla\boldsymbol{h}, A\boldsymbol{w}) + (\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{z}, A\boldsymbol{w}) + (\alpha(\boldsymbol{f}-\boldsymbol{f}_\infty), A\boldsymbol{w}), \qquad (37)\\
&\frac{1}{2}\frac{d}{dt}\|\nabla \boldsymbol{z}\|^2 + \gamma\|A\boldsymbol{z}\|^2 \\
= \;& -(\boldsymbol{w}\cdot\nabla\boldsymbol{h}, A\boldsymbol{z}) - (\boldsymbol{u}_\infty\cdot\nabla\boldsymbol{z}, A\boldsymbol{z}) \\
&+(\boldsymbol{z}\cdot\nabla\boldsymbol{u}, A\boldsymbol{z}) + (\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{w}, A\boldsymbol{z}). \qquad (38)
\end{aligned}$$

Next, we estimate the terms on the right-hand sides:

$$|-\alpha(\boldsymbol{u}_\infty\cdot\nabla\boldsymbol{w}, A\boldsymbol{w})| \le \alpha\|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)}\|A\boldsymbol{w}\| \le C_\epsilon\alpha\|\nabla\boldsymbol{u}_\infty\|^4\|\nabla\boldsymbol{w}\|^2 + \epsilon\alpha\|A\boldsymbol{w}\|^2,$$

$$|(\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{z}, A\boldsymbol{w})| \le \|\boldsymbol{h}_\infty\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{z}\|_{\boldsymbol{L}^3(\Omega)}\|A\boldsymbol{w}\| \le C_\epsilon\|\nabla\boldsymbol{z}\|^2\|\nabla\boldsymbol{h}_\infty\|^4 + \epsilon\|A\boldsymbol{w}\|^2 + \delta\|A\boldsymbol{z}\|^2,$$

$$|-(\boldsymbol{u}_\infty\cdot\nabla\boldsymbol{z}, A\boldsymbol{z})| \le \|\boldsymbol{u}_\infty\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{z}\|_{\boldsymbol{L}^3(\Omega)}\|A\boldsymbol{z}\| \le C_\delta\|\nabla\boldsymbol{z}\|^2\|\nabla\boldsymbol{u}_\infty\|^4 + \delta\|A\boldsymbol{z}\|^2,$$

$$|(\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{w}, A\boldsymbol{z})| \le \|\boldsymbol{h}_\infty\|_{\boldsymbol{L}^6(\Omega)}\|\nabla\boldsymbol{w}\|_{\boldsymbol{L}^3(\Omega)}\|A\boldsymbol{z}\| \le C_{\delta,\epsilon}\|\nabla\boldsymbol{w}\|^2\|\nabla\boldsymbol{h}_\infty\|^4 + \epsilon\|A\boldsymbol{w}\|^2 + \delta\|A\boldsymbol{z}\|^2,$$

$$\begin{aligned}
|-(\boldsymbol{w}\cdot\nabla\boldsymbol{u}, A\boldsymbol{w})| &= |-(\boldsymbol{w}\cdot\nabla\boldsymbol{w}, A\boldsymbol{w})| + |-(\boldsymbol{w}\cdot\nabla\boldsymbol{u}_\infty, A\boldsymbol{w})| \\
&\le \|\boldsymbol{w}\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{w}\|\|A\boldsymbol{w}\| + \|\nabla\boldsymbol{u}_\infty\|\|\boldsymbol{w}\|_{\boldsymbol{L}^\infty}\|A\boldsymbol{w}\| \\
&\le C_\epsilon\|\nabla\boldsymbol{w}\|^6 + C_\epsilon\|\nabla\boldsymbol{u}_\infty\|^4\|\nabla\boldsymbol{w}\|^2 + \epsilon\|A\boldsymbol{w}\|^2,
\end{aligned}$$

$$\begin{aligned}
|(\boldsymbol{z}\cdot\nabla\boldsymbol{h}, A\boldsymbol{w})| &= |(\boldsymbol{z}\cdot\nabla\boldsymbol{z}, A\boldsymbol{w})| + |(\boldsymbol{z}\cdot\nabla\boldsymbol{h}_\infty, A\boldsymbol{w})| \\
&\le \|\boldsymbol{z}\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{z}\|\|A\boldsymbol{w}\| + \|\boldsymbol{z}\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{h}_\infty\|\|A\boldsymbol{w}\| \\
&\le C_{\epsilon,\delta}\|\nabla\boldsymbol{h}_\infty\|^4\|\nabla\boldsymbol{z}\|^2 + C_{\epsilon,\delta}\|\nabla\boldsymbol{z}\|^6 + \epsilon\|A\boldsymbol{w}\|^2 + \delta\|A\boldsymbol{z}\|^2,
\end{aligned}$$

$$\begin{aligned}
|-(\boldsymbol{w}\cdot\nabla\boldsymbol{h}, A\boldsymbol{z})| &\le |-(\boldsymbol{w}\cdot\nabla\boldsymbol{z}, A\boldsymbol{z})| + |-(\boldsymbol{w}\cdot\nabla\boldsymbol{h}_\infty, A\boldsymbol{z})| \\
&\le C_{\epsilon,\delta}\|\nabla\boldsymbol{h}_\infty\|^4\|\nabla\boldsymbol{w}\|^2 + C_{\epsilon,\delta}\|\nabla\boldsymbol{w}\|^2\|\nabla\boldsymbol{z}\|^4 + \epsilon\|A\boldsymbol{w}\|^2 + \delta\|A\boldsymbol{z}\|^2,
\end{aligned}$$

$$\begin{aligned}
|(\boldsymbol{z}\cdot\nabla\boldsymbol{u}, A\boldsymbol{z})| &= |(\boldsymbol{z}\cdot\nabla\boldsymbol{w}, A\boldsymbol{z})| + |(\boldsymbol{z}\cdot\nabla\boldsymbol{u}_\infty, A\boldsymbol{z})| \\
&\le C_\delta\|\nabla\boldsymbol{u}_\infty\|^4\|\nabla\boldsymbol{z}\|^2 + C_\delta\|\nabla\boldsymbol{w}\|^4\|\nabla\boldsymbol{z}\|^2 + \delta\|A\boldsymbol{z}\|^2.
\end{aligned}$$

Consequently

$$\begin{aligned}
&\frac{d}{dt}(\alpha\|\nabla\boldsymbol{w}\|^2 + \|\nabla\boldsymbol{z}\|^2) + \nu\|A\boldsymbol{w}\|^2 + \gamma\|A\boldsymbol{z}\|^2 \\
&\qquad \le C\|\nabla\boldsymbol{w}\|^6 + C\|\nabla\boldsymbol{z}\|^6 + C\|\nabla\boldsymbol{w}\|^2\|\nabla\boldsymbol{z}\|^4 + C\|\nabla\boldsymbol{w}\|^4\|\nabla\boldsymbol{z}\|^2 \\
&\qquad +C\|\nabla\boldsymbol{w}\|^2(\|\nabla\boldsymbol{u}_\infty\|^4 + \|\nabla\boldsymbol{h}_\infty\|^4\|) + C\|\nabla\boldsymbol{z}\|^2(\|\nabla\boldsymbol{u}_\infty\|^4 + \|\nabla\boldsymbol{h}_\infty\|^4\|) \\
&\qquad +C\|\boldsymbol{f}-\boldsymbol{f}_\infty\|^2.
\end{aligned} \qquad (39)$$

Now we observe that

$$\nu\|A\boldsymbol{w}\|^2 + \gamma\|A\boldsymbol{z}\|^2 \ge \min(\frac{\nu}{\alpha},\gamma)C_{e2}\ (\alpha\|\nabla\boldsymbol{w}\|^2 + \|\nabla\boldsymbol{z}\|^2) \ge \kappa\ (\alpha\|\nabla\boldsymbol{w}\|^2 + \|\nabla\boldsymbol{z}\|^2)$$

for every $\kappa \in (0,\kappa_0]$, where

$$\kappa_0 = \min\{\min(\frac{\nu}{\alpha},\gamma)C_{e2},\ \beta_0\},$$

Above C_{e2} denotes the embedding constant of $\boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega) \hookrightarrow \boldsymbol{H}_0^1(\Omega)$ and β_0 is given by (32).

Thus (39) implies

$$\begin{aligned}
\frac{d}{dt}e^{\kappa t}(\alpha\|\nabla\boldsymbol{w}\|^2 + \|\nabla\boldsymbol{z}\|^2) \;\le\;& Ce^{\kappa t}\|\nabla\boldsymbol{w}\|^6 + Ce^{\kappa t}\|\nabla\boldsymbol{z}\|^6 \\
+\;& Ce^{\kappa t}\|\nabla\boldsymbol{w}\|^2\|\nabla\boldsymbol{z}\|^4 + Ce^{\kappa t}\|\nabla\boldsymbol{w}\|^4\|\nabla\boldsymbol{z}\|^2 \\
+\;& Ce^{\kappa t}\|\nabla\boldsymbol{w}\|^2(\|\nabla\boldsymbol{u}_\infty\|^4 + \|\nabla\boldsymbol{h}_\infty\|^4\|) \qquad (40)\\
+\;& Ce^{\kappa t}\|\nabla\boldsymbol{z}\|^2(\|\nabla\boldsymbol{u}_\infty\|^4 + \|\nabla\boldsymbol{h}_\infty\|^4\|) \\
+\;& Ce^{\kappa t}\|\boldsymbol{f}-\boldsymbol{f}_\infty\|^2.
\end{aligned}$$

Now, integrating (40) from 0 to t, we have

$$
\begin{aligned}
& \alpha\|\nabla \boldsymbol{w}(t)\|^2+\|\nabla \boldsymbol{z}(t)\|^2 \\
\leq\ & Ce^{-\kappa t}(\alpha\|\nabla \boldsymbol{w}(0)\|^2+\|\nabla \boldsymbol{z}(0)\|^2) \\
+\ & Ce^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla \boldsymbol{w}(s)\|^6 ds+Ce^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla \boldsymbol{z}(s)\|^6 ds \\
+\ & Ce^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla \boldsymbol{w}(s)\|^2\|\nabla \boldsymbol{z}(s)\|^4 ds+Ce^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla \boldsymbol{w}(s)\|^4\|\nabla \boldsymbol{z}(s)\|^2 ds \\
+\ & Ce^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla \boldsymbol{w}(s)\|^2(\|\nabla \boldsymbol{u}_\infty\|^4+\|\nabla \boldsymbol{h}_\infty\|^4\|)ds \qquad (41)\\
+\ & Ce^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla \boldsymbol{z}(s)\|^2(\|\nabla \boldsymbol{u}_\infty\|^4+\|\nabla \boldsymbol{h}_\infty\|^4\|)ds \\
+\ & Ce^{-\kappa t}\int_0^t e^{\kappa s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2 ds.
\end{aligned}
$$

We also note that

$$\|\nabla \boldsymbol{z}(t)\|^4 \leq \|\nabla \boldsymbol{h}(t)-\nabla \boldsymbol{h}_\infty\|^4 \leq \|\nabla \boldsymbol{h}(t)\|^4+\|\nabla \boldsymbol{h}_\infty\|^4,$$

and, using (10) and (35), we obtain

$$\|\nabla \boldsymbol{z}(t)\|^4 \leq M_2^4+\frac{\alpha^4}{\gamma^2\nu^2}\|\boldsymbol{f}_\infty\|^4=M_3.$$

Similarly,

$$\|\nabla \boldsymbol{w}(t)\|^4 \leq M_2^4+\frac{4\alpha^4}{\nu^4}\|\boldsymbol{f}_\infty\|^4=M_4.$$

Thus

$$Ce^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla \boldsymbol{w}(s)\|^2\|\nabla \boldsymbol{z}(s)\|^4 ds \leq CM_3e^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla \boldsymbol{w}(s)\|^2 ds$$

and

$$Ce^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla \boldsymbol{w}(s)\|^6 ds \leq CM_4e^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla \boldsymbol{w}(s)\|^2 ds.$$

Using the above in (41), we conclude that

$$
\begin{aligned}
& \alpha\|\nabla \boldsymbol{w}(t)\|^2+\|\nabla \boldsymbol{z}(t)\|^2 \\
& \leq Ce^{-\kappa t}(\alpha\|\nabla \boldsymbol{w}_0\|^2+\|\nabla \boldsymbol{z}_0\|^2)+C(M_3+M_4)e^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla \boldsymbol{w}(s)\|^2 ds \\
& +C(M_3+M_4)e^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla \boldsymbol{z}(s)\|^2 ds+Ce^{-\kappa t}\int_0^t e^{\kappa s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2 ds
\end{aligned}
$$

Finally, the second and third terms on the right-hand side of the last inequality can be estimated using (33), which yields the estimate (36). □

In the next section we will need the following estimates.

Proposition 8. *Let $(\boldsymbol{u},\boldsymbol{h})$, $(\boldsymbol{u}_\infty,\boldsymbol{h}_\infty)$ and $\kappa>0$ be as in Proposition 7. Then the following inequalities hold:*

$$e^{-\kappa t}\int_0^t e^{\kappa s}\{\nu\|A\boldsymbol{u}(s)-A\boldsymbol{u}_\infty\|^2+\gamma\|A\boldsymbol{h}(s)-A\boldsymbol{h}_\infty\|^2\}ds$$

$$\begin{aligned} &\leq C_3 e^{-\kappa t}(\alpha\|\boldsymbol{u}_0-\boldsymbol{u}_\infty\|^2+\|\boldsymbol{h}_0-\boldsymbol{h}_\infty\|^2)\\ &+C_4 e^{-\kappa t}(\nu\|\nabla\boldsymbol{u}_0-\nabla\boldsymbol{u}_\infty\|^2+\gamma\|\nabla\boldsymbol{h}_0-\nabla\boldsymbol{h}_\infty\|^2)\\ &+C_5 e^{-\kappa t}\int_0^t e^{\kappa s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2 ds. \end{aligned} \tag{42}$$

$$\begin{aligned} \alpha e^{-\kappa t}\int_0^t e^{\kappa s}\|\boldsymbol{w}_t(s)\|^2 ds \quad &\leq +C_6 e^{-\kappa t}(\alpha\|\boldsymbol{u}_0-\boldsymbol{u}_\infty\|^2+\|\boldsymbol{h}_0-\boldsymbol{h}_\infty\|^2)\\ &+C_7 e^{-\kappa t}(\nu\|\nabla\boldsymbol{u}_0-\nabla\boldsymbol{u}_\infty\|^2+\gamma\|\nabla\boldsymbol{h}_0-\nabla\boldsymbol{h}_\infty\|^2)\\ &+C_8 e^{-\kappa t}\int_0^t e^{\kappa s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2 ds. \end{aligned} \tag{43}$$

and

$$\begin{aligned} e^{-\kappa t}\int_0^t e^{\kappa s}\|\boldsymbol{z}_t(s)\|^2 ds \quad &\leq +C_9 e^{-\kappa t}(\alpha\|\boldsymbol{u}_0-\boldsymbol{u}_\infty\|^2+\|\boldsymbol{h}_0-\boldsymbol{h}_\infty\|^2)\\ &+C_{10} e^{-\kappa t}(\nu\|\nabla\boldsymbol{u}_0-\nabla\boldsymbol{u}_\infty\|^2+\gamma\|\nabla\boldsymbol{h}_0-\nabla\boldsymbol{h}_\infty\|^2)\\ &+C_{11} e^{-\kappa t}\int_0^t e^{\kappa s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2 ds. \end{aligned} \tag{44}$$

Proof. By (39), we have

$$\begin{aligned} &\frac{d}{dt}e^{\kappa t}(\alpha\|\nabla\boldsymbol{w}\|^2+\|\nabla\boldsymbol{z}\|^2)+\nu e^{\kappa t}\|A\boldsymbol{w}\|^2+\gamma e^{\kappa t}\|A\boldsymbol{z}\|^2\\ \leq\ & Ce^{\kappa t}\|\nabla\boldsymbol{w}\|^6+Ce^{\kappa t}\|\nabla\boldsymbol{z}\|^6\\ +\ & Ce^{\kappa t}\|\nabla\boldsymbol{w}\|^2\|\nabla\boldsymbol{z}\|^4+Ce^{\kappa t}\|\nabla\boldsymbol{w}\|^4\|\nabla\boldsymbol{z}\|^2\\ +\ & Ce^{\kappa t}\|\nabla\boldsymbol{w}\|^2(\|\nabla\boldsymbol{u}_\infty\|^4+\|\nabla\boldsymbol{h}_\infty\|^4\|)\\ +\ & Ce^{\kappa t}\|\nabla\boldsymbol{z}\|^2(\|\nabla\boldsymbol{u}_\infty\|^4+\|\nabla\boldsymbol{h}_\infty\|^4\|)\\ +\ & Ce^{\kappa t}\|\boldsymbol{f}-\boldsymbol{f}_\infty\|^2+\kappa e^{\kappa t}(\alpha\|\nabla\boldsymbol{w}\|^2+\|\nabla\boldsymbol{z}\|^2). \end{aligned} \tag{45}$$

Now, integrating the above with respect to time from 0 to t, we obtain

$$\begin{aligned} &\nu e^{-\kappa t}\int_0^t e^{\kappa s}\|A\boldsymbol{w}(s)\|^2 ds+\gamma e^{-\kappa t}\int_0^t e^{\kappa s}\|A\boldsymbol{z}(s)\|^2 ds\\ &\leq Ce^{-\kappa t}(\alpha\|\nabla\boldsymbol{u}_0-\nabla\boldsymbol{u}_\infty\|^2+\|\nabla\boldsymbol{h}_0-\nabla\boldsymbol{h}_\infty\|^2)+C_M e^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla\boldsymbol{w}(s)\|^2 ds\\ &+C_M e^{-\kappa t}\int_0^t e^{\kappa s}\|\nabla\boldsymbol{w}(s)\|^2 ds+Ce^{-\kappa t}\int_0^t e^{\kappa s}\|\boldsymbol{f}(s)-\boldsymbol{f}_\infty\|^2 ds\\ &+\kappa e^{-\kappa t}\int_0^t e^{\kappa s}(\alpha\|\nabla\boldsymbol{w}(s)\|^2+\|\nabla\boldsymbol{z}(s)\|^2)ds, \end{aligned}$$

Using (33) above with $\beta=\kappa$ and $C_M>0$ depending on M_3, M_4, we obtain the required estimate (42).

Next, from (25) and (26), we obtain

$$\alpha\,\boldsymbol{w}_t=P(-\alpha\boldsymbol{w}\cdot\nabla\boldsymbol{u}+\alpha\boldsymbol{u}_\infty\cdot\nabla\boldsymbol{w}+\boldsymbol{z}\cdot\nabla\boldsymbol{h}+\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{z})-\nu A\boldsymbol{w}+P(\boldsymbol{f}-\boldsymbol{f}_\infty) \tag{46}$$

and

$$\boldsymbol{z}_t=P(-\boldsymbol{w}\cdot\nabla\boldsymbol{h}-\boldsymbol{u}_\infty\cdot\nabla\boldsymbol{z}+\boldsymbol{z}\cdot\nabla\boldsymbol{u}+\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{w})-\gamma A\boldsymbol{z}. \tag{47}$$

These expressions imply that

$$\alpha\|\boldsymbol{w}_t\|^2\leq C\{\|\boldsymbol{w}\cdot\nabla\boldsymbol{u}\|^2+\|\boldsymbol{u}_\infty\cdot\nabla\boldsymbol{w}\|^2+\|\boldsymbol{z}\cdot\nabla\boldsymbol{h}\|^2+\|\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{z}\|^2+\nu^2\|A\boldsymbol{u}\|^2+\|\boldsymbol{f}-\boldsymbol{f}_\infty\|^2\}, \tag{48}$$

$$\|\boldsymbol{z}_t\|^2\leq C\{\|\boldsymbol{w}\cdot\nabla\boldsymbol{h}\|^2+\|\boldsymbol{u}_\infty\cdot\nabla\boldsymbol{z}\|^2+\|\boldsymbol{z}\cdot\nabla\boldsymbol{u}\|^2+\|\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{w}\|^2+\gamma\|A\boldsymbol{z}\|^2\}. \tag{49}$$

Next, using (35), the embeddings $L^r(\Omega) \hookrightarrow H^1(\Omega)$, where $r = 3$ or 6, $H^2(\Omega) \hookrightarrow L^\infty(\Omega)$ and the equivalence of norms $\|\boldsymbol{v}\|_{\boldsymbol{H}^2(\Omega)}$ and $\|A\boldsymbol{u}\|$, (48) yields

$$\begin{aligned}
&\alpha\|\boldsymbol{w}_t\|^2 \\
&\leq C\{\|\nabla \boldsymbol{u}\|^2 + \nu^2 + \|\boldsymbol{u}_\infty\|^2_{\boldsymbol{L}^3(\Omega)}\}\|A\boldsymbol{w}\|^2 + C(\|\nabla \boldsymbol{h}\|^2 + \|\boldsymbol{h}_\infty\|^2_{\boldsymbol{L}^3(\Omega)})\|A\boldsymbol{z}\|^2 \\
&+C\|\boldsymbol{f} - \boldsymbol{f}_\infty\|^2 \\
&\leq C\|A\boldsymbol{w}\|^2 + C\|A\boldsymbol{z}\|^2 + C\|\boldsymbol{f} - \boldsymbol{f}_\infty\|^2.
\end{aligned} \tag{50}$$

Similarly, from (49) we obtain

$$\|\boldsymbol{z}_t\|^2 \leq C\|A\boldsymbol{w}\|^2 + C\|A\boldsymbol{z}\|^2. \tag{51}$$

Consequently,

$$\begin{aligned}
&\alpha e^{-\kappa t}\int_0^t e^{\kappa s}\|\boldsymbol{w}_t(s)\|^2 ds \\
&\leq Ce^{-\kappa t}\int_0^t e^{\kappa s}\|A\boldsymbol{w}(s)\|^2 ds + Ce^{-\kappa t}\int_0^t e^{\kappa s}\|A\boldsymbol{z}(s)\|^2 ds + Ce^{-\kappa t}\int_0^t e^{\kappa s}\|\boldsymbol{f}(s) - \boldsymbol{f}_\infty\|^2 ds,
\end{aligned}$$

Using (42), the above yields the required estimate (43).

Estimate (44) is obtained similarly, using (49). □

Finally, using estimate (36) and arguing exactly as in the proof of Theorem 3.2, we can prove the following stability result.

Theorem 4.4. **($\boldsymbol{H}^1$-stability)** *Assume that* $\lim_{t\to\infty}\|\boldsymbol{f}(t) - \boldsymbol{f}_\infty\| = 0$ *and that (34) and (35) hold. Then* $\|\boldsymbol{u}(t) - \boldsymbol{u}_\infty\|_{\boldsymbol{H}_0^1(\Omega)} \to 0$ *and* $\|\boldsymbol{h}(t) - \boldsymbol{h}_\infty\|_{\boldsymbol{H}_0^1(\Omega)} \to 0$ *as* $t \to +\infty$.

The following is an immediate corollary of the theorem.

Corollary 2. *If in Theorem 4.4, we set* $\boldsymbol{f}(t) = \boldsymbol{f}_\infty$, *then the convergence rate there is exponential in the* H^1*-norm.*

5. H^2-stability.

If one assumes greater regularity of the initial data, stability in the H^2-norm is attained. To show this, we assume throughout this section that

$$\begin{cases}
\boldsymbol{u}_0, \boldsymbol{h}_0 \in D(A) \\
\boldsymbol{f},\ \boldsymbol{f}_t \in L^\infty([0,\infty); \boldsymbol{L}^2(\Omega)) \\
\|A\boldsymbol{u}_0\| + \|A\boldsymbol{h}_0\| + \sup\limits_{t\geq 0}(\|\boldsymbol{f}(t)\| + \|\boldsymbol{f}_t(t)\|) \leq C.
\end{cases} \tag{52}$$

The following result was proved in [18].

Theorem 5.1. *Let the hypotheses be as in Theorem 4.3 and assume, in addition, that (52) holds. Then* $\boldsymbol{u}, \boldsymbol{h} \in C([0,\infty); \boldsymbol{V} \cap \boldsymbol{H}^2(\Omega)) \cap C^1([0,\infty); \boldsymbol{H})$, *where* $\boldsymbol{u}$ *and* $\boldsymbol{h}$ *are as in Theorem 4.3. Further, there exists a finite positive constant* C *such that*

$$\sup_{t\geq 0}\{\|A\boldsymbol{u}(t)\|, \|A\boldsymbol{h}(t)\|\} \leq C, \tag{53}$$

$$\sup_{t\geq 0}\{\|\boldsymbol{u}_t(t)\|, \|\boldsymbol{h}_t(t)\|\} \leq C. \tag{54}$$

Using the preceding statements, we obtain the following estimates:

Proposition 9. *Assume that (35), (52) and the uniqueness condition in the stationary system hold and let* $(\boldsymbol{u}, \boldsymbol{h})$ *be a strong solution as in Theorem 5.1. Then* *there exists a positive constant $\tilde{\beta}_0 > 0$ such that, for every $\tilde{\beta} \in (0, \tilde{\beta}_0]$, the following holds*

$$\begin{aligned} &\alpha\|\boldsymbol{w}_t(t)\|^2 + \|\boldsymbol{z}_t(t)\|^2 \\ &\leq e^{-\tilde{\beta}t}(\alpha\|\boldsymbol{w}_t(0)\|^2 + \|\boldsymbol{z}_t(0)\|^2) + Ce^{-\tilde{\beta}t}\int_0^t Ce^{\tilde{\beta}s}\|\boldsymbol{w}_t(s)\|^2 ds \\ &+Ce^{-\tilde{\beta}t}\int_0^t Ce^{\tilde{\beta}s}\|\boldsymbol{z}_t(s)\|^2 ds + Ce^{-\tilde{\beta}t}\int_0^t e^{\tilde{\beta}s}\|\boldsymbol{f}_t(s)\|^2 ds. \end{aligned} \tag{55}$$

Here, $\alpha\|\boldsymbol{w}_t(0)\|^2 \leq C\|A\boldsymbol{w}(0)\|^2 + C\|A\boldsymbol{z}(0)\|^2 + C\|\boldsymbol{f}(0) - \boldsymbol{f}_\infty\|^2$ *and* $\|\boldsymbol{z}_t(0)\|^2 \leq C\|A\boldsymbol{w}(0)\|^2 + C\|A\boldsymbol{z}(0)\|^2$.

Proof. We differentiate (25) and (26) with respect to t and set $\boldsymbol{v} = \boldsymbol{w}_t$ and $\boldsymbol{b} = \boldsymbol{z}_t$ there. Noting that $\boldsymbol{w}_t = \boldsymbol{u}_t$ and $\boldsymbol{z}_t = \boldsymbol{h}_t$, we conclude that

$$\begin{aligned} \frac{\alpha}{2}\frac{d}{dt}\|\boldsymbol{w}_t\|^2 + \nu\|\nabla\boldsymbol{w}_t\|^2 &= -\alpha(\boldsymbol{w}_t\cdot\nabla\boldsymbol{u}, \boldsymbol{w}_t) + (\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{z}_t, \boldsymbol{w}_t) \\ &+ (\boldsymbol{z}_t\cdot\nabla\boldsymbol{h}, \boldsymbol{w}_t) + (\boldsymbol{z}\cdot\nabla\boldsymbol{h}_t, \boldsymbol{w}_t) + (\alpha\boldsymbol{f}_t, \boldsymbol{w}_t) \end{aligned} \tag{56}$$

and

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\boldsymbol{z}_t\|^2 + \gamma\|\nabla\boldsymbol{z}_t\|^2 &= -(\boldsymbol{w}_t\cdot\nabla\boldsymbol{h}, \boldsymbol{z}_t) + (\boldsymbol{z}_t\cdot\nabla\boldsymbol{u}, \boldsymbol{z}_t) \\ &+ (\boldsymbol{z}\cdot\nabla\boldsymbol{u}_t, \boldsymbol{z}_t) + (\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{w}_t, \boldsymbol{z}_t). \end{aligned} \tag{57}$$

Next, we estimate the terms on the right-hand sides:

$$|-\alpha(\boldsymbol{w}_t\cdot\nabla\boldsymbol{u}, \boldsymbol{w}_t)| \leq \alpha\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^3(\Omega)}\|\boldsymbol{w}_t\|_{\boldsymbol{L}^6(\Omega)}\|\boldsymbol{w}_t\| \leq C_\epsilon\alpha\|\nabla\boldsymbol{u}\|^2\|A\boldsymbol{u}\|^2\|\boldsymbol{w}_t\|^2 + \epsilon\|\nabla\boldsymbol{w}_t\|^2,$$

$$|(\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{z}_t, \boldsymbol{w}_t)| \leq \|\boldsymbol{h}_\infty\|_{\boldsymbol{L}^\infty(\Omega)}\|\nabla\boldsymbol{z}_t\|\|\boldsymbol{w}_t\| \leq C_\delta\|\boldsymbol{w}_t\|^2\|A\boldsymbol{h}_\infty\|^2 + \delta\|\nabla\boldsymbol{z}_t\|^2,$$

$$\begin{aligned} |(\boldsymbol{z}_t\cdot\nabla\boldsymbol{h}, \boldsymbol{w}_t)| &= |-(\boldsymbol{z}_t\cdot\nabla\boldsymbol{w}_t, \boldsymbol{h})| \\ &\leq C\|\boldsymbol{z}_t\|_{\boldsymbol{L}^3(\Omega)}\|\nabla\boldsymbol{w}_t\|\|\boldsymbol{h}\|_{\boldsymbol{L}^6(\Omega)} \\ &\leq C\|\boldsymbol{z}_t\|^{1/2}\|\nabla\boldsymbol{z}_t\|^{1/2}\|\nabla\boldsymbol{w}_t\|\|\nabla\boldsymbol{h}\| \\ &\leq C_{\epsilon,\delta}\|\nabla\boldsymbol{h}\|^4\|\boldsymbol{z}_t\|^2 + \epsilon\|\nabla\boldsymbol{w}_t\|^2 + \delta\|\nabla\boldsymbol{z}_t\|^2, \end{aligned}$$

$$|(\boldsymbol{z}\cdot\nabla\boldsymbol{h}_t, \boldsymbol{w}_t)| = |(\boldsymbol{z}\cdot\nabla\boldsymbol{z}_t, \boldsymbol{w}_t)| \leq C_\delta\|A\boldsymbol{z}\|^2\|\boldsymbol{w}_t\|^2 + \delta\|\nabla\boldsymbol{z}_t\|^2,$$

$$|-(\boldsymbol{w}_t\cdot\nabla\boldsymbol{h}, \boldsymbol{z}_t)| = |(\boldsymbol{w}_t\cdot\nabla\boldsymbol{z}_t, \boldsymbol{h})| \leq C_\delta\|A\boldsymbol{h}\|^2\|\boldsymbol{w}_t\|^2 + \delta\|\nabla\boldsymbol{z}_t\|^2,$$

$$|(\boldsymbol{z}_t\cdot\nabla\boldsymbol{u}, \boldsymbol{z}_t)| = |(\boldsymbol{z}_t\cdot\nabla\boldsymbol{z}_t, \boldsymbol{u})| \leq C_\delta\|A\boldsymbol{u}\|^2\|\boldsymbol{z}_t\|^2 + C_\delta\|\nabla\boldsymbol{z}_t\|^2,$$

$$|-(\boldsymbol{z}\cdot\nabla\boldsymbol{u}_t, \boldsymbol{z}_t)| = |(\boldsymbol{z}\cdot\nabla\boldsymbol{w}_t, \boldsymbol{z}_t)| \leq C_\epsilon\|A\boldsymbol{z}\|^2\|\boldsymbol{z}_t\|^2 + C_\epsilon\|\nabla\boldsymbol{w}_t\|^2,$$

$$|(\boldsymbol{h}_\infty\cdot\nabla\boldsymbol{w}_t, \boldsymbol{z}_t)| \leq C_\epsilon\|A\boldsymbol{h}_\infty\|^2\|\boldsymbol{z}_t\|^2 + \epsilon\|\nabla\boldsymbol{w}_t\|^2.$$

Now, by adding the equalities (56) and (57) and using the last obtained estimates, we get

$$\frac{d}{dt}(\alpha\|\boldsymbol{w}_t\|^2 + \|\boldsymbol{z}_t\|^2) + \frac{\nu}{\alpha}\alpha\|\nabla\boldsymbol{w}_t\|^2 + \gamma\|\nabla\boldsymbol{z}_t\|^2 \leq C\|\boldsymbol{w}_t\|^2\phi_1(t) + C\|\boldsymbol{z}_t\|^2\phi_2(t) + C\|\boldsymbol{f}_t\|^2, \tag{58}$$

where

$$\begin{aligned} \phi_1(t) &= \|A\boldsymbol{u}(t)\|^4 + \|A\boldsymbol{h}_\infty\|^2 + \|A\boldsymbol{h}(t)\|^2 + \|A\boldsymbol{z}(t)\|^2, \\ \phi_2(t) &= \|A\boldsymbol{u}(t)\|^2 + \|A\boldsymbol{h}(t)\|^4 + \|A\boldsymbol{z}(t)\|^2 + \|A\boldsymbol{h}_\infty\|^2 \end{aligned}$$

are bounded functions.

By working similarly with (58), whose left-hand is analogous to the corresponding one of (31), we obtain (55) for $\tilde{\beta} \in (0, \tilde{\beta}_0]$ with

$$\tilde{\beta}_0 = \min\{\frac{\nu}{\alpha}, \gamma\}C_{e1}$$

where as before C_{e1} is the embedding constant of $\boldsymbol{H}_0^1(\Omega) \hookrightarrow \boldsymbol{L}^2(\Omega)$.

Finally, the stated estimates for $\alpha\|\boldsymbol{w}_t(0)\|$ and $\|\boldsymbol{z}_t(0)\|$ are consequences of (50) and (51), respectively. □

Next, we have

Proposition 10. *Let the hypotheses be as in Proposition 9. Then, there exists a positive constant* $\tilde{\kappa}_0$*, which only depends* Ω *and on the given parameters of the problem, such that, for every* $\tilde{\kappa} \in (0, \tilde{\kappa}_0]$*, we have:*

$$\begin{aligned}\|A\boldsymbol{w}(t)\| \quad &\leq Ce^{-\tilde{\kappa}t}(\|A\boldsymbol{w}(0)\|^2 + \|A\boldsymbol{z}(0)\|^2) + Ce^{-\tilde{\kappa}t}\int_0^t e^{\tilde{\kappa}s}\|\boldsymbol{f}(s) - \boldsymbol{f}_\infty\|^2 ds \\ &+Ce^{-\tilde{\kappa}t}\int_0^t e^{\tilde{\kappa}s}\|\boldsymbol{f}_t(s)\|^2 ds + \|\boldsymbol{f}(t) - \boldsymbol{f}_\infty\|\end{aligned} \tag{59}$$

and

$$\begin{aligned}\|A\boldsymbol{z}(t)\| \quad &\leq Ce^{-\tilde{\kappa}t}(\|A\boldsymbol{w}(0)\|^2 + \|A\boldsymbol{z}(0)\|^2) + Ce^{-\tilde{\kappa}t}\int_0^t e^{\tilde{\kappa}t}\|\boldsymbol{f}(s) - \boldsymbol{f}_\infty\|^2 ds \\ &+Ce^{-\tilde{\kappa}t}\int_0^t e^{\tilde{\kappa}s}\|\boldsymbol{f}_t(s)\|^2 ds.\end{aligned} \tag{60}$$

Proof. Let κ_0 and $\tilde{\beta}_0$ be the positive constants in Propositions 7 and 9, respectively, and set

$$\tilde{\kappa}_0 = \min\{\tilde{\beta}_0, \kappa_0\}.$$

It follows from (46) that

$$\begin{aligned}\|\nu A\boldsymbol{w}(t)\| \quad &\leq \|\boldsymbol{w}_t(t)\| + \|\nabla\boldsymbol{w}\|\|A\boldsymbol{u}\| + \alpha\|A\boldsymbol{u}_\infty\|\|\nabla\boldsymbol{w}\| \\ &+\|\nabla\boldsymbol{z}\|\|A\boldsymbol{h}\| + \|A\boldsymbol{h}_\infty\|\|\nabla\boldsymbol{z}\| + \|\boldsymbol{f}(t) - \boldsymbol{f}_\infty\|.\end{aligned}$$

Now, taking $\tilde{\kappa} \leq \tilde{\kappa}_0$ and using (55) together with the hypotheses on $(\boldsymbol{u}_\infty, \boldsymbol{h}_\infty)$ and $(\boldsymbol{u}, \boldsymbol{h})$ in this last inequality, we obtain

$$\begin{aligned}\|\nu A\boldsymbol{w}(t)\| \quad &\leq e^{-\tilde{\kappa}t}(\alpha\|\boldsymbol{w}_t(0)\|^2 + \|\boldsymbol{z}_t(0)\|^2) + Ce^{-\tilde{\kappa}t}\int_0^t e^{\tilde{\kappa}s}\|\boldsymbol{w}_t(s)\|^2 ds \\ &+Ce^{-\tilde{\kappa}t}\int_0^t Ce^{\tilde{\kappa}s}\|\boldsymbol{z}_t(s)\|^2 ds + Ce^{-\tilde{\kappa}t}\int_0^t e^{\tilde{\kappa}s}\|\boldsymbol{f}_t(s)\|^2 ds. \\ &+Ce^{-\tilde{\kappa}t}(\alpha\|\nabla\boldsymbol{u}_0 - \nabla\boldsymbol{u}_\infty\|^2 + \|\nabla\boldsymbol{h}_0 - \nabla\boldsymbol{h}_\infty\|^2) \\ &+Ce^{-\tilde{\kappa}t}\int_0^t e^{\tilde{\kappa}s}\|\boldsymbol{f}(s) - \boldsymbol{f}_\infty\|^2 ds \\ &+\|\boldsymbol{f}(t) - \boldsymbol{f}_\infty\|\end{aligned}$$

Using the estimates for $\alpha\|\boldsymbol{w}_t(0)\|$ and $\alpha\|\boldsymbol{z}_t(0)\|$ given in (43), (44) and Proposition 9, we obtain estimate (59) from the preceding inequality.

Next, (47) implies that

$$\begin{aligned}\|\gamma A\boldsymbol{z}(t)\| \quad &\leq \|\boldsymbol{z}_t(t)\| + \|\nabla\boldsymbol{w}\|\|A\boldsymbol{h}\| + \|A\boldsymbol{u}_\infty\|\|\nabla\boldsymbol{z}\| \\ &+\|\nabla\boldsymbol{z}\|\|A\boldsymbol{u}\| + \|A\boldsymbol{h}_\infty\|\|\nabla\boldsymbol{w}\|.\end{aligned}$$

Arguing as before, the above inequality yields estimate (60). □

Finally, arguing exactly as in the proof of Theorem 3.2, estimates (59) and (60) with a fixed $\tilde{\kappa} \in (0, \tilde{\kappa}_0]$ yield following stability result.

Theorem 5.2. *Let the hypotheses be as in Proposition 10 and assume, in addition, that* $\lim_{t\to\infty} \|\boldsymbol{f}(t)-\boldsymbol{f}_\infty\| = 0$ *and* $\lim_{t\to\infty} \|\boldsymbol{f}_t(t)\| = 0$. *Then* $\|\boldsymbol{u}(t)-\boldsymbol{u}_\infty\|_{\boldsymbol{H}^2(\Omega)} \to 0$ *and* $\|\boldsymbol{h}(t) - \boldsymbol{h}_\infty\|_{\boldsymbol{H}^2(\Omega)} \to 0$ *as* $t \to +\infty$.

The following is an immediate corollary of the theorem.

Corollary 3. *If we set* $\boldsymbol{f}(t) = \boldsymbol{f}_\infty$ *in Theorem 5.2, then the convergence rate there is exponential in the* H^2*-norm.*

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