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THE WELL-POSEDNESS AND REGULARITY OF A ROTATING BLADES EQUATION

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ABSTRACT. In this paper, a rotating blades equation is considered. The arbitrary pre-twisted angle, arbitrary pre-setting angle and arbitrary rotating speed are taken into account when establishing the rotating blades model. The nonlinear PDEs of motion and two types of boundary conditions are derived by the extended Hamilton principle and the first-order piston theory. The well-posedness of weak solution (global in time) for the rotating blades equation with Clamped-Clamped (C-C) boundary conditions can be proved by compactness method and energy method. Strong energy estimates are derived under additional assumptions on the initial data. In addition, the existence and regularity of weak solutions (global in time) for the rotating blades equation with Clamped-Free (C-F) boundary conditions are proved as well.

1. **Introduction.** Rotating blades (thin-walled beam) are important structures widely used in mechanical and aerospace engineering as aviation engine blades, various cooling fans, windmill blades, helicopter rotor blades, airplane propellers etc. The study of the dynamics of rotating blades is important to design purposes, optimization, and control.

If the shear effect is not considered, the Euler-Bernoulli beam equation is used to model vibration of thin-walled beam. Chen et al.[6] studied the boundary feedback stabilization of a linear Euler-Bernoulli beam equation, they proved that the total energy of the equation decays uniformly and exponentially with t, when the beam is clamped at the left end and subjected to a feedback boundary conditions at the right end. Then Guo et al.[12] proved the well-posedness and stability of the system proposed by Chen et al.[14] considered a nonlinear Euler-Bernoulli beam equation with a feedback force applied at the free end, the existence of the weak and classical solutions were proved. Other relevant studies are referred to Refs. [4, 18, 19, 17, 13]. Note that the above literatures studied the longitudinal vibration in one direction.

A very extensive work devoted to the longitudinal vibration in two directions were done by Librescu and Song[21, 32, 33] and their co-workers[25, 28]. Under assumption of the cross-section to be rigid in its own plane, they modelled the rotating blades by 1-D linear governing equations. The influence of many factors on rotating blades, such as the anisotropy and heterogeneity of constituent materials,

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functionally graded materials (FGM), temperature, shear effects, primary and secondary warping phenomena (Vlasov effect), centrifugal and Coriolis forces etc have been taken into account in the 1-D linear governing equations.

Following Librescu's approach, various blades models were derived. Georgiades et al.[11] modelled a rotating blades by means of linear strain-displacement relationships, considering arbitrary pitch (presetting) angle and non-constant rotating speed. Choi et al.[8] studied bending vibration control of the pre-twisted rotating composite blades, who emphasized the important of piezoelectric effect in single cell composite blades. Fazelzadeh et al.[9] considered a thin-walled blades made of FGM which is used in turbomachinery under aerothermodynamics loading. In the paper, quasi-steady aerodynamic pressure loadings was determined by the first-order piston theory, and steady beam surface temperature was obtained from gas dynamics theory. Fazelzadeh et al.[10] studied the governing equations which included the effects of the presetting angle and the rotary inertia. The effects of steady wall temperature and quasi-steady aerodynamic pressure loadings due to flow motion were also taken into account.

The models in Refs.[21, 32, 33, 25, 28, 11, 8, 9, 10] are linear. When the engine blades rotate at a low speed, the linear approximation can completely meet the needs of practical application. However, when the blade rotate at a high speed, the simple linear approximation can not accurately describe the dynamic behavior of the system. So the non-linear analysis of rotating blades has attracted considerable attention.

The nonlinear governing equations of a rotating blades at constant angular velocity was presented by Anderson[1], and the author linearized the equation under the assumption that a small perturbed motion occurred at an initially stressed equilibrium configuration. Chen et al.[7] considered the effects of geometric nonlinearity, shear deformation and rotary inertia. Arvin et al.[2] builded a nonlinear governing equation for rotating blades considering centrifugal forces by means of von-Karmans strain-displacement relationships under assumption of the constant rotation speed and zero pitch (presetting) angle. Yao et al.[39] employed the Hamilton's principle to derive the nonlinear governing equations with periodic rotating speed, arbitrary pitch (presetting) angle and linear pre-twist angle. Under the assumption that the location of shear centre is different from the centre of gravity, Avramov et al.[3] obtained results of the investigations on flexural-flexural-torsional nonlinear vibrations of twisted rotating blades described by the model of three nonlinear integro-differential equations. Other nonlinear models can be found in Refs.[34, 29, 27, 37, 15, 31, 36].

To the best of the author's knowledge, all the above literatures about the longitudinal vibration in two directions skipped the existence proof of solutions to the governing equations, and directly used the finite element method to study the influence of various parameters on blades vibration. To address this situation, we first try to model a governing equations of the blades with arbitrary rotating speed, arbitrary pre-setting angle and pre-twist angle. In the process of building the model, we take into account the free vibration at the right end of the blade and the nonlinear relationship between stress and strain. In the paper we aim to investigate the well-posedness and regularity of the governing equations. The well-posedness of other nonlinear blade vibration models can be found in Refs. [5, 38, 26, 35, 16].

2. **Model building.** Let us consider a slender, straight blades mounted on a rigid hub of radius R_0 rotating with the angular velocity $\omega(t)$. The length of the blades is denoted by l, its wall thickness by h, the length and the width of the cross section of the blades are a and b, respectively. The sum of pre-twist angle $\alpha(x)$ and pre-setting angle β is defined by $\theta(x)$, as shown in Figure 1 and 2.

2.1. **Assumptions.** To derive the model of the rotating blades, the following kinematic and static assumptions are postulated:

(i) The blades is perfectly elastic bodies, the blades material is isotropic and is not affected by temperature,

(ii) The cross section of the blades is rectangular and all its geometrical dimensions remain invariant in its plane,

(iii) The ratio of wall thickness h to the radius of curvature r at any point of the blade wall is negligibly small while compared to unity,

(iv) The transverse shear effect of the cross section is neglected,

(v) The axial displacement w is much smaller than u or v and the derivatives of w can be neglected in the strain-displacement relations. where u, v, w represent the displacement of the middle line of cross-sections along the y, z, x axis, respectively.

2.2. **Four coordinate systems are defined to describe the motion of the blades.** Inertial Cartesian coordinate systems (X, Y, Z) attached to the center of the hub O, the unit vectors of the inertial coordinate systems (X, Y, Z) is defined as (I, J, K).

Rotating coordinate systems (x, y, z) located at the blade root, the origin o of the rotating coordinate systems is set at the center of the beam cross section, the unit vectors of the inertial coordinate systems (x, y, z) is defined as (i, j, k).

Transformation between XYZ and xyz frames can be written in the form

$$(X, Y, Z) = (x, y, z) + (R_0, 0, 0). \tag{1}$$

Local coordinates systems (x^p, y^p, z^p) also located at the blade root and oriented with respect to plane of rotation (y, z) at angle $\omega(t)$. So $(x, y, z)' = \mathrm{B}(x^p, y^p, z^p)'$, where the rotation matrix B is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{pmatrix}.$$

Local, curvilinear coordinate systems (x, n, s) related to blade cross section (Figure 2). Its origin is set conveniently at the point on a mid-line contour. s and n are the circumferential and thickness coordinate variables, the unit vectors of axis n and s are defined as $\mathbf{e}_n, \mathbf{e}_t$, respectively.

In order to determine the relationship between the two coordinate systems (x, y, z) and (x, n, s), one defines the position vector $\mathbf{r}(\equiv \mathbf{r}(s, x))$ from the reference x-axis of the blades to an arbitrary point A located on the middle surface as

$$\mathbf{r} = x\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k}. \tag{2}$$

The position vector $\mathbf{r}^*$ of an arbitrary point A^* off the mid-surface of the blades can be expressed as

$$\mathbf{r}^* = \mathbf{r} + n\mathbf{e}_n, \tag{3}$$

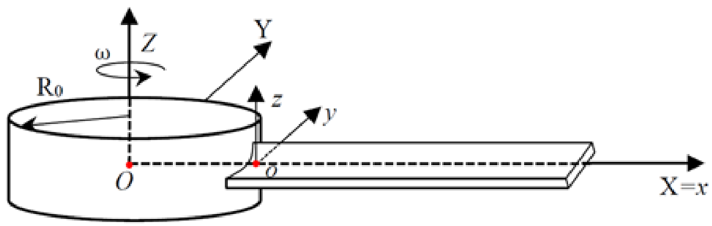

FIGURE 1. Rotating blades

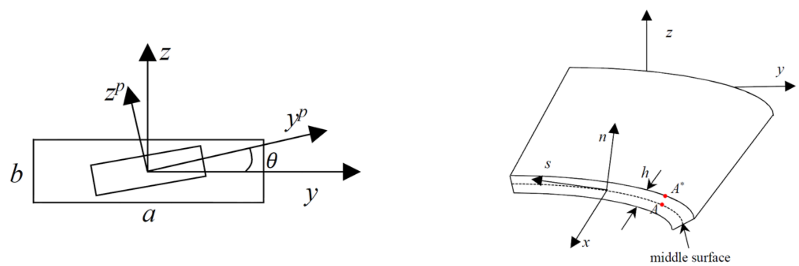

FIGURE 2. The cross-section of rotating blades

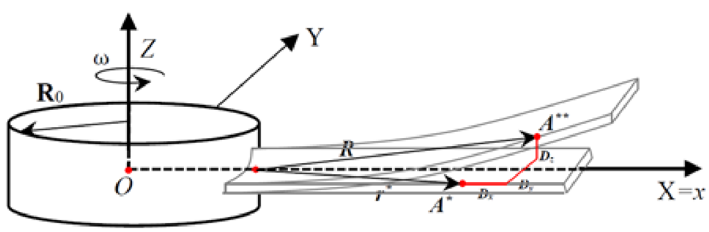

FIGURE 3. Position vectors of ponit A^* in reference frames

As a result

$$\mathbf{e}_t = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s} = \frac{\mathrm{d}y(s)}{\mathrm{d}s}\mathbf{j} + \frac{\mathrm{d}z(s)}{\mathrm{d}s}\mathbf{k}, \tag{4}$$

$$\mathbf{e}_n = \mathbf{e}_t \times \mathbf{i} = \frac{\mathrm{d}z(s)}{\mathrm{d}s}\mathbf{j} - \frac{\mathrm{d}y(s)}{\mathrm{d}s}\mathbf{k}. \tag{5}$$

Moreover, in order to avoid confusion, the notation (x, y, z) represents the points associated with the middle surface, the notation $A^* = (x^*, y^*, z^*)$ represents the points off the middle surface, the two notations (x, y, z) and (x^*, y^*, z^*) are presented as follows:

$$x^* = x, \quad y^* = \mathbf{r}^* \cdot \mathbf{j} = y + n\frac{\mathrm{d}z}{\mathrm{d}s}, \quad z^* = \mathbf{r}^* \cdot \mathbf{k} = z - n\frac{\mathrm{d}y}{\mathrm{d}s}. \tag{6}$$

2.3. **Displacement field.** Based on the assumptions (iv) and (v), the axial displacement D_x, D_y, D_z (see Figure 3) are expressed as (see Ref. [21])

$$D_x = \phi_y(x,t)(z(s) - n\frac{\mathrm{d}y}{\mathrm{d}s}) + \phi_z(x,t)(y(s) + n\frac{\mathrm{d}z}{\mathrm{d}s}),\ D_y = u,\ D_z = v. \tag{7}$$

where ϕ_y, ϕ_z denote rotation of the cross-section about y and z axis. For non-shearable blades, The expression of ϕ_y and ϕ_z can be written in the form

$$\phi_y = -v_x, \phi_z = -u_x. \tag{8}$$

where u_x, v_x denote derivatives with respect to x.

2.4. **Strains and stress.** Based on the assumptions (v), the displacement-strain relationships is expressed as follows[20]:

$$\varepsilon_{xx} = \frac{\partial D_x}{\partial x} + \frac{1}{2}\Big[\Big(\frac{\partial D_y}{\partial x}\Big)^2 + \Big(\frac{\partial D_z}{\partial x}\Big)^2\Big]. \tag{9}$$

Thanks to (8), we can get

$$\varepsilon_{xx} = \bar{\varepsilon}_{xx} + \bar{\bar{\varepsilon}}_{xx} n,$$

where

$$\bar{\varepsilon}_{xx} = \frac{1}{2}(u_x^2 + v_x^2) - (u_{xx}y(s) + v_{xx}z(s)),\quad \bar{\bar{\varepsilon}}_{xx} = -u_{xx}\frac{\mathrm{d}z}{\mathrm{d}s} + v_{xx}\frac{\mathrm{d}y}{\mathrm{d}s}. \tag{10}$$

The shear strain ε_{nx} can be expressed as

$$\varepsilon_{nx} = \frac{1}{2}\Big(\frac{\partial U_n}{\partial x} + \frac{\partial w}{\partial n}\Big), \tag{11}$$

where U_n represents the components of $\big(D_x, D_y, D_z\big)$ along the n axis,

$$U_n = \big(D_x, D_y, D_z\big)\cdot \mathbf{e}_n = u\frac{\mathrm{d}z}{\mathrm{d}s} - v\frac{\mathrm{d}y}{\mathrm{d}s}. \tag{12}$$

Substituting (7) and (12) into (11), we deduce $\varepsilon_{nx} = 0$. Using the coordinate transformation of strain component, taking into account assumptions (iv), the shear strain ε_{sx} can be expressed as

$$\varepsilon_{sx} = \frac{1}{2}\big(\frac{\mathrm{d}y}{\mathrm{d}s}\gamma_{xy} + \frac{\mathrm{d}z}{\mathrm{d}s}\gamma_{xz}\big) = 0. \tag{13}$$

where γ_{xy}, γ_{xz} are the transverse shear effect of the cross section.

Since these materials are isotropic, the corresponding thermoelastic constitutive law adapted to the case of structures is expressed as

$$\begin{pmatrix} \sigma_{ss} \\ \sigma_{xx} \\ \sigma_{xn} \\ \sigma_{ns} \\ \sigma_{sx} \end{pmatrix} = \begin{pmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{pmatrix} \begin{pmatrix} \varepsilon_{ss} \\ \varepsilon_{xx} \\ \varepsilon_{xn} \\ \varepsilon_{ns} \\ \varepsilon_{sx} \end{pmatrix},$$

Herein, the reduced thermoelastic coefficients are defined as:

$$Q_{11} = Q_{22} = \frac{E}{1-\nu^2}, Q_{12} = Q_{21} = \frac{E\nu}{1-\nu^2}, Q_{44} = Q_{55} = \frac{k^2 E}{2(1+\nu)}, Q_{66} = \frac{E}{2(1+\nu)},$$

where E is Young's modulus, ν is Poisson's ratio, k^2 is the transverse shear correction factor.

According to assumption (ii), the cross section is rigid, then we can derive $\sigma_{ns} = 0$. Considering the assumption of the hoop stress σ_{ss} to be negligible, we can get

$$(\sigma_{ss}, \sigma_{xx}, \sigma_{xn}, \sigma_{ns}, \sigma_{sx}) = (0, E\varepsilon_{xx}, 0, 0, 0).$$

2.5. **The centrifugal force and the perturbed gas pressure.** The centrifugal force can be represented as

$$F_c = \int_x^l \rho \bar{A} \omega^2 (R_0 + x) \mathrm{d}x = \rho \bar{A} \omega^2 R(x),$$

where ρ is the density of the blades, $\bar{A}$ is the cross section area of the blades, $R(x) = R_0(l - x) + \frac{1}{2}(l^2 - x^2)$.

In the paper, we use the first-order piston theory (see Ref.[24]) to evaluate the perturbed gas pressure. The pressure on the principal plane of the blade can be obtained as

$$P_{y^p} = C_\infty \rho_\infty \Big(\frac{\partial u^p}{\partial t} + U^t_{y^p} \frac{\partial u^p}{\partial x} \Big), P_{z^p} = C_\infty \rho_\infty \Big(\frac{\partial v^p}{\partial t} + U^t_{z^p} \frac{\partial v^p}{\partial x} \Big), \tag{14}$$

where

$$U^t_{y^p} = U_\infty \cos\theta, \quad U^t_{z^p} = U_\infty \sin\theta, \tag{15}$$

C_∞ represents the speed of sound, ρ_∞ and U_∞, respectively, denote the density and the velocity of the free stream air, $U^t_{y^p}$ and $U^t_{z^p}$ are, respectively, the tangential components of the fluid velocity on the positive y^p and z^p plane, and u^p and v^p denote the displacement components along the principal axes y^p and z^p, respectively.

The transformation relationship between (u^p, v^p) and (u, v) is given by

$$u^p = u\cos\theta + v\sin\theta, \;\; v^p = -u\sin\theta + v\cos\theta. \tag{16}$$

Therefore, the external forces per unit axial length in the y direction and the z direction can be obtained as

$$p_y = aP_{z^p}\sin\theta - bP_{y^p}\cos\theta, \;\; p_z = -aP_{z^p}\cos\theta - bP_{y^p}\sin\theta. \tag{17}$$

Combining (14), (15), (16), p_y, p_z can be expressed as a linear function of u, v, u_t, v_t, u_x, v_x, respectively.

$$p_y = b_1 u_x + b_2 v_x + b_3 u + b_4 v + b_5 u_t + b_6 v_t \tag{18}$$

$$p_z = e_1 u_x + e_2 v_x + e_3 u + e_4 v + e_5 u_t + e_6 v_t \tag{19}$$

where

$$\begin{aligned}
b_1 &= -C_\infty \rho_\infty U_\infty (a\sin^3\theta + b\cos^3\theta) \\
b_2 &= -C_\infty \rho_\infty U_\infty \sin\theta\cos\theta(-a\sin\theta + b\cos\theta) \\
b_3 &= -C_\infty \rho_\infty U_\infty \theta_x \sin\theta\cos\theta(a\sin\theta - b\cos\theta) \\
b_4 &= -C_\infty \rho_\infty U_\infty \theta_x (a\sin^3\theta + b\cos^3\theta) \\
b_5 &= -C_\infty \rho_\infty (a\sin^2\theta + b\cos^2\theta) \\
b_6 &= C_\infty \rho_\infty \sin\theta\cos\theta(a - b) \\
e_1 &= -C_\infty \rho_\infty U_\infty \sin\theta\cos\theta(-a\sin\theta + b\cos\theta) \\
e_2 &= -C_\infty \rho_\infty U_\infty \sin\theta\cos\theta(a\cos\theta + b\sin\theta) \\
e_3 &= C_\infty \rho_\infty U_\infty \theta_x \sin\theta\cos\theta(a\sin\theta + b\cos\theta) \\
e_4 &= -C_\infty \rho_\infty U_\infty \theta_x \sin\theta\cos\theta(-a\sin\theta + b\cos\theta)
\end{aligned}$$

$$e_5 = C_\infty \rho_\infty \sin\theta \cos\theta (a-b)$$
$$e_6 = -C_\infty \rho_\infty (a\cos^2\theta + b\sin^2\theta)$$

2.6. **The velocity vector and the acceleration.** In order to calculate the kinetic energy, the velocity vector and the acceleration should be given first. Based on the assumption (v), the position vector $\mathbf{R}$ of a point A^{**} of the deformed blades is expressed in the form

$$\mathbf{R}(X,Y,Z,t) = x^*\mathbf{i} + (y^*+u)\mathbf{j} + (z^*+v)\mathbf{k} + R_0\mathbf{i}.$$

Keep in mind that the rotation takes place solely in the XY plan, it results

$$\mathbf{i}_t = \omega\mathbf{j}; \quad \mathbf{j}_t = -\omega\mathbf{i}; \quad \mathbf{k}_t = 0.$$

Then the velocity vector and the acceleration of an arbitrary point A^{**} are obtained:

$$\mathbf{R}_t = -\omega(y^*+u)\mathbf{i} + (\omega(R_0+x)+u_t)\mathbf{j} + v_t\mathbf{k}, \tag{20}$$
$$\begin{aligned}\mathbf{R}_{tt} = &-[2\omega u_t + \omega_t(y^*+u) + \omega^2(R_0+x)]\mathbf{i}\\ &+[u_{tt} - \omega_t(R_0+x) - \omega^2(y^*+u)]\mathbf{j} + v_{tt}\mathbf{k},\end{aligned} \tag{21}$$

where subscript t denotes derivatives with respect to the time.

2.7. **Rotating blades model.** In order to derive the blades model and the associated boundary conditions, the extended Hamilton's principle is used. This can be formulated as

$$\int_{t_1}^{t_2} (\delta K - \delta U + \delta W)\mathrm{d}t = 0,\ \delta u = 0, \delta v = 0 \text{ at } t = t_1, t_2. \tag{22}$$

where K and U, respectively, denote the kinetic energy and the strain energy, W is the virtual work of external forces, δ is the variation operator.

Thanks to the cross section of the blades is rectangular, we get

$$\oint y(s)\mathrm{d}s = \oint z(s)\mathrm{d}s = 0. \tag{23}$$

Utilizing (23), the kinetic energy is obtained

$$\begin{aligned}K =& \frac{1}{2}\int_\tau \rho \mathbf{R}_t^2 \mathrm{d}\tau\\
=& \frac{1}{2}\rho\int_\tau u_t^2 + v_t^2 + 2\omega(R_0+x)u_t\\
&\qquad + \omega^2\big(u^2 + 2(y+n\frac{dz}{ds})u + y + n\frac{dz}{ds} + (R_0+x)^2\big)\mathrm{d}\tau\\
=& \frac{1}{2}\rho\bar{A}\int_0^l u_t^2 + v_t^2 + 2\omega(R_0+x)u_t\\
&\qquad + \omega^2\big(u^2 + 2\frac{h\oint y\mathrm{d}s}{\bar{A}}u + \frac{h\oint y\mathrm{d}s}{\bar{A}} + (R_0+x)^2\big)\mathrm{d}x\\
=& \frac{1}{2}\rho\bar{A}\int_0^l u_t^2 + v_t^2 + 2\omega(R_0+x)u_t + \omega^2\big(u^2 + (R_0+x)^2\big)\mathrm{d}x,\end{aligned}$$

where $\mathrm{d}\tau$ denotes the differential volume element. Then

$$\int_{t_1}^{t_2} \delta K \mathrm{d}t = -\rho\bar{A}\int_{t_1}^{t_2}\int_0^l \big\{[u_{tt} + \omega_t(R_0+x) - u\omega^2]\delta u + v_{tt}\delta v\big\}\mathrm{d}x\mathrm{d}t, \tag{24}$$

Due to the rotating motion of the blades, the total strain energy consists of two parts. The strain energy caused by the centrifugal force can be obtained as

$$U_1 = \frac{1}{2}\int_0^l \oint \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_x^l \rho\omega^2(R_0+\varsigma)\varepsilon_{xx}\mathrm{d}\varsigma\mathrm{d}n\mathrm{d}s\mathrm{d}x \tag{25}$$

where ρ is the bulk density of the blades, $\int_x^l \rho\omega^2(R_0+\varsigma)\mathrm{d}\varsigma$ is the centrifugal force per unit cross section. Then,

$$\begin{aligned}\delta U_1 =&\frac{1}{2}\int_0^l \oint \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_x^l \rho\omega^2(R_0+\varsigma)\delta\varepsilon_{xx}\mathrm{d}\varsigma\mathrm{d}n\mathrm{d}s\mathrm{d}x\\ =&\frac{1}{2}\int_0^l \oint \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_x^l \rho\omega^2(R_0+\varsigma)\delta\bar{\varepsilon}_{xx}\mathrm{d}\varsigma\mathrm{d}n\mathrm{d}s\mathrm{d}x\\ =&\rho\bar{A}\omega^2\int_0^l R(x)(u_x\delta u_x+v_x\delta v_x)\mathrm{d}x\\ &-\frac{\rho\omega^2 h}{2}\int_0^l\int_x^l(R_0+\varsigma)(\oint y(s)\mathrm{d}s\delta u_{xx}+\oint z(s)\mathrm{d}s\delta v_{xx})\mathrm{d}\varsigma\mathrm{d}x,\end{aligned} \tag{26}$$

where $\bar{A}$ is the area of the cross section of the blades, $R(x)=R_0(l-x)+\frac{1}{2}(l^2-x^2)$.

Thanks to (23), the variation of the strain energy caused by the centrifugal force can be rewritten as

$$\delta U_1 = \rho\bar{A}\omega^2\big[R(x)(u_x\delta u+v_x\delta v)\big]\Big|_0^l - \int_0^l (R(x)u_x)_x\delta u + (R(x)v_x)_x\delta v\mathrm{d}x \tag{27}$$

The strain energy induced by the deformation of the rotating blades can be expressed as

$$U_2 = \frac{1}{2}\int_0^l\oint\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_{xx}\varepsilon_{xx}\mathrm{d}n\mathrm{d}s\mathrm{d}x. \tag{28}$$

Substituting the expressions of the stress and strain resultants into (28) yields

$$\begin{aligned}\delta U_2 =&E\int_0^l\oint\int_{-\frac{h}{2}}^{\frac{h}{2}}\varepsilon_{xx}\delta\varepsilon_{xx}dndsdx\\ =&\frac{Eh}{4}\int_0^l\oint(u_x^2+v_x^2)\delta(u_x^2+v_x^2)dsdx\\ &-\frac{Eh}{2}\int_0^l\oint(u_x^2+v_x^2)\delta(u_{xx}y(s)+v_{xx}z(s))dsdx\\ &-\frac{Eh}{2}\int_0^l\oint(u_{xx}y(s)+v_{xx}z(s))\delta(u_x^2+v_x^2)dsdx\\ &+Eh\int_0^l\oint(u_{xx}y(s)+v_{xx}z(s))\delta(u_{xx}y(s)+v_{xx}z(s))dsdx\\ &+\frac{Eh^3}{12}\int_0^l\oint(-u_{xx}\frac{\mathrm{d}z}{\mathrm{d}s}+v_{xx}\frac{\mathrm{d}y}{\mathrm{d}s})\delta(-u_{xx}\frac{\mathrm{d}z}{\mathrm{d}s}+v_{xx}\frac{\mathrm{d}y}{\mathrm{d}s})dsdx\end{aligned} \tag{29}$$

$$
\begin{aligned}
\delta U_2 &= E\int_0^l \oint \int_{-\frac{h}{2}}^{\frac{h}{2}} \varepsilon_{xx}\delta\varepsilon_{xx}\mathrm{d}n\mathrm{d}s\mathrm{d}x \\
&= \rho\bar{A}\Bigg\{ \int_0^l [-\frac{a_5}{2}((u_x^2+v_x^2)u_x)_x + (a_6u_{xx}-a_3v_{xx})_{xx} \\
&\qquad - \frac{a_1}{2}(u_x^2+v_x^2)_{xx} + (a_1u_{xx}+a_2v_{xx})u_x)_x]\mathrm{d}x\Bigg\}\delta u \\
&\quad + \rho\bar{A}\Bigg\{ \int_0^l [-\frac{a_5}{2}((u_x^2+v_x^2)v_x)_x + (a_4v_{xx}-a_3u_{xx})_{xx} \\
&\qquad - \frac{a_2}{2}(u_x^2+v_x^2)_{xx} + (a_1u_{xx}+a_2v_{xx})v_x)_x]\mathrm{d}x\Bigg\}\delta v \\
&\quad + \rho\bar{A}\Bigg\{[(a_6u_{xx}-a_3v_{xx}) - \frac{a_1}{2}(u_x^2+v_x^2)]\delta u_x\Bigg\}\Bigg|_0^l \\
&\quad + \rho\bar{A}\Bigg\{[(a_4v_{xx}-a_3u_{xx}) - \frac{a_2}{2}(u_x^2+v_x^2)]\delta v_x\Bigg\}\Bigg|_0^l \\
&\quad + \rho\bar{A}\Bigg\{[\frac{a_5}{2}(u_x^2+v_x^2)u_x - (a_6u_{xx}-a_3v_{xx})_x \\
&\qquad + \frac{a_1}{2}(u_x^2+v_x^2)_x + a_1u_{xx}+a_2v_{xx}u_x]\delta u\Bigg\}\Bigg|_0^l \\
&\quad + \rho\bar{A}\Bigg\{[\frac{a_5}{2}(u_x^2+v_x^2)v_x - (a_4v_{xx}-a_3u_{xx})_x \\
&\qquad + \frac{a_2}{2}(u_x^2+v_x^2)_x + a_1u_{xx}+a_2v_{xx}v_x]\delta v\Bigg\}\Bigg|_0^l
\end{aligned} \tag{30}
$$

where

$$
\begin{aligned}
a_1 &= \frac{Eh}{\bar{A}\rho}\oint y\mathrm{d}s = 0; \\
a_2 &= \frac{Eh}{\bar{A}\rho}\oint z\mathrm{d}s = 0; \\
a_3(x) &= \frac{Eh}{\bar{A}\rho}\oint \frac{h^2}{12}\frac{\mathrm{d}y}{\mathrm{d}s}\frac{\mathrm{d}z}{\mathrm{d}s} - yz\mathrm{d}s; \\
a_4(x) &= \frac{Eh}{\bar{A}\rho}\oint \frac{h^2}{12}(\frac{\mathrm{d}z}{\mathrm{d}s})^2 + y^2\mathrm{d}s; \\
a_5 &= \frac{E}{\rho} \\
a_6(x) &= \frac{Eh}{\bar{A}\rho}\oint \frac{h^2}{12}(\frac{\mathrm{d}y}{\mathrm{d}s})^2 + z^2\mathrm{d}s.
\end{aligned}
$$

The work of the non-conservative external forces can be obtained as

$$
W = \int_0^l p_y u + p_z v\mathrm{d}x. \tag{31}
$$

Then

$$\begin{aligned}\int_{t_1}^{t_2} \delta W \mathrm{d}t = &\int_{t_1}^{t_2}\int_0^l \tilde{p}_y \delta u + \tilde{p}_z \delta v \mathrm{d}x\mathrm{d}t \\ &+ \int_{t_1}^{t_2} \Big\{(b_1 u + e_1 v)\delta u + (b_2 u + e_2 v)\delta v\Big\}\Big|_0^l \mathrm{d}t.\end{aligned} \tag{32}$$

where

$$\tilde{p}_y = 2b_3 u + (e_3 + b_4)v \tag{33}$$

$$\tilde{p}_z = (e_3 + b_4)u + 2e_4 v \tag{34}$$

Inserting variation of potential energy (27) and (30), variation of kinetic energy (24) and variation of external work equation (32) into the extended Hamilton's principle (22), collecting the terms associated with the same variations, invoking the stationarity of the functional within the time interval $[t_0, t_1]$, and the fact that the variations $(\delta u, \delta v)$ are independent and arbitrary, their coefficients in the two integrands must vanish independently [see [21]]. The partial differential equations with respect to variation of problem's independent variables and the associated boundary conditions are obtained as

$$\begin{aligned}u_{tt} + \Big(a_6 u_{xx} - a_3 v_{xx}\Big)_{xx} - \frac{a_5}{2}((u_x^2 + v_x^2)u_x)_x \\ -\omega^2 (R(x)u_x)_x - \omega^2 u - \overline{p}_1 + \omega_t (R_0 + x) = 0,\end{aligned} \tag{35}$$

$$v_{tt} + \Big(a_4 v_{xx} - a_3 u_{xx}\Big)_{xx} - \frac{a_5}{2}((u_x^2 + v_x^2)v_x)_x - \omega^2 (R(x)v_x)_x - \overline{p}_2 = 0, \tag{36}$$

where

$$\overline{p}_1 = \frac{\tilde{p}_y}{\rho \bar{A}}, \quad \overline{p}_2 = \frac{\tilde{p}_z}{\rho \bar{A}}. \tag{37}$$

The following two types of boundary conditions are generated due to the different design of engine blades.

$$u, u_x = 0, v, v_x = 0, \qquad x = 0, \tag{38}$$

$$u, u_x = 0, v, v_x = 0, \qquad x = l. \tag{39}$$

and

$$u, u_x = 0, v, v_x = 0, \qquad x = 0, \tag{40}$$

$$a_6 u_{xx} - a_3 v_{xx} = 0, \qquad x = l, \tag{41}$$

$$a_4 v_{xx} - a_3 u_{xx} = 0, \qquad x = l, \tag{42}$$

$$\frac{a_5}{2}(u_x^2 + v_x^2)u_x - (a_6 u_{xx} - a_3 v_{xx})_x - b_1 u - e_1 v = 0, \qquad x = l, \tag{43}$$

$$\frac{a_5}{2}(u_x^2 + v_x^2)v_x - (a_4 v_{xx} - a_3 u_{xx})_x - b_2 u - e_2 v = 0, \qquad x = l. \tag{44}$$

The conditions (38)-(39) represent C-C boundary condition. The conditions (40)-(44) represent C-F boundary condition. Now we study the well-posedness and regularity of the solution for C-C and C-F blades.

3. **Preliminary.** We write $Q = \Omega \times (0, T)$, where $\Omega = (0, l)$ and $T > 0$,

$$H_0^2 = \{\psi \in H^2(\Omega) \mid \psi(x) = 0, \psi_x(x) = 0, x \in \partial\Omega\},$$
$$H_f^2 = \{\psi \in H^2(\Omega) \mid \psi(0) = 0, \psi_x(0) = 0, x = 0\}.$$

We list Gagliardo-Nirenberg inequality for bounded domains (see [23]) to be used in the subsequent sections

Lemma 3.1. *Let $\Omega \subset R^n$ be a bounded domain with smooth boundary. Let $1 \leq p, q, r \leq \infty$ be real numbers and $j \leq m$ be non-negative integers. If a real number α satisfies*

$$\frac{1}{p} - \frac{j}{n} = \alpha\left(\frac{1}{r} - \frac{m}{n}\right) + (1-\alpha)\frac{1}{q}, \quad \frac{j}{m} \leq \alpha \leq 1$$

Then

$$\left\|\mathrm{D}^j f\right\|_{L^p(\Omega)} \leq C_1 \left\|\mathrm{D}^m f\right\|^{\alpha}_{L^r(\Omega)} \|f\|^{1-\alpha}_{L^q(\Omega)} + C_2\|f\|_{L^s(\Omega)}$$

where $s > 0$, and the constants C_1 and C_2 depend upon Ω and the indices p, q, r, m, j, s only.

Now, we give the Aubin-Lions Lemma (see [22]).

Lemma 3.2. *Suppose B_0, B, B_1 are Banach Space, if*
(i) $\{u_i\}_{i=1}^{\infty}$ is bounded in $L^{p_0}(0, T; B_0)$;
(ii) $\{u_{i,t}\}_{i=1}^{\infty}$ is bounded in $L^{p_1}(0, T; B_1)$;
(iii) $B_0 \hookrightarrow\hookrightarrow B \hookrightarrow B_1$,
then $\{u_i\}_{i=1}^{\infty}$ admits a strongly converging subsequence in $L^{p_0}(0, T; B)$, provided $p_0 < \infty, p_1 > 1$.

4. Mathematical results of C-C blades.

Without loss of generality, we assume $a_5 = 1$. For the sake of brevity, the initial boundary-value problems of C-C blades are rewritten as:

$$\begin{cases} u_{tt} + (a_6 u_{xx})_{xx} - (a_3 v_{xx})_{xx} - \dfrac{1}{2}((u_x^2 + v_x^2)u_x)_x & \\ \qquad - \omega^2 (R u_x)_x - p_1 = 0 & \text{in } Q, \\ v_{tt} - (a_3 u_{xx})_{xx} + (a_4 v_{xx})_{xx} - \dfrac{1}{2}((u_x^2 + v_x^2)v_x)_x & \\ \qquad - \omega^2 (R v_x)_x - p_2 = 0 & \text{in } Q, \\ u, u_x = 0, v, v_x = 0 & \text{on } \partial\Omega \times [0, T], \\ u = u_0(x), u_t = u_1(x), v = v_0(x), v_t = v_1(x) & \text{on } \Omega \times \{t = 0\}. \end{cases} \tag{45}$$

where

$$p_1 = \overline{p}_1 + \omega^2 u - \omega_t (R_0 + x), \quad p_2 = \overline{p}_2. \tag{46}$$

Definition 4.1. We say function (u, v), $u, v \in L^\infty(0, T; H_0^2(\Omega))$, with

$$u_t, v_t \in L^\infty(0, T; L^2(\Omega)), u_{tt}, v_{tt} \in L^\infty(0, T; H^{-2}(\Omega)),$$

is a weak solution of the initial boundary value problem (45) provided
(i)

$$\begin{aligned}(u_{tt}, \varphi) + (a_6 u_{xx}, \varphi_{xx}) - (a_3 v_{xx}, \varphi_{xx}) + \frac{1}{2}((u_x^2 + v_x^2)u_x, \varphi_x) \\ + \omega^2 (R u_x, \varphi_x) - (p_1, \varphi) = 0.\end{aligned} \tag{47}$$

$$\begin{aligned}(v_{tt}, \varphi) - (a_3 u_{xx}, \varphi_{xx}) + (a_4 v_{xx}, \varphi_{xx}) + \frac{1}{2}((u_x^2 + v_x^2)v_x, \varphi_x) \\ + \omega^2 (R v_x, \varphi_x) - (p_2, \varphi) = 0.\end{aligned} \tag{48}$$

for each $\varphi \in H_0^2(\Omega)$, and a.e. time $0 \le t \le T$, and
(ii)

$$u(0) = u_0, u_t(0) = u_1; v(0) = v_0, v_t(0) = v_1. \tag{49}$$

Remark 1. From the definition, we know $u, v \in C([0,T]; L^2(\Omega))$ and $u_t, v_t \in C([0,T]; H^{-2}(\Omega))$.

Remark 2. By the product Minkowshi inequality, we can obtain:

$$a_4 a_6 - a_3^2 > 0, \; x \in \Omega. \tag{50}$$

Theorem 4.2. *(Existence for weak solution of* (45)*) Assume*

$$\omega \in C^1(0,T), a_3, a_4, a_6 \in L^\infty(\Omega), u_0, v_0 \in H_0^2(\Omega), u_1, v_1 \in L^2(\Omega). \tag{51}$$

there exists a weak solution of (45)*.*

We now briefly outline the proof of Theorem 4.2 in the following:

Step 1. employing Galerkin's method to construct solutions of certain finite-dimensional approximations to (45) (correspond to Lemma 6.1 in chapter 6);

Step 2. using the energy method to find the uniform estimates of the finite-dimensional approximations solutions (correspond to Lemma 6.2 in chapter 6);

Step 3. using compactness method to obtain the weak solutions of (45).

Now we give the smoothness of weak solutions of (45).

Theorem 4.3. *(Improved regularity) Assume*

$$\begin{cases} a_3, a_4, a_6 \in L^\infty(\Omega), \omega \in C^1(0,T), \omega_{tt} \in L^\infty(0,T), \\ u_0 \in H_0^2(\Omega) \cap H^4(\Omega), v_0 \in H_0^2(\Omega) \cap H^4(\Omega), u_1 \in H_0^2(\Omega), v_1 \in H_0^2(\Omega), \end{cases} \tag{52}$$

the weak solution of (45) *satisfies*

$$u_t, v_t \in L^\infty(0,T; H_0^2(\Omega)), u_{tt}, v_{tt} \in L^\infty(0,T; L^2(\Omega)). \tag{53}$$

Theorem 4.4. *(Interior regularity) Under the condition* (52)*, for any $\varphi(x) \in C_0^\infty(\Omega), \varphi(x) > 0, x \in \Omega$, then the weak solution of* (45) *satisfies*

$$\varphi u_{xxx}, \varphi v_{xxx} \in L^\infty(0,T; L^2(\Omega)).$$

Remark 3. Multiplying the first and second equation of (45) by $a_4(x)u_{tt}, a_3(x)v_{tt}$ respectively, integrating with over Ω', and summing the two equations, we can obtain that the weak solution of (45) satisfies

$$u_{xxxx}, v_{xxxx} \in L^\infty(0,T; L^2(\Omega')).$$

where Ω' is a bounded open interval and $\overline{\Omega}' \subset \Omega$.

If the pre-twist angle $\alpha(x)$ is neglected, we can get $\theta(x), a_3(x), a_4(x), a_6(x)$ are all constant. Then we give the following theorem.

Theorem 4.5. *(Improved regularity when $\alpha(x) = 0$) Under the condition* (52)*, assume $\alpha(x) = 0$, then the weak solution of* (45) *satisfies*

$$u, v \in L^\infty(0,T; H^4(\Omega) \cup H_0^2(\Omega)).$$

Now, we study the uniqueness and stability of (45), denote operator π satisfying

$$\pi : \{a_3, a_4, a_6, \omega, \theta, u_0, v_0, u_1, v_1\} \to \{u, v\},$$

and

$$W(Q) = \{\psi | \psi \in L^\infty(0,T; H_0^2(\Omega)), \psi_t \in L^\infty(0,T; L^2(\Omega))\}$$

with the norm

$$||\psi||_{W(Q)} = ||\psi||_{L^\infty(0,T;H_0^2(\Omega))} + ||\psi_t||_{L^\infty(0,T;L^2(\Omega))}.$$

Then $W(Q)$ is Banach space.

Theorem 4.6. *Undering the condition (52),*
$\pi : \{(L^\infty(\Omega))^3 \times W^{1,1}(0,T) \cap C^1(0,T) \times L^1(\Omega) \times (H_0^2(\Omega))^2 \times (L^2(\Omega))^2\} \to (W(Q))^2$ *is continuous.*

5. Mathematical results of C-F blades.

The initial boundary-value problems of C-F blades are rewritten as :

$$\begin{cases} u_{tt} + (a_6 u_{xx})_{xx} - (a_3 v_{xx})_{xx} - \dfrac{1}{2}((u_x^2 + v_x^2)u_x)_x & \\ \qquad - \omega^2 (R u_x)_x - p_1 = 0 & \text{in } Q, \\ v_{tt} - (a_3 u_{xx})_{xx} + (a_4 v_{xx})_{xx} - \dfrac{1}{2}((u_x^2 + v_x^2)v_x)_x & \\ \qquad - \omega^2 (R v_x)_x - p_2 = 0 & \text{in } Q, \\ u, u_x = 0, v, v_x = 0, & x = 0, \\ a_6 u_{xx} - a_3 v_{xx} = 0, & x = l, \\ a_4 v_{xx} - a_3 u_{xx} = 0, & x = l, \\ \dfrac{1}{2}(u_x^2 + v_x^2)u_x - (a_6 u_{xx} - a_3 v_{xx})_x - b_1 u - e_1 v = 0, & x = l, \\ \dfrac{1}{2}(u_x^2 + v_x^2)v_x - (a_4 v_{xx} - a_3 u_{xx})_x - b_2 u - e_2 v = 0, & x = l, \\ u = u_0(x), u_t = u_1(x), v = v_0(x), v_t = v_1(x) & \text{on } \Omega \times \{t = 0\}. \end{cases} \tag{54}$$

Definition 5.1. We say function (u, v), $u, v \in L^\infty(0,T; H_f^2(\Omega))$, with

$$u_t, v_t \in L^\infty(0,T; L^2(\Omega)), u_{tt}, v_{tt} \in L^\infty(0,T; H^{-2}(\Omega)),$$

is a weak solution of the initial boundary value problem (54) provided

(i)

$$\begin{aligned} (u_{tt}, \varphi) + (a_6 u_{xx}, \varphi_{xx}) - (a_3 v_{xx}, \varphi_{xx}) + \frac{1}{2}((u_x^2 + v_x^2)u_x, \varphi_x) + \omega^2 (R u_x, \varphi_x) \\ -(p_1, \varphi) - \big(b_1 u(l) + e_1 v(l)\big)\varphi(l) = 0. \end{aligned} \tag{55}$$

$$\begin{aligned} (v_{tt}, \varphi) - (a_3 u_{xx}, \varphi_{xx}) + (a_4 v_{xx}, \varphi_{xx}) + \frac{1}{2}((u_x^2 + v_x^2)v_x, \varphi_x) + \omega^2 (R v_x, \varphi_x) \\ -(p_2, \varphi) - \big(b_2 u(l) + e_2 v(l)\big)\varphi(l) = 0. \end{aligned} \tag{56}$$

for each $\varphi \in H_f^2(\Omega)$, and a.e. time $0 \le t \le T$, and

(ii)

$$u(0) = u_0, u_t(0) = u_1; v(0) = v_0, v_t(0) = v_1.$$

Similar as above, we derive the following conclusion of the initial boundary value problem (54).

Theorem 5.2. *(Existence for the weak solutions of (54)) Assume*

$$\omega \in C^1(0,T), a_3, a_4, a_6 \in L^\infty(\Omega), u_0, v_0 \in H_f^2(\Omega), u_1, v_1 \in L^2(\Omega), \tag{57}$$

there exists a weak solution of (54).

Theorem 5.3. *(Regularity weak solution of (54)) Assume*

$$\begin{cases} a_3, a_4, a_6 \in L^\infty(\Omega), \omega \in C^1(0,T), \omega_{tt} \in L^\infty(0,T), \\ \qquad u_0, v_0 \in H_f^2(\Omega) \cap H^4(\Omega), u_1, v_1 \in H_f^2(\Omega). \end{cases} \tag{58}$$

Then there exists $T^* < T$ such that, the weak solution of (54) satisfies

$$u_t, v_t \in L^\infty(0,T^*; H_f^2(\Omega)), u_{tt}, v_{tt} \in L^\infty(0,T^*; L^2(\Omega)). \tag{59}$$

Remark 4. If pre-twist angle $\alpha(x)$ is neglected, according to the first and second equations of (54), we can obtain that the weak solution of (54) satisfies

$$u, v \in L^\infty(0,T^*; H^4(\Omega) \cup H_f^2(\Omega)). \tag{60}$$

where $u_{tt}, v_{tt} \in L^\infty(0,T^*; L^2(\Omega))$ and Gagliardo-Nirenberg inequality are used.

6. **Proof of mathematical results.** We construct weak solution of the initial boundary-value problem (45) by first solving a finite dimensional approximation. We thus employ Galerkin's method by selecting smooth functions $\varphi_k = \varphi_k(x)(k = 1, \cdots)$ such that

$$\{\varphi_k\}_{k=1}^\infty \text{is an orthogonal basis of} H_0^2(\Omega), \tag{61}$$

and

$$\{\varphi_k\}_{k=1}^\infty \text{is an orthonormal basis of} L^2(\Omega). \tag{62}$$

Fix a positive integer m, and write

$$u_m = \sum_{k=1}^m d_{0m}^k(t)\varphi_k,\ \ v_m = \sum_{k=1}^m d_{1m}^k(t)\varphi_k, \tag{63}$$

where we intend to select the coefficients $d_{0m}^k(t), d_{1m}^k(t)(0 \le t \le T, k = 1, \cdots, m)$ to satisfy

$$d_{0m}^k(0) = (u_0, \varphi_k), d_{1m}^k(0) = (v_0, \varphi_k), k = 1, \cdots, m, \tag{64}$$

$$d_{0m,t}^k(0) = (u_1, \varphi_k), d_{1m,t}^k(0) = (v_1, \varphi_k), k = 1, \cdots, m, \tag{65}$$

and

$$\begin{aligned} (u_{m,tt}, \varphi_k) + (a_6 u_{m,xx}, \varphi_{k,xx}) - (a_3 v_{m,xx}, \varphi_{k,xx}) + \frac{1}{2}((u_{m,x}^2 + v_{m,x}^2)u_{m,x}, \varphi_{k,x}) \\ + \omega^2(R u_{m,x}, \varphi_{k,x}) - (p_{1m}, \varphi_k) = 0, \end{aligned} \tag{66}$$

$$\begin{aligned} (v_{m,tt}, \varphi_k) - (a_3 u_{m,xx}, \varphi_{k,xx}) + (a_4 v_{m,xx}, \varphi_{k,xx}) + \frac{1}{2}((u_{m,x}^2 + v_{m,x}^2)v_{m,x}, \varphi_{k,x}) \\ + \omega^2(R v_{m,x}, \varphi_{k,x}) - (p_{2m}, \varphi_k) = 0. \end{aligned} \tag{67}$$

where

$$p_{1m} = \frac{1}{\rho \bar{A}}\big(2b_3 u_m + (e_3 + b_4)v_m\big) + \omega^2 u_m - \omega_t(R_0 + x),$$

$$p_{2m} = \frac{1}{\rho \bar{A}}\big((e_3 + b_4)u_m + 2e_4 v_m\big).$$

In order to proof Theorem 4.2, we need the following Lemma.

Lemma 6.1. *Under the condition (51), for each integer $m = 1, 2, \cdots$, there exists a unique function u_m, v_m of the form (63) satisfying (64)-(67) for $0 \le t \le t_m$.*

Proof. Assuming u_m, v_m to be given by (63), by using (62), equation (66) and (67) become the nonlinear system of ODE

$$\begin{aligned} &d^k_{0m,tt} + \sum_{j=1}^m d^j_{0m}(a_6\varphi_{j,xx}, \varphi_{k,xx}) - \sum_{j=1}^m d^j_{1m}(a_3\varphi_{j,xx}, \varphi_{k,xx}) \\ &+ \frac{1}{2}\sum_{j=1}^m d^j_{0m}\Big(\big((\sum_{i=1}^m d^i_{0m}\varphi_{i,x})^2 + (\sum_{i=1}^m d^i_{1m}\varphi_{i,x})^2\big)\varphi_{j,x}, \varphi_{k,x}\Big) \\ &+ \omega^2 \sum_{j=1}^m d^j_{0m}(R\varphi_{j,x}, \varphi_{k,x}) - \frac{1}{\rho\bar{A}}\big(2b_3 d^k_{0m} + (e_3+e_4)d^k_{1m}\big) \\ =&\omega^2 d^k_{0m} - \omega_t((R_0 + x), \varphi_k), \end{aligned} \tag{68}$$

$$\begin{aligned} &d^k_{1m,tt} + \sum_{j=1}^m d^j_{1m}(a_4\varphi_{j,xx}, \varphi_{k,xx}) - \sum_{j=1}^m d^j_{0m}(a_3\varphi_{j,xx}, \varphi_{k,xx}) \\ &+ \frac{1}{2}\sum_{j=1}^m d^j_{1m}\Big(\big((\sum_{i=1}^m d^i_{0m}\varphi_{i,x})^2 + (\sum_{i=1}^m d^i_{1m}\varphi_{i,x})^2\big)\varphi_{j,x}, \varphi_{k,x}\Big) \\ &+ \omega^2 \sum_{j=1}^m d^j_{1m}(R\varphi_{j,x}, \varphi_{k,x}) - \frac{1}{\rho\bar{A}}\big((e_3+e_4)d^k_{0m} + 2e_4 d^k_{1m}\big) = 0. \end{aligned} \tag{69}$$

subject to the initial conditions (64), (65). According to standard theory for ODE, there exists unique C^2 funcion $\mathbf{d}_{0m}(t) = (d^1_{0m}, d^2_{0m}, \cdots, d^m_{0m}), \mathbf{d}_{1m}(t) = (d^1_{1m}, d^2_{1m}, \cdots, d^m_{1m})$, satisfying (64), (65), and solving (68), (69) for $0 \le t \le t_m$, where $t_m := t(m)$ is a function of m. □

We propose now to send m to infinity and to show a subsequence of the solutions u_m, v_m of the approximate problem (64)-(67) converges to a weak solution of (45). For this we will need the following uniform estimates.

Lemma 6.2. *Undering the condition (51), there exists positive constant $C(\Omega, T)$, such that*

$$\begin{aligned} &||u_{m,t}||^2_{L^2(\Omega)} + ||v_{m,t}||^2_{L^2(\Omega)} + ||u_m||^2_{H^2_0(\Omega)} + ||v_m||^2_{H^2_0(\Omega)} \\ \le& C(||u_1||^2_{L^2(\Omega)} + ||v_1||^2_{L^2(\Omega)} + ||u_0||^2_{H^2_0(\Omega)} + ||v_0||^2_{H^2_0(\Omega)}) + C \end{aligned} \tag{70}$$

for $m = 1, 2, \cdots$.

Proof. Multiplying equality (66) by $d^k_{0m,t}$, summing $k = 1, 2, \cdots$, we deduce

$$\begin{aligned} &(u_{m,tt}, u_{m,t}) + (a_6 u_{m,xx}, u_{m,txx}) - (a_3 v_{m,xx}, u_{m,txx}) \\ &+ \frac{1}{2}((u^2_{m,x} + v^2_{m,x})u_{m,x}, u_{m,tx}) + \omega^2(R u_{m,x}, u_{m,tx}) - (p_{1m}, u_{m,t}) = 0 \end{aligned} \tag{71}$$

for a.e. $0 \le t \le t_m$.

Multiplying equality (67) by $d^k_{1m,t}$, summing $k = 1, 2, \cdots$, we deduce

$$\begin{aligned} &(v_{m,tt}, v_{m,t}) - (a_3 u_{m,xx}, v_{m,txx}) + (a_4 v_{m,xx}, v_{m,txx}) \\ &+ \frac{1}{2}((u^2_{m,x} + v^2_{m,x})v_{m,x}, v_{m,tx}) + \omega^2(R v_{m,x}, v_{m,tx}) - (p_{2m}, v_{m,t}) = 0 \end{aligned} \tag{72}$$

for a.e. $0 \le t \le t_m$.

To simplify the equation (71) and (72), we can get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{m,t}\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{a_6}u_{m,xx}\|^2_{L^2(\Omega)} - (a_3 v_{m,xx}, u_{m,txx})\\ &+ \frac{1}{2}((u^2_{m,x} + v^2_{m,x})u_{m,x}, u_{m,tx}) + \frac{\omega^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{R}u_{m,x}\|^2_{L^2(\Omega)} - (p_{1m}, u_{m,t}) = 0.\end{aligned} \tag{73}$$

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|v_{m,t}\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{a_4}v_{m,xx}\|^2_{L^2(\Omega)} - (a_3 u_{m,xx}, v_{m,txx})\\ &+ \frac{1}{2}((u^2_{m,x} + v^2_{m,x})v_{m,x}, v_{m,tx}) + \frac{\omega^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{R}v_{m,x}\|^2_{L^2(\Omega)} - (p_{2m}, v_{m,t}) = 0.\end{aligned} \tag{74}$$

Summing the equations (73) and (74), we discover

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{||u_{m,t}||^2_{L^2(\Omega)} + ||v_{m,t}||^2_{L^2(\Omega)} + ||\sqrt{a_6}u_{m,xx}||^2_{L^2(\Omega)} + ||\sqrt{a_4}v_{m,xx}||^2_{L^2(\Omega)}\\ &+ \frac{\omega^2}{2}(||\sqrt{R}u_{m,x}||^2_{L^2(\Omega)} + ||\sqrt{R}v_{m,x}||^2_{L^2(\Omega)})\} - \frac{\mathrm{d}}{\mathrm{d}t}||a_3 u_{m,xx}v_{m,xx}||_{L^1(\Omega)}\\ &+ \frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}t}||u^2_{m,x} + v^2_{m,x}||^2_{L^2(\Omega)}\\ =&\omega\omega_t(||\sqrt{R}u_{m,x}||^2_{L^2(\Omega)} + ||\sqrt{R}v_{m,x}||^2_{L^2(\Omega)}) + (p_{1m}, u_{m,t}) + (p_{2m}, v_{m,t}).\end{aligned} \tag{75}$$

Since $\omega \in C^1(0,T)$, the equality (75) implies

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{||u_{m,t}||^2_{L^2(\Omega)} + ||v_{m,t}||^2_{L^2(\Omega)} + ||\sqrt{a_6}u_{m,xx}||^2_{L^2(\Omega)} + ||\sqrt{a_4}v_{m,xx}||^2_{L^2(\Omega)}\\ &+ \frac{\omega^2}{2}(||\sqrt{R}u_{m,x}||^2_{L^2(\Omega)} + ||\sqrt{R}v_{m,x}||^2_{L^2(\Omega)})\} - \frac{\mathrm{d}}{\mathrm{d}t}||a_3 u_{m,xx}v_{m,xx}||_{L^1(\Omega)}\\ &+ \frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}t}||u^2_{m,x} + v^2_{m,x}||^2_{L^2(\Omega)}\\ \le&C(||u_{m,t}||^2_{L^2(\Omega)} + ||v_{m,t}||^2_{L^2(\Omega)} + ||u_m||^2_{H^1_0(\Omega)} + ||v_m||^2_{H^1_0(\Omega)}) + C.\end{aligned} \tag{76}$$

where we used Young inequality.

Integrating (76) with respect to t, we discover

$$\begin{aligned}&||u_{m,t}||^2_{L^2(\Omega)} + ||v_{m,t}||^2_{L^2(\Omega)} + ||\sqrt{a_6}u_{m,xx}||^2_{L^2(\Omega)} + ||\sqrt{a_4}v_{m,xx}||^2_{L^2(\Omega)}\\ &- 2||a_3 u_{m,xx}v_{m,xx}||_{L^1(\Omega)} + \frac{\omega^2}{2}(||\sqrt{R}u_{m,x}||^2_{L^2(\Omega)} + ||\sqrt{R}v_{m,x}||^2_{L^2(\Omega)})\\ &+ \frac{1}{4}||u^2_{m,x} + v^2_{m,x}||^2_{L^2(\Omega)}\\ \le&C(\Omega,T)\{||u_{1m}||^2_{L^2(\Omega)} + ||v_{1m}||^2_{L^2(\Omega)} + ||u_{0m}||^2_{H^2(\Omega)} + ||v_{0m}||^2_{H^2(\Omega)}\}\\ &+ C\int_0^t ||u_{m,t}||^2_{L^2(\Omega)} + ||v_{m,t}||^2_{L^2(\Omega)}\mathrm{d}t + ||u_m||^2_{H^1_0(\Omega)} + ||v_m||^2_{H^1_0(\Omega)} + C.\end{aligned} \tag{77}$$

Thanks to (50), there exists a constant C, such that

$$\begin{aligned}&||\sqrt{a_6}u_{m,xx}||^2_{L^2(\Omega)} + ||\sqrt{a_4}v_{m,xx}||^2_{L^2(\Omega)} - 2||a_3 u_{m,xx}v_{m,xx}||_{L(\Omega)}\\ \ge&C(||u_m||^2_{H^2_0(\Omega)} + ||v_m||^2_{H^2_0(\Omega)}).\end{aligned} \tag{78}$$

Substituting (78) into the inequality (77), By using Poincaré inequality, we find

$$\begin{aligned} &||u_{m,t}||^2_{L^2(\Omega)} + ||v_{m,t}||^2_{L^2(\Omega)} + ||u_m||^2_{H^2_0(\Omega)} + ||v_m||^2_{H^2_0(\Omega)} \\ \le& C(||u_{1m}||^2_{L^2(\Omega)} + ||v_{1m}||^2_{L^2(\Omega)} + ||u_{0m}||^2_{H^2_0(\Omega)} + ||v_{0m}||^2_{H^2_0(\Omega)}) \\ &+ C\int_0^t ||u_{m,t}||^2_{L^2(\Omega)} + ||v_{m,t}||^2_{L^2(\Omega)} \mathrm{d}t + ||u_m||^2_{H^2_0(\Omega)} + ||v_m||^2_{H^2_0(\Omega)} + C. \end{aligned} \tag{79}$$

Then, by using Gronwall inequality, we obtain

$$\begin{aligned} ||u_{m,t}||^2_{L^2(\Omega)} +& ||v_{m,t}||^2_{L^2(\Omega)} + ||u_m||^2_{H^2_0(\Omega)} + ||v_m||^2_{H^2_0(\Omega)} \\ &\le C(||u_{m,t}(0)||^2_{L^2(\Omega)} + ||v_{m,t}(0)||^2_{L^2(\Omega)} \\ &\quad + ||u_m(0)||^2_{H^2_0(\Omega)} + ||v_m(0)||^2_{H^2_0(\Omega)}) + C \\ &\le C(||u_1||^2_{L^2(\Omega)} + ||v_1||^2_{L^2(\Omega)} + ||u_0||^2_{H^2_0(\Omega)} + ||v_0||^2_{H^2_0(\Omega)}) + C \end{aligned}$$

for $m = 1, 2, \cdots$. □

Remark 5. From the Energy estimates, we can obtain: $t_m \to T$, as $m \to \infty$.

Thanks to Lemma 6.1 and Lemma 6.2, we can obtain the existence for the weak solutions of the initial boundary value problem (45).

6.1. Proof of Theorem 4.2.

Proof. (i) According to the energy estimates (70), we see that

$$\{u_m\}_{m=1}^\infty, \{v_m\}_{m=1}^\infty \quad \text{is bounded in } L^\infty(0,T;H^2_0(\Omega)); \tag{80}$$

$$\{u_{m,t}\}_{m=1}^\infty, \{v_{m,t}\}_{m=1}^\infty \quad \text{is bounded in } L^\infty(0,T;L^2(\Omega)); \tag{81}$$

$$\{u_{m,tt}\}_{m=1}^\infty, \{v_{m,tt}\}_{m=1}^\infty \quad \text{is bounded in } L^\infty(0,T;H^{-2}(\Omega)). \tag{82}$$

As a consequence there exists subsequence $\{u_\mu, v_\mu\}_{m=1}^\infty$ and $u, v \in L^\infty(0,T;$ $H^2_0(\Omega))$, with $u_t, v_t \in L^\infty(0,T;L^2(\Omega))$, $u_{tt}, v_{tt} \in L^\infty(0,T;H^{-2}(\Omega))$, such that

$$\begin{cases} u_\mu \to u, v_\mu \to v & \text{weakly } * \text{ in } \ L^\infty(0,T;H^2_0(\Omega)) \\ u_{\mu,t} \to u_t, v_{\mu,t} \to v_t & \text{weakly } * \text{ in } \ L^\infty(0,T;L^2(\Omega)) \\ u_{\mu,tt} \to u_{tt}, v_{\mu,tt} \to v_{tt} & \text{weakly } * \text{ in } \ L^\infty(0,T;H^{-2}(\Omega)). \end{cases} \tag{83}$$

(ii) By Gagliardo-Nirenberg inequality, we can see

$$||u_{\mu,x}||_{L^\infty(Q)} \le C||u_{\mu,xx}||_{L^\infty(0,T;L^2(\Omega))} \le C. \tag{84}$$

Otherwise,

$$\begin{aligned} ||u^2_{\mu,x} + v^2_{\mu,x}||_{L^\infty(0,T;L^2(\Omega))} &\le ||u_{\mu,x}||^2_{L^\infty(0,T;L^4(\Omega))} + ||v_{\mu,x}||^2_{L^\infty(0,T;L^4(\Omega))} \\ &\le C||u_\mu||^2_{L^\infty(0,T;H^2_0(\Omega))} + c||v_\mu||^2_{L^\infty(0,T;H^2_0(\Omega))} \\ &\le C. \end{aligned} \tag{85}$$

Combining (84) and (85), we discover

$$||(u^2_{\mu,x} + v^2_{\mu,x})u_{\mu,x}||_{L^\infty(0,T;L^2(\Omega))} \le C||u^2_{\mu,x} + v^2_{\mu,x}||_{L^\infty(0,T;L^2(\Omega))} \le C. \tag{86}$$

Moreover, there exists a function $\chi \in L^\infty(0,T;L^2(\Omega))$ satisfy

$$(u^2_{\mu,x} + v^2_{\mu,x})u_{\mu,x} \to \chi \ \text{ weakly } * \text{ in } \ L^\infty(0,T;L^2(\Omega)). \tag{87}$$

By Lemma 3.2, we can find

$$u_\mu \to u, v_\mu \to v \;\text{ strongly in }\; L^2(0,T;H_0^1(\Omega)). \tag{88}$$

And so

$$u_{\mu,x} \to u_x, v_{\mu,x} \to v_x \;\text{ strongly in }\; L^2(Q), \tag{89}$$

Thus

$$(u_{\mu,x}^2 + v_{\mu,x}^2)u_{\mu,x} \to (u_x^2 + v_x^2)u_x. \tag{90}$$

Combining (87) and (90), we can obtain

$$(u_{\mu,x}^2 + v_{\mu,x}^2)u_{\mu,x} \to (u_x^2 + v_x^2)u_x \;\text{ weakly in }\; L^2(Q), \tag{91}$$

where we used the Lemma 1.3 of Chapter 1 in [22]. Furthermore, we have $\chi = (u_x^2 + v_x^2)u_x$.

Meanwhile,

$$((u_{\mu,x}^2 + v_{\mu,x}^2)u_{\mu,x}, \varphi_{k,x}) \to ((u_x^2 + v_x^2)u_x, \varphi_{k,x}) \;\text{ weakly * in }\; L^\infty(0,T). \tag{92}$$

In the same way,

$$((u_{\mu,x}^2 + v_{\mu,x}^2)v_{\mu,x}, \varphi_{k,x}) \to ((u_x^2 + v_x^2)v_x, \varphi_{k,x}) \;\text{ weakly * in }\; L^\infty(0,T). \tag{93}$$

Next fix an integer k, we select $\mu \geq k$, from (66) we can get

$$\begin{aligned}&(u_{\mu,tt}, \varphi_k) + (a_6 u_{\mu,xx}, \varphi_{k,xx}) - (a_3 v_{\mu,xx}, \varphi_{k,xx}) + \frac{1}{2}((u_{\mu,x}^2 + v_{\mu,x}^2)u_{\mu,x}, \varphi_{k,x})\\ &- \omega^2(Ru_{\mu,x}, \varphi_{k,x}) - (p_{1\mu}, \varphi_k) = 0.\end{aligned} \tag{94}$$

Thanks to (83), we can get

$$\begin{cases} (u_{\mu,tt}, \varphi_k) \to (u_{tt}, \varphi_k) & \text{weakly * in } \; L^\infty(0,T),\\ (a_6 u_{\mu,xx}, \varphi_{k,xx}) \to (a_6 u_{xx}, \varphi_{k,xx}) & \text{weakly * in } \; L^\infty(0,T),\\ (a_3 v_{\mu,xx}, \varphi_{k,xx}) \to (a_3 v_{xx}, \varphi_{k,xx}) & \text{weakly * in } \; L^\infty(0,T),\\ (p_{1\mu}, \varphi_k) \to (p_1, \varphi_k) & \text{weakly * in } \; L^\infty(0,T). \end{cases} \tag{95}$$

From (92) and (95), we can discover

$$\begin{aligned}&+v_x^2)u_x, \varphi_{k,x})\\ &+\omega^2(Ru_x, \varphi_{k,x}) - (p_1, \varphi_k) = 0\end{aligned} \tag{96}$$

for all fixed k.

Note

$$\{\varphi_k\}_{k=1}^\infty \text{ is an orthogonal basis of } H_0^2(\Omega),$$

then,

$$\begin{aligned}(u_{tt}, \varphi) + (a_6 u_{xx}, \varphi_{xx}) - (a_3 v_{xx}, \varphi_{xx}) + \frac{1}{2}((u_x^2 + v_x^2)u_x, \varphi_x)&\\ +\omega^2(Ru_x, \varphi_x) - (p_1, \varphi) = 0&\end{aligned} \tag{97}$$

for arbitrary $\varphi \in H_0^2(\Omega)$.

In the same way,

$$\begin{aligned}(v_{tt}, \varphi) + (a_4 v_{xx}, \varphi_{xx}) - (a_3 u_{xx}, \varphi_{xx}) + \frac{1}{2}((u_x^2 + v_x^2)v_x, \varphi_x)&\\ +\omega^2(Rv_x, \varphi_x) - (p_2, \varphi) = 0&\end{aligned} \tag{98}$$

for arbitrary $\varphi \in H_0^2(\Omega)$.

Synthesizes the above analysis, there exists u, v satisfying (47), (48), and

$$\begin{aligned} u, v &\in L^\infty(0,T; H_0^2(\Omega)), \\ u_t, v_t &\in L^\infty(0,T; L^2(\Omega)), \\ u_{tt}, v_{tt} &\in L^\infty(0,T; H^{-2}(\Omega)). \end{aligned}$$

(iii) Now let's prove the initial conditions.

Since, $u_\mu, v_\mu, u_{\mu,t}, v_{\mu,t}$ are bounded in $L^2(Q)$, by Lemma 1.2 of Chapter 1 in [22], we can discover

$$u_\mu(x,0) \to u(x,0) \text{ weakly in } L^2(\Omega). \tag{99}$$

Otherwise,

$$u_m(x,0) \to u_0(x) \text{ in } H_0^2(\Omega). \tag{100}$$

Combining identities (99) and (100), we can get

$$u(x,0) = u_0(x).$$

Next, according to (83), we can obtain

$$(u_{\mu,t}, \varphi_k) \to (u_t, \varphi_k) \text{ weakly * in } L^\infty(0,T),$$

$$(u_{\mu,tt}, \varphi_k) \to (u_{tt}, \varphi_k) \text{ weakly * in } L^\infty(0,T).$$

Then, we can discover

$$(u_{\mu,t}(x,0), \varphi_k) \to (u_t, \varphi_k)|_{t=0} = (u_t(x,0), \varphi_k). \tag{101}$$

Otherwise,

$$(u_{m,t}(x,0), \varphi_k) \to (u_1(x), \varphi_k), \tag{102}$$

Comparing identities (101) and (102), we can get

$$(u_t(x,0), \varphi_k) = (u_1(x), \varphi_k), \text{ for arbitrary } k.$$

So

$$u_t(x,0) = u_1(x).$$

In the same way, we can obtain

$$v(x,0) = v_0(x), v_t(x,0) = v_1(x).$$

□

6.2. **Proof of Theorem 4.3.**

Proof. Differentiating the first equation of (45) with respect to t, multiplying by u_{tt} and integrating with respect to x, we discover

$$\begin{aligned} &(u_{ttt}, u_{tt}) + ((a_6 u_{txx})_{xx}, u_{tt}) - ((a_3 v_{txx})_{xx}, u_{tt}) - \frac{1}{2}((((u_x^2 + v_x^2) u_{tx})_x, u_{tt}) \\ &- ((u_x^2 u_{tx})_x, u_{tt}) - ((u_x v_x v_{tx})_x, u_{tt}) - 2\omega\omega_t((Ru_x)_x, u_{tt}) \\ &- \omega^2((Ru_{tx})_x, u_{tt}) - (p_{1,t}, u_{tt}) = 0. \end{aligned} \tag{103}$$

Differentiating the second equation of (45) with respect to t, multiplying by v_{tt} and integrating with respect to x, we discover

$$\begin{aligned}&(v_{ttt},v_{tt})+((a_4v_{txx})_{xx},v_{tt})-((a_3u_{txx})_{xx},v_{tt})-\frac{1}{2}(((u_x^2+v_x^2)v_{tx})_x,v_{tt})\\&-((v_x^2v_{tx})_x,v_{tt})-((u_xv_xu_{tx})_x,v_{tt})-2\omega\omega_t((Rv_x)_x,v_{tt})\\&-\omega^2((Rv_{tx})_x,v_{tt})-(p_{2,t},v_{tt})=0.\end{aligned}\tag{104}$$

Summing the equation (103) and (104), we discover after integrating by parts:

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||u_{tt}||^2_{L^2(\Omega)}+||v_{tt}||^2_{L^2(\Omega)})+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\sqrt{a_6}u_{txx}||^2_{L^2(\Omega)}+||\sqrt{a_4}v_{txx}||^2_{L^2(\Omega)})\\&-\frac{\mathrm{d}}{\mathrm{d}t}||a_3u_{txx}v_{txx}||_{L^1(\Omega)}+\frac{\omega^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\sqrt{R(x)}u_{tx}||^2_{L^2(\Omega)}+\frac{\omega^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}||\sqrt{R(x)}v_{tx}||^2_{L^2(\Omega)}\\=&\frac{1}{2}((u_x^2+v_x^2)u_{txx},u_{tt})+(u_x^2u_{txx},u_{tt})+(u_xv_xv_{txx},u_{tt})+\frac{1}{2}((u_x^2+v_x^2)v_{txx},v_{tt})\\&+(v_x^2v_{txx},v_{tt})+(u_xv_xu_{txx},v_{tt})+3(u_xu_{xx}u_{tx},u_{tt})+(v_xv_{xx}u_{tx},u_{tt})\\&+(u_{xx}v_xv_{tx},u_{tt})+(u_xv_{xx}v_{tx},u_{tt})+(u_xu_{xx}v_{tx},v_{tt})+3(v_xv_{xx}v_{tx},v_{tt})\\&+(u_{xx}v_xu_{tx},v_{tt})+(u_xv_{xx}u_{tx},v_{tt})+2\omega\omega_t\big((R_xu_x,u_{tt})+(Ru_{xx},u_{tt})\\&+(R_xv_x,v_{tt})+(Rv_{xx},v_{tt})\big)+(p_{1,t},u_{tt})+(p_{2,t},v_{tt}).\end{aligned}\tag{105}$$

Obviously, by Young inequality and Hölder inequality, we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||u_{tt}||^2_{L^2(\Omega)}+||v_{tt}||^2_{L^2(\Omega)})+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\sqrt{a_6}u_{txx}||^2_{L^2(\Omega)}+||\sqrt{a_4}v_{txx}||^2_{L^2(\Omega)})\\&-\frac{\mathrm{d}}{\mathrm{d}t}||a_3u_{txx}v_{txx}||_{L(\Omega)}+\frac{\mathrm{d}}{\mathrm{d}t}[\frac{\omega^2}{2}(||\sqrt{R}u_{tx}||^2_{L^2(\Omega)}+||\sqrt{R}v_{tx}||^2_{L^2(\Omega)})]\\\le&C(||u_t||^2_{H_0^2(\Omega)}+||v_t||^2_{H_0^2(\Omega)}+||u_{tt}||^2_{L^2(\Omega)}+||v_{tt}||^2_{L^2(\Omega)}+||u||^2_{H_0^2(\Omega)}+||v||^2_{H_0^2(\Omega)}+1).\end{aligned}\tag{106}$$

Next integrate (106) with respect to t,

$$\begin{aligned}&||u_{tt}||^2_{L^2(\Omega)}+||v_{tt}||^2_{L^2(\Omega)}+||\sqrt{a_6}u_{txx}||^2_{L^2(\Omega)}+||\sqrt{a_4}v_{txx}||^2_{L^2(\Omega)}\\&-2||a_3u_{txx}v_{txx}||_{L(\Omega)}+\omega^2(||\sqrt{R}u_{tx}||^2_{L^2(\Omega)}+||\sqrt{R}v_{tx}||^2_{L^2(\Omega)})\\\le&C\int_0^t||u_t||^2_{H_0^2(\Omega)}+||v_t||^2_{H_0^2(\Omega)}+||u_{tt}||^2_{L^2(\Omega)}+||v_{tt}||^2_{L^2(\Omega)}+||u||^2_{H_0^2(\Omega)}+||v||^2_{H_0^2(\Omega)}\mathrm{d}t\\&+||u_{tt}(x,0)||^2_{L^2(\Omega)}+||u_{tx}(x,0)||^2_{L^2(\Omega)}+||u_{txx}(x,0)||^2_{L^2(\Omega)}\\&+||v_{tt}(x,0)||^2_{L^2(\Omega)}+||v_{tx}(x,0)||^2_{L^2(\Omega)}+||v_{txx}(x,0)||^2_{L^2(\Omega)}+C.\end{aligned}\tag{107}$$

Since $u_1,v_1\in H_0^2(\Omega)$, we have

$$||u_{txx}(x,0)||_{L^2(\Omega)},||v_{txx}(x,0)||_{L^2(\Omega)},||u_{tx}(x,0)||_{L^2(\Omega)},||v_{tx}(x,0)||_{L^2(\Omega)},\tag{108}$$

are bounded. On the other hand, multiplying the first equation and second equation of (45) by u_{tt},v_{tt}, respectively, and integrating with respect to x, we discover

$$||u_{tt}(x,0)||_{L^2(\Omega)}\le C(||u_0||_{H^4(\Omega)}+||v_0||_{H^4(\Omega)})+C\le C,$$

$$||v_{tt}(x,0)||_{L^2(\Omega)}\le C(||u_0||_{H^4(\Omega)}+||v_0||_{H^4(\Omega)})+C\le C.$$

where the condition (52), Young inequality and Poincaré inequality are used.

As before, we have

$$\begin{aligned}&||u_{tt}||^2_{L^2(\Omega)}+||v_{tt}||^2_{L^2(\Omega)}+||u_t||^2_{H_0^2(\Omega)}+||v_t||^2_{H_0^2(\Omega)}\\&\qquad\le C\int_0^t(||u_{tt}||^2_{L^2(\Omega)}+||v_{tt}||^2_{L^2(\Omega)}+||u_t||^2_{H_0^2(\Omega)}+||v_t||^2_{H_0^2(\Omega)})\mathrm{d}t+C.\end{aligned}\tag{109}$$

Applying the Gronwall inequality to (109) gives

$$\begin{aligned}u_t,v_t&\quad\text{is bounded in}\quad L^\infty(0,T;H_0^2(\Omega)),\\u_{tt},v_{tt}&\quad\text{is bounded in}\quad L^\infty(0,T;L^2(\Omega)).\end{aligned}\tag{110}$$

Moreover, the weak solutions of (45) satisfies

$$\begin{aligned}&u_t\in L^\infty(0,T;H_0^2(\Omega)),\\&v_t\in L^\infty(0,T;H_0^2(\Omega)),\\&u_{tt}\in L^\infty(0,T;L^2(\Omega)),\\&v_{tt}\in L^\infty(0,T;L^2(\Omega)).\end{aligned}$$

□

6.3. **Proof of Theorem 4.4.**

Proof. Multiplying the first equation and the second equation of (45) by $-\varphi^2u_{txx}$, $-\varphi^2v_{txx}$, respectively, summing the two equations, and integrating with respect to x, we discover

$$\begin{aligned}&(u_{tt},-\varphi^2u_{txx})+(v_{tt},-\varphi^2v_{txx})+((a_6u_{xx})_{xx},-\varphi^2u_{txx})\\&+((a_4v_{xx})_{xx},-\varphi^2v_{txx})-((a_3v_{xx})_{xx},-\varphi^2u_{txx})-((a_3u_{xx})_{xx},-\varphi^2v_{txx})\\&-\frac12(((u_x^2+v_x^2)u_x)_x,-\varphi^2u_{txx})-\frac12(((u_x^2+v_x^2)v_x)_x,-\varphi^2v_{txx})\\&-\omega^2((Ru_x)_x,-\varphi^2u_{txx})-\omega^2((Rv_x)_x,-\varphi^2v_{txx})\\&-(p_1,-\varphi^2u_{txx})-(p_2,-\varphi^2v_{txx})=0.\end{aligned}\tag{111}$$

Thanks to integration by parts and the properties of φ, we obtain the following equality

$$I_1+I_2+I_3+I_4+I_5=0.\tag{112}$$

where $I_i, i=1,\cdots,5$, are given as follows respectively

$$\begin{aligned}I_1&=(u_{tt},-\varphi^2u_{txx})+(v_{tt},-\varphi^2v_{txx})\\I_2&=((a_6u_{xx})_{xx},-\varphi^2u_{txx})+((a_4v_{xx})_{xx},-\varphi^2v_{txx})\\I_3&=-((a_3v_{xx})_{xx},-\varphi^2u_{txx})-((a_3u_{xx})_{xx},-\varphi^2v_{txx})\\I_4&=-\frac12(((u_x^2+v_x^2)u_x)_x,-\varphi^2u_{txx})-\frac12(((u_x^2+v_x^2)v_x)_x,-\varphi^2v_{txx})\\I_5&=-(p_1,-\varphi^2u_{txx})-(p_2,-\varphi^2v_{txx}).\end{aligned}$$

To conclude, we need to estimate each of $I_i, i=1,\cdots,5$. By using integration by parts and conclusion (53), we find

$$\begin{aligned}I_1=&\frac12\frac{\mathrm{d}}{\mathrm{d}t}(||\varphi u_{tx}||^2_{L^2(\Omega)}+||\varphi v_{tx}||^2_{L^2(\Omega)})+2(\varphi\varphi_xu_{tt},u_{tx})+2(\varphi\varphi_xv_{tt},v_{tx})\\\ge&\frac12\frac{\mathrm{d}}{\mathrm{d}t}(||\varphi u_{tx}||^2_{L^2(\Omega)}+||\varphi v_{tx}||^2_{L^2(\Omega)})-C\end{aligned}\tag{113}$$

For I_2, we get

$$
\begin{aligned}
I_2 =&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\sqrt{a_6}\varphi u_{xxx}||^2_{L^2(\Omega)}+||\sqrt{a_4}\varphi v_{xxx}||^2_{L^2(\Omega)})+(a_{6,x}u_{xx},\varphi^2 u_{txxx})\\
&+(a_{4,x}v_{xx},\varphi^2 v_{txxx})+2(a_6u_{xxx},\varphi\varphi_x u_{txx})+2(a_4v_{xxx},\varphi\varphi_x v_{txx})\\
&+2(a_{6,x}u_{xx},\varphi\varphi_x u_{txx})+2(a_{4,x}v_{xx},\varphi\varphi_x v_{txx})\\
\geq&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\sqrt{a_6}\varphi u_{xxx}||^2_{L^2(\Omega)}+||\sqrt{a_4}\varphi v_{xxx}||^2_{L^2(\Omega)})\\
&-(a_{6,x}u_{xxx},\varphi^2 u_{txx})-(a_{4,x}v_{xxx},\varphi^2 v_{txx})\\
&+2(a_6u_{xxx},\varphi\varphi_x u_{txx})+2(a_4v_{xxx},\varphi\varphi_x v_{txx})-C\\
\geq&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(||\sqrt{a_6}\varphi u_{xxx}||^2_{L^2(\Omega)}+||\sqrt{a_4}\varphi v_{xxx}||^2_{L^2(\Omega)})\\
&-C||\varphi u_{xxx}||^2_{L^2(\Omega)}-C||\varphi v_{xxx}||^2_{L^2(\Omega)}-C
\end{aligned}
\tag{114}
$$

where Lemma 3.1 and (53) are used. Similarly from (114), we deduce

$$
\begin{aligned}
I_3 =&-(a_3v_{xxx},\varphi^2 u_{txxx})-(a_3u_{xxx},\varphi^2 v_{txxx})-(a_{3,x}v_{xx},\varphi^2 u_{txxx})\\
&-(a_{3,x}u_{xx},\varphi^2 v_{txxx})-2(a_3v_{xxx},\varphi\varphi_x u_{txx})-2(a_3u_{xxx},\varphi\varphi_x v_{txx})\\
&-2(a_{3,x}v_{xx},\varphi\varphi_x u_{txx})-2(a_{3,x}u_{xx},\varphi\varphi_x v_{txx})\\
=&-\frac{\mathrm{d}}{\mathrm{d}t}||a_3\varphi^2 u_{xxx}v_{xxx}||_{L^1(\Omega)}-(a_{3,x}v_{xx},\varphi^2 u_{txxx})-(a_{3,x}u_{xx},\varphi^2 v_{txxx})\\
&-2(a_3v_{xxx},\varphi\varphi_x u_{txx})-2(a_3u_{xxx},\varphi\varphi_x v_{txx})\\
&-2(a_{3,x}v_{xx},\varphi\varphi_x u_{txx})-2(a_{3,x}u_{xx},\varphi\varphi_x v_{txx})\\
=&-\frac{\mathrm{d}}{\mathrm{d}t}||a_3\varphi^2 u_{xxx}v_{xxx}||_{L^1(\Omega)}+(a_{3,xx}v_{xx},\varphi^2 u_{txx})+(a_{3,xx}u_{xx},\varphi^2 v_{txx})\\
&+(a_{3,x}v_{xxx},\varphi\varphi_x u_{txx})+(a_{3,x}u_{xxx},\varphi\varphi_x v_{txx})+2(a_{3,x}v_{xx},\varphi\varphi_x u_{txx})\\
&+2(a_{3,x}u_{xx},\varphi\varphi_x v_{txx})-2(a_3v_{xxx},\varphi\varphi_x u_{txx})-2(a_3u_{xxx},\varphi\varphi_x v_{txx})\\
&-2(a_{3,x}v_{xx},\varphi\varphi_x u_{txx})-2(a_{3,x}u_{xx},\varphi\varphi_x v_{txx})\\
\geq&-\frac{\mathrm{d}}{\mathrm{d}t}||a_3\varphi^2 u_{xxx}v_{xxx}||_{L^1(\Omega)}-C||\varphi u_{xxx}||^2_{L^2(\Omega)}-C||\varphi v_{xxx}||^2_{L^2(\Omega)}-C
\end{aligned}
\tag{115}
$$

By using Lemma 3.1 and (53), we find

$$I_4=-C||u_{txx}||^2_{L^2(\Omega)}-C||v_{txx}||^2_{L^2(\Omega)}-C||u_{xx}||^2_{L^2(\Omega)}-c||v_{xx}||^2_{L^2(\Omega)}\geq -C \tag{116}$$

$$I_5=-C\big(||u_t||^2_{H_0^2(\Omega)}+||v_t|^2_{H_0^2(\Omega)}+||u||^2_{H_0^2(\Omega)}+||v||^2_{H_0^2(\Omega)}\big)-C\geq -C \tag{117}$$

Putting (113)-(117) into (112), this yields

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(||\varphi u_{tx}||^2_{L^2(\Omega)}+||\varphi v_{tx}||^2_{L^2(\Omega)}+||\sqrt{a_6}\varphi u_{xxx}||^2_{L^2(\Omega)}+||\sqrt{a_4}\varphi v_{xxx}||^2_{L^2(\Omega)}\\
&\qquad -2||a_3\varphi^2 u_{xxx}v_{xxx}||_{L^1(\Omega)}\Big)\\
\leq& C\big(||\varphi u_{xxx}||^2_{L^2(\Omega)}+||\varphi v_{xxx}||^2_{L^2(\Omega)}\big)+C
\end{aligned}
\tag{118}
$$

Integrating with respect to $t\in(0,t)$, taking into account the condition (50), we discover

$$||\varphi u_{tx}||^2_{L^2(\Omega)}+||\varphi v_{tx}||^2_{L^2(\Omega)}+||\varphi u_{xxx}||^2_{L^2(\Omega)}+||\varphi v_{xxx}||^2_{L^2(\Omega)}$$

$$\begin{aligned}\leq& C\int_0^t ||\varphi u_{xxx}||^2_{L^2(\Omega)} + ||\varphi v_{xxx}||^2_{L^2(\Omega)}\mathrm{d}t\\ &+ C\big(||u_1||^2_{H^1(\Omega)} + ||v_1||^2_{H^1(\Omega)} + ||u_0||^2_{H^3(\Omega)} + ||v_0||^2_{H^3(\Omega)}\big) + C\end{aligned} \tag{119}$$

Thanks to the Gronwall inequality, we deduce

$$||\varphi u_{xxx}||^2_{L^2(\Omega)} + ||\varphi v_{xxx}||^2_{L^2(\Omega)} \leq C \tag{120}$$

□

6.4. **Proof of Theorem 4.5.**

Proof. Utilizing to the first and second equations of (45), we can deduce

$$\begin{aligned}\big((a_6a_4 - a_3^2)u_{xxxx}, u_{xxxx}\big) =& \Big(- a_4u_{tt} - a_3v_{tt} + \omega^2\big(R(a_4u_x + a_3v_x)\big)_x, u_{xxxx}\Big)\\ &+ \frac{1}{2}\Big(\big((u_x^2 + v_x^2)(a_4u_x + a_3v_x)\big)_x, u_{xxxx}\Big)\\ &+ \big((a_4p_1 + a_3p_2), u_{xxxx}\big).\end{aligned} \tag{121}$$

Thanks to (50) and Hölder inequality, we see that

$$||u_{xxxx}||_{L^2(\Omega)} \leq C\Big(||u_{tt} + v_{tt} + u_t + v_t||_{L^2(\Omega)} + ||u + v||_{H_0^2(\Omega)}\Big) \leq C. \tag{122}$$

Furthermore,

$$||u_{xxx}||_{L^2(\Omega)} \leq C||u_{xxxx}||^{\frac{4}{7}}_{L^2(\Omega)}||u||^{\frac{3}{7}}_{L^2(\Omega)} + C||u||_{L^2(\Omega)} \leq C, \tag{123}$$

where Gagliardo-Nirenberg inequality for bounded domains is used. Similarly, we deduce

$$||v_{xxx}||_{L^2(\Omega)} \leq C, ||v_{xxxx}||_{L^2(\Omega)} \leq C. \tag{124}$$

Combining with the conclusions in Theorem (4.2) and Theorem (4.3), we can find that

$$u, v \in L^\infty(0, T; H^4(\Omega) \cup H_0^2(\Omega)).$$

□

6.5. **Proof of Theorem 4.6.**

Proof. Denote

$$\{\tilde{u}, \tilde{v}\} = \pi(\{\tilde{a}_3, \tilde{a}_4, \tilde{a}_6, \tilde{\omega}, \tilde{\theta}, \tilde{u}_0, \tilde{v}_0, \tilde{u}_1, \tilde{v}_1\}),$$
$$\eta = u - \tilde{u}, \quad \zeta = v - \tilde{v}.$$

Then η, ζ satisfy

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(||\eta_t||^2_{L^2(\Omega)} + ||\zeta_t||^2_{L^2(\Omega)} + ||\sqrt{a_6}\eta_{xx}||^2_{L^2(\Omega)}\\ &\qquad + ||\sqrt{a_4}\zeta_{xx}||^2_{L^2(\Omega)} - 2||a_3\eta_{xx}\zeta_{xx}||_{L(\Omega)}\Big)\\ =&\frac{1}{2}(((u_x^2 + v_x^2)u_x - (\tilde{u}_x^2 + \tilde{v}_x^2)\tilde{u}_x)_x, \eta_t) + \frac{1}{2}(((u_x^2 + v_x^2)v_x - (\tilde{u}_x^2 + \tilde{v}_x^2)\tilde{v}_x)_x, \zeta_t)\\ &+ \omega^2(((R(x)\eta_x)_x, \eta_t) + ((R(x)\zeta_x)_x, \zeta_t)) + \omega^2(\eta, \eta_t) + (\overline{p}_1, \eta_t) + (\overline{p}_2, \zeta_t)\\ &- ((a_6 - \tilde{a}_6)\tilde{u}_{xx}, \eta_{txx}) - ((a_4 - \tilde{a}_4)\tilde{v}_{xx}, \zeta_{txx}) + ((a_3 - \tilde{a}_3)\tilde{v}_{xx}, \eta_{txx})\\ &+ ((a_3 - \tilde{a}_3)\tilde{u}_{xx}, \zeta_{txx}) - (\omega^2 - \tilde{\omega}^2)(R\tilde{u}_x, \eta_{tx}) - (\omega^2 - \tilde{\omega}^2)(R\tilde{v}_x, \zeta_{tx})\\ &+ (\omega^2 - \tilde{\omega}^2)(\tilde{u}, \eta_t) + (\omega_t - \tilde{\omega}_t)(R_0 + x, \eta_t).\end{aligned} \tag{125}$$

In equation (125), the nonlinear term satisfies

$$\begin{aligned}
&((u_x^2+v_x^2)u_x-(\tilde{u}_x^2+\tilde{v}_x^2)\tilde{u}_x)_x\\
=&((u_x^2+v_x^2)\eta_x)_x+((u_x^2+v_x^2-\tilde{u}_x^2-\tilde{v}_x^2)\tilde{u}_x)_x\\
=&((u_x^2+v_x^2)\eta_x)_x+(((u_x+\tilde{u}_x)\eta_x+(v_x+\tilde{v}_x)\zeta_x)\tilde{u}_x)_x\\
=&2(u_xu_{xx}+v_xv_{xx})\eta_x+(u_x^2+v_x^2)\eta_{xx}+((u_x+\tilde{u}_x)\eta_{xx}+(v_x+\tilde{v}_x)\zeta_{xx})\tilde{u}_x\\
&+((u_{xx}+\tilde{u}_{xx})\eta_x+(v_{xx}+\tilde{v}_{xx})\zeta_x)\tilde{u}_x+((u_x+\tilde{u}_x)\eta_x+(v_x+\tilde{v}_x)\zeta_x)\tilde{u}_{xx}.
\end{aligned} \tag{126}$$

By Hölder inequality and Sobolev inequality, we can obtain

$$\begin{aligned}
&(((u_x^2+v_x^2)u_x-(\tilde{u}_x^2+\tilde{v}_x^2)\tilde{u}_x)_x,\eta_t)\\
\le& C||u_{xx}+v_{xx}||_{L^2(\Omega)}||\eta_x||_{L^\infty(\Omega)}||\eta_t||_{L^2(\Omega)}+C||\eta_{xx}||_{L^2(\Omega)}||\eta_t||_{L^2(\Omega)}+||\eta_{xx}\\
&+(||u_{xx}+\tilde{u}_{xx}||_{L^2(\Omega)}||\eta_x||_{L^\infty(\Omega)}+||v_{xx}+\tilde{v}_{xx}||_{L^2(\Omega)}||\zeta_x||_{L^\infty(\Omega)})||\eta_t||_{L^2(\Omega)}\\
&+(|\eta_x||_{L^\infty(\Omega)}+||\zeta_x||_{L^\infty(\Omega)})||\tilde{u}_{xx}||_{L^2(\Omega)}||\eta_t||_{L^2(\Omega)}+\zeta_{xx}||_{L^2(\Omega)}||\eta_t||_{L^2(\Omega)}\\
\le& C||\eta||_{H_0^2(\Omega)}||\eta_t||_{L^2(\Omega)}+C||\eta||_{H_0^2(\Omega)}||\eta_t||_{L^2(\Omega)}\\
&+C||\eta||_{H_0^2(\Omega)}||\eta_t||_{L^2(\Omega)}+C||\zeta||_{H_0^2(\Omega)}||\eta_t||_{L^2(\Omega)}\\
\le& C(||\eta_t||^2_{L^2(\Omega)}+||\eta||^2_{H_0^2(\Omega)}+||\zeta||^2_{H_0^2(\Omega)}).
\end{aligned} \tag{127}$$

In the same way, we have

$$(((u_x^2+v_x^2)v_x-(\tilde{u}_x^2+\tilde{v}_x^2)\tilde{v}_x)_x,\zeta_t)\le C(||\zeta_t||^2_{L^2(\Omega)}+||\eta||^2_{H_0^2(\Omega)}+||\zeta||^2_{H_0^2(\Omega)}). \tag{128}$$

On the other hand, we easily have

$$\begin{aligned}
&(\overline{p}_1,\eta_t)+(\overline{p}_2,\zeta_t)\\
\le& C\big(||\theta-\tilde{\theta}||_{L^1(\Omega)}+||\eta_t||^2_{L^2(\Omega)}+||\zeta_t||^2_{L^2(\Omega)}+||\eta||^2_{H_0^1(\Omega)}+||\zeta||^2_{H_0^1(\Omega)}\big).
\end{aligned} \tag{129}$$

Substituting (127)-(129) into (125), we deduce

$$\begin{aligned}
&||\eta_t||^2_{L^2(\Omega)}+||\zeta_t||^2_{L^2(\Omega)}+||\eta||^2_{H_0^2(\Omega)}+||\zeta||^2_{H_0^2(\Omega)}\\
\le& C\int_0^t ||\eta_t||^2_{L^2(\Omega)}+||\zeta_t||^2_{L^2(\Omega)}+||\eta||^2_{H_0^2(\Omega)}+||\zeta||^2_{H_0^2(\Omega)}\mathrm{d}t\\
&+||u_1-\tilde{u}_1||^2_{L^2(\Omega)}+||v_1-\tilde{v}_1||^2_{L^2(\Omega)}+||u_0-\tilde{u}_0||^2_{H_0^2(\Omega)}+||v_0-\tilde{v}_0||^2_{H_0^2(\Omega)}\\
&+C\Big(\sum_{i=3}^5||a_i-\tilde{a}_i||_{L^\infty(\Omega)}+||\theta-\tilde{\theta}||_{L^1(\Omega)}+||\omega-\tilde{\omega}||_{W^{1,1}(0,T)}\Big).
\end{aligned} \tag{130}$$

where we used the inequalities

$$\begin{aligned}
||\tilde{u}||_{L^\infty(Q)}\le C,\ ||\tilde{u}_x||_{L^\infty(Q)}\le C, ||\tilde{v}||_{L^\infty(Q)}\le C,\ ||\tilde{v}_x||_{L^\infty(Q)}\le C,\\
||\tilde{u}_{xx}||_{L^\infty(0,T,L^2(\Omega))}\le C, ||\tilde{v}_{xx}||_{L^\infty(0,T,L^2(\Omega))}\le C,\\
||\eta_{tx}||_{L^\infty(Q)}\le C, ||\zeta_{tx}||_{L^\infty(Q)}\le C,
\end{aligned}$$

which are deduced from Theorem 4.2 and Theorem 4.3. □

6.6. **Proof of Theorem 5.2.**

Proof. Assume

$$\{\tilde{\varphi}_k\}_{k=1}^\infty \text{is an orthogonal basis of} H_f^2(\Omega), \tag{131}$$

and

$$\{\tilde{\varphi}_k\}_{k=1}^\infty \text{is an orthonormal basis of} L^2(\Omega). \tag{132}$$

Fix a positive integer m, and write

$$u_m = \sum_{k=1}^{m} \tilde{d}_{0m}^k(t)\tilde{\varphi}_k,\ v_m = \sum_{k=1}^{m} \tilde{d}_{1m}^k(t)\tilde{\varphi}_k, \tag{133}$$

where we intend to select the coefficients $\tilde{d}_{0m}^k(t), \tilde{d}_{1m}^k(t)(0 \le t \le T, k = 1, \cdots, m)$ to satisfy

$$\tilde{d}_{0m}^k(0) = (u_0, \tilde{\varphi}_k), \tilde{d}_{1m}^k(0) = (v_0, \tilde{\varphi}_k), k = 1, \cdots, m, \tag{134}$$

$$\tilde{d}_{0m,t}^k(0) = (u_1, \tilde{\varphi}_k), \tilde{d}_{1m,t}^k(0) = (v_1, \tilde{\varphi}_k), k = 1, \cdots, m, \tag{135}$$

and

$$\begin{aligned}&(u_{m,tt}, \tilde{\varphi}_k) + (a_6 u_{m,xx}, \tilde{\varphi}_{k,xx}) - (a_3 v_{m,xx}, \tilde{\varphi}_{k,xx}) + \frac{1}{2}((u_{m,x}^2 + v_{m,x}^2)u_{m,x}, \tilde{\varphi}_{k,x})\\ &+ \omega^2(R u_{m,x}, \tilde{\varphi}_{k,x}) - (p_{1m}, \tilde{\varphi}_k) - \big(b_1 u_m(l) + e_1 v_m(l)\big)\tilde{\varphi}(l) = 0,\end{aligned} \tag{136}$$

$$\begin{aligned}&(v_{m,tt}, \tilde{\varphi}_k) - (a_3 u_{m,xx}, \tilde{\varphi}_{k,xx}) + (a_4 v_{m,xx}, \tilde{\varphi}_{k,xx}) + \frac{1}{2}((u_{m,x}^2 + v_{m,x}^2)v_{m,x}, \tilde{\varphi}_{k,x})\\ &+ \omega^2(R v_{m,x}, \tilde{\varphi}_{k,x}) - (p_{2m}, \tilde{\varphi}_k) - \big(b_2 u_m(l) + e_2 v_m(l)\big)\tilde{\varphi}(l) = 0.\end{aligned} \tag{137}$$

As in earlier treatments of C-C boundary condition, we can conclude the following two conclude without difficulty.

(i) For each integer $m = 1, 2, \cdots$, there exists a unique Galerkin approximations function u_m, v_m of the from (133) satisfying (134)-(137) for $0 \le t \le T$.

(ii) $u(x, 0) = u_0(x), v(x, 0) = v_0(x), u_t(x, 0) = u_1(x), v_t(x, 0) = v_1(x)$.

Then, we proof the following estimate

$$\begin{aligned}&\|u_{m,t}\|_{L^2(\Omega)}^2 + \|v_{m,t}\|_{L^2(\Omega)}^2 + \|u_m\|_{H_f^2(\Omega)}^2 + \|v_m\|_{H_f^2(\Omega)}^2\\ &\le C\left(\|u_1\|_{L^2(\Omega)}^2 + \|v_1\|_{L^2(\Omega)}^2 + \|u_0\|_{H_f^2(\Omega)}^2 + \|v_0\|_{H_f^2(\Omega)}^2\right) + C\end{aligned} \tag{138}$$

Similarly from (75), we can deduce

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{||u_{m,t}||_{L^2(\Omega)}^2 + ||v_{m,t}||_{L^2(\Omega)}^2 + ||\sqrt{a_6}u_{m,xx}||_{L^2(\Omega)}^2 + ||\sqrt{a_4}v_{m,xx}||_{L^2(\Omega)}^2\\ &\quad + \frac{\omega^2}{2}(||\sqrt{R}u_{m,x}||_{L^2(\Omega)}^2 + ||\sqrt{R}v_{m,x}||_{L^2(\Omega)}^2)\} - \frac{\mathrm{d}}{\mathrm{d}t}||a_3 u_{m,xx}v_{m,xx}||_{L^1(\Omega)}\\ &\quad + \frac{1}{8}\frac{\mathrm{d}}{\mathrm{d}t}||u_{m,x}^2 + v_{m,x}^2||_{L^2(\Omega)}^2 - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\{b_1 u_m^2(l) + e_2 v_m^2(l) + 2e_1 u_m(l)v_m(l)\}\\ &= \omega\omega_t(||\sqrt{R}u_{m,x}||_{L^2(\Omega)}^2 + ||\sqrt{R}v_{m,x}||_{L^2(\Omega)}^2) + (p_{1m}, u_{m,t}) + (p_{2m}, v_{m,t}),\end{aligned} \tag{139}$$

where $b_2 = e_1$ is used.

Then we integrate (139) with respect to t, to discover

$$\begin{aligned}&||u_{m,t}||_{L^2(\Omega)}^2 + ||v_{m,t}||_{L^2(\Omega)}^2 + ||u_m||_{H_f^2(\Omega)}^2 + ||v_m||_{H_f^2(\Omega)}^2\\ &- b_1 u_m^2(l) - e_2 v_m^2(l) - 2e_1 u_m(l)v_m(l)\\ \le& C(||u_{m,t}(0)||_{L^2(\Omega)}^2 + ||v_{m,t}(0)||_{L^2(\Omega)}^2 + ||u_m(t)||_{H_f^2(\Omega)}^2 + ||v_m(t)||_{H_f^2(\Omega)}^2)\\ &- b_1 u_{0m}^2(l) - e_2 v_{0m}^2(l) - 2e_1 u_{0m}(l)v_{0m}(l)\\ &+ C\int_0^t ||u_{m,t}||_{L^2(\Omega)}^2 + ||v_{m,t}||_{L^2(\Omega)}^2 \mathrm{d}t + ||u_m||_{H_f^2(\Omega)}^2 + ||v_m||_{H_f^2(\Omega)}^2 + C.\end{aligned} \tag{140}$$

By simple calculation, we deduce

$$-b_1u_m^2(l) - e_2v_m^2(l) - 2e_1u_m(l)v_m(l) \geq 0. \tag{141}$$

where $b_1e_2 \geq e_1^2$ is used.

On the other hand, according to $u_{0m}(x), v_{0m}(x) \in H_f^2(\Omega)$, by Sobolev inequality, we obtain

$$u_{0m}(x), v_{0m}(x) \in C(\overline{\Omega}).$$

Thus,

$$u_{0m}(l), v_{0m}(l) \leq C. \tag{142}$$

Substituting (141) and (142) into (140), applying Gronwall inequality, we can deduce (138).

Now we pass to limits in our Galerkin approximations, applying estimate (138),we can discover (92), (93), (95). In order to complete the proof of the theorem, we just have to proof

$$u_\mu(l) \to u(l), v_\mu(l) \to v(l), \text{ strongly in } L^\infty(0,T). \tag{143}$$

where u_μ, v_μ are the convergent subsequence of u_m, v_m, respectively.

To verify this, recalling (138), we observe that

$$u_\mu \to u, v_\mu \to v \text{ strongly in } C(0,T;H_f^1(\Omega)) \tag{144}$$

where the Corollary 4 of Chapter 8 in [30] is used.

Furthermore, thanks to the conditions $u_m(0) = u(0) = 0$, we obtain

$$\begin{aligned} &||u_\mu(l) - u(l)||_{L^\infty(0,T)} = ||(u_\mu(l) - u(l)) - (u_\mu(0) - u(0))||_{L^\infty(0,T)} \\ =&|| \int_0^l \big(u_\mu(x) - u(x)\big)_x dx||_{L^\infty(0,T)} \leq \sqrt{l}||\big(u_\mu(x) - u(x)\big)_x||_{L^\infty(0,T;L^2(\Omega))} \\ \leq&\sqrt{l}||u_\mu(x) - u(x)||_{L^\infty(0,T;H_f^1(\Omega))} \end{aligned} \tag{145}$$

Thanks to (144), we can deduce

$$u_\mu(l) \to u(l) \text{ strongly in } L^\infty(0,T),$$

Similarly, we have

$$v_\mu(l) \to v(l) \text{ strongly in } L^\infty(0,T).$$

□

6.7. **Proof of Theorem 5.3.**

Proof. Similarly as (105), we can get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E + \frac{1}{2}P_1 + P_2 = P_3. \tag{146}$$

where

$$\begin{aligned} E =&||u_{tt}||^2_{L^2(\Omega)} + ||v_{tt}||^2_{L^2(\Omega)} + ||\sqrt{a_6}u_{txx}||^2_{L^2(\Omega)} + ||\sqrt{a_4}v_{txx}||^2_{L^2(\Omega)} \\ &- 2||a_3u_{txx}v_{txx}||_{L(\Omega)} + \omega^2||\sqrt{R(x)}u_{tx}||^2_{L^2(\Omega)} + \omega^2||\sqrt{R(x)}v_{tx}||^2_{L^2(\Omega)} \end{aligned}$$

$$
\begin{aligned}
P_1 =&\Big(\big((u_x^2+v_x^2)u_x\big)_t, u_{ttx}\Big) + \Big(\big((u_x^2+v_x^2)v_x\big)_t, v_{ttx}\Big)\\
P_2 =&-\big(b_1u(l)+e_1v(l)\big)_t u_{tt}(l) - \big(b_2u(l)+e_2v(l)\big)_t v_{tt}(l)\\
P_3 =&2\omega\omega_t\Big(\big((Ru_x)_x, u_{tt}\big) + \big((Rv_x)_x, v_{tt}\big) + ||\sqrt{R}u_{tx}||^2_{L^2(\Omega)} + ||\sqrt{R}v_{tx}||^2_{L^2(\Omega)}\Big)\\
&+ (p_{1,t}, u_{tt}) + (p_{2,t}, v_{tt})
\end{aligned}
$$

By calculation, we can deduce

$$
\begin{aligned}
P_1 =& \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{||\sqrt{u_x^2+v_x^2}u_{tx}||^2_{L^2(\Omega)} + ||\sqrt{u_x^2+v_x^2}v_{tx}||^2_{L^2(\Omega)} + 2||u_xu_{tx}||^2_{L^2(\Omega)}\\
&\qquad +2||v_xv_{tx}||^2_{L^2(\Omega)} + 4||u_xv_xu_{tx}v_{tx}||_{L^1(\Omega)}\Big\}\\
&\qquad -3\int_0^l u_xu_{tx}^3 + v_xv_{tx}^3 + u_xu_{tx}v_{tx}^2 + v_xv_{tx}u_{tx}^2\mathrm{d}x \qquad (147)\\
P_2 =& -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\{b_1u_t^2(l) + e_2v_t^2(l) + 2e_1u_t(l)v_t(l)\big\} \qquad (148)\\
P_3 \le& ||u_{tt}||^2_{L^2(\Omega)} + ||v_{tt}||^2_{L^2(\Omega)} + ||u||^2_{H_f^2(\Omega)} + ||v||^2_{H_f^2(\Omega)}\\
&+||u_{tx}||^2_{L^2(\Omega)} + ||v_{tx}||^2_{L^2(\Omega)} \qquad (149)
\end{aligned}
$$

Substituting (147), (148) and (149) into (146), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{E + ||\sqrt{u_x^2+v_x^2}u_{tx}||^2_{L^2(\Omega)} + ||\sqrt{u_x^2+v_x^2}v_{tx}||^2_{L^2(\Omega)} + 2||u_xu_{tx}||^2_{L^2(\Omega)}\\
&\qquad + 2||v_xv_{tx}||^2_{L^2(\Omega)} + 4||u_xv_xu_{tx}v_{tx}||_{L^1(\Omega)}\\
&\qquad - \big(b_1u_t^2(l) + e_2v_t^2(l) + 2e_1u_t(l)v_t(l)\big)\Big\}\\
\le& ||u_{tt}||^2_{L^2(\Omega)} + ||v_{tt}||^2_{L^2(\Omega)} + ||u||^2_{H_f^2(\Omega)} + ||v||^2_{H_f^2(\Omega)} + ||u_{tx}||^2_{L^2(\Omega)} + ||v_{tx}||^2_{L^2(\Omega)}\\
&\qquad + 3\int_0^l u_xu_{tx}^3 + v_xv_{tx}^3 + u_xu_{tx}v_{tx}^2 + v_xv_{tx}u_{tx}^2\mathrm{d}x
\end{aligned}
\qquad (150)
$$

Then integrate (150) with respect to t, to discover

$$
\begin{aligned}
E \le& \int_0^t ||u_{tt}||^2_{L^2(\Omega)} + ||v_{tt}||^2_{L^2(\Omega)} + ||u||^2_{H_f^2(\Omega)} + ||v||^2_{H_f^2(\Omega)}\\
&\qquad + ||u_{tx}||^2_{L^2(\Omega)} + ||v_{tx}||^2_{L^2(\Omega)}\mathrm{d}t\\
&\qquad + 3\int_0^t\int_0^l u_xu_{tx}^3 + v_xv_{tx}^3 + u_xu_{tx}v_{tx}^2 + v_xv_{tx}u_{tx}^2\mathrm{d}x\mathrm{d}t\\
&\qquad + ||u_{tt}(x,0)||^2_{L^2(\Omega)} + ||u_1||^2_{H_f^2(\Omega)} + ||v_{tt}(x,0)||^2_{L^2(\Omega)} + ||v_1||^2_{H_f^2(\Omega)}.
\end{aligned}
$$

where

$$-b_1u_t^2(l) - e_2v_t^2(l) - 2e_1u_t(l)v_t(l) \ge 0$$

and

$$||u_xu_{tx}||^2_{L^2(\Omega)} + ||v_xv_{tx}||^2_{L^2(\Omega)} + 2||u_xv_xu_{tx}v_{tx}||_{L^1(\Omega)} \ge 0.$$

are used.

Similarly as (108), we discover

$$||u_{tt}(x,0)||_{L^2(\Omega)}, ||v_{tt}(x,0)||_{L^2(\Omega)} \le c. \qquad (151)$$

According to (151), we get

$$E \leq c\int_0^t ||u_{tt}||^2_{L^2(\Omega)} + ||v_{tt}||^2_{L^2(\Omega)} + ||u_{txx}||^2_{L^2(\Omega)} + ||v_{txx}||^2_{L^2(\Omega)}\mathrm{d}t$$
$$+ 3\int_0^t\int_0^l u_x u_{tx}^3 + v_x v_{tx}^3 + u_x u_{tx} v_{tx}^2 + v_x v_{tx} u_{tx}^2 \mathrm{d}x\mathrm{d}t + c.$$

By Hölder inequality and Sobolev inequality, we have

$$\begin{aligned} E \leq c\int_0^t \Big\{ & ||u_{tt}||^2_{L^2(\Omega)} + ||u_{txx}||^2_{L^2(\Omega)} + ||u_{txx}||^3_{L^2(\Omega)} + ||u_{txx}||^4_{L^2(\Omega)} \\ & ||v_{tt}||^2_{L^2(\Omega)} + ||v_{txx}||^2_{L^2(\Omega)} + ||v_{txx}||^3_{L^2(\Omega)} + ||v_{txx}||^4_{L^2(\Omega)} \Big\}\mathrm{d}t + c. \end{aligned} \tag{152}$$

Then, we obtain Theorem 5.3 by Gronwall inequality. □

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