

THE EXISTENCE OF SOLUTIONS FOR A SHEAR THINNING COMPRESSIBLE NON-NEWTONIAN MODELS

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ABSTRACT. This paper is concerned with the initial boundary value problem for a shear thinning fluid-particle interaction non-Newtonian model with vacuum. The viscosity term of the fluid and the non-Newtonian gravitational force are fully nonlinear. Under Dirichlet boundary for velocity and the no-flux condition for density of particles, the existence and uniqueness of strong solutions is investigated in one dimensional bounded intervals.

1. INTRODUCTION

Fluid-particle interaction model arises in many practical applications, and is of primary importance in the sedimentation analysis of disperse suspensions of particles in fluids. This model is one of the commonly used models nowadays in biotechnology, medicine, mineral processing and chemical engineering [27]-[25]. Usually, the fluid flow is governed by the Navier-Stokes equations for a compressible fluid while the evolution of the particle densities is given by the Smoluchowski equation [4], the system has the form:

$$(1) \quad \begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) + \nabla(P(\rho) + \eta) - \mu \Delta u - \lambda \nabla \operatorname{div} u = -(\eta + \beta \rho) \nabla \Phi, \\ \eta_t + \operatorname{div}(\eta(u - \nabla \Phi)) - \Delta \eta = 0, \end{cases}$$

where $\rho, u, \eta, P(\rho) = a\rho^\gamma, \Phi(x)$ denote the fluid density, velocity, the density of particles in the mixture, pressure, and the external potential respectively, $a > 0, \gamma > 1, \mu > 0$ is the viscosity coefficient, and $3\lambda + 2\mu \geq 0$ are non-negative constants satisfied the physical requirements.

There are many kinds of literatures on the study of the existence and behavior of solutions to Navier-Stokes equations (See [1]-[17]). Taking system (1) as an example, Carrillo *et al* [4] discussed the the global existence and asymptotic behavior of the weak solutions providing a rigorous mathematical theory based on the principle of balance laws, following the framework of Lions [18] and Feireisl *et al* [11, 12]. Motivated by the stability arguments in [5], the authors also investigated the numerical analysis in [6]. Ballew and Trivisa [1] constructed suitable weak solutions and low stratification singular limit for a fluid particle interaction model. In addition, Mellet and Vasseur [20] proved the global existence of weak solutions

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of equations by using the entropy method on the asymptotic regime corresponding to a strong drag force and strong brownian motion. Zhang *et al* [31] establish the existence and uniqueness of classical solution to the system (1).

Despite the important progress, there are few results of non-Newtonian fluid-particle interaction model. As we know, the Navier Stokes equations are generally accepted as a right governing equations for the compressible or incompressible motion of viscous fluids, which is usually described as

$$\begin{cases} \rho_t + \operatorname{div}(\rho u) = 0, \\ (\rho u)_t + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\Gamma) + \nabla P = \rho f, \end{cases}$$

where Γ denotes the viscous stress tensor, which depends on $E_{ij}(\nabla u)$, and

$$E_{ij}(\nabla u) = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i},$$

is the rate of strain. If the relation between the stress and rate of strain is linear, namely, $\Gamma = \mu E_{ij}(\nabla u)$, where μ is the viscosity coefficient, then the fluid is called Newtonian. If the relation is not linear, the fluid is called non-Newtonian. The simplest model of the stress-strain relation for such fluids given by the power laws, which states that

$$\Gamma = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)^q,$$

for $0 < q < 1$ (see[3]). In [16], Ladyzhenskaya proposed a special form for Γ on the incompressible model:

$$\Gamma_{ij} = (\mu_0 + \mu_1 |\mathbb{E}(\nabla_x u)|^{p-2}) \mathbb{E}_{ij}(\nabla_x u).$$

For $\mu_0 = 0$, if $p < 2$ it is a pseudo-plastic fluid. In the view of physics, the model captures the shear thinning fluid for the case of $1 < p < 2$ (see[19]).

Non-Newtonian fluid flows are frequently encountered in many physical and industrial processes [8, 9], such as porous flows of oils and gases [7], biological fluid flows of blood [30], saliva and mucus, penetration grouting of cement mortar and mixing of massive particles and fluids in drug production [13]. The possible appearance of the vacuum is one of the major difficulties when trying to prove the existence and strong regularity results. On the other hand, the constitutive behavior of non-Newtonian fluid flow is usually more complex and highly non-linear, which may bring more difficulties to study such flows.

In recent years, there has been many research in the field of non-Newtonian flows, both theoretically and experimentally (see [14]-[26]). For example, in [14], Guo and Zhu studied the partial regularity of the generalized solutions to an incompressible monopolar non-Newtonian fluids. In [32], the trajectory attractor and global attractor for an autonomous non-Newtonian fluid in dimension two was studied. The existence and uniqueness of solutions for non-Newtonian fluids were established in [29] by applying Ladyzhenskaya's viscous stress tensor model.

In this paper, followed by Ladyzhenskaya's model of non-Newtonian fluid, we consider the following system

$$(2) \quad \begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + \rho \Psi_x - \lambda(|u_x|^{p-2} u_x)_x + (P + \eta)_x = -\eta \Phi_x, & (x, t) \in \Omega_T \\ (|\Psi_x|^{q-2} \Psi_x)_x = 4\pi g \left(\rho - \frac{1}{|\Omega|} \int_{\Omega} \rho dx \right), \\ \eta_t + (\eta(u - \Phi_x))_x = \eta_{xx}, \end{cases}$$

with the initial and boundary conditions

$$(3) \quad \begin{cases} (\rho, u, \eta)|_{t=0} = (\rho_0, u_0, \eta_0), & x \in \Omega, \\ u|_{\partial\Omega} = \Psi|_{\partial\Omega} = 0, & t \in [0, T], \end{cases}$$

and the no-flux condition for the density of particles

$$(4) \quad (\eta_x + \eta\Phi_x)|_{\partial\Omega} = 0, \quad t \in [0, T],$$

where $\rho, u, \eta, P(\rho) = a\rho^\gamma, \Phi(x)$ denote the fluid density, velocity, the density of particles in the mixture, pressure, and the external potential respectively, $a > 0, \gamma > 1, \frac{4}{3} < p, q < 2$. $\lambda > 0$ is the viscosity coefficient, Ω is a one-dimensional bounded interval, for simplicity we only consider $\Omega = (0, 1)$, $\Omega_T = \Omega \times [0, T]$.

The system describes a compressible shear thinning fluid-particle interaction system for the evolution of particles dispersed in a viscous non-Newtonian fluid and the particle is driven by non-Newtonian gravitational potential. To our knowledge, there still no existence results for (2)-(4) when $1 < p, q < 2$. The aim of this paper is to study the existence and uniqueness of strong solutions to this system. Throughout the paper we assume that $a = \lambda = 1$ for simplicity. In the following sections, we will use simplified notations for standard Sobolev spaces and Bochner spaces, such as $L^p = L^p(\Omega)$, $H_0^1 = H_0^1(\Omega)$, $C([0, T]; H^1) = C([0, T]; H^1(\Omega))$.

We state the definition of strong solution as follows:

Definition 1.1. The (ρ, u, Ψ, η) is called a strong solution to the initial boundary value problem(2)-(4), if the following conditions are satisfied:

(i)

$$\begin{aligned} \rho &\in L^\infty(0, T_*; H^1(\Omega)), u \in L^\infty(0, T_*; W_0^{1,p}(\Omega) \cap H^2(\Omega)), \\ \Psi &\in L^\infty(0, T_*; H^2(\Omega)), \eta \in L^\infty(0, T_*; H^2(\Omega)), \rho_t \in L^\infty(0, T_*; L^2(\Omega)), \\ u_t &\in L^2(0, T_*; H_0^1(\Omega)), \Psi_t \in L^\infty(0, T_*; H^1(\Omega)), \eta_t \in L^\infty(0, T_*; L^2(\Omega)), \\ \sqrt{\rho}u_t &\in L^\infty(0, T_*; L^2(\Omega)), (|u_x|^{p-2}u_x)_x \in C(0, T_*; L^2(\Omega)). \end{aligned}$$

(ii) For all $\phi \in L^\infty(0, T_*; H^1(\Omega))$, $\phi_t \in L^\infty(0, T_*; L^2(\Omega))$, for a.e. $t \in (0, T)$, we have

$$(5) \quad \int_{\Omega} \rho\phi(x, t)dx - \int_0^t \int_{\Omega} (\rho\phi_t + \rho u\phi_x)(x, s)dxds = \int_{\Omega} \rho_0\phi(x, 0)dx,$$

(iii) For all $\varphi \in L^\infty(0, T_*; W_0^{1,p}(\Omega) \cap H^2(\Omega))$, $\varphi_t \in L^2(0, T_*; H_0^1(\Omega))$, for a.e. $t \in (0, T)$, we have

$$(6) \quad \begin{aligned} \int_{\Omega} \rho u\varphi(x, t)dx - \int_0^t \int_{\Omega} \{ \rho u\varphi_t + \rho u^2\varphi_x - \rho\Psi_x\varphi - \lambda|u_x|^{p-2}u_x\varphi_x \\ + (P + \eta)\varphi_x - \eta\Phi_x\varphi \}(x, s)dxds = \int_{\Omega} \rho_0 u_0\varphi(x, 0)dx, \end{aligned}$$

(iv) For all $\psi \in L^\infty(0, T_*; H^2(\Omega))$, $\psi_t \in L^\infty(0, T_*; H^1(\Omega))$, for a.e. $t \in (0, T)$, we have

$$(7) \quad - \int_0^t \int_{\Omega} |\Psi_x|^{q-2}\Psi_x\psi_x(x, s)dxds = \int_0^t \int_{\Omega} 4\pi g(\rho - \frac{1}{|\Omega|} \int_{\Omega} \rho dx)\psi(x, 0)dxds,$$

(v) For all $\vartheta \in L^\infty(0, T_*; H^2(\Omega))$, $\vartheta_t \in L^\infty(0, T_*; L^2(\Omega))$, for a.e. $t \in (0, T)$, we have

$$(8) \quad \int_{\Omega} \eta\vartheta(x, t)dx - \int_0^t \int_{\Omega} [\eta(u - \Phi_x) - \eta_x]\vartheta_x(x, s)dxds = \int_{\Omega} \eta_0\vartheta(x, 0)dx.$$

The main result of this paper is stated in the following theorem.

1.1. Main theorem.

Theorem 1.2. *Let $\Phi \in C^2(\Omega)$, $\frac{4}{3} < p, q < 2$ and assume that the initial data (ρ_0, u_0, η_0) satisfy the following conditions*

$$0 \leq \rho_0 \in H^1(\Omega), u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \eta_0 \in H^2(\Omega),$$

and the compatibility condition

$$(9) \quad -(|u_{0x}|^{p-2}u_{0x})_x + (P(\rho_0) + \eta_0)_x + \eta_0\Phi_x = \rho_0^{\frac{1}{2}}(g + \Phi_x),$$

for some $g \in L^2(\Omega)$. Then there exist a $T_* \in (0, +\infty)$ and a unique strong solution (ρ, u, Ψ, η) to (2)-(4) such that

$$\begin{aligned} \rho &\in L^\infty(0, T_*; H^1(\Omega)), u \in L^\infty(0, T_*; W_0^{1,p}(\Omega) \cap H^2(\Omega)), \\ \Psi &\in L^\infty(0, T_*; H^2(\Omega)), \eta \in L^\infty(0, T_*; H^2(\Omega)), \rho_t \in L^\infty(0, T_*; L^2(\Omega)), \\ u_t &\in L^2(0, T_*; H_0^1(\Omega)), \Psi_t \in L^\infty(0, T_*; H^1(\Omega)), \eta_t \in L^\infty(0, T_*; L^2(\Omega)), \\ \sqrt{\rho}u_t &\in L^\infty(0, T_*; L^2(\Omega)), (|u_x|^{p-2}u_x)_x \in C(0, T_*; L^2(\Omega)). \end{aligned}$$

Remark 1. By using exactly the similar argument, we can prove the result also hold for the case $1 < p, q \leq \frac{4}{3}$. We omit the details here.

2. A PRIORI ESTIMATES FOR SMOOTH SOLUTIONS

In this section, we will prove the local existence of strong solutions. From the continuity equation (2)₁, we can deduce the conservation of mass

$$\int_{\Omega} \rho(t)dx = \int_{\Omega} \rho_0 dx := m_0, \quad (t > 0, m_0 > 0)$$

Because equation (2)₂ possesses always with singularity, we overcome this difficulty by introduce a regularized process, then by taking the limiting process back to the original problem. Namely, we consider the following system

$$(10) \quad \rho_t + (\rho u)_x = 0,$$

$$(11) \quad (\rho u)_t + (\rho u^2)_x + \rho \Psi_x - \left[\left(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon} \right)^{\frac{2-p}{2}} u_x \right]_x + (P + \eta)_x = -\eta \Phi_x,$$

$$(12) \quad \left[\left(\frac{\varepsilon \Psi_x^2 + 1}{\Psi_x^2 + \varepsilon} \right)^{\frac{2-q}{2}} \Psi_x \right]_x = 4\pi g(\rho - m_0),$$

$$(13) \quad \eta_t + (\eta(u - \Phi_x))_x = \eta_{xx},$$

with the initial and boundary conditions.

$$(14) \quad (\rho, u, \eta)|_{t=0} = (\rho_0, u_0, \eta_0), \quad x \in \Omega,$$

$$(15) \quad u|_{\partial\Omega} = \Psi|_{\partial\Omega} = (\eta_x + \eta\Phi_x)|_{\partial\Omega} = 0, \quad t \in [0, T],$$

and $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ is the smooth solution of the boundary value problem

$$(16) \quad \begin{cases} - \left[\left(\frac{\varepsilon u_{0x}^2 + 1}{u_{0x}^2 + \varepsilon} \right)^{\frac{2-p}{2}} u_{0x} \right]_x + (P(\rho_0) + \eta_0)_x + \eta_0\Phi_x = \rho_0^{\frac{1}{2}}(g + \Phi_x), \\ u_0|_{\partial\Omega} = 0. \end{cases}$$

Provided that (ρ, u, η) is a smooth solution of (10)-(15) and $\rho_0 \geq \delta$, where $0 < \delta \ll 1$ is a positive number. We denote by $M_0 = 1 + \mu_0 + \mu_0^{-1} + |\rho_0|_{H^1} + |g|_{L^2}$.

We first get the estimate of $|u_{0xx}|_{L^2}$. From (16), we have

$$u_{0xx} = \left(\frac{\varepsilon u_{0x}^2 + 1}{u_{0x}^2 + \varepsilon} \right)^{\frac{p}{2}} \frac{(u_{0x}^2 + \varepsilon)^2 [(P(\rho_0) + \eta_0)_x + \eta_0 \Phi_x - \rho_0^{\frac{1}{2}}(g + \Phi_x)]}{(\varepsilon u_{0x}^2 + 1)(u_{0x}^2 + \varepsilon) - (2-p)(1-\varepsilon^2)u_{0x}^2}.$$

Then

$$\begin{aligned} |u_{0xx}|_{L^2} &\leq \frac{1}{p-1} \left| \left(\frac{u_{0x}^2 + \varepsilon}{\varepsilon u_{0x}^2 + 1} \right)^{1-\frac{p}{2}} \right|_{L^\infty} |(P(\rho_0) + \eta_0)_x + \eta_0 \Phi_x - \rho_0^{\frac{1}{2}}(g + \Phi_x)|_{L^2} \\ &\leq \frac{1}{p-1} (|u_{0x}|_{L^\infty}^2 + 1)^{1-\frac{p}{2}} (|(P(\rho_0) + \eta_0)_x + \eta_0 \Phi_x - \rho_0^{\frac{1}{2}}(g + \Phi_x)|_{L^2}) \\ &\leq \frac{1}{p-1} (|u_{0xx}|_{L^2}^2 + 1)^{1-\frac{p}{2}} (|P_x(\rho_0)|_{L^2} + |\eta_{0x}|_{L^2} + |\eta_0|_{L^\infty} |\Phi_x|_{L^2} \\ &\quad + |\rho_0|_{L^\infty}^{\frac{1}{2}} |g|_{L^2} + |\rho_0|_{L^\infty}^{\frac{1}{2}} |\Phi_x|_{L^2}). \end{aligned}$$

Applying Young's inequality, we have

$$\begin{aligned} |u_{0xx}|_{L^2} &\leq C (|P_x(\rho_0)|_{L^2} + |\eta_{0x}|_{L^2} + |\eta_0|_{L^\infty} |\Phi_x|_{L^2} + |\rho_0|_{L^\infty}^{\frac{1}{2}} |g|_{L^2} \\ &\quad + |\rho_0|_{L^\infty}^{\frac{1}{2}} |\Phi_x|_{L^2})^{\frac{1}{p-1}} \leq C, \end{aligned}$$

thus

$$(17) \quad |u_0|_{L^\infty} + |u_{0x}|_{L^\infty} + |u_{0xx}|_{L^2} \leq C,$$

where C is a positive constant, depending only on M_0 .

Next, we introduce an auxiliary function

$$Z(t) = \sup_{0 \leq s \leq t} (1 + |\rho(s)|_{H^1} + |u(s)|_{W_0^{1,p}} + |\sqrt{\rho} u_t(s)|_{L^2} + |\eta_t(s)|_{L^2} + |\eta(s)|_{H^1}).$$

We will derive some useful estimate to each term of $Z(t)$ in terms of some integrals of $Z(t)$, then apply arguments of Gronwall's inequality to prove $Z(t)$ is locally bounded.

2.1. Preliminaries. In order to prove the main Theorem, we first give some useful lemmas for later use.

Lemma 2.1. *Let $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $\rho_0 \in H^1(\Omega)$, $\eta_0 \in H^2(\Omega)$, $\Phi \in C^2(\Omega)$, $g \in L^2(\Omega)$, u_0^ε is a solution of the boundary value problem*

$$(18) \quad \begin{cases} - \left[\left(\frac{\varepsilon (u_{0x}^\varepsilon)^2 + 1}{(u_{0x}^\varepsilon)^2 + \varepsilon} \right)^{\frac{2-p}{2}} u_{0x}^\varepsilon \right]_x + (P(\rho_0) + \eta_0)_x + \eta_0 \Phi_x = \rho_0^{\frac{1}{2}}(g + \Phi_x), \\ u_{0x}^\varepsilon(0) = u_{0x}^\varepsilon(1) = 0. \end{cases}$$

Then there are a subsequence $\{u_0^{\varepsilon_j}\}$, $j = 1, 2, 3, \dots$, of $\{u_0^\varepsilon\}$ and $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ such that as $\varepsilon_j \rightarrow 0$,

$$\begin{aligned} u_0^{\varepsilon_j} &\rightarrow u_0 \text{ in } H_0^1(\Omega) \cap H^2(\Omega), \\ \left[\left(\frac{\varepsilon_j (u_{0x}^{\varepsilon_j})^2 + 1}{(u_{0x}^{\varepsilon_j})^2 + \varepsilon_j} \right)^{\frac{2-p}{2}} u_{0x}^{\varepsilon_j} \right]_x &\rightarrow (|u_{0x}|^{p-2} u_{0x})_x \text{ in } L^2(\Omega). \end{aligned}$$

Proof. According to (18), we have

$$u_{0xx}^\varepsilon = \left(\frac{\varepsilon (u_{0x}^\varepsilon)^2 + 1}{(u_{0x}^\varepsilon)^2 + \varepsilon} \right)^{\frac{p}{2}} \frac{((u_{0x}^\varepsilon)^2 + \varepsilon)^2 ((P(\rho_0) + \eta_0)_x + \eta_0 \Phi_x + \rho_0^{\frac{1}{2}}(g + \Phi_x))}{(\varepsilon (u_{0x}^\varepsilon)^2 + 1)((u_{0x}^\varepsilon)^2 + \varepsilon) - (2-p)(1-\varepsilon^2)(u_{0x}^\varepsilon)^2}.$$

Taking it by the L^2 norm, we have

$$\begin{aligned} |u_{0xx}^\varepsilon|_{L^2} &\leq \left| \left(\frac{\varepsilon(u_{0x}^\varepsilon)^2 + 1}{(u_{0x}^\varepsilon)^2 + \varepsilon} \right)^{1-\frac{p}{2}} \right|_{L^\infty} |(P(\rho_0) + \eta_0)_x + \eta_0 \Phi_x + \rho_0^{\frac{1}{2}}(g + \Phi_x)|_{L^2} \\ &\leq (|u_{0x}^\varepsilon|_{L^\infty}^2 + 1)^{1-\frac{p}{2}} |(P(\rho_0) + \eta_0)_x + \eta_0 \Phi_x + \rho_0^{\frac{1}{2}}(g + \Phi_x)|_{L^2}, \end{aligned}$$

then

$$(19) \quad |u_{0xx}^\varepsilon|_{L^2} \leq C(1 + |(P(\rho_0) + \eta_0)_x + \eta_0 \Phi_x + \rho_0^{\frac{1}{2}}(g + \Phi_x)|_{L^2})^{\frac{1}{p-1}} \leq C.$$

Therefore, by the above inequality, as $\varepsilon_j \rightarrow 0$,

$$\begin{aligned} u_0^{\varepsilon_j} &\rightarrow u_0 \text{ in } C^{\frac{3}{2}}(\Omega), \\ u_{0xx}^{\varepsilon_j} &\rightarrow u_{0xx} \text{ in } L^2(\Omega) \text{ weakly.} \end{aligned}$$

Thus, we can obtain $\{u_{0x}^{\varepsilon_j}\}$ is a Cauchy subsequence of $C^{\frac{3}{2}}(\Omega)$, for all $\alpha_1 > 0$, we find N , as $i, j > N$, and

$$|u_{0x}^{\varepsilon_i} - u_{0x}^{\varepsilon_j}|_{L^\infty(\Omega)} < \alpha_1.$$

Now, we prove that $\{u_{0xx}^\varepsilon\}$ has a Cauchy sequence in L_2 norm.

Let

$$\phi_i = \phi((u_{0x}^{\varepsilon_i})^2) = \left(\frac{\varepsilon_i(u_{0x}^{\varepsilon_i})^2 + 1}{(u_{0x}^{\varepsilon_i})^2 + \varepsilon_i} \right)^{\frac{p}{2}} \frac{((u_{0x}^{\varepsilon_i})^2 + \varepsilon_i)^2}{(\varepsilon_i(u_{0x}^{\varepsilon_i})^2 + 1)((u_{0x}^{\varepsilon_i})^2 + \varepsilon_i) - (2-p)(1-\varepsilon_i^2)(u_{0x}^{\varepsilon_i})^2}.$$

For all $\alpha > 0$, there exists N , as $i, j > N$, we can deduce that

$$|u_{0xx}^{\varepsilon_i} - u_{0xx}^{\varepsilon_j}|_{L^2(\Omega)} \leq |\phi_i - \phi_j|_{L^\infty(\Omega)} |(P(\rho_0) + \eta_0)_x + \eta_0 \Phi_x - \rho_0^{\frac{1}{2}}(g + \Phi_x)|_{L^2(\Omega)}.$$

With the assumption, we can obtain

$$|(P(\rho_0) + \eta_0)_x + \eta_0 \Phi_x - \rho_0^{\frac{1}{2}}(g + \Phi_x)|_{L^2(\Omega)} \leq C,$$

where C is a positive constant, depending only on $|\rho_0|_{H^1(\Omega)}$, $|g|_{L^2(\Omega)}$ and $|\eta_0|_{H^2(\Omega)}$. Using the following inequality,

$$(20) \quad |\phi_i - \phi_j|_{L^\infty(\Omega)} \leq \left| \int_0^1 \phi'(\theta(u_{0x}^{\varepsilon_i})^2 + (1-\theta)(u_{0x}^{\varepsilon_j})^2) d\theta ((u_{0x}^{\varepsilon_i})^2 - (u_{0x}^{\varepsilon_j})^2) \right|_{L^\infty(\Omega)},$$

where $0 < \theta < 1$.

By the simple calculation, we can get

$$\phi'(s) \leq \frac{2}{p-1} (1 + s^{-\frac{p}{2}}),$$

where C depending only on p , then

$$\begin{aligned} &|\phi_i - \phi_j|_{L^\infty(\Omega)} \\ &\leq \frac{2}{p-1} \left| \left(1 + \int_0^1 (\theta(u_{0x}^{\varepsilon_i})^2 + (1-\theta)(u_{0x}^{\varepsilon_j})^2) d\theta \right) ((u_{0x}^{\varepsilon_i})^2 - (u_{0x}^{\varepsilon_j})^2) \right|_{L^\infty(\Omega)} \\ &\leq \frac{2}{p-1} |u_{0x}^{\varepsilon_i} - u_{0x}^{\varepsilon_j}|_{L^\infty(\Omega)} |u_{0x}^{\varepsilon_i} + u_{0x}^{\varepsilon_j}|_{L^\infty(\Omega)} \\ &\quad + \frac{4}{(2-p)(p-1)} |u_{0x}^{\varepsilon_i} - u_{0x}^{\varepsilon_j}|_{L^\infty(\Omega)}^{\frac{2-p}{2}} |u_{0x}^{\varepsilon_i} + u_{0x}^{\varepsilon_j}|_{L^\infty(\Omega)}^{\frac{2-p}{2}} \leq \alpha. \end{aligned}$$

Substituting this into (18), we have

$$|u_{0xx}^{\varepsilon_i} - u_{0xx}^{\varepsilon_j}|_{L^\infty(\Omega)} < \alpha,$$

then there is a subsequence $\{u_{0xx}^{\varepsilon_j}\}$ and $\{u_{0xx}^\varepsilon\}$, such that

$$\{u_{0xx}^{\varepsilon_j}\} \rightarrow \chi \text{ in } L^2(\Omega).$$

By the uniqueness of the weak convergence, we have

$$\chi = \{u_{0xx}^\varepsilon\}.$$

Since $(P(\rho_0) + \eta)_x + \eta_0 \Phi_x - \rho_0^{\frac{1}{2}}(g + \Phi_x)$ are independent of ε , the same that we obtain, as $\varepsilon_j \rightarrow 0$,

$$\left[\left(\frac{\varepsilon_j (u_{0x}^{\varepsilon_j})^2 + 1}{(u_{0x}^{\varepsilon_j})^2 + \varepsilon_j} \right)^{\frac{2-p}{2}} u_{0x}^{\varepsilon_j} \right]_x \rightarrow (|u_{0x}|^{p-2} u_{0x})_x \text{ in } L^2(\Omega).$$

This completes the proof of Lemma 2.1. \square

Lemma 2.2.

$$(21) \quad \sup_{0 \leq t \leq T} |\rho(t)|_{H^1}^2 \leq C \exp\left(C \int_0^t Z^{\frac{6\gamma}{(3p-4)(q-1)}}(s) ds\right),$$

where C is a positive constant, depending only on M_0 .

Proof. We estimates for u and η for later use. It follows from (11) that

$$\left[\left(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon} \right)^{\frac{2-p}{2}} u_x \right]_x = \rho u_t + \rho u u_x + \rho \Psi_x + (P + \eta)_x + \eta \Phi_x.$$

We note that

$$\begin{aligned} |u_{xx}| &\leq \frac{1}{p-1} (u_x^2 + \varepsilon)^{1-\frac{p}{2}} |\rho u_t + \rho u u_x + \rho \Psi_x + (P + \eta)_x + \eta \Phi_x| \\ &\leq \frac{1}{p-1} (|u_x|^{2-p} + 1) |\rho u_t + \rho u u_x + \rho \Psi_x + (P + \eta)_x + \eta \Phi_x|. \end{aligned}$$

Taking it by the L^2 norm and using Young's inequality, we have

$$\begin{aligned} |u_{xx}|_{L^2}^{p-1} &\leq C(1 + |\rho u_t|_{L^2} + |\rho u u_x|_{L^2} + |\rho \Psi_x|_{L^2} + |(P + \eta)_x|_{L^2} + |\eta \Phi_x|_{L^2}) \\ &\leq C(1 + |\rho|^{\frac{1}{2}}_{L^\infty} |\sqrt{\rho} u_t|_{L^2} + |\rho|_{L^\infty} |u|_{L^\infty} |u_x|_{L^p}^{\frac{p}{2}} |u_x|_{L^\infty}^{1-\frac{p}{2}} + |\rho|_{L^\infty}^{\gamma-1} |\rho_x|_{L^2} \\ &\quad + |\eta_x|_{L^2} + |\eta|_{L^\infty} |\Phi_x|_{L^2} + |\rho|_{L^2} |\Psi_{xx}|_{L^2}) \\ &\leq C[1 + |\rho|^{\frac{1}{2}}_{L^\infty} |\sqrt{\rho} u_t|_{L^2} + (|\rho|_{L^\infty} |u|_{L^\infty} |u_x|_{L^p}^{\frac{p}{2}})^{\frac{2(p-1)}{3p-4}} + |\rho|_{L^\infty}^{\gamma-1} |\rho_x|_{L^2} \\ (22) \quad &\quad + |\eta_x|_{L^2} + |\eta|_{L^\infty} |\Phi_x|_{L^2} + |\rho|_{L^2} |\Psi_{xx}|_{L^2}] + \frac{1}{2} |u_{xx}|_{L^2}^{p-1}. \end{aligned}$$

On the other hand, by (12), we have

$$|\Psi_{xx}| \leq \frac{1}{q-1} (|\Psi_x|^{2-q} + 1) |4\pi g(\rho - m_0)|.$$

Taking it by L^2 -norm, using Young's inequality, which gives

$$(23) \quad |\Psi_{xx}|_{L^2} \leq CZ^{\frac{1}{q-1}}(t).$$

This implies that

$$\begin{aligned} |u_{xx}|_{L^2} &\leq CZ^{\max\{\frac{q}{q-1}, \frac{(p-1)(4+p)}{3p-4}\gamma\}}(t) \\ (24) \quad &\leq CZ^{\frac{6\gamma}{(3p-4)(q-1)}}(t). \end{aligned}$$

By (13), taking it by the L^2 norm, we have

$$\begin{aligned}
|\eta_{xx}|_{L^2} &\leq |\eta_t + (\eta(u - \Phi_x))_x|_{L^2} \\
&\leq |\eta_t|_{L^2} + |\eta_x|_{L^2}|u|_{L^\infty} + |\eta_x|_{L^2}|\Phi_x|_{L^\infty} + |\eta|_{L^2}|u_{xx}|_{L^2} + |\eta|_{L^\infty}|\Phi_{xx}|_{L^2} \\
(25) \quad &\leq CZ^{\frac{6\gamma+2}{(3p-4)(q-1)}}(t).
\end{aligned}$$

Multiplying (10) by ρ , integrating over Ω , we deduce that

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\rho|^2 ds + \int_{\Omega} (\rho u)_x \rho dx = 0.$$

Integrating it by parts, using Sobolev inequality, we obtain

$$(26) \quad \frac{d}{dt} |\rho(t)|_{L^2}^2 \leq \int_{\Omega} |u_x| |\rho|^2 dx \leq |u_{xx}|_{L^2} |\rho|_{L^2}^2.$$

Differentiating (10) with respect to x , and multiplying it by ρ_x , integrating over Ω , and using Sobolev inequality, we have

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} |\rho_x|^2 dx &= - \int_{\Omega} \left[\frac{3}{2} u_x (\rho_x)^2 + \rho \rho_x u_{xx} \right] (t) dx \\
&\leq C[|u_x|_{L^\infty} |\rho_x|_{L^2}^2 + |\rho|_{L^\infty} |\rho_x|_{L^2} |u_{xx}|_{L^2}] \\
(27) \quad &\leq C|\rho|_{H^1}^2 |u_{xx}|_{L^2}.
\end{aligned}$$

From (26) and (27) and the Gronwall's inequality, then lemma 2.2 holds. \square

Lemma 2.3.

$$(28) \quad |\eta|_{H^1}^2 + |\eta_t|_{L^2}^2 + \int_0^t (|\eta_x|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta_{xt}|_{L^2}^2)(s) ds \leq C(1 + \int_0^t Z^4(s) ds),$$

where C is a positive constant, depending only on M_0 .

Proof. Multiplying (13) by η , integrating the resulting equation over Ω_T , using the boundary conditions (4) and Young's inequality, we have

$$\begin{aligned}
\int_0^t |\eta_x(s)|_{L^2}^2 ds + \frac{1}{2} |\eta(t)|_{L^2}^2 &\leq \iint_{\Omega_T} (|\eta u \eta_x| + |\eta \Phi_x \eta_x|) dx ds \\
&\leq \frac{1}{4} \int_0^t |\eta_x(s)|_{L^2}^2 ds + C \int_0^t |u_x|_{L^p}^2 |\eta|_{H^1}^2 ds + C \int_0^t |\eta|_{H^1}^2 ds + C \\
(29) \quad &\leq \frac{1}{4} \int_0^t |\eta_x(s)|_{L^2}^2 ds + C(1 + \int_0^t Z^4(t) ds).
\end{aligned}$$

Multiplying (13) by η_t , integrating (by parts) over Ω_T , using the boundary conditions (4) and Young's inequality, we have

$$\begin{aligned}
\int_0^t |\eta_t(s)|_{L^2}^2 ds + \frac{1}{2} |\eta_x(t)|_{L^2}^2 &\leq \iint_{\Omega_T} |\eta(u - \Phi_x) \eta_{xt}| dx ds \\
&\leq \frac{1}{4} \int_0^t |\eta_{xt}(s)|_{L^2}^2 ds + C \int_0^t |\eta|_{H^1}^2 |u_x|_{L^p}^2 ds + C \int_0^t |\eta|_{H^1}^2 ds + C \\
(30) \quad &\leq \frac{1}{4} \int_0^t |\eta_{xt}(s)|_{L^2}^2 ds + C(1 + \int_0^t Z^4(t) ds).
\end{aligned}$$

Differentiating (13) with respect to t , multiplying the resulting equation by η_t , integrating (by parts) over Ω_T , we get

$$\begin{aligned}
\int_0^t |\eta_{xt}(s)|_{L^2}^2 ds + \frac{1}{2} |\eta_t(t)|_{L^2}^2 &= \iint_{\Omega_T} (\eta(u - \Phi_x))_t \eta_{xt} dx ds \\
&\leq C + \iint_{\Omega_T} (|\eta_t u \eta_{xt}| + |\eta_t \Phi_x \eta_{xt}| + |\eta_x u_t \eta_t| + |\eta u_{xt} \eta_t|) dx ds \\
&\leq C(1 + \int_0^t (|\eta_t|_{L^2}^2 \|u_x\|_{L^p}^2 + |\eta_t|_{L^2}^2 + |\eta_x|_{L^2}^2 |\eta_t|_{L^2}^2 + |\eta|_{H^1}^2 |\eta_t|_{L^2}^2) dx) \\
&\quad + \frac{1}{2} \int_0^t |\eta_{xt}|_{L^2}^2 + \frac{1}{2} \int_0^t |u_{xt}|_{L^2}^2 \\
(31) \quad &\leq C(1 + \int_0^t Z^4(s) ds).
\end{aligned}$$

Combining (29)-(31), we obtain the desired estimate of Lemma 2.3. \square

Lemma 2.4.

$$(32) \quad \int_0^t |\sqrt{\rho} u_t(s)|_{L^2}^2 ds + |u_x(t)|_{L^p}^p \leq C(1 + \int_0^t Z^{\frac{10+4\gamma}{(3p-4)(q-1)}}(s) ds),$$

where C is a positive constant, depending only on M_0 .

Proof. Using (10), we rewritten the (11) as

$$(33) \quad \rho u_t + (\rho u) u_x + \rho \Psi_x - \left[\left(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon} \right)^{\frac{2-p}{2}} u_x \right]_x + (P + \eta)_x = -\eta \Phi_x.$$

Multiplying (33) by u_t , integrating (by parts) over Ω_T , we have

$$\begin{aligned}
\iint_{\Omega_T} \rho |u_t|^2 dx ds + \iint_{\Omega_T} \left(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon} \right)^{\frac{2-p}{2}} u_x u_{xt} dx ds \\
(34) \quad = - \iint_{\Omega_T} (\rho u u_x + \rho \Psi_x + P_x + \eta_x + \eta \Phi_x) u_t dx ds.
\end{aligned}$$

We deal with each term as follows:

$$\begin{aligned}
\int_{\Omega} \left(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon} \right)^{\frac{2-p}{2}} u_x u_{xt} dx &= \frac{1}{2} \int_{\Omega} \left(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon} \right)^{\frac{2-p}{2}} (u_x^2)_t dx \\
&= \frac{1}{2} \frac{d}{dt} \int_{\Omega} \left(\int_0^{u_x^2} \left(\frac{\varepsilon s + 1}{s + \varepsilon} \right)^{\frac{2-p}{2}} ds \right) dx,
\end{aligned}$$

$$\begin{aligned}
\int_0^{u_x^2} \left(\frac{\varepsilon s + 1}{s + \varepsilon} \right)^{\frac{2-p}{2}} ds &\geq \int_0^{u_x^2} (s + 1)^{\frac{2-p}{2}} ds \\
&= \frac{2}{p} [(u_x^2 + 1)^{\frac{p}{2}} - 1],
\end{aligned}$$

$$\begin{aligned}
- \iint_{\Omega_T} P_x u_t dx ds &= \iint_{\Omega_T} P u_{xt} dx ds \\
&= \frac{d}{dt} \iint_{\Omega_T} P u_x dx ds - \iint_{\Omega_T} P_t u_x dx ds.
\end{aligned}$$

By virtue of (10), we have

$$(35) \quad \begin{aligned} P_t &= -\gamma P u_x - P_x u, \\ -\iint_{\Omega_T} \eta_x u_t dx ds &= \iint_{\Omega_T} \eta u_{xt} dx ds = \frac{d}{dt} \iint_{\Omega_T} \eta u_x dx ds - \iint_{\Omega_T} \eta_t u_x dx ds, \\ -\iint_{\Omega_T} \eta \Phi_x u_t dx ds &= -\frac{d}{dt} \iint_{\Omega_T} \eta \Phi_x u dx ds + \iint_{\Omega_T} \eta_t \Phi_x u dx ds. \end{aligned}$$

Substituting the above into (34), using Sobolev inequality and Young's inequality, we have

$$(36) \quad \begin{aligned} &\int_0^t |\sqrt{\rho} u_t(s)|_{L^2}^2 ds + |u_x(t)|_{L^p}^p \\ &\leq \iint_{\Omega_T} (|\rho u u_x u_t| + |\rho \Psi_x u_t| + |\gamma P u_x^2| + |P_x u u_x| + |\eta_t u_x| + |\eta_t \Phi_x u|) dx ds \\ &\quad + \int_{\Omega} (|P u_x| + |\eta u_x| + |\eta \Phi_x u|) dx + C \\ &\leq C + \int_0^t (|\rho|_{L^\infty}^{\frac{1}{2}} |u|_{L^\infty} |u_x|_{L^p}^{\frac{p}{2}} |u_x|_{L^\infty}^{1-\frac{p}{2}} |\sqrt{\rho} u_t|_{L^2} + |\rho|_{L^\infty}^{\frac{1}{2}} |\Psi_x|_{L^\infty} |\sqrt{\rho} u_t|_{L^2}) ds \\ &\quad + \int_0^t (\gamma |P|_{L^2} |u_x|_{L^p}^{\frac{p}{2}} |u_x|_{L^\infty}^{1-\frac{p}{2}} |u_{xx}|_{L^2} + a\gamma |\rho|_{L^\infty}^{\gamma-1} |\rho_x|_{L^2} |u|_{L^\infty} |u_x|_{L^\infty} + |\eta_t|_{L^2} |u_x|_{L^p}^{\frac{p}{2}} |u_x|_{L^\infty}^{1-\frac{p}{2}} \\ &\quad + |\eta_t|_{L^2} |\Phi_x|_{L^2} |u|_{L^\infty}) ds + |P|_{L^{\frac{p}{p-1}}} |u_x|_{L^p} + |\eta|_{L^{\frac{p}{p-1}}} |u_x|_{L^p} + |\eta|_{L^{\frac{p}{p-1}}} |\Phi_x|_{L^p} |u|_{L^\infty} \\ &\leq C(1 + \int_0^t (|\rho|_{L^\infty} |u_x|_{L^p}^{2+p} |u_{xx}|_{L^2}^{2-p} + |\rho|_{H^1} |\Psi_{xx}|_{L^2}^2 + |P|_{L^\infty} |u_x|_{L^p}^{\frac{p}{2}} |u_{xx}|_{L^2}^{2-\frac{p}{2}} \\ &\quad + |\rho|_{L^\infty}^{\gamma-1} |\rho_x|_{L^2} |u_x|_{L^p} |u_{xx}|_{L^2} + |\eta_t|_{L^2} |u_x|_{L^p}^{\frac{p}{2}} |u_{xx}|_{L^2}^{1-\frac{p}{2}} + |\eta_t|_{L^2} |u_x|_{L^p}) ds) \\ &\quad + |P|_{L^{\frac{p}{p-1}}}^{\frac{p}{p-1}} + |\eta|_{L^{\frac{p}{p-1}}}^{\frac{p}{p-1}} + \frac{1}{2} \int_0^t |\sqrt{\rho} u_t(s)|_{L^2}^2 ds + \frac{1}{2} |u_x(t)|_{L^p}^p. \end{aligned}$$

To estimate (36), combining (35) we have the following estimates

$$(37) \quad \begin{aligned} \int_{\Omega} |P(t)|_{L^{\frac{p}{p-1}}} dx &= \int_{\Omega} |P(0)|_{L^{\frac{p}{p-1}}} dx + \int_0^t \frac{\partial}{\partial s} \left(\int_{\Omega} P(s)_{L^{\frac{p}{p-1}}} dx \right) ds \\ &\leq \int_{\Omega} |P(0)|_{L^{\frac{p}{p-1}}} dx + \frac{p}{p-1} \int_0^t \int_{\Omega} a\gamma \rho^{\gamma-1} P(s)_{L^{\frac{p}{p-1}}} (-\rho_x u - \rho u_x) dx ds \\ &\leq C + C \int_0^t |\rho|_{L^\infty}^{\gamma-1} |P|_{L^\infty}^{\frac{1}{p-1}} |\rho|_{H^1} |u_x|_{L^p} ds \\ &\leq C(1 + \int_0^t Z^{\frac{\gamma}{p-1} + \gamma + 1}(s) ds), \end{aligned}$$

In exactly the same way, we also have

$$(38) \quad \int_{\Omega} |\eta(t)|_{L^{\frac{p}{p-1}}} dx \leq C(1 + \int_0^t Z^{\frac{1}{p-1} + 1}(s) ds),$$

which, together with (36) and (37), implies (32) holds. \square

Lemma 2.5.

$$(39) \quad |\sqrt{\rho} u_t(t)|_{L^2}^2 + \int_0^t |u_{xt}|_{L^2}^2(s) ds \leq C(1 + \int_0^t Z^{\frac{26\gamma}{(3p-4)(q-1)}}(s) ds),$$

where C is a positive constant, depending only on M_0 .

Proof. Differentiating equation (11) with respect to t , multiplying the result equation by u_t , and integrating it over Ω , we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\Omega} \rho |u_t|^2 dx + \int_{\Omega} \left[\left(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon} \right)^{\frac{2-p}{2}} u_x \right]_t u_{xt} dx \\ (40) \quad & = \int_{\Omega} [(\rho u)_x (u_t^2 + u u_x u_t + \Psi_x u_t) - \rho u_x u_t^2 + (P + \eta)_t u_{xt} - \eta_t \Phi_x u_t - \rho \Psi_{xt} u_t] dx. \end{aligned}$$

Note that

$$\begin{aligned} & \int_{\Omega} \left[\left(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon} \right)^{\frac{2-p}{2}} u_x \right]_t u_{xt} dx \\ & = \int_{\Omega} \left[\left(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon} \right)^{-\frac{p}{2}} u_x \right] \frac{(\varepsilon u_x^2 + 1)(u_x^2 + \varepsilon) - (2-p)(1 - \varepsilon^2)u_x^2}{(u_x^2 + \varepsilon)^2} u_{xt}^2 dx \\ (41) \quad & \geq (p-1) \int_{\Omega} (u_x^2 + 1)^{\frac{p-2}{2}} |u_{xt}|^2 dx, \end{aligned}$$

Let

$$\omega = (u_x^2 + 1)^{\frac{p-2}{4}},$$

from (24), it follows that

$$\begin{aligned} |\omega^{-1}|_{L^\infty} &= |(u_x^2 + 1)^{\frac{2-p}{4}}|_{L^\infty} \\ &\leq C(|u_{xx}|_{L^{\frac{2}{2-p}}} + 1) \\ &\leq CZ^{\frac{2\gamma}{(3p-4)(q-1)}}(t). \end{aligned}$$

Combining (35), (40) can be rewritten into

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} |\rho |u_t|^2 dx + \int_{\Omega} |\omega u_{xt}|^2 dx \\ & \leq 2 \int_{\Omega} \rho |u| |u_t| |u_{xt}| dx + \int_{\Omega} \rho |u| |u_x|^2 |u_t| dx + \int_{\Omega} |\rho_x| |u|^2 |u_x| |u_t| dx \\ & \quad + \int_{\Omega} |\rho_x| |u| |\Psi_x| |u_t| dx + \int_{\Omega} \rho |u_x| |\Psi_x| |u_t| dx + \int_{\Omega} \rho |u_x| |u_t|^2 dx \\ & \quad + \int_{\Omega} \gamma P |u_x| |u_{xt}| dx + \int_{\Omega} |P_x| |u| |u_{xt}| dx + \int_{\Omega} |\eta_t| |u_{xt}| dx \\ & \quad + \int_{\Omega} |\eta_t| |\Phi_x| |u_t| dx + \int_{\Omega} \rho |\Psi_{xt}| |u_t| dx \\ (42) \quad & = \sum_{j=1}^{11} I_j. \end{aligned}$$

Using Sobolev inequality, Young's inequality, (11),(24) and (25), we obtain

$$\begin{aligned} I_1 &\leq 2|\rho|_{L^\infty}^{\frac{1}{2}} |u|_{L^\infty} |\sqrt{\rho} u_t|_{L^2} |\omega u_{xt}|_{L^2} |\omega^{-1}|_{L^\infty} \\ &\leq CZ^{\frac{14\gamma}{(3p-4)(q-1)}}(t) + \frac{1}{7} |\omega u_{xt}|_{L^2}^2, \end{aligned}$$

$$\begin{aligned}
I_2 &\leq |\rho|_{L^2}^{\frac{1}{2}} |u|_{L^\infty} |u_x|_{L^\infty}^2 |\sqrt{\rho} u_t|_{L^2} \\
&\leq |\rho|_{H^1}^{\frac{1}{2}} |u_x|_{L^p} |u_{xx}|_{L^2}^2 |\sqrt{\rho} u_t|_{L^2} \\
&\leq CZ^{\frac{11\gamma}{(3p-4)(q-1)}}(t), \\
I_3 &\leq |\rho_x|_{L^2} |u|_{L^\infty}^2 |u_x|_{L^p} |u_x|_{L^\infty}^{1-\frac{p}{2}} |u_t|_{L^\infty} \\
&\leq |\rho|_{H^1} |u_x|_{L^p}^3 |u_{xx}|_{L^2}^{1-\frac{p}{2}} |\omega u_{xt}|_{L^2} |\omega^{-1}|_{L^\infty} \\
&\leq CZ^{\frac{24\gamma}{(3p-4)(q-1)}}(t) + \frac{1}{7} |\omega u_{xt}|_{L^2}^2, \\
I_4 &\leq |\rho_x|_{L^2} |u|_{L^\infty} |\Psi_x|_{L^2} |u_t|_{L^\infty} \\
&\leq |\rho|_{H^1} |u_x|_{L^p} |\Psi_{xx}|_{L^2} |\omega u_{xt}|_{L^2} |\omega^{-1}|_{L^\infty} \\
&\leq CZ^{\frac{16\gamma}{(3p-4)(q-1)}}(t) + \frac{1}{7} |\omega u_{xt}|_{L^2}^2, \\
I_5 &\leq |\rho|_{L^\infty}^{\frac{1}{2}} |u_x|_{L^\infty} |\Psi_x|_{L^2} |\sqrt{\rho} u_t|_{L^2} \\
&\leq |\rho|_{H^1}^{\frac{1}{2}} |u_{xx}|_{L^2} |\Psi_{xx}|_{L^2} |\sqrt{\rho} u_t|_{L^2} \\
&\leq CZ^{\frac{26\gamma}{(3p-4)(q-1)}}(t), \\
I_6 &\leq |u_x|_{L^\infty} |\sqrt{\rho} u_t|_{L^2}^2 \\
&\leq CZ^{\frac{10\gamma}{(3p-4)(q-1)}}(t), \\
I_7 &\leq C |P|_{L^2} |u_x|_{L^\infty} |\omega u_{xt}|_{L^2} |\omega^{-1}|_{L^\infty} \\
&\leq CZ^{\frac{20\gamma}{(3p-4)(q-1)}}(t) + \frac{1}{7} |\omega u_{xt}|_{L^2}^2, \\
I_8 &\leq |P_x|_{L^2} |u|_{L^\infty} |\omega u_{xt}|_{L^2} |\omega^{-1}|_{L^\infty} \\
&\leq CZ^{\frac{8\gamma}{(3p-4)(q-1)}}(t) + \frac{1}{7} |\omega u_{xt}|_{L^2}^2, \\
I_9 &\leq |\eta_t|_{L^2} |\omega u_{xt}|_{L^2} |\omega^{-1}|_{L^\infty} \\
&\leq CZ^{\frac{12\gamma}{(3p-4)(q-1)}}(t) + \frac{1}{7} |\omega u_{xt}|_{L^2}^2, \\
I_{10} &\leq |\eta_t|_{L^2} |\Phi_x|_{L^\infty} |u_t|_{L^\infty} \\
&\leq C |\eta_t|_{L^2} |\omega u_{xt}|_{L^2} |\omega^{-1}|_{L^\infty} \\
&\leq CZ^{\frac{12\gamma}{(3p-4)(q-1)}}(t) + \frac{1}{7} |\omega u_{xt}|_{L^2}^2, \\
I_{11} &\leq |\rho|_{L^\infty}^{\frac{1}{2}} |\Psi_{xt}|_{L^2} |\sqrt{\rho} u_t|_{L^2}.
\end{aligned}$$

In order to estimate I_{11} , we need to deal with the estimate of $|\Psi_{xt}|_{L^2}$. Differentiating (12) with respect to t , multiplying it by Ψ_t and integrating over Ω , we have

$$(43) \quad \int_{\Omega} \left[\left(\frac{\epsilon \Psi_x^2 + 1}{\Psi_x^2 + \epsilon} \right)^{\frac{2-q}{2}} \Psi_x \right]_t \Psi_{xt} dx = -4\pi g \int_{\Omega} (\rho u)_x \Psi_t dx,$$

and

$$(44) \quad \int_{\Omega} \left[\left(\frac{\epsilon \Psi_x^2 + 1}{\Psi_x^2 + \epsilon} \right)^{\frac{2-q}{2}} \Psi_x \right]_t \Psi_{xt} dx \geq (q-1) \int_{\Omega} (\Psi_x^2 + 1)^{\frac{q-2}{2}} |\Psi_{xt}|^2 dx.$$

Let

$$\beta^q = (\Psi_x^2 + 1)^{\frac{q-2}{4}}$$

then

$$\begin{aligned} |(\beta^q)^{-1}|_{L^\infty} &= |(\Psi_x^2 + 1)^{\frac{2-q}{4}}|_{L^\infty} \\ &\leq C(|\Psi_{xx}|_{L^2}^{\frac{2-q}{2}} + 1) \\ &\leq CZ^{\frac{2-q}{2(q-1)}}(t). \end{aligned}$$

Then (43) can be rewritten into

$$\begin{aligned} \int_{\Omega} |\beta^q \Psi_{xt}|^2 dx &\leq C \int_{\Omega} (\rho u) \Psi_{xt} dx \\ &\leq C |\rho|_{L^2} |u|_{L^\infty} |\beta^q \Psi_{xt}|_{L^2} |(\beta^q)^{-1}|_{L^\infty}. \end{aligned}$$

Using Young's inequality, combining the above estimates we deduce that

$$\begin{aligned} I_{11} &\leq |\rho|_{L^\infty}^{\frac{1}{2}} |\sqrt{\rho} u_t|_{L^2} |\beta^q \Psi_{xt}|_{L^2} |(\beta^q)^{-1}|_{L^\infty} \\ &\leq CZ^{\frac{5q-3}{2(q-1)}}(t). \end{aligned}$$

Substituting $I_j (j = 1, 2, \dots, 11)$ into (42), and integrating over $(\tau, t) \subset (0, T)$ on the time variable, we have

$$(45) \quad |\sqrt{\rho} u_t(t)|_{L^2}^2 + \int_0^t |\omega u_{xt}|_{L^2}^2(s) ds \leq |\sqrt{\rho} u_t(\tau)|_{L^2}^2 + \int_0^t Z^{\frac{26\gamma}{(3p-4)(q-1)}}(s) ds.$$

To obtain the estimate of $|\sqrt{\rho} u_t(t)|_{L^2}^2$, we need to estimate $\lim_{\tau \rightarrow 0} |\sqrt{\rho} u_t(\tau)|_{L^2}^2$.

Multiplying (33) by u_t and integrating over Ω , we get

$$\int_{\Omega} \rho |u_t|^2 dx \leq 2 \int_{\Omega} (\rho |u|^2 |u_x|^2 + \rho |\Psi_x|^2 + \rho^{-1}) - [(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon})^{\frac{2-p}{2}} u_x]_x + (P + \eta)_x + \eta |\Phi_x|^2 dx.$$

According to the smoothness of (ρ, u, η) , we have

$$\begin{aligned} &\lim_{\tau \rightarrow 0} \int_{\Omega} (\rho |u|^2 |u_x|^2 + \rho |\Psi_x|^2 + \rho^{-1}) - [(\frac{\varepsilon u_x^2 + 1}{u_x^2 + \varepsilon})^{\frac{2-p}{2}} u_x]_x + (P + \eta)_x + \eta |\Phi_x|^2 dx \\ &= \int_{\Omega} (\rho_0 |u_0|^2 |u_{0x}|^2 + \rho_0 |\Psi_x|^2 + \rho_0^{-1}) - [(\frac{\varepsilon u_{0x}^2 + 1}{u_{0x}^2 + \varepsilon})^{\frac{2-p}{2}} u_{0x}]_x + (P_0 + \eta_0)_x + \eta_0 |\Phi_x|^2 dx \\ &\leq |\rho_0|_{L^\infty} |u_0|_{L^\infty}^2 |u_{0x}|_{L^2}^2 + |\rho_0|_{L^\infty} |\Psi_x|^2 + |g|_{L^2}^2 + |\Phi_x|_{L^2}^2 \leq C. \end{aligned}$$

Then, taking a limit on τ in (45), as $\tau \rightarrow 0$, we can easily obtain

$$(46) \quad |\sqrt{\rho} u_t(t)|_{L^2}^2 + \int_0^t |u_{xt}|_{L^2}^2(s) ds \leq C(1 + \int_0^t Z^{\frac{26\gamma}{(3p-4)(q-1)}}(s) ds),$$

This complete the proof of Lemma 2.5. \square

With the help of Lemma 2.2 to Lemma 2.5, and the definition of $Z(t)$, we conclude that

$$(47) \quad Z(t) \leq C \exp(\tilde{C} \int_0^t Z^{\frac{26\gamma}{(3p-4)(q-1)}}(s) ds),$$

where C, \tilde{C} are positive constants, depending only on M_0 . This means that there exist a time $T_1 > 0$ and a constant C , such that

$$(48) \quad \begin{aligned} & \operatorname{ess\,sup}_{0 \leq t \leq T_1} (|\rho|_{H^1} + |u|_{W_0^{1,p} \cap H^2} + |\eta|_{H^2} + |\eta_t|_{L^2} + |\sqrt{\rho}u_t|_{L^2} + |\rho_t|_{L^2}) \\ & + \int_0^{T_1} (|\sqrt{\rho}u_t|_{L^2}^2 + |u_{xt}|_{L^2}^2 + |\eta_x|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta_{xt}|_{L^2}^2) ds \leq C, \end{aligned}$$

where C is a positive constant, depending only on M_0 .

3. PROOF OF THE MAIN THEOREM

In this section, the existence of strong solutions can be established by a standard argument. We construct the approximate solutions by using the iterative scheme, derive uniform bounds and thus obtain solutions of the original problem by passing to the limit. Our proof will be based on the usual iteration argument and some ideas developed in [10]. Precisely, we first define $u^0 = 0$ and assuming that u^{k-1} was defined for $k \geq 1$, let ρ^k, u^k, η^k be the unique smooth solution to the following system

$$(49) \quad \rho_t^k + \rho_x^k u^{k-1} + \rho^k u_x^{k-1} = 0,$$

$$(50) \quad \rho^k u_t^k + \rho^k u^{k-1} u_x^k + \rho^k \Psi_x^k + L_p u^k + P_x^k + \eta_x^k = -\eta^k \Phi_x,$$

$$(51) \quad L_q \Psi^k = 4\pi g(\rho^k - m_0),$$

$$(52) \quad \eta_t^k + (\eta^k (u^{k-1} - \Phi_x))_x = \eta_{xx}^k,$$

with the initial and boundary conditions

$$(53) \quad (\rho^k, u^k, \eta^k)|_{t=0} = (\rho_0, u_0, \eta_0),$$

$$(54) \quad u^k|_{\partial\Omega} = (\eta_x^k + \eta^k \Phi_x)|_{\partial\Omega} = 0,$$

where

$$L_p \theta^k = - \left[\left(\frac{\varepsilon (\theta_x^k)^2 + 1}{(\theta_x^k)^2 + \varepsilon} \right)^{\frac{2-p}{2}} \theta_x^k \right]_x.$$

With the process, the nonlinear coupled system has been deduced into a sequence of decoupled problems and each problem admits a smooth solution. And the following estimates hold

$$(55) \quad \begin{aligned} & \operatorname{ess\,sup}_{0 \leq t \leq T_1} (|\rho^k|_{H^1} + |u^k|_{W_0^{1,p} \cap H^2} + |\eta^k|_{H^2} + |\eta_t^k|_{L^2} + |\sqrt{\rho^k}u_t^k|_{L^2} + |\rho_t^k|_{L^2}) \\ & + \int_0^{T_1} (|\sqrt{\rho^k}u_t^k|_{L^2}^2 + |u_{xt}^k|_{L^2}^2 + |\eta_x^k|_{L^2}^2 + |\eta_t^k|_{L^2}^2 + |\eta_{xt}^k|_{L^2}^2) ds \leq C, \end{aligned}$$

where C is a generic constant depending only on M_0 , but independent of k .

In addition, we first find ρ^k from the initial problem

$$\rho_t^k + u^{k-1} \rho_x^k + u_x^{k-1} \rho^k = 0,$$

$$\rho^k|_{t=0} = \rho_0,$$

with smooth function u^{k-1} , obviously, there is a unique solution ρ^k on the above problem and also we could obtain that

$$\rho^k(x, t) \geq \delta \exp \left[- \int_0^{T_1} |u_x^{k-1}(\cdot, s)|_{L^\infty} ds \right] > 0, \text{ for all } t \in (0, T_1).$$

Next, we will prove the approximate solution (ρ^k, u^k, η^k) converges to a limit $(\rho^\varepsilon, u^\varepsilon, \eta^\varepsilon)$ in a strong sense. To this end, let us define

$$\bar{\rho}^{k+1} = \rho^{k+1} - \rho^k, \quad \bar{u}^{k+1} = u^{k+1} - u^k, \quad \bar{\eta}^{k+1} = \eta^{k+1} - \eta^k, \quad \bar{\Psi}^{k+1} = \Psi^{k+1} - \Psi^k.$$

By a direct calculation, we can verify that the functions $\bar{\rho}^{k+1}, \bar{u}^{k+1}, \bar{\eta}^{k+1}$ satisfy the system of equations

$$(56) \quad \bar{\rho}_t^{k+1} + (\bar{\rho}^{k+1} u^k)_x + (\rho^k \bar{u}^k)_x = 0,$$

$$(57) \quad \begin{aligned} & \rho^{k+1} \bar{u}_t^{k+1} + \rho^{k+1} u^k \bar{u}_x^{k+1} + (L_p u^{k+1} - L_p u^k) = -\bar{\rho}^{k+1} (u_t^k + u^k u_x^k + \Psi_x^{k+1}) \\ & - (P^{k+1} - P^k)_x - \bar{\eta}_x^{k+1} + \rho^k (\bar{u}^k u_x^k - \bar{\Psi}_x^{k+1}) - \bar{\eta}^{k+1} \Phi_x, \end{aligned}$$

$$(58) \quad L_q \Psi^{k+1} - L_q \Psi^k = 4\pi g \bar{\rho}^{k+1},$$

$$(59) \quad \bar{\eta}_t^{k+1} + (\eta^k \bar{u}^k)_x + (\bar{\eta}^{k+1} (u^k - \Phi_x))_x = \bar{\eta}_{xx}^{k+1}.$$

Multiplying (56) by $\bar{\rho}^{k+1}$, integrating over Ω and using Young's inequality, we obtain

$$(60) \quad \begin{aligned} \frac{d}{dt} |\bar{\rho}^{k+1}|_{L^2}^2 & \leq C |\bar{\rho}^{k+1}|_{L^2}^2 |u_x^k|_{L^\infty} + |\rho^k|_{H^1} |\bar{u}_x^k|_{L^2} |\bar{\rho}^{k+1}|_{L^2} \\ & \leq C |u_{xx}^k|_{L^2} |\bar{\rho}^{k+1}|_{L^2}^2 + C_\xi |\rho^k|_{H^1}^2 |\bar{\rho}^{k+1}|_{L^2}^2 + \xi |\bar{u}_x^k|_{L^2}^2 \\ & \leq C_\xi |\bar{\rho}^{k+1}|_{L^2}^2 + \xi |\bar{u}_x^k|_{L^2}^2, \end{aligned}$$

where C_ξ is a positive constant, depending on M_0 and ζ for all $t < T_1$ and $k \geq 1$.

Multiplying (57) by \bar{u}^{k+1} , integrating over Ω and using Young's inequality, we obtain

$$(61) \quad \begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_\Omega \rho^{k+1} |\bar{u}^{k+1}|^2 dx + \int_\Omega (L_p u^{k+1} - L_p u^k) \bar{u}^{k+1} dx \\ & \leq C \int_\Omega [|\bar{\rho}^{k+1}| (|u_t^k| + |u^k u_x^k| + |\Psi_x^{k+1}|) + |P_x^{k+1} - P_x^k| + |\bar{\eta}_x^{k+1}| + |\rho^k| |\bar{u}^k| |u_x^k| \\ & \quad + |\rho^k| |\bar{\Psi}_x^{k+1}| + |\bar{\eta}^{k+1} \Phi_x|] |\bar{u}^{k+1}| dx \\ & \leq C (|\bar{\rho}^{k+1}|_{L^2} |u_{xt}^k|_{L^2} |\bar{u}_x^{k+1}|_{L^2} + |\bar{\rho}^{k+1}|_{L^2} |u_x^k|_{L^p} |u_{xx}^k|_{L^2} |\bar{u}_x^{k+1}|_{L^2} + |\bar{\rho}^{k+1}|_{L^2} |\Psi_x^{k+1}|_{L^2} |\bar{u}_x^{k+1}|_{L^2} \\ & \quad + |P^{k+1} - P^k|_{L^2} |\bar{u}_x^{k+1}|_{L^2} + |\bar{\eta}^{k+1}|_{L^2} |\bar{u}_x^{k+1}|_{L^2} + |\rho^k|_{L^2}^{\frac{1}{2}} |\sqrt{\rho^k} \bar{u}^k|_{L^2} |u_{xx}^k|_{L^2} |\bar{u}_x^{k+1}|_{L^2} \\ & \quad + |\rho^k|_{H^1} |\bar{\Psi}_x^{k+1}|_{L^2} |\bar{u}_x^{k+1}|_{L^2} + |\bar{\eta}^{k+1}|_{L^2} |\bar{u}_x^{k+1}|_{L^2}). \end{aligned}$$

Let

$$\sigma(s) = \left(\frac{\varepsilon s^2 + 1}{s^2 + \varepsilon} \right)^{\frac{2-p}{2}} s,$$

then

$$\begin{aligned} \sigma'(s) & = \left(\frac{\varepsilon s^2 + 1}{s^2 + \varepsilon} \right)^{-\frac{p}{2}} \frac{(\varepsilon s^2 + 1)(s^2 + \varepsilon) - (2-p)(1 - \varepsilon^2)s^2}{(s^2 + \varepsilon)^2} \\ & \geq \frac{p-1}{(s^2 + \varepsilon)^{\frac{2-p}{2}}}. \end{aligned}$$

To estimate the second term of (61), we have

$$(62) \quad \begin{aligned} \int_\Omega (L_p u^{k+1} - L_p u^k) \bar{u}^{k+1} dx & = \int_\Omega \int_0^1 \sigma'(\theta u_x^{k+1} + (1-\theta)u_x^k) d\theta |\bar{u}_x^{k+1}|^2 dx \\ & \geq \int_\Omega \left[\int_0^1 \frac{d\theta}{|\theta u_x^{k+1} + (1-\theta)u_x^k|_{L^\infty}^{2-p} + 1} \right] (\bar{u}_x^{k+1})^2 \\ & \geq C^{-1} \int_\Omega |\bar{u}_x^{k+1}|^2 dx. \end{aligned}$$

On the other hand, multiplying (58) by $\bar{\Psi}^{k+1}$, integrating over Ω , we obtain

$$(63) \quad \int_{\Omega} (L_q \Psi^{k+1} - L_q \Psi^k) \bar{\Psi}^{k+1} dx = 4\pi g \int_{\Omega} \bar{\rho}^{k+1} \bar{\Psi}^{k+1} dx.$$

Since

$$\int_{\Omega} (L_q \Psi^{k+1} - L_q \Psi^k) \bar{\Psi}_x^{k+1} dx = (q-1) \int_{\Omega} \left(\int_0^1 |\theta \Psi_x^{k+1} + (1-\theta) \Psi_x^k|^{q-2} d\theta \right) (\bar{\Psi}_x^{k+1})^2 dx,$$

and

$$\begin{aligned} \int_0^1 |\theta \Psi_x^{k+1} + (1-\theta) \Psi_x^k|^{q-2} d\theta &= \int_0^1 \frac{1}{|\theta \Psi_x^{k+1} + (1-\theta) \Psi_x^k|^{2-q}} d\theta \\ &\geq \int_0^1 \frac{1}{(|\Psi_x^{k+1}| + |\Psi_x^k|)^{2-q}} d\theta \\ &= \frac{1}{(|\Psi_x^{k+1}| + |\Psi_x^k|)^{2-q}}, \end{aligned}$$

then

$$\int_{\Omega} [|\Psi_x^{k+1}|^{q-2} \Psi_x^{k+1} - |\Psi_x^k|^{q-2} \Psi_x^k] \bar{\Psi}_x^{k+1} dx \geq \frac{1}{(|\Psi_x^{k+1}(t)|_{L^\infty} + |\Psi_x^k(t)|_{L^\infty})^{2-q}} \int_{\Omega} (\bar{\Psi}_x^{k+1})^2 dx,$$

which implies

$$(64) \quad \int_{\Omega} (\bar{\Psi}_x^{k+1})^2 dx \leq C |\bar{\rho}^{k+1}|_{L^2}^2.$$

From (55), (62) and (64), (61) can be re-written as

$$(65) \quad \begin{aligned} &\frac{d}{dt} \int_{\Omega} \rho^{k+1} |\bar{u}^{k+1}|^2 dx + C^{-1} \int_{\Omega} |\bar{u}_x^{k+1}|^2 dx \\ &\leq B_\xi(t) |\bar{\rho}^{k+1}|_{L^2}^2 + C(|\sqrt{\rho^k} \bar{u}^k|_{L^2}^2 + |\bar{\eta}^{k+1}|_{L^2}^2) + \xi |\bar{u}_x^{k+1}|_{L^2}^2, \end{aligned}$$

where $B_\xi(t) = C(1 + |u_{xt}^k(t)|_{L^2}^2)$, for all $t \leq T_1$ and $k \geq 1$. Using (55) we derive

$$\int_0^t B_\xi(s) ds \leq C + Ct.$$

Multiplying (59) by $\bar{\eta}^{k+1}$, integrating over Ω , using (55) and Young's inequality, we have

$$(66) \quad \begin{aligned} &\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\bar{\eta}^{k+1}|^2 dx + \int_{\Omega} |\bar{\eta}_x^{k+1}|^2 dx \\ &\leq \int_{\Omega} |\bar{\eta}^{k+1}| |u^k - \Phi_x| |\bar{\eta}_x^{k+1}| dx + \int_{\Omega} (|\eta^k| |\bar{u}^k|)_x |\bar{\eta}^{k+1}| dx \\ &\leq |\bar{\eta}^{k+1}|_{L^2} |u^k - \Phi_x|_{L^\infty} |\bar{\eta}_x^{k+1}|_{L^2} + |\eta_x^k|_{L^2} |\bar{u}^k|_{L^\infty} |\bar{\eta}^{k+1}|_{L^2} + |\eta^k|_{L^\infty} |\bar{u}_x^k|_{L^2} |\bar{\eta}^{k+1}|_{L^2} \\ &\leq C_\xi |\bar{\eta}^{k+1}|_{L^2}^2 + \xi |\bar{\eta}_x^{k+1}|_{L^2}^2 + \xi |\bar{u}_x^k|_{L^2}^2. \end{aligned}$$

Combining (60), (65) and (66), we have

$$(67) \quad \begin{aligned} &\frac{d}{dt} \left(|\bar{\rho}^{k+1}(t)|_{L^2}^2 + |\sqrt{\rho^{k+1}} \bar{u}^{k+1}(t)|_{L^2}^2 + |\bar{\eta}^{k+1}(t)|_{L^2}^2 \right) + |\bar{u}_x^{k+1}(t)|_{L^2}^2 + |\bar{\eta}_x^{k+1}|_{L^2}^2 \\ &\leq E_\xi(t) |\bar{\rho}^{k+1}(t)|_{L^2}^2 + C |\sqrt{\rho^k} \bar{u}^k|_{L^2}^2 + C_\xi |\bar{\eta}^{k+1}|_{L^2}^2 + \xi |\bar{u}_x^k|_{L^2}^2, \end{aligned}$$

where $E_\zeta(t)$ is depending only on $B_\zeta(t)$ and C_ξ , for all $t \leq T_1$ and $k \geq 1$. Using (55), we obtain

$$\int_0^t E_\xi(s) ds \leq C + C_\xi t.$$

Integrating (67) over $(0, t) \subset (0, T_1)$ with respect to t , using Gronwall's inequality, we have

$$\begin{aligned} & |\bar{\rho}^{k+1}(t)|_{L^2}^2 + |\sqrt{\rho^{k+1}} \bar{u}^{k+1}(t)|_{L^2}^2 + |\bar{\eta}^{k+1}(t)|_{L^2}^2 + \int_0^t |\bar{u}_x^{k+1}(t)|_{L^2}^2 ds + \int_0^t |\bar{\eta}_x^{k+1}|_{L^2}^2 ds \\ (68) \quad & \leq C \exp(C_\xi t) \int_0^t (|\sqrt{\rho^k} \bar{u}^k(s)|_{L^2}^2 + |\bar{u}_x^k(s)|_{L^2}^2) ds. \end{aligned}$$

From the above recursive relation, choose $\xi > 0$ and $0 < T_* < T_1$ such that $C \exp(C_\xi T_*) < \frac{1}{2}$, using Gronwall's inequality, we deduce that

$$\begin{aligned} & \sum_{k=1}^K \left[\sup_{0 \leq t \leq T_*} (|\bar{\rho}^{k+1}(t)|_{L^2}^2 + |\sqrt{\rho^{k+1}} \bar{u}^{k+1}(t)|_{L^2}^2 + |\bar{\eta}^{k+1}(t)|_{L^2}^2) \right. \\ (69) \quad & \left. + \int_0^{T_*} |\bar{u}_x^{k+1}(t)|_{L^2}^2 + \int_0^{T_*} |\bar{\eta}_x^{k+1}(t)|_{L^2}^2 dt \right] < C, \end{aligned}$$

where C is a positive constant, depending only on M_0 .

Therefore, as $k \rightarrow +\infty$, the sequence (ρ^k, u^k, η^k) converges to a limit $(\rho^\varepsilon, u^\varepsilon, \eta^\varepsilon)$ in the following strong sense

$$(70) \quad \rho^k \rightarrow \rho^\varepsilon \quad \text{in } L^\infty(0, T_*; L^2(\Omega)),$$

$$(71) \quad u^k \rightarrow u^\varepsilon \quad \text{in } L^\infty(0, T_*; L^2(\Omega)) \cap L^2(0, T_*; H_0^1(\Omega)),$$

$$(72) \quad \eta^k \rightarrow \eta^\varepsilon \quad \text{in } L^\infty(0, T_*; L^2(\Omega)) \cap L^2(0, T_*; H^1(\Omega)).$$

By virtue of the lower semi-continuity of various norms, we deduce from the uniform estimate (55) that $(\rho^\varepsilon, u^\varepsilon, \eta^\varepsilon)$ satisfies the following uniform estimate

$$\begin{aligned} & \text{ess sup}_{0 \leq t \leq T_1} (|\rho^\varepsilon|_{H^1} + |u^\varepsilon|_{W_0^{1,p} \cap H^2} + |\eta^\varepsilon|_{H^2} + |\eta_t^\varepsilon|_{L^2} + |\sqrt{\rho^\varepsilon} u_t^\varepsilon|_{L^2} + |\rho_t^\varepsilon|_{L^2}) \\ (73) \quad & + \int_0^{T_*} (|\sqrt{\rho^\varepsilon} u_t^\varepsilon|_{L^2}^2 + |u_{xt}^\varepsilon|_{L^2}^2 + |\eta_x^\varepsilon|_{L^2}^2 + |\eta_t^\varepsilon|_{L^2}^2 + |\eta_{xt}^\varepsilon|_{L^2}^2) ds \leq C. \end{aligned}$$

Since all of the constants are independent of ε , there exists a subsequence $(\rho^{\varepsilon_j}, u^{\varepsilon_j}, \eta^{\varepsilon_j})$ of $(\rho^\varepsilon, u^\varepsilon, \eta^\varepsilon)$, without loss of generality, we denote to $(\rho^\delta, u^\delta, \eta^\delta)$. Let $\varepsilon \rightarrow 0$, we can get the following convergence

$$(74) \quad \rho^\varepsilon \rightarrow \rho^\delta \quad \text{in } L^\infty(0, T_*; L^2(\Omega)),$$

$$(75) \quad u^\varepsilon \rightarrow u^\delta \quad \text{in } L^\infty(0, T_*; L^2(\Omega)) \cap L^2(0, T_*; H_0^1(\Omega)),$$

$$(76) \quad \eta^\varepsilon \rightarrow \eta^\delta \quad \text{in } L^\infty(0, T_*; L^2(\Omega)) \cap L^2(0, T_*; H^1(\Omega)),$$

and there also holds

$$\begin{aligned} & \text{ess sup}_{0 \leq t \leq T_1} (|\rho^\delta|_{H^1} + |u^\delta|_{W_0^{1,p} \cap H^2} + |\eta^\delta|_{H^2} + |\eta_t^\delta|_{L^2} + |\sqrt{\rho^\delta} u_t^\delta|_{L^2} + |\rho_t^\delta|_{L^2}) \\ (77) \quad & + \int_0^{T_*} (|\sqrt{\rho^\delta} u_t^\delta|_{L^2}^2 + |u_{xt}^\delta|_{L^2}^2 + |\eta_x^\delta|_{L^2}^2 + |\eta_t^\delta|_{L^2}^2 + |\eta_{xt}^\delta|_{L^2}^2) ds \leq C. \end{aligned}$$

For each small $\delta > 0$, let $\rho_0^\delta = J_\delta * \rho_0 + \delta$, where J_δ is a mollifier on Ω , and $u_0^\delta \in H_0^1(\Omega) \cap H^2(\Omega)$ is a smooth solution of the boundary value problem

$$(78) \quad \begin{cases} L_p u_0^\delta + (P(\rho_0^\delta) + \eta_0^\delta)_x + \eta_0^\delta \Phi_x = (\rho_0^\delta)^{\frac{1}{2}}(g^\delta + \Phi_x), \\ u_0^\delta|_{\partial\Omega} = 0, \end{cases}$$

where $g^\delta \in C_0^\infty$ and satisfies $|g^\delta|_{L^2} \leq |g|_{L^2}$, $\lim_{\delta \rightarrow 0^+} |g^\delta - g|_{L^2} = 0$.

We deduce that $(\rho^\delta, u^\delta, \eta^\delta)$ is a solution of the following initial boundary value problem

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2)_x + \rho \Psi_x - \lambda(|u_x|^{p-2} u_x)_x + (P + \eta)_x = -\eta \Phi_x, \\ (|\Psi_x|^{q-2} \Psi_x)_x = 4\pi g(\rho - \frac{1}{|\Omega|} \int_\Omega \rho dx), \\ \eta_t + (\eta(u - \Phi_x))_x = \eta_{xx}, \\ (\rho, u, \eta)|_{t=0} = (\rho_0^\delta, u_0^\delta, \eta_0^\delta), \\ u|_{\partial\Omega} = (\eta_x + \eta \Phi_x)|_{\partial\Omega} = 0, \end{cases}$$

where $\rho_0^\delta \geq \delta$, $\frac{4}{3} < p, q < 2$.

By the proof of Lemma 2.1, there exists a subsequence $\{u_0^{\delta_j}\}$ of $\{u_0^\delta\}$, as $\delta_j \rightarrow 0^+$, $u_0^{\delta_j} \rightarrow u_0$ in $H_0^1(\Omega) \cap H^2(\Omega)$, $-(|u_{0x}^{\delta_j}|^{p-2} u_{0x}^{\delta_j})_x \rightarrow -(|u_{0x}|^{p-2} u_{0x})_x$ in $L^2(\Omega)$. Hence, u_0 satisfies the compatibility condition (9) of Theorem 1.2. By virtue of the lower semi-continuity of various norms, we deduce that (ρ, u, η) satisfies the following uniform estimate

$$(79) \quad \begin{aligned} & \operatorname{ess\,sup}_{0 \leq t \leq T_1} (|\rho|_{H^1} + |u|_{W_0^{1,p} \cap H^2} + |\eta|_{H^2} + |\eta_t|_{L^2} + |\sqrt{\rho} u_t|_{L^2} + |\rho_t|_{L^2}) \\ & + \int_0^{T^*} (|\sqrt{\rho} u_t|_{L^2}^2 + |u_{xt}|_{L^2}^2 + |\eta_x|_{L^2}^2 + |\eta_t|_{L^2}^2 + |\eta_{xt}|_{L^2}^2) ds \leq C, \end{aligned}$$

where C is a positive constant, depending only on M_0 . The uniqueness of solution can also be obtained by the same method as the above proof of convergence, we omit the details here. This completes the proof.

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