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WEAK APPROXIMATIVE COMPACTNESS OF HYPERPLANE AND ASPLUND PROPERTY IN MUSIELAK-ORLICZ-BOCHNER FUNCTION SPACES

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Abstract. In this paper, some criteria for weakly approximative compactness and approximative compactness of weak* hyperplane for Musielak-Orlicz-Bochner function spaces are given. Moreover, we also prove that, in Musielak-Orlicz-Bochner function spaces generated by strongly smooth Banach space, $L^0_M(X)$ (resp $L_M(X)$) is an Asplund space if and only if M and N satisfy condition Δ. As a corollary, we obtain that $L^0_M(R)$ (resp $L_M(R)$) is an Asplund space if and only if M and N satisfy condition Δ.

1. Introduction and preliminaries

The study of Orlicz function space originated in the last century. Orlicz function space is an important class of Banach spaces. Orlicz function space has important applications in the field of partial differential equations. However, with the development of differential equation theory, Orlicz function space can no longer satisfy the development of theory of differential equation (see [1], [6]-[11] and [15]-[21]). Mathematicians began to pay attention to the extended form of Orlicz function space. Musielak-Orlicz-Bochner function space is an important extension of Orlicz function space. The development of theory of Musielak-Orlicz-Bochner function space provides theoretical basis for the development of differential equations. In this paper, some criteria for weakly approximative compactness and approximative compactness of weak* hyperplane for Musielak-Orlicz-Bochner function spaces are given. Moreover, we also prove that, in Musielak-Orlicz-Bochner function spaces generated by strongly smooth Banach space, $L^0_M(X)$ (resp $L_M(X)$) is an Asplund space if and only if $M \in \Delta$ and $N \in \Delta$. As a corollary, we obtain that $L^0_M(R)$ (resp $L_M(R)$) is an Asplund space if and only if $M \in \Delta$ and $N \in \Delta$.

Let $(X, \|\cdot\|)$ be a real Banach space, $S(X)$ and $B(X)$ denote the unit sphere and unit ball of X, respectively. By X^* we denote the dual space of X. Let N, R and R^+ denote the sets natural numbers, reals and nonnegative reals, respectively. Let us take a point $x \in S(X)$ and let $H_x = \{x^* \in X^* : x^*(x) = 1\}$. Let $C \subset X$ be a nonempty subset of X. Then the set-valued mapping $P_C : X \to C$

$$P_C(x) = \left\{ z \in C : \|x - z\| = \text{dist}\,(x, C) = \inf_{y \in C} \|x - y\| \right\}$$

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is called the metric projection operator from X onto C. First let us recall some definitions and results that will be used in the further part of the paper.

Definition 1.1. A nonempty subset C of X is said to be approximatively compact (weakly approximatively compact) if for any sequence $\{y_n\}_{n=1}^{\infty} \subset C$ and any $x \in X$ satisfying $\|x - y_n\| \to \inf_{y \in C} \|x - y\|$ as $n \to \infty$, there exists a subsequence converging (weakly) to an element of C.

Definition 1.2. A point $x \in S(X)$ is called a smooth point if it has a unique supporting functional $f_x \in S(X^*)$. If every $x \in S(X)$ is a smooth point, then X is called a smooth space.

Consider a convex subset A of a Banach space X. A point $x \in A$ is said to be an extreme point of A if $2x = y + z$ and $y, z \in A$ imply $y = z$. The set of all extreme points of A is denoted by $ExtA$. If $ExtB(X) = S(X)$, then X is said to be strictly convex. Moreover, it is well known that if X^* is strictly convex, then X is smooth.

Definition 1.3. A point $x \in S(X)$ is said to be a strongly smooth point of X if there exist $\{x_n^*\}_{n=1}^{\infty} \subset S(X^*)$ and $x_0^* \in S(X^*)$ such that $x_n^* \to x_0^*$ whenever $x_n^*(x) \to 1$. A Banach space X is said to be strongly smooth if every point of $S(X)$ is strongly smooth point of X.

Definition 1.4. A Banach space X is said to have the Radon-Nikodym property if let (T, Σ, μ) be nonatomic measurable space. v is a measure and v is bounded variation and absolutely continuous with respect to μ, then there exists an integrable function f such that for any $A \in \Sigma$, we have

$$v(A) = \int_A f d\mu.$$

It is well known that if X is strongly smooth, then X has the Radon-Nikodym property. Moreover, it is easy to see that if X is a strongly smooth space, then X is smooth. Let f be a real continuous convex function on X. Recall that f is said to be Gâteaux differentiable at the point x in X if the limit

$$(*) \qquad df(x)(y) = \lim_{t \to 0} \frac{1}{t}[f(x + ty) - f(x)]$$

exists for all $y \in X$. If the difference quotient in $(*)$ converges to $df(x)(y)$ uniformly for y in the unit ball $B(X)$, then f is said to be Frechet differentiable at x. X is called an Asplund space if for every continuous convex function on X, there exists a dense G_δ subset G of X such that it is Frechet differentiable at each point of G. It is well known that if X is an Asplund space, then X^* is separable if and only if X is separable. Moreover, It is well known that X is an Asplund space if and only if X^* has the Radon-Nikodym property.

Let (T, Σ, μ) be a complete nonatomic measurable space. Suppose that a function $M : T \times [0, \infty) \to [0, \infty]$ statisfies the following conditions.

(1) for μ-a.e, $t \in T$, $M(t, 0) = 0$, $\lim\limits_{u \to \infty} M(t, u) = \infty$ and $M(t, u') < \infty$ for some $u' > 0$.

(2) for μ-a.e, $t \in T$, $M(t, u)$ is convex in $[0, \infty)$ with respect to u.

(3) for each $u \in [0, \infty)$, $M(t, u)$ is a Σ-measurable function of t on T.

Every such a function M is called a Musielak-Orlicz-function. Let $p(t, \cdot)$ denote the right derivative of $M(t, \cdot)$ at $u \in R^+$ (where $p(t, u) = \infty$ if $M(t, u) = \infty$) and let $q(t, \cdot)$ be the generalized inverse function of $p(t, \cdot)$ defined on R^+ by

$$q(t, v) = \sup\nolimits_{u \geq 0} \{u \geq 0 : p(t, u) \leq v\}.$$

Then the function $N(t,v)$ defined by $N(t,v)=\int_0^{|v|}q(t,s)ds$ for any $v\in R$ and Σ-a.e. $t\in T$ is called the complementary function to M in the sense of Young. It is well known that the Young inequality $uv\le M(t,u)+N(t,v)$ holds for all $u,v,\in R$ and μ-a.e. $t\in T$. Moreover, for any $u\in R$ the equality $uv=M(t,u)+N(t,v)$ holds if and only if $v\in[p_-(t,u),p(t,u)]$. Let

$$e(t)=\sup\{u>0:M(t,u)=0\}\quad\text{and}\quad E(t)=\sup\{u>0:M(t,u)<\infty\}.$$

For a fixed $t\in T$ and $v\ge 0$, if there exists $\varepsilon\in(0,1)$ such that

$$M(t,v)=\frac{1}{2}M(t,v+\varepsilon)+\frac{1}{2}M(t,v-\varepsilon)<\infty,$$

then v is called a point of affinity of $M(t,\cdot)$. The set of all points of affinity of $M(t,\cdot)$ for a fixed $t\in T$ is denoted by K_t.

Definition 1.5. (see[2]) We say that a Musielak-Orlicz function M satisfies condition $\Delta(M\in\Delta)$ if there exist $K\ge 1$ and a measureable nonnegative function $\delta(t)$ on T such that $\int_T M(t,\delta(t))\,dt<\infty$ and $M(t,2u)\le KM(t,u)$ for almost all $t\in T$ and all $u\ge\delta(t)$.

Definition 1.6. (see[2]) A Musielak-Orlicz function $M(t,u)$ is said to be strictly convex with respect to u for $t\in T$ if for almost every $t\in T$ and any $u,v\in R$, $u\ne v$, we have

$$M\left(t,\frac{u+v}{2}\right)<\frac{1}{2}M(t,u)+\frac{1}{2}M(t,v).$$

Given any Banach space $(X,\|\cdot\|)$, we denote by X_T the set of all strongly Σ-measurable functions from T to X, and for each $u\in X_T$, we define the modular of u by

$$\rho_M(u)=\int_T M(t,\|u(t)\|)dt.$$

Let us define the Musielak-Orlicz-Bochner function space $L_M(X)$ by

$$L_M(X)=\{u\in X_T:\rho_M(\lambda u)<\infty\ \text{ for some }\ \lambda>0\}$$

and its subspace

$$E_M(X)=\{u\in X_T:\rho_M(\lambda u)<\infty\ \text{ for all }\ \lambda>0\}.$$

It is well known that the spaces $L_M(X)$ and their subspaces $E_M(X)$ are Banach spaces when they are equipped with the Luxemburg norm

$$\|u\|=\inf\left\{\lambda>0:\rho_M\left(\frac{u}{\lambda}\right)\le 1\right\}$$

or with the Orlicz norm

$$\|u\|^0=\inf_{k>0}\frac{1}{k}[1+\rho_M(ku)].$$

It is well known that the Luxemburg norm and the Orlicz norm are equivalent. The spaces $(L_M(X),\|\cdot\|),(L_M(X),\|\cdot\|^0),(E_M(X),\|\cdot\|_M)$ and $(E_M(X),\|\cdot\|^0)$ are denoted shortly by $L_M(X),L_M^0(X),E_M(X)$ and $E_M^0(X)$, respectively. $L_M(R)$ and $L_M^0(R)$ are called Musielak-Orlicz function spaces, and $E_M(R)$ and $E_M^0(R)$ are the subspaces of $L_M(R)$ and $L_M^0(R)$, respectively. Moreover, it is well known that $(E_M(R))^*=L_N^0(R)$ and $\left(E_M^0(R)\right)^*=L_N(R)$ (see[4], [20]). Moreover, by [2], we know that $E_M(R)=L_M(R)$ if and only if $M\in\Delta$.

Lemma 1.7. *(see[2])* $\|u\|\le 1\Rightarrow\rho_M(u)\le\|u\|$ *and* $\|u\|>1\Rightarrow\rho_M(u)>\|u\|$.

Lemma 1.8. *(see [2]) $M \notin \Delta \Leftrightarrow$ for any $\varepsilon > 0$, there exists $u \in L_M(X)$ such that $\rho_M(u) = \varepsilon$ and $\|u(t)\| < E(t)$ for almost all $t \in T$.*

2. Strongly smooth point and approximative compactness in Musielak-Orlicz-Bochner function spaces

Theorem 2.1. *Suppose that X is a strongly smooth space and $v \in S(E_N^0(X))$. Then the following statements are equivalent:*

(1) The point v is a strongly smooth point of $E_N^0(X)$;
(2) The hyperplane H_v of $L_M(X^*)$ is approximatively compact;
(3) The hyperplane H_v of $L_M(X^*)$ is weakly approximatively compact;
(4) $M \in \Delta$ and $\mu\{t \in T : \|u(t)\| \in K_t\} = 0$ whenever $\langle u, v\rangle = \|u\|$.

In order to prove the theorem, we first give some lemmas.

Lemma 2.2. *Suppose that X is a Banach space and $x \in S(X)$. Then the following statements are equivalent:*

(1) The hyperplane H_x of X^* is approximatively (weakly approximatively) compact.

(2) If $y_n^* \in S(X^*)$ and $y_n^*(x) \to 1$ as $n \to \infty$, then the sequence $\{y_n^*\}_{n=1}^\infty$ is relatively (weakly) compact.

Proof. (2) $\Rightarrow$ (1). By Theorem 2.1 of [18], we obtain that the hyperplane H_x is a proximinal set. We will prove that if $\|x^* - y_n^*\| \to \operatorname{dist}(x^*, H_x)$ as $n \to \infty$, then the sequence $\{y_n^*\}_{n=1}^\infty$ is relatively (weakly) compact, where $\{y_n^*\}_{n=1}^\infty \subset H_x$ and $x^* \notin H_x$. Since $\|x^* - y_n^*\| \to \operatorname{dist}(x^*, H_x)$ as $n \to \infty$, we have $\|0 - (y_n^* - x^*)\| \to \operatorname{dist}(0, H_x - x^*)$. Since $H_x - x^*$ is weakly* closed set, we have $\operatorname{dist}(0, H_x - x^*) = r > 0$. Pick $y_0^* \in P_{H_x - x^*}(0)$. Then

$$r = \operatorname{dist}(0, P_{H_x - x^*}(0)) = \|y_0^*\|, \quad B(0, r) \cap (H_x - x^*) = \emptyset$$

and

$$\overline{B(0, r)} \cap (H_x - x^*) = P_{H_x - x^*}(0).$$

For clarity, we will divide the proof into two cases.

Case I. $k = \sup\{x(y^*) : y^* \in H_x - x^*\} \le 0$. We claim that

$$k = \sup\{x(y^*) : y^* \in H_x - x^*\} \le \inf\{x(y^*) : y^* \in B(0, r)\} = -\|x\| \cdot \|y_0^*\|.$$

In fact, suppose that there exists $y_1^* \in B(0, r)$ such that $x(y_1^*) < k$. Then there exists $\lambda \in (0, 1)$ such that $x(\lambda y_1^*) = k$. It is easy to see that $\lambda y_1^* \in B(0, r)$ and $\lambda y_1^* \in H_x$, a contradiction. Since $y_0^* \in P_{H_x - x^*}(0) \subset H_x - x^*$, we have

$$\begin{aligned} -\|x\| \cdot \|y_0^*\| \le x(y_0^*) &\le \sup\{x(y^*) : y^* \in H_x - x^*\} \\ &\le \inf\{x(y^*) : y^* \in B(0, r)\} \\ &= -\|x\| \cdot \|y_0^*\| \end{aligned}$$

and

$$-\|x\| \cdot \|y_0^*\| = x(y_0^*) = \sup\{x(y^*) : y^* \in H_x - x^*\}.$$

This means that the inequality $x(y_0^*) \ge x(y_n^* - x^*)$ holds. Therefore

$$\begin{aligned} \|y_0^*\| &= x(0 - y_0^*) \le x(x^* - y_n^*) \le \|x^* - y_n^*\| \to \operatorname{dist}(x^*, H_x) \\ &= \operatorname{dist}(0, H_x - x^*) = \|0 - y_0^*\| = \|y_0^*\|. \end{aligned}$$

This implies that $\|x^* - y_n^*\| \to \|y_0^*\|$ and $x(x^* - y_n^*) \to \|y_0^*\|$ as $n \to \infty$. Moreover, we have

$$\lim_{n\to\infty} x\left(-\frac{x^* - y_n^*}{\|x^* - y_n^*\|} + \frac{x^* - y_n^*}{\|y_0^*\|}\right) = \lim_{n\to\infty}\left(\frac{1}{\|y_0^*\|} - \frac{1}{\|x^* - y_n^*\|}\right)\cdot x(x^* - y_n^*) = 0.$$

Therefore, by $x(x^* - y_n^*) \to \|y_0^*\|$ as $n \to \infty$, we have

$$\lim_{n\to\infty} x\left(-\frac{x^* - y_n^*}{\|x^* - y_n^*\|}\right) = 1.$$

Hence the sequence $\{-(x^* - y_n^*)/\|(x^* - y_n^*)\|\}_{n=1}^\infty$ is relatively (weakly) compact. Since $\|x^* - y_n^*\| \to \|y_0^*\|$ as $n \to \infty$, we obtain that $\{y_n^*\}_{n=1}^\infty$ is relatively (weakly) compact. Hence H_x is approximatively (weakly approximatively) compact.

Case II. $\sup\{x(y^*) : y^* \in H_x - x^*\} > 0$. This implies that $k = \sup\{-x(y^*) : y^* \in H_x - x^*\} < 0$. Analogous to the proof of Case I, we have

$$k = \sup\{-x(y^*) : y^* \in H_x - x^*\} \le \inf\{-x(y^*) : y^* \in B(0, r)\} = -\|x\| \cdot \|y_0^*\|.$$

Analogous to the proof of Case I, we obtain that $\{y_n^*\}_{n=1}^\infty$ is relatively (weakly) compact. Hence weak* hyperplane H_x is approximatively (weakly approximatively) compact.

(1) $\Rightarrow$ (2). Let $\{y_n^*\}_{n=1}^\infty \subset S(X^*)$ and $x \in S(X)$ satisfy $y_n^*(x) \to 1$ as $n \to \infty$. Since H_x is a weak* closed set, by Theorem 2.1 of [18], we obtain that H_x is a proximinal set. Hence there exists $z_n^* \in H_x$ such that $\|y_n^* - z_n^*\| = \mathrm{dist}(y_n^*, H_x)$. Pick $x_0^* \in H_x \cap S(X^*)$. Since

$$\begin{aligned}\lim_{n\to\infty}\|y_n^* - z_n^*\| &= \lim_{n\to\infty}\mathrm{dist}\,(y_n^*, H_x) = \lim_{n\to\infty}\mathrm{dist}\,(y_n^* - x_0^*, H_x - x_0^*)\\ &= \lim_{n\to\infty}[x(x_0^* - y_n^*)] = \lim_{n\to\infty}[1 - x(y_n^*)] = 0,\end{aligned}$$

we have the following formula

$$1 \le \limsup_{n\to\infty}\|z_n^*\| \le \lim_{n\to\infty}\|y_n^*\| + \lim_{n\to\infty}\|y_n^* - z_n^*\| = 1 = \mathrm{dist}\,(0, H_x).$$

Therefore, by formula $\|z_n^*\| \ge 1$, we have $\|0 - z_n^*\| \to \mathrm{dist}\,(0, H_x)$ as $n \to \infty$. This implies that the sequence $\{z_n^*\}_{n=1}^\infty$ is relatively (weakly) compact. Consequently $\{y_n^*\}_{n=1}^\infty$ is relatively (weakly) compact by $y_n^* = z_n^* + (y_n^* - z_n^*)$, which completes the proof. □

Lemma 2.3. *Suppose that $u \in S(L_M(X^*))$ is norm attainable on $S(E_N^0(X))$ and $M \in \Delta$. Then (2)⇒(1) is true, where*

(1) $\mu\{t \in T : \|u(t)\| \in K_t\} = 0$;

(2) If $u = \sum_{i=1}^\infty t_i u_i$, where $u_i \in B(L_M(X^*))$, $t_i \in (0,1)$ and $\sum_{i=1}^\infty t_i = 1$, then the sequence $\{u_i\}_{i=1}^\infty$ is relatively weakly compact.

Proof. (2)⇒(1). Let $u \in L_M(X^*)$ and $u = \sum_{i=1}^\infty t_i u_i$, where $u_i \in B(L_M(X^*))$, $t_i \in (0,1)$ and $\sum_{i=1}^\infty t_i = 1$. Suppose that $\mu H' > 0$, where

$$H' = \{t \in T_0 : 2M(t, \|u(t)\|) = M(t, \|u(t)\| + \varepsilon_t) + M(t, \|u(t)\| - \varepsilon_t),\ \varepsilon_t \in (0,1)\},$$

$T_0 = \{t \in T : \|u(t)\| > 0\}$ and $\|u(t)\| - \varepsilon_t > 0$. Since $H' \subset \cup_{n=2}^\infty H_n$, where

$$H_n = \left\{t \in T_0 : 2M(t, \|u(t)\|) = M(t, (1 - \frac{1}{n})\|u(t)\|) + M(t, (1 - \frac{1}{n})\|u(t)\|)\right\},$$

there exists a natural number $n_0 \in N$ such that $\mu H_{n_0} > 0$, where

$$H_{n_0} = \left\{t \in T_0 : 2M(t, \|u(t)\|) = M(t, (1 - \frac{1}{n_0})\|u(t)\|) + M(t, (1 - \frac{1}{n_0})\|u(t)\|)\right\}.$$

Since u is norm attainable on $S(E_N^0(X))$, there exists a point $v \in S(E_N^0(X))$ such that $\int_T \langle u, v\rangle dt = \|u\| = \|v\|^0 = 1$. Let $H = H_{n_0}$. Then, decompose H into H_1^1 and H_1^2 such that $H_1^1 \cup H_2^1 = H$, $H_1^1 \cap H_2^1 = \emptyset$ and $\int_{H_1^1} \langle u, v\rangle dt = \int_{H_2^1} \langle u, v\rangle dt$. Decompose H_1^1 into H_1^2 and H_2^2 such that $H_1^2 \cup H_2^2 = H_1^1$, $H_1^2 \cap H_2^2 = \emptyset$ and $\int_{H_1^2} \langle u, v\rangle dt = \int_{H_2^2} \langle u, v\rangle dt$. Decompose H_2^1 into H_3^2 and H_4^2 such that $H_3^2 \cup H_4^2 = H_2^1$, $H_3^2 \cap H_4^2 = \emptyset$ and $\int_{H_3^2} \langle u, v\rangle dt = \int_{H_4^2} \langle u, v\rangle dt$. Generally, decompose H_i^{n-1} into H_{2i-1}^n and H_{2i}^n such that

$$H_{2i-1}^n \cup H_{2i}^n = H_i^{n-1},\ H_{2i-1}^n \cap H_{2i}^n = \emptyset \quad \text{and} \quad \int\limits_{H_{2i-1}^n} \langle u, v\rangle dt = \int\limits_{H_{2i}^n} \langle u, v\rangle dt, \tag{2.1}$$

where $n = 1, 2, 3, ...$ and $i = 1, 2, ..., 2^n$. Then we define two function sequences $\{u_n\}_{n=1}^\infty$ and $\{u_n^1\}_{n=1}^\infty$, where

$$u_n(t) = \begin{cases} u(t) & t \in T\backslash H \\ (1 - r_0)\, u(t) & t \in H_1^n \\ (1 + r_0)\, u(t) & t \in H_2^n \\ ... & ... \\ (1 - r_0)\, u(t) & t \in H_{2^n-1}^n \\ (1 + r_0)\, u(t) & t \in H_{2^n}^n \end{cases} \qquad u_n^1(t) = \begin{cases} u(t) & t \in T\backslash H \\ (1 - r_0)\, u(t) & t \in H_1^n \\ (1 - r_0)\, u(t) & t \in H_2^n \\ ... & ... \\ (1 + r_0)\, u(t) & t \in H_{2^n-1}^n \\ (1 - r_0)\, u(t) & t \in H_{2^n}^n \end{cases}$$

and $r_0 = 1/n_0$. Moreover, by formula (2.1) and the definition of H, we have

$$\begin{aligned}
&\rho_M(u_n)+\rho_M(u_n^1) \\
&= \int\limits_{T\backslash H} M(t, \|u(t)\|)dt + \int\limits_{H_1^n} M\left(t, (1 - \frac{1}{n_0})\, \|u(t)\|\right) dt + \int\limits_{H_2^n} M\left(t, (1 + \frac{1}{n_0})\, \|u(t)\|\right) dt \\
&+\cdots+ \int\limits_{H_{2^n-1}^n} M\left(t, (1 - \frac{1}{n_0})\, \|u(t)\|\right) dt + \int\limits_{H_{2^n}^n} M\left(t, (1 + \frac{1}{n_0})\, \|u(t)\|\right) dt \\
&+ \int\limits_{T\backslash H} M(t, \|u(t)\|)dt + \int\limits_{H_1^n} M\left(t, (1 + \frac{1}{n_0})\, \|u(t)\|\right) dt + \int\limits_{H_2^n} M\left(t, (1 - \frac{1}{n_0})\, \|u(t)\|\right) dt \\
&+\cdots+ \int\limits_{H_{2^n-1}^n} M\left(t, (1 + \frac{1}{n_0})\, \|u(t)\|\right) dt + \int\limits_{H_{2^n}^n} M\left(t, (1 - \frac{1}{n_0})\, \|u(t)\|\right) dt \\
&= 2\int\limits_{T\backslash H} M(t, \|u(t)\|)dt + 2\int\limits_{H_1^n} M\left(t, \|u(t)\|\right) dt + 2\int\limits_{H_2^n} M\left(t, \|u(t)\|\right) dt \\
&\quad +\cdots+ 2\int\limits_{H_{2^n-1}^n} M\left(t, \|u(t)\|\right) dt + 2\int\limits_{H_{2^n}^n} M\left(t, \|u(t)\|\right) dt = 2
\end{aligned}$$

and

$$\begin{aligned}
&\langle u_n, v\rangle \\
&= \int\limits_{T\backslash H} (u_n(t), v(t))dt + \int\limits_{H_1^n} \left((1 - \frac{1}{n_0})u(t), v(t)\right) dt + \int\limits_{H_2^n} \left((1 - \frac{1}{n_0})u(t), v(t)\right) dt \\
&\quad +\cdots+ \int\limits_{H_{2^n-1}^n} \left((1 - \frac{1}{n_0})u(t), v(t)\right) dt + \int\limits_{H_{2^n}^n} \left((1 + \frac{1}{n_0})u(t), v(t)\right) dt
\end{aligned}$$

$$= \langle u, v\rangle + \frac{-1}{n_0}\int\limits_{H_1^n}(u(t),v(t))dt + \frac{1}{n_0}\int\limits_{H_2^n}(u(t),v(t))dt + \cdots + \frac{-1}{n_0}\int\limits_{H_{2^n-1}^n}(u(t),v(t))dt$$

$$+\frac{-1}{n_0}\int\limits_{H_{2^n}^n}(u(t),v(t))dt = \langle u, v\rangle = 1.$$

This implies that $\|u_n\| \geq 1$. Hence $\rho_M(u_n) \geq 1$. Similarly, we have $\rho_M(u_n^1) \geq 1$. Since $\rho_M(u_n) + \rho_M(u_n^1) = 2$, we get that $\rho_M(u_n) = \rho_M(u_n^1) = 1$. This implies that $\|u_n\| = \left\|u_n^1\right\| = 1$. Hence we define a new function sequence $\{z_n\}_{n=1}^{\infty}$ such that

$$\{z_n(t)\}_{n=1}^{\infty} = \left(u_1(t), u_1^1(t), u_2(t), u_2^1(t), \ldots, u_n(t), u_n^1(t), \ldots\right).$$

Moreover, it is easy to see that

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\frac{1}{2^n}u_n + \frac{1}{2}\frac{1}{2^n}u_n^1\right) = \sum_{n=1}^{\infty}\frac{1}{2^{n+1}}\left(u_n + u_n^1\right) = \sum_{n=1}^{\infty}\frac{2}{2^{n+1}}u = u,$$

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\cdot\frac{1}{2^n} + \frac{1}{2}\cdot\frac{1}{2^n}\right) = \sum_{n=1}^{\infty}\frac{1}{2^n} = 1.$$

Since $\int_T (u(t), v(t))dt = \|u\| = \|v\|^0 = 1$ and $M \in \Delta$, by formula $\mu H > 0$, we have $(u(t), v(t)) > 0$ for μ-a.e, $t \in H$. Therefore, by formula $\mu H > 0$, we get that $\int_H (u(t), v(t))dt > 0$. Moreover, since the sequence $\{u_n\}_{n=1}^{\infty}$ is a subsequence of $\{z_n\}_{n=1}^{\infty}$, we have the following formula

$$\int\limits_T (u_n(t) - u_m(t), v(t))dt = \frac{1}{2}\int\limits_H (u(t), v(t))dt > 0$$

whenever $m \neq n$. Hence the sequence $\{z_n\}_{n=1}^{\infty}$ is not relatively weakly compact, a contradiction. This implies that $\mu\{t \in T : \|u(t)\| \in K_t\} = 0$, which completes the proof. □

Lemma 2.4. *Suppose that $v_0 \in E_N^0(X)$ and the w^*-hyperplane H_{v_0} of $L_M(X^*)$ is weakly approximatively compact and X is a strongly smooth space. Then $M \in \Delta$.*

Proof. Since X is a strongly smooth space, we obtain that X^* has the Radon-Nikodym property. Then $(E_N^0(X))^* = L_M(X^*)$. Hence there exists $u_0 \in L_M(X^*)$ such that $\|u_0\| = 1$ and $\langle u_0, v_0\rangle = 1$. Suppose that $M \notin \Delta$. Then, by Lemma 1.8, there exists a point $u \in L_M(X^*)$ such that $\|u\| = 1$ and $\rho_M(u) < 1$. This implies that

$$\lambda_0 = \inf\left\{\lambda \in R^+ : \rho_M\left(\frac{u}{\lambda}\right) < +\infty\right\} = 1.$$

Hence, for any $L > 1$, we have $\rho_M(Lu) = \infty$. Indeed, suppose that there exists $L_1 > 1$ such that $\rho_M(L_1 u) < \infty$. Moreover, we know that the function $F(k) = \int_T M(t, k\|u(t)\|)dt$ is continuous on $[1, L_1]$. Then there exists $L_2 > 1$ such that $\rho_M(L_2 u) = 1$. This implies that $\|u\| \leq 1/L_2$, contradicting $\|u\| = 1$.

Decompose T into E_1 and G_1 such that $\mu E_1 = \mu G_1$. Then for any $L > 1$, we obtain that $\int_{E_1} M(t, L\|u(t)\|)dt = \infty$ or $\int_{G_1} M(t, L\|u(t)\|)dt = \infty$. Hence we may assume without loss of generality that $\int_{E_1} M(t, L\|u(t)\|)dt = \infty$. Decompose E_1 into E_2 and G_2 such that $\mu E_2 = \mu G_2$. Then, for any $L > 1$, we have $\int_{E_2} M(t, L\|u(t)\|)dt = \infty$ or $\int_{G_2} M(t, L\|u(t)\|)dt = \infty$. Hence we may assume without loss of generality that $\int_{E_2} M(t, L\|u(t)\|)dt = \infty$. In generical, decompose E_n into E_{n+1} and G_{n+1} such that $\mu E_{n+1} = \mu G_{n+1}$. Then, for any $L > 1$, we have

$\int_{E_{n+1}} M(t, L\|u(t)\|)dt = \infty$ or $\int_{G_{n+1}} M(t, L\|u(t)\|)dt = \infty$. Hence we may assume without loss of generality that $\int_{E_{n+1}} M(t, L\|u(t)\|)dt = \infty$. Hence

$$E_1 \supset E_2 \supset E_3 \supset \cdots,\ \mu E_i = \frac{1}{2}\mu E_{i+1} \quad \text{and} \quad \|u\chi_{E_i}\| = 1,\ i = 1, 2, ...$$

This implies that $\|u\chi_{E_i}\| = 1$. Let $u_n = u_0 + u\chi_{T\setminus E_i}$ and $D = \overline{co}\{u_n\}_{n=1}^{\infty}$. Then, for every $v = u_0 + \alpha_1 u\chi_{T\setminus E_{n(1)}} + \cdots + \alpha_k u\chi_{T\setminus E_{n(k)}} \in co\{u_n\}_{n=1}^{\infty}$, there exists a natural number $i(0) > \max\{n(1), ..., n(k)\}$ such that

$$\|u_0 - v\| = \left\|\alpha_1 u\chi_{E_{n(1)}} + \cdots + \alpha_k u\chi_{E_{n(k)}}\right\| \geq \left\|u\chi_{E_{i(0)}}\right\| = 1,$$

where $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$. This implies that $\operatorname{dist}(u_0, D) = 1$. Therefore, by the separable theorem, there exist $f \in \left(L_M^0(X)\right)^*$ and $r > 0$ such that

$$f(u_0) - r > \sup\left\{f(v) : v \in \overline{co}\{u_n\}_{n=1}^{\infty}\right\}. \tag{2.2}$$

Moreover, by formula $\langle u_0, v_0\rangle = 1$ and the definition of u_n, we have the following inequalities

$$\begin{aligned} 0 \leq \limsup_{n\to\infty} |\langle u_n, v_0\rangle - 1| &= \limsup_{n\to\infty} |\langle u_n - u_0, v_0\rangle| \\ &= \limsup_{n\to\infty} \int_{E_n} |(u(t), v_0(t))|\, dt \\ &\leq \liminf_{n\to\infty} \left[\|u\| \, \|v_0\chi_{E_n}\|^0\right] = 0. \end{aligned}$$

Hence we have $\langle u_n, v_0\rangle \to 1$ as $n \to \infty$. Since weak* hyperplane H_{v_0} of $L_M(X^*)$ is weakly approximatively compact, by Lemma 2.2, we get that $\{u_n\}_{n=1}^{\infty}$ is relatively weakly compact. Moreover, for any $v \in E_N^0(X)$, we have

$$\limsup_{n\to\infty} |\langle u_n - u_0, v\rangle| \leq \limsup_{n\to\infty} \int_{E_n} |(u(t), v(t))|\, dt \leq \liminf_{n\to\infty} \left[\|u\| \, \|v\chi_{E_n}\|^0\right] = 0.$$

This implies that $u_n \xrightarrow{w^*} u_0$ as $n \to \infty$. Since the sequence $\{u_n\}_{n=1}^{\infty}$ is relatively weakly compact, we get that $u_n \xrightarrow{w} u_0$ as $n \to \infty$. However, by formula (2.2), we get that $u_n \xrightarrow{w} u_0$ is impossible, a contradiction, which completes the proof. □

We next prove that Theorem 2.1.

Proof. By Lemma 2.2, it is easy to see that (1)⇒(2) and (2)⇒(3) are true. We next will prove that (3)⇒(4). By Lemma 2.4, we obtain that $M \in \Delta$. Moreover, if $u \in S(L_M(X^*))$ is norm attainable at point v_0 and $u = \sum_{n=1}^{\infty} t_n u_n$, where $u_n \in B(L_M(X^*))$, $t_n \in (0, 1)$ and $\sum_{n=1}^{\infty} t_n = 1$, then

$$1 = \langle u, v_0\rangle = \left\langle \sum_{n=1}^{\infty} t_n u_n, v_0 \right\rangle = \sum_{n=1}^{\infty} t_n \langle u_n, v_0\rangle.$$

This implies that $\langle u_n, v_0\rangle = 1$ for all $n \in N$. Therefore, by Lemma 2.2, we obtain that $\{u_n\}_{n=1}^{\infty}$ is relatively weakly compact. Therefore, by Lemma 2.3, we get that $\mu\{t \in T : \|u(t)\| \in K_t\} = 0$.

(4)⇒(1). First we will prove that v is a smooth point of $E_N^0(X)$. In fact, since X is strongly smooth, we obtain that X^* has the Radon-Nikodym property. Then $(E_N^0(X))^* = L_M(X^*)$. Suppose that $\langle u_1, v\rangle = \langle u_2, v\rangle = 1$ and $\|u_1\| = \|u_2\| = 1$. Then $\langle u, v\rangle = 1$, where $2u = u_1 + u_2$. Hence $\mu\{t \in T : \|u(t)\| \in K_t\} = 0$. Then $\|u(t)\| \in L_M(R)$ and $\|v(t)\| \in E_N^0(R)$. Since $M \in \Delta$ and $\mu\{t \in T : \|u(t)\| \in K_t\} =$

0, by Theorem 5.10 of [2], we get that $\|u\|$ is an extreme point of $L_M(R)$. Moreover, by $\|u_1\| = \|u_2\| = 1$ and $v \in S(E_N^0(X))$, we have

$$1 \geq \int_T \|u_1(t)\| \, \|v(t)\| dt = \int_T (u_1(t), v(t)) \, dt = \langle u_1, v \rangle = 1$$

and

$$1 \geq \int_T \|u_2(t)\| \, \|v(t)\| dt = \int_T (u_2(t), v(t)) \, dt = \langle u_2, v \rangle = 1.$$

This implies that

$$\int_T \left(\frac{1}{2} \|u_1(t)\| + \|u_2(t)\| \right) \|v(t)\| dt = \frac{1}{2} \langle u_1, v \rangle + \frac{1}{2} \langle u_2, v \rangle = 1,$$

$(u_1(t), v(t)) = \|u_1(t)\| \, \|v(t)\|$ and $(u_2(t), v(t)) = \|u_2(t)\| \, \|v(t)\|$ for almost all $t \in T$. Moreover, by $\langle u, v \rangle = 1$ and $2u = u_1 + u_2$, we get that $\|u\| = 1$. Hence

$$1 \geq \int_T \left(\frac{1}{2} \|u_1(t) + u_2(t)\| \right) \|v(t)\| dt = \int_T (u(t), v(t)) \, dt = \langle u, v \rangle = 1.$$

This implies that $2\|u(t)\| = \|u_1(t)\| + \|u_2(t)\|$ for almost all $t \in T$. Since $\|u\|$ is an extreme point of $L_M(R)$, we have $\|u_1(t)\| = \|u_2(t)\|$ for almost all $t \in T$. Since $(u_1(t), v(t)) = \|u_1(t)\| \, \|v(t)\|$ and $(u_2(t), v(t)) = \|u_2(t)\| \, \|v(t)\|$ for almost all $t \in T$, by the smoothness of X and $\|u_1(t)\| = \|u_2(t)\|$ for almost all $t \in T$, we get that $u_1(t) = u_2(t)$ for almost all $t \in T$. This implies that $u_1 = u_2$. Hence we obtain that v is a smooth point of $E_N^0(X)$.

Next we will prove that the point v is a strongly smooth point of $E_N^0(X)$. Let $\langle u, v \rangle = 1$ and $\langle u_n, v \rangle \to 1$ as $n \to \infty$, where $\{u_n\}_{n=1}^\infty \in S(L_M(X^*))$ and $u \in S(L_M(X^*))$. Since $B(L_M(R))$ is weakly* sequentially compact, by $(E_N^0(R))^* = L_M(R)$ and $u_n \in L_M(X^*)$, we may assume without loss of generality that there exists a functional $h \in S(L_M(R))$ such that $\int_T \|u_n(t)\| \, w(t) dt \to \int_T h(t) w(t) dt$ whenever $w \in E_N^0(R)$. This implies that $\int_T h(t) \, \|v(t)\| \, dt = 1$. Since v is a smooth point of $E_N^0(X)$, it is easy to see $\|v\|$ is a smooth point of $E_N^0(R)$. Therefore, by formula

$$1 \geq \int_T \|u(t)\| \, \|v(t)\| \, dt = \int_T (u(t), v(t)) \, dt = \int_T h(t) \, \|v(t)\| \, dt = 1,$$

we obtain that $\|u(t)\| = h(t)$ μ-a.e. on T. Hence we have the following formula

$$\lim_{n\to\infty} \int_T \|u_n(t)\| \, w(t) dt = \int_T \|u(t)\| \, w(t) dt \tag{2.3}$$

whenever $w \in E_N^0(R)$. Therefore, by formula (2.3), we obtain that $\{\|u_n\|\}_{n=1}^\infty$ converges weakly* to $\|u\|$. We claim that $\rho_M(u_n \chi_E) \to \rho_M(u \chi_G)$ for all $G \subset T$. In fact, let

$$E_m = \{t \in G : m \, \|v(t)\| \geq p(t, \|u(t)\|)\}$$

and

$$E_m^n = \{t \in E_m : \|u_n(t)\| \geq \|u(t)\|\}$$

for any $m, n \in N$. Since $v \in E_N^0(X)$, we get that $p(u\chi_{E_n^m}) \in E_N^0(R)$. Therefore, by formula (2.3), we have the following formula

$$\begin{aligned}
&\int_{E_m} [M(t,\|u_n(t)\|) - M(t,\|u(t)\|)]dt \\
&= \int_{E_m^n} [M(t,\|u_n(t)\|) - M(t,\|u(t)\|)]dt - \int_{E_m\setminus E_m^n} [M(t,\|u_n(t)\|) - M(t,\|u(t)\|)]dt \\
&\geq \int_{E_m^n} (\|u_n(t)\| - \|u(t)\|)p(t,\|u(t)\|)dt - \int_{E_m\setminus E_m^n} (\|u(t)\| - \|u_n(t)\|)p(t,\|u(t)\|)dt \\
&= \int_{E_m^n} (\|u_n(t)\| - \|u(t)\|)p(t,\|u(t)\|)dt \\
&= \int_T (\|u_n(t)\| - \|u(t)\|)p(t,\|u(t)\|)\chi_{E_m^n}dt \to 0,\ n \to \infty.
\end{aligned}$$

This implies that $\liminf_{n\to\infty}\rho_M(u_n\chi_{E_m}) \geq \rho_M(u\chi_{E_m})$. Let $m \to \infty$. Then, for any $G \subset T$, we have $\liminf_{n\to\infty}\rho_M(u_n\chi_G) \geq \rho_M(u\chi_G)$. Since $\rho_M(u_n) = \rho_M(u) = 1$, we get that $\rho_M(u_n\chi_G) \to \rho_M(u\chi_G)$ for any $G \subset T$.

We next will prove that $\|u_n(t)\| \to \|u(t)\|$ in measure on T. Let $\{r(i,t)\}_{i=1}^{\infty}$ be a set of all the extreme points of linear interval of $M(t,u)$ and let

$$F = \{t \in T : M(t,\|u(t)\|) \notin \{r(i,t)\}_{i=1}^{\infty}\}.$$

Since $\mu\{t \in T : \|u(t)\| \in K_t\} = 0$, we get that $M(t,\|u(t)\|) > 0$ whenever $t \in F$. We claim that $\|u_n(t)\| \to \|u(t)\|$ in measure on F. Otherwise, we may assume without loss of generality that for each $n \in N$, there exists $E_n \subseteq F$, $\varepsilon_0 > 0$ and $\sigma_0 > 0$ such that $\mu E_n \geq \varepsilon_0$, where $E_n = \{t \in F : |\|u_n(t)\| - \|u(t)\|| \geq \sigma_0\}$. Let us define the sets

$$A_n = \left\{t \in T : M(t,\|u_n(t)\|) > \frac{8}{\varepsilon_0}\right\} \quad \text{and} \quad B = \left\{t \in T : M(t,\|u(t)\|) > \frac{8}{\varepsilon_0}\right\}.$$

Then

$$1 = \int_T M(t,\|u_n(t)\|)dt \geq \int_{A_n} M(t,\|u_n(t)\|)dt \geq \frac{8}{\varepsilon_0}\mu A_n \Rightarrow \mu A_n \leq \frac{\varepsilon_0}{8}.$$

Similarly, we have $\mu B \leq \varepsilon_0/8$. Hence, for μ-a.e. $t \in T$, we define the bounded closed sets

$$C_t = \left\{(u,v) \in R^2 : M(t,u) \leq \frac{8}{\varepsilon_0},\ M(t,v) \leq \frac{8}{\varepsilon_0},\ |u-v| \geq \frac{1}{8}\sigma_0,\ u = \|u(t)\|\right\}$$

in the two dimensional space R^2. Since C_t is compact, there exists $(u_t, v_t) \in C_t$ such that

$$1 > \frac{2M(t,\frac{1}{2}(u_t+v_t))}{M(t,u_t)+M(t,v_t)} \geq \frac{2M(t,\frac{1}{2}(u+v))}{M(t,u)+M(t,v)} \tag{2.4}$$

for any $(u,v) \in C_t$ and for μ-a.e. $t \in T$. Hence we define the function

$$1-\delta(t) = \frac{2M(t,\frac{1}{2}(u_t+v_t))}{M(t,u_t)+M(t,v_t)}. \tag{2.5}$$

We claim that $\delta(t)$ is Σ-measurable. In fact, pick a dense subset $\{r_i\}_{i=1}^{\infty}$ of $(0,\infty)$ and define the function

$$1-\delta_{r_i,r_j}(t)=\begin{cases}\dfrac{2M(t,\frac{1}{2}(r_i+r_j))}{M(t,r_i)+M(t,r_j)} & M(t,r_i)\le\dfrac{8}{\varepsilon_0}\ \text{ and }\ M(t,r_j)\le\dfrac{8}{\varepsilon_0}\\ 0 & M(t,r_i)>\dfrac{8}{\varepsilon_0}\ \text{ or }\ M(t,r_j)>\dfrac{8}{\varepsilon_0},\end{cases}$$

then by the definition of $M(t,u)$, it is easy to see that $1-\delta_{r_i,r_j}(t)$ is Σ-measurable and

$$1-\delta(t)\ge\sup\left\{1-\delta_{r_i,r_j}(t):|r_i-r_j|\ge\frac{1}{8}\sigma_0\right\}.$$

On the other hand, since $\{r_i\}_{i=1}^{\infty}$ is dense in $(0,\infty)$ then $\{(r_i,r_j)\}_{i=1,j=1}^{\infty}$ is dense in $(0,\infty)\times(0,\infty)$. Therefore, by the definition of the function $1-\delta(t)$, for μ-a.e. $t\in T,\varepsilon>0$, there exists

$$(r_i,r_j)\in\left\{(u,v)\in R^2:M(t,u)\le\frac{8}{\varepsilon_0},M(t,v)\le\frac{8}{\varepsilon_0},|u-v|\ge\frac{1}{8}\sigma_0,u=\|u(t)\|\right\}$$

such that

$$1-\delta(t)-\varepsilon<1-\delta_{r_i,r_j}(t)\le\sup\left\{1-\delta_{r_i,r_j}(t):|r_i-r_j|\ge\frac{1}{8}\sigma_0\right\}$$

μ-a.e. on T. Since ε is arbitrary, we have the following formula

$$1-\delta(t)\le\sup\left\{1-\delta_{r_i,r_j}(t):|r_i-r_j|\ge\frac{1}{8}\sigma_0\right\}$$

μ-a.e. on T. Hence $1-\delta(t)=\sup\left\{1-\delta_{r_i,r_j}(t):|r_i-r_j|\ge\sigma_0/8\right\}$ μ-a.e. on T. This implies that $\delta(t)$ is Σ-measurable. Since $\delta(t)>0$, there exists a real number $\delta_0\in(0,1)$ such that $\mu G<\varepsilon_0/8$, where $G=\{t\in T:\delta(t)\le\delta_0\}$. Moreover, by $M(t,\|u(t)\|)>0,\ t\in F$, there exist $F_0\subset F$ and $r>0$ such that

$$M(t,\|u(t)\|)>r,\ t\in F\backslash F_0\ \text{ and }\ \mu F_0<\frac{\varepsilon_0}{16}.$$

Let $H_n=E_n\backslash(A_n\cup B\cup G\cup F_0)$. Then $\mu H_n\ge\varepsilon_0/8$ and

$$M\left(t,\frac{\|u_n(t)\|+\|u(t)\|}{2}\right)\le\frac{1}{2}(1-\delta_0)\left[M\left(t,\|u_n(t)\|\right)+M\left(t,\|u(t)\|\right)\right]$$

whenever $t\in H_n$. This implies that

$$\begin{aligned}&\rho_M(u_n)+\rho_M(u)-2\rho_M\left(\frac{u_n+u}{2}\right)\\ \ge\ &\int_T M\left(t,\|u_n(t)\|\right)+M\left(t,\|u(t)\|\right)-2M\left(t,\frac{\|u_n(t)\|+\|u(t)\|}{2}\right)dt\\ \ge\ &\int_{H_n} M\left(t,\|u_n(t)\|\right)+M\left(t,\|u(t)\|\right)-2M\left(t,\frac{\|u_n(t)\|+\|u(t)\|}{2}\right)dt\\ \ge\ &\delta_0\int_{H_n}\left[M\left(t,\|u_n(t)\|\right)+M\left(t,\|u(t)\|\right)\right]dt\\ \ge\ &\delta_0\int_{H_n}M\left(t,\|u(t)\|\right)dt\ge\delta_0 r\frac{\varepsilon_0}{8}.\end{aligned}$$

Moreover, by formula $\langle u_n, v\rangle \to 1$ and $\langle u, v\rangle = 1$, we get that $\|u\| = 1$, $\|u_n\| \to 1$ and $\|u_n + u\|/2 \to 1$ as $n \to \infty$. Therefore, by $M \in \Delta$, we obtain that $\rho_M(u) = 1$, $\rho_M(u_n) \to 1$ and $\rho_M\left((u_n + u)/2\right) \to 1$ as $n \to \infty$. This implies that

$$\lim_{n\to\infty}\left[\rho_M(u_n) + \rho_M(u) - 2\rho_M\left(\frac{u_n + u}{2}\right)\right] = 0,$$

this is a contradiction. Hence we obtain that $\|u_n(t)\| \to \|u(t)\|$ in measure on F.

Let $\{r_1(i,t)\}_{i=1}^{\infty}$ denote the set of all the right extreme points of linear interval of $M(t,u)$. Define the set

$$G = \left\{t \in T : M(t, \|u(t)\|) \in \{r_1(i,t)\}_{i=1}^{\infty}\right\}.$$

We will prove that $\|u_n(t)\| \to \|u(t)\|$ in measure on G. Otherwise, we may assume without loss of generality that for each $n \in N$, there exists $E_n \subseteq G$, $\varepsilon_0 > 0$ and $\sigma_0 > 0$ such that $\mu E_n \geq \varepsilon_0$, where $E_n = \{t \in G : |\|u_n(t)\| - \|u(t)\|| \geq \sigma_0\}$. Hence we may assume that $E_n = \{t \in T : \|u_n(t)\| - \|u(t)\| \geq \sigma_0\}$. Let

$$F_n = \{t \in T : \|u(t)\| \leq \|u_n(t)\| < \|u(t)\| + \sigma_0\}$$

and

$$H_n = \{t \in T : \|u(t)\| > \|u_n(t)\|\}.$$

For clarity, we will divide the proof into two cases.

Case I. Let $\limsup_{n\to\infty}\mu(G_0 \cap E_n) = \eta_0 > 0$, where $t \in G_0$ if and only if $M(t, \|u(t)\|)$ is a right extreme point and not a left extreme point. Then $G_0 \subset G$. Therefore, by formula $M(t,u) = \int_0^u p(t,u)dt$, we have the following inequalities

$$\begin{aligned} h_n(t) &= M(t, \|u_n(t)\|) - M(t, \|u(t)\|) - p(t, \|u(t)\|)\left[\|u_n(t)\| - \|u(t)\|\right] \\ &\geq M(t, \|u(t)\| + \sigma_0) - M(t, \|u(t)\|) - \sigma_0 p(t, \|u(t)\|) > 0 \end{aligned}$$

whenever $t \in E_n$. Hence there exist $H_n \subset G_0$ and $h > 0$ such that $\mu H_n < \eta_0/16$ and $h_n(t) > h$ whenever $t \in E_n \setminus H_n$. Moreover, there exists a natural number $m \in N$ such that $\mu(T\backslash T_m) < \eta_0/16$, where

$$T_m = \{t \in T : m\|v(t)\| \geq p(t, \|u(t)\|)\}.$$

Let $G_n = (G_0 \cap T_m)\backslash H_n$. Then, by $\limsup_{n\to\infty}\mu(G_0 \cap E_n) = \eta_0$ and $\mu(T\backslash T_m) < \eta_0/16$, we may assume that $\mu(G_n \cap E_n) > \eta_0/8$. Moreover, by the definition of T_m, we have $p(t, \|u(t)\|)\chi_{G_n} \in E_N^0(X)$. Therefore, by formula (2.3), we have

$$\begin{aligned} 0 \leftarrow & \int_{G_n} M(t, \|u_n(t)\|)dt - \int_{G_n} M(t, \|u(t)\|)dt \\ \geq & \int_{G_n\cap E_n} p(t, \|u(t)\|)\left[\|u_n(t)\| - \|u(t)\|\right]dt + \int_{G_n\cap F_n} p(t, \|u(t)\|)\left[\|u_n(t)\| - \|u(t)\|\right]dt \\ & + \int_{G_n\cap H_n} p(t, \|u(t)\|)\left[\|u_n(t)\| - \|u(t)\|\right]dt + \int_{G_n\cap E_n} h_n(t)dt \\ = & \int_{G_n} p(t, \|u(t)\|)\left[\|u_n(t)\| - \|u(t)\|\right]dt + \int_{G_n\cap E_n} h_n(t)dt \geq h\frac{\eta_0}{16} \end{aligned}$$

for n large enough, this is a contradiction.

Case II. Let $\limsup_{n\to\infty}\mu(G_0 \cap E_n) = \eta_0 > 0$, where $t \in G_0$ if and only if $M(t, \|u(t)\|)$ is a left extreme point and is a right extreme point of linear interval of $M(t,u)$. Then $G_0 \subset G$ and $p(t, \|u(t)\|) - p_-(t, \|u(t)\|) > 0$ whenever $t \in G_0$.

Hence there exist a set $H \subset G_0$ and a real number $h > 0$ such that $\mu H < \eta_0/16$ and $p(t, \|u(t)\|) - p_-(t, \|u(t)\|) > h$ whenever $t \in G_0 \setminus H$. Moreover, there exists $m \in N$ such that $\mu(T \backslash T_m) < \eta_0/16$. Let $F = (G_0 \cap T_m) \backslash H$. Then, by formula $\limsup_{n\to\infty} \mu(G_0 \cap E_n) = \eta_0$, we may assume that $\mu(F \cap E_n) > \eta_0/8$. Therefore, by formula (2.3), we get the following inequalities

$$
\begin{aligned}
0 \leftarrow & \int\limits_F M(t, \|u_n(t)\|)dt - \int\limits_F M(t, \|u(t)\|)dt \\
\geq & \int\limits_{F\cap E_n} p(t, \|u(t)\|)\left[\|u_n(t)\| - \|u(t)\|\right]dt + \int\limits_{F\cap F_n} p(t, \|u(t)\|)\left[\|u_n(t)\| - \|u(t)\|\right]dt \\
& + \int\limits_{F\cap H_n} p_-(t, \|u(t)\|)\left[\|u_n(t)\| - \|u(t)\|\right]dt \\
\geq & \int\limits_{F\cap E_n} \left[p(t, \|u(t)\|) - p_-(t, \|u(t)\|)\right]\left[\|u_n(t)\| - \|u(t)\|\right]dt \\
& + \int\limits_F p_-(t, \|u(t)\|)\left[\|u_n(t)\| - \|u(t)\|\right]dt \\
\geq & \int\limits_{F\cap E_n} \left[p(t, \|u(t)\|) - p_-(t, \|u(t)\|)\right]\sigma_0 dt + \int\limits_F p(t, \|u(t)\|)\left[\|u_n(t)\| - \|u(t)\|\right]dt \\
\geq & \; h\frac{\eta_0}{16}
\end{aligned}
$$

for n large enough, this is a contradiction. Hence we get that $\|u_n(t)\| \to \|u(t)\|$ in measure on G. Let $\{r_1(i,t)\}_{i=1}^{\infty}$ denote the set of all the left extreme points of linear interval of $M(t,u)$. Define the set

$$G = \{t \in T : M(t, \|u(t)\|) \in \{r_1(i,t)\}_{i=1}^{\infty}\}.$$

Similarly, we get that $\|u_n(t)\| \to \|u(t)\|$ in measure on G. In summary, we have $\|u_n(t)\| \to \|u(t)\|$ in measure on T. Therefore, by the Riesz Theorem, there exists a subsequence $\{n\}$ of $\{n\}$ such that $\|u_n(t)\| \to \|u(t)\|$ $\mu-$a.e. in T. Noticing that

$$|(u_n(t), v(t))| \leq \|u_n(t)\| \cdot \|v(t)\|, \quad \lim_{n\to\infty} \int_T (u_n(t), v(t))dt = 1$$

and

$$\int_T \|u_n(t)\| \cdot \|v(t)\|\, dt \leq \|u_n\| \cdot \|v\|^0 \leq 1,$$

we get the following formula

$$\lim_{n\to\infty} \int_T \|u_n(t)\| \cdot \|v(t)\|\, dt = 1, \qquad \lim_{n\to\infty} \int_T \left[\|u_n(t)\| \cdot \|v(t)\| - (u_n(t), v(t))\right] dt = 0.$$

Moreover, it is easy to see that

$$\lim_{n\to\infty} \int_T \left|\|u_n(t)\| \cdot \|v(t)\| - (u_n(t), v(t))\right| dt = 0.$$

This implies that $\|u_n(t)\| \cdot \|v(t)\| - (u_n(t), v(t)) \xrightarrow{\mu} 0$ in measure. Therefore, by the Riesz theorem, there exists a subsequence $\{n\}$ of $\{n\}$ such that $\|u_n(t)\| \cdot \|v(t)\| - (u_n(t), v(t)) \to 0$ $\mu-$a.e. in T. Since $\|u_n(t)\| \to \|u(t)\|$ μ-a.e. on T, we get that

$(u_n(t), v(t)) \to \|u(t)\| \cdot \|v(t)\|$ $\mu-$a.e. on T. Hence we may assume without loss of generality that

$$\lim_{n\to\infty}\left(\frac{u_n(t)}{\|u(t)\|}, \frac{v(t)}{\|v(t)\|}\right) = 1$$

on $\{t \in T : \|u(t)\| \cdot \|v(t)\| \neq 0\}$. Since $\langle u, v\rangle = 1$, we get that $\mu T_1 = 0$, where

$$T_1 = \{t \in T : \|v(t)\| = 0\} \cap \{t \in T : \|u(t)\| \neq 0\}.$$

Hence we may assume without loss of generality that

$$\lim_{n\to\infty}\left(\frac{u_n(t)}{\|u(t)\|}, \frac{v(t)}{\|v(t)\|}\right) = 1, \quad t \in \{t \in T : \|u(t)\| \neq 0\}.$$

Since X is a strongly smooth space, we obtain that sequence $\{u_n(t)/\|u(t)\|\}_{n=1}^{\infty}$ is convergent. Hence there exists $x(t) \in S(X)$ such that $u_n(t)/\|u(t)\| \to x(t)$ on $\{t \in T : \|u(t)\| \neq 0\}$. Let

$$u_0(t) = \begin{cases} \|u(t)\|\, x(t), & t \in \{t \in T : \|u(t)\| \neq 0\} \\ 0, & t \in \{t \in T : \|u(t)\| = 0\}. \end{cases}$$

Then it is easy to see that $\|u_0\|^0 = 1$ and $u_n(t) \to u_0(t)$ μ-a.e. on T. Therefore, by the Fatou Lemma, we obtain the following inequalities

$$\begin{aligned} &\rho_M(u_0) \\ &= \int_H \lim_{n\to\infty}\left[\frac{1}{2}M\left(t, \|u_n(t)\|\right) + \frac{1}{2}M\left(t, \|u_0(t)\|\right) - M\left(t, \frac{\|u_n(t) - u_0(t)\|}{\varepsilon}\right)\right]dt \\ &\leq \liminf_{n\to\infty}\int_T \left[\frac{1}{2}M(t, \|u_n(t)\|) + \frac{1}{2}M(t, \|u_0(t)\|) - M\left(t, \frac{\|u_n(t) - u_0(t)\|}{2}\right)\right]dt \\ &= \rho_M(u_0) - \limsup_{n\to\infty}\rho_M\left(\frac{1}{2}(u_n - u)\right). \end{aligned}$$

This implies that $\rho_M((u_n - u)/2) \to 0$ as $n \to \infty$. Moreover, from the previous proof, we obtain that $\rho_M(2u_n\chi_E/\varepsilon) \to \rho_M(2u_0\chi_E/\varepsilon)$ for any $E \subset T$. We next will prove that $\|u_n - u\| \to 0$ as $n \to \infty$. Let $\varepsilon \in (0, 1/4)$. Since $M \in \Delta$, there exists $\delta > 0$ such that $\rho_M(2u_0\chi_E/\varepsilon) < 1/2$ whenever $\mu E < 4\delta$. Moreover, there exists $r > 0$ such that $\mu\{t \in T : 0 < \|u(t)\| < r\} < \delta$. Since $u_n(t) \to u_0(t)$ μ-a.e. on T, by the Egorov theorem, there exist a natural number $n_0 \in N$ and $F \subset T$ such that $\mu F < \delta$ and $\|u_n(t) - u_0(t)\| < \varepsilon^2$, $t \in T\backslash F$ whenever $n > n_0$. This implies that $\left\|(u_n - u_0)\chi_{B\backslash F}\right\| < 2\varepsilon$ whenever $n > n_0$. Let $B = \{t \in T : \|u_0(t)\| \geq \varepsilon\}$, $D = \{t \in T : 0 < \|u(t)\| < \varepsilon\}$ and $H = \{t \in T : \|u_0(t)\| = 0\} \cup F \cup D$.

$$\rho_M\left(\frac{(u_n - u_0)\chi_{B\backslash F}}{2\varepsilon}\right) \leq \int_{B\backslash F} M\left(t, \frac{\varepsilon\,\|u_0(t)\|}{2\varepsilon}\right)dt = \int_{B\backslash F} M\left(t, \frac{\|u_0(t)\|}{2}\right)dt \leq 1.$$

Moreover, by $\rho_M(2u_n\chi_E/\varepsilon) \to \rho_M(2u_0\chi_E/\varepsilon)$ for any $E \subset T$ and $\mu(F \cup D) < 2\delta$, there exists a natural number $n_1 > n_0$ such that

$$\begin{aligned} \rho_M\left(\frac{(u_n - u_0)\chi_H}{2\varepsilon}\right) &\leq \frac{1}{2}\rho_M\left(\frac{u_n\chi_H}{\varepsilon}\right) + \frac{1}{2}\rho_M\left(\frac{u_0\chi_H}{\varepsilon}\right) \\ &\leq \rho_M\left(\frac{u_0\chi_H}{\varepsilon}\right) + \varepsilon \\ &= \rho_M\left(\frac{u_0\chi_{F\cup D}}{\varepsilon}\right) + \varepsilon < \frac{1}{2} + \varepsilon < 1 \end{aligned}$$

whenever $n > n_1$. This implies that $\|(u_n - u_0)\chi_H\| < 2\varepsilon$ whenever $n > n_1$. Since $T = H \cup (B \backslash F)$, we have

$$\|u_n - u_0\| \le \|(u_n - u_0)\chi_H\| + \left\|(u_n - u_0)\chi_{B\backslash F}\right\| < 2\varepsilon + 2\varepsilon = 4\varepsilon$$

whenever $n > n_1$. Hence we have $\|u_n - u\| \to 0$ as $n \to \infty$. This implies that v is a strongly smooth point of $E_N^0(X)$, which completes the proof. □

Corollary 2.5. *Suppose that $v \in S(E_N^0(R))$. Then the following statements are equivalent:*

(1) The point v is a strongly smooth point of $E_N^0(R)$;
(2) The hyperplane H_v of $L_M(R)$ is approximatively compact;
(3) The hyperplane H_v of $L_M(R)$ is weakly approximatively compact;
(4) $M \in \Delta$ and $\mu\{t \in T : |u(t)| \in K_t\} = 0$ whenever $\langle u, v\rangle = \|u\|$.

Corollary 2.6. *Suppose that X is a strongly smooth space. Then the following statements are equivalent:*

(1) $E_N^0(X)$ is a strongly smooth space;
(2) Every weak hyperplane of $L_M(X^*)$ is approximatively compact;*
(3) Every weak hyperplane of $L_M(X^*)$ is weakly approximatively compact.*

3. Asplund property and Radon-Nikodym property in Musielak-Orlicz-Bochner function spaces

Theorem 3.1. *Suppose that X is a strongly smooth space. Then the following statements are equivalent:*

(1) $L_M^0(X)$ is an Asplund space;
(2) $M \in \Delta$ and $N \in \Delta$.

Proof. (1)⇒(2). Suppose that $M \notin \Delta$. Then we define the functional $\theta(u) = \inf\{\lambda \in R^+ : \rho_M(u/\lambda) < +\infty\}$. Therefore, by the definition of $\theta(\cdot)$, we get that $\theta(ku) = k\theta(u)$ whenever $k > 0$. Moreover, by the convexity of M, we have

$$\begin{aligned}
&\rho_M\left(\frac{u_1 + u_2}{\theta(u_1) + \theta(u_2) + 2\varepsilon}\right) \\
=\ &\rho_M\left(\frac{\theta(u_1) + \varepsilon}{\theta(u_1) + \theta(u_2) + 2\varepsilon}\frac{u_1}{\theta(u_1) + \varepsilon} + \frac{\theta(u_2) + \varepsilon}{\theta(u_1) + \theta(u_2) + 2\varepsilon}\frac{u_2}{\theta(u_2) + \varepsilon}\right) \\
\le\ &\frac{\theta(u_1) + \varepsilon}{\theta(u_1) + \theta(u_2) + 2\varepsilon}\rho_M\left(\frac{u_1}{\theta(u_1) + \varepsilon}\right) + \frac{\theta(u_2) + \varepsilon}{\theta(u_1) + \theta(u_2) + 2\varepsilon}\rho_M\left(\frac{u_2}{\theta(u_2) + \varepsilon}\right) \\
<\ &+\infty.
\end{aligned}$$

This implies that $\theta(u_1 + u_1) \le \theta(u_1) + \theta(u_2) + 2\varepsilon$. Since ε is arbitrary, we have $\theta(u_1 + u_1) \le \theta(u_1) + \theta(u_2)$. It is easy to see that $\theta(u)$ is a convex function. We claim that if $\|u_n\| \to 0$ then $\theta(u_n) \to 0$ as $n \to \infty$. In fact, for any $\varepsilon > 0$, we have $\|u_n/\varepsilon\| \to 0$ as $n \to \infty$. Hence there exists a natural number n_0 such that $\rho_M(u_n/\varepsilon) \le \|u_n/\varepsilon\| \le 1$ whenever $n > n_0$. This implies that $\theta(u_n) < \varepsilon$ whenever $n > n_0$. Hence $\theta(u_n) \to 0$ as $n \to \infty$.

Next we will prove that $\theta(\cdot)$ is continuous. Otherwise, there exist $\varepsilon_0 > 0$, $u \in L_M(X)$ and $\{u_n\}_{n=1}^{\infty} \subset L_M(X)$ such that $\|u_n - u\| \to 0$ and $|\theta(u_n) - \theta(u)| \ge 2\varepsilon_0$. Then we may assume that $\theta(u) \ge \theta(u_n) + 2\varepsilon_0$ or $\theta(u_n) \ge \theta(u) + 2\varepsilon_0$. Hence, if $\theta(u) \ge \theta(u_n) + 2\varepsilon_0$, then

$$\theta(u) \le \limsup_{n\to\infty} [\theta(u_n) + \theta(u - u_n)] = \limsup_{n\to\infty} \theta(u_n) \le \theta(u) - \varepsilon_0,$$

a contradiction. Moreover, if $\theta(u_n) \geq \theta(u) + 2\varepsilon_0$, then

$$\limsup_{n\to\infty} \theta(u_n) \leq \limsup_{n\to\infty} [\theta(u) + \theta(u_n - u)] = \theta(u) < \limsup_{n\to\infty} [\theta(u_n) - \varepsilon_0],$$

a contradiction. Hence $\theta(\cdot)$ is a continuous function. Pick $u \in L^0_M(X) \setminus E^0_M(X)$. Then $\theta(u) > 0$. We next will prove that there exist $E \subset T$ and $F \subset T$ such that $E \cup F = T$, $E \cap F = \emptyset$ and $\theta(u\chi_E) = \theta(u\chi_F) = \theta(u)$. Define $G(n) = \{t \in T : n-1 \leq \|u(t)\| < n\}$ and for each $n \in N$, decompose $G(n)$ into $G_1(n)$ and $G_2(n)$ such that

$$2 \int_{G_i(n)} M\left(t, \frac{n-1}{\theta - 2\varepsilon}\right) dt = \int_{G(n)} M\left(t, \frac{n-1}{\theta - 2\varepsilon}\right) dt,$$

where $i = 1, 2$. We claim that $u_i = \sum_{n=1}^{\infty} u\chi_{G_i(n)}$ satisfy $\theta(u_i) = \theta(u)$, where $i = 1, 2$. In fact, let $\theta = \theta(u)$. Then for any $\varepsilon \in (0, \theta/2)$, there exists a natural number m such that

$$\frac{m-1}{m} \cdot \frac{1}{\theta - 2\varepsilon} \geq \frac{1}{\theta - \varepsilon}.$$

Since

$$t, s \in G(n) \Rightarrow \|u(t)\| < n \leq \frac{n-1}{n} \|u(s)\|,$$

for all $n \geq m$ and all $t \in G_i(n)$, we have the following inequalities

$$\frac{\|u_i(t)\|}{\theta - 2\varepsilon} = \frac{\|u(t)\|}{\theta - 2\varepsilon} \geq \frac{n-1}{\theta - 2\varepsilon} \geq \frac{n-1}{n} \cdot \frac{\|u(t)\|}{\theta - 2\varepsilon} > \frac{\|u(t)\|}{\theta - \varepsilon}.$$

Therefore, by the definition of $\theta(\cdot)$, we have the following inequalities

$$\begin{aligned}\rho_M\left(\frac{u_i}{\theta - 2\varepsilon}\right) &\geq \sum_{n \geq m} \int_{G_i(n)} M\left(t, \frac{\|u(t)\|}{\theta - 2\varepsilon}\right) dt \geq \sum_{n \geq m} \int_{G_i(n)} M\left(t, \frac{n-1}{\theta - 2\varepsilon}\right) dt \\ &= \frac{1}{2} \sum_{n \geq m} \int_{G(n)} M\left(t, \frac{n-1}{\theta - 2\varepsilon}\right) dt \geq \frac{1}{2} \sum_{n \geq m} \int_{G(n)} M\left(t, \frac{\|u(t)\|}{\theta - \varepsilon}\right) dt = \infty.\end{aligned}$$

This implies that $\theta(u_i) = \theta(u)$, where $i = 1, 2$. Hence there exist a set $E \subset T$ and $F \subset T$ such that $E \cup F = T$, $E \cap F = \emptyset$ and $\theta(u\chi_E) = \theta(u\chi_F) = \theta(u)$. Let $v = u\chi_E - u\chi_F$. Then, if $t > 0$ then

$$\theta(u + tv) = \theta((1+t)u\chi_E + (1-t)u\chi_F) = \theta((1+t)u\chi_E) = (1+t)\theta(u).$$

This implies that

$$\frac{\theta(u + tv) - \theta(u)}{t} = \frac{(1+t)\theta(u) - \theta(u)}{t} = \frac{t\theta(u)}{t} = \theta(u)$$

whenever $t > 0$. Moreover, if $t < 0$ then

$$\theta(u + tv) = \theta((1+t)u\chi_E + (1-t)u\chi_F) = \theta((1-t)u\chi_F) = (1-t)\theta(u).$$

This implies that

$$\frac{\theta(u + tv) - \theta(u)}{t} = \frac{(1-t)\theta(u) - \theta(u)}{t} = \frac{-t\theta(u)}{t} = -\theta(u)$$

whenever $t < 0$. Hence, for any $u \in L^0_M(X) \setminus E^0_M(X)$, we obtain that $\theta(\cdot)$ is not differentiable at u, a contradiction. Then $M \in \Delta$.

Suppose that $N \notin \Delta$. Then there exists a point $v \in L_N(X^*)$ such that $\|v\| = 1$ and $\rho_N(v) < 1$. Pick a real number $l > 1$. Then $\rho_N(lv) = \infty$. Define

$$G = \{t \in T : N(t, \|v(t)\|) = \infty\} \quad \text{and} \quad G(n) = \{t \in T : n-1 \leq N(t, \|v(t)\|) < n\}$$

for all $n \in N$. Decompose $G(n)$ into $G_1(n)$ and $G_2(n)$ such that $G_1(n) \cup G_2(n) = G(n)$ and $\rho_N(lv\chi_{G_1(n)}) = \rho_N(lv\chi_{G_2(n)})$. Decompose G into G_1 and G_2 such that $G_1 \cup G_2 = G$ and $\rho_N(lv\chi_{G_1}) = \rho_N(lv\chi_{G_2})$. Let

$$E = G_1 \cup \left(\bigcup_{n=1}^{\infty} G_1(n) \right) \quad \text{and} \quad F = G_2 \cup \left(\bigcup_{n=1}^{\infty} G_2(n) \right).$$

Then $\rho_N(lv\chi_E) = \rho_N(lv\chi_F)$. This implies that $\|v\chi_E\| \geq 1/l$ and $\|v\chi_F\| \geq 1/l$. Hence there exists a sequence of set $\{E_n\}_{n=1}^{\infty}$ such that $\|v\chi_{E_n}\| \geq 1/l$ and $E_i \cap E_j = \emptyset$ whenever $i \neq j$. Define the set

$$C = \left\{ \sum_{i=1}^{\infty} \varepsilon_i v \chi_{E_i} : \{\varepsilon_i\}_{i=1}^{\infty} \in B(c_0) \right\}.$$

Then it is easy to see that C is a bounded closed convex set of $L_N(X^*)$. We claim that C has no extreme points. In fact, Pick $u = \sum_{i=1}^{\infty} \varepsilon_i v \chi_{E_i} \in C$. Then there exists a natural number $j \in N$ such that $|\varepsilon_j| < 1/4$. Define

$$u_1 = \left(\varepsilon_j + \frac{1}{2} \right) + \sum_{i \neq j} \varepsilon_i \quad \text{and} \quad u_2 = \left(\varepsilon_j - \frac{1}{2} \right) + \sum_{i \neq j} \varepsilon_i.$$

Then $u_1, u_2 \in C$, $u_1 \neq u_2$ and $2u = u_1 + u_2$. This implies that u is not an extreme point. Since u is arbitrary, we have $ExtC = \emptyset$. Hence $L_N(X^*)$ has not the Krein-Milman property. Then $L_N(X^*)$ has not the Radon-Nikodym property. Hence $E_M^0(X)$ is not an Asplund space. However, since $L_M^0(X)$ is an Asplund space, we obtain that $E_M^0(X)$ is an Asplund space, a contradiction. Then $N \in \Delta$.

(2)⇒(1). By Lemma 1.15 of [2], there exists a function M_1 such that

$$M(t,u) \leq M_1(t,u) \leq 2M(t,u), \quad u \in R \tag{3.1}$$

and right derivative of $p_1(t,u)$ of $M_1(t,u)$ is continuous with respect to u for almost all $t \in T$. Therefore, by formula 3.1, we obtain that $u \in L_M(X)$ for any $u \in L_{M_1}(X)$. Since $M \in \Delta$, we have the following inequalities

$$\int_T M_1(t, \lambda \|u(t)\|) dt \leq \int_T 2M(t, \lambda \|u(t)\|) dt < +\infty$$

for any $\lambda > 0$. This implies that $M_1 \in \Delta$. Moreover, if $v \in L_{N_1}(X^*)$, then there exists $\lambda_0 > 0$ such that $\rho_{N_1}(\lambda_0 v) < +\infty$. Since

$$N(t,v) = \sup_{u>0} \{uv - M(t,u)\} \leq \frac{1}{2} \sup_{u>0} \{2uv - M_1(t,u)\} = \frac{1}{2} N_1(t, 2v), \ v \geq 0,$$

we have the following inequalities

$$\int_T N\left(t, \frac{\lambda_0}{2} \|v(t)\|\right) dt \leq \frac{1}{2} \int_T N_1 \left(t, 2 \cdot \frac{\lambda_0}{2} \|v(t)\|\right) dt = \frac{1}{2} \rho_{N_1}(\lambda_0 v) < +\infty.$$

This implies that $v \in L_N(X^*)$. Therefore, by $N \in \Delta$, we obtain that $v \in E_N(X^*)$. Therefore, by formula

$$N_1(t,v) = \sup_{u>0} \{uv - M_1(t,u)\} \leq \sup_{u>0} \{uv - M(t,u)\} = N(t,v), \quad v \geq 0,$$

we have the following inequalities

$$\int_T N_1 \left(t, \lambda \|v(t)\|\right) dt \leq \int_T N \left(t, \lambda \|v(t)\|\right) dt < +\infty$$

for every $\lambda > 0$. This implies that $N_1 \in \Delta$. Therefore, by theorem 2.7, we obtain that $L^0_{M_1}(X)$ is a strongly smooth space. This implies that $L^0_{M_1}(X)$ is an Asplund space. Moreover, we have

$$\|u\|^0 = \inf_{k>0} \frac{1}{k}[1 + \int_T M(t, \|ku(t)\|)dt] \leq \inf_{k>0} \frac{1}{k}[1 + \int_T M_1(t, \|ku(t)\|)dt] = \|u\|^0_1$$

and

$$\|u\|^0_1 = \inf_{k>0} \frac{1}{k}[1 + \int_T M_1(t, \|ku(t)\|)dt] \leq \inf_{k>0} \frac{1}{k}[1 + \int_T 2M(t, \|ku(t)\|)dt] \leq 2\|u\|^0$$

for any $u \in L^0_M(X)$. This means that $L^0_M(X)$ is an Asplund space, which completes the proof. □

By Theorem 3.1, we obtain that Corollary 3.2 and Corollary 3.3.

Corollary 3.2. *Suppose that X is a strongly smooth space. Then the following statements are equivalent:*
(1) $L_M(X)$ is an Asplund space;
(2) $M \in \Delta$ and $N \in \Delta$.

Corollary 3.3. *$L^0_M(R)(L_M(R))$ is an Asplund space if and only if $M \in \Delta$ and $N \in \Delta$.*

Theorem 3.4. *Suppose that*
(1) $M \in \Delta$ and X is a strongly smooth space;
(2) $p(t, u)$ is continuous with respect to u for almost all $t \in T$.
Then $E^0_N(X)$ is a strongly smooth space.

Proof. By Theorem 2.1, it is easy to see that Theorem 3.4 is true, which completes the proof. □

Theorem 3.5. *Suppose that X is a strongly smooth space. Then $L_M(X^*)$ has the Radon-Nikodym property if and only if $M \in \Delta$.*

Proof. Sufficiency. Let X be a strongly smooth space. Then, by Lemma 1.15 of [2], there exists a function M_1 such that

$$M(t, u) \leq M_1(t, u) \leq 2M(t, u), \quad u \in R$$

and right derivative of $p_1(t, u)$ of $M_1(t, u)$ is continuous with respect to u for almost all $t \in T$. Moreover, by the proof of Theorem 3.1, we get that $M_1 \in \Delta$. Therefore, by Theorem 3.4, we obtain that $E^0_N(X)$ is a strongly smooth space. Hence $E^0_N(X)$ is an Asplund space. This implies that $L_M(X^*)$ has the Radon-Nikodym property.

Necessity. Suppose that $M \notin \Delta$. Then, by the proof of Theorem 3.1, we obtain that $L_M(X^*)$ has not the Krein-Milman property. Hence $L_M(X^*)$ has not the Radon-Nikodym property, a contradiction. This implies that $M \in \Delta$, which completes the proof. □

Corollary 3.6. *Suppose that X is a strongly smooth space. Then $L^0_M(X^*)$ has the Radon-Nikodym property if and only if $M \in \Delta$.*

Corollary 3.7. *$L^0_M(R)(L_M(R))$ has the Radon-Nikodym property if and only if $M \in \Delta$.*

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