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Research article

Alarm probabilities of simple tests of merger control: An analytic derivation

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Abstract: For ten simple horizontal merger tests, I analytically derive the probability of blocking a merger. The assumptions are the symmetry of diversion ratios, price-cost margins, and efficiency gains, and the uniform joint distribution of these variables. The robustness analyses suggest that the limitations arising from the simplifying assumptions are only moderate. Based on these results, I provide an easy-to-use Excel calculator (without VBA macro), with which the probabilities associated with any product market and price increase threshold can be quickly calculated. Additionally, I propose a paired testing protocol for unknown demand systems by linking linear- and isoelastic-demand indicators through the threshold or the alarm probability.

Keywords: horizontal merger control; price-effect indicator; UPP; GUPPI; CMCR; alarm probability; Excel calculator; paired testing protocol

JEL Codes: C46, C65, L41

1. Introduction

When firms merge, antagonistic forces emerge which affect the future pricing policy of the combined company. Disappearing competition between merging parties boosts, while decreasing marginal cost restrains the incentive for a price increase. The boost is due to the fact that, after a price increase, a portion of the diverted customers are captured by other product(s) of the now same owner. The importance of this recapture is scaled up by the price-cost margin(s). The potential restraint is due to the well-known synergies (economies of scale and scope etc.).[1] In other words, after the merger, the objective functions of the companies are not separately but globally optimised. It follows that the larger the portion of the recaptured consumers, or the larger the pre-merger price-cost margins of the companies' products, or the smaller the efficiency gain, the greater the incentive for a post-merger price increase. These are the typical unilateral (non-coordinated) price issues under Bertrand competition with differentiated products.

A key element of a merger control procedure is the estimation of the resultant of the abovementioned incentives. Scientific debates have forged several simple indicators as well as more complex methods for this purpose. The spectrum of indicators is quite diverse. An important divide is whether the manner of the pass-through of costs to prices along a demand system is incorporated or not. If it is, the price increase incentive is scaled, that is, the indicator can provide a price-effect (i.e. how much a firm would increase the price of its product post-merger), and not just some outline of the magnitude. Another important aspect is the depth of the optimisation process. The indicators completely or partially lack the price-setting loops of the merging parties (i.e. the internal feedback); moreover, they completely miss the price reactions of the non-merging parties (i.e. the external feedback). Consequently, the estimate of an indicator with pass-through factor is an approximation of a full-fledged simulation's estimate. On the other hand, the indicators' need for data and time is typically less, and so they are better suited for screening purposes.

In this article, I provide analytically the subpopulations that are flagged by the indicators as problematic (and so the alarm probabilities), with a customisable sample space and threshold. For several reasons, these formulae provide valuable prior information for the competition authorities.

First, each authority has its own idea of what is an acceptable price increase in a given market, i.e. what is the threshold "standard". (e.g. 5 percent). Now they can finally face the potential consequence, i.e. the portion of the (market-specific) merger population they probably screen out in the long run during the initial inspection. Second, it may make them rethink their standards on the basis of the revealed trade-offs. If an indicator is thought to be acceptably accurate (a conscientious agent would not use it otherwise, would he?), and it flags, say, 95 (or 5) out of 100 mergers with a conventional threshold, the test is too strict (or too soft), and calls for higher (or lower) threshold standards. Third, such extreme (high or low) probabilities with economically reasonable thresholds may also indicate the serious inaccuracy of the indicator, but I do not address this issue in this article. Fourth, the expected rates of price increase under linear and isoelastic demands tend to be the boundary cases. In practice, however, extremes are rare. By mapping the related linear and isoelastic indicators through a specific threshold (or alarm probability), I also show that a rough interval of alarm probability (or threshold) can be estimated, behind which a foggy spectrum of alternative demand systems lurks. To sum it up, if the theories behind the indicators (and thus the accuracy) are thought to be more or less correct, the alarm probabilities provide valuable pieces of information for policymakers and theorists.

The structure of the article is as follows. Section 2 briefly outlines the indicators. Section 3 derives their alarm probabilities analytically. Section 4 discusses the results through an arbitrary sample space. Section 5 checks the robustness of the results to the assumptions on symmetry and distribution. Section 6 concludes.

[1] For a detailed survey, see Röller et al. (2006).

2. Overview of simple indicators

A group of indicators does not incorporate the pass-through factor. In his pioneering work, Werden (1996) derived a formula that shows the marginal cost reduction needed to offset the incentive of a merging firm to increase the price (CMCR, Compensating Marginal Cost Reductions). It was derived under the approach that the prices of the merging parties' products are mutually adjusted. Farrell and Shapiro (2010a) presented a simpler approach, called UPP (Upward Pricing Pressure), about which they claimed to have an advantage in terms of transparency.[2] It assumes that the price of the merging partner remains unchanged, that is, in contrast with CMCR, it lacks the in-house feedback. Schmalensee (2009) proposed an amendment to UPP, namely the incorporation of the cost change of the merging partner. Farrell and Shapiro (2010b) argued again that, with this complement, this indicator is more advanced at the expense of transparency. Moresi (2010) presented an indicator based on UPP, called GUPPI (Gross Upward Pricing Pressure Index), which does not account for the downward pressure of efficiencies but is accompanied by an exogenous threshold.

The other group of indicators, which I call quasi-simulations, incorporates the pass-through factor. Shapiro (2010) derived a formula assuming linear demand and that the slope (own-price derivatives) for both products is −1. It is the generalisation of Schmalensee's (2009) indicator called PCAL (Price Change Assuming Linearity) to asymmetric products (i.e. to non-identical prices, marginal costs, diversion ratios and efficiencies). Additionally, Shapiro (2010) provided another formula assuming constant elasticity of demand and full symmetry of variables. Based on linear demand, Hausman et al. (2011) derived a formula assuming identical cross-price derivatives for the demand functions (Slutsky symmetry). They argued that their assumption is more realistic than Shapiro's (2010); however, their expression does not cover efficiency gains. Hausman et al. (2011) actually provided an additional, more general formula, of which the one with Slutsky symmetry is a special case. All these metrics are called IPR (Illustrative or Indicative Price Rise). In a brief note, Salop and Moresi (2009) described how GUPPI should be transformed to measure the "first-round" price increase, both in the case of linear and isoelastic demand. In the linear case, Shapiro (2010) and Hausman et al. (2011) explicitly showed the difference between the transformed GUPPI and their more complex formulae, i.e. the effect of the feedback.

Table 1 summarises the indicators. With the assumption of price symmetry, I can express the asymmetric formulae in terms of margins rather than prices and costs. Variable m_i is the pre-merger price-cost margin of product i (Firm 1), $m_i = (p_i - c_i)/p_i$. Variable d_{ij} is the diversion ratio from product i to product j (Firm 2), $d_{ij} = -\left(\partial q_j/\partial p_i\right)/(\partial q_i/\partial p_i)$ where q represents the demand. Variable e_i is the efficiency gain as a proportion of the pre-merger marginal cost, $e_i = \left(c_i - c_i^{post}\right)/c_i$. Note that originally the CMCR-equation was arranged to show the marginal cost reduction necessary to restore pre-merger prices; that is, acronym CMCR stood for the e_i itself. I have rearranged the original equation to one side, and I call this expression "CMCR" for simplicity.

In the case of UPPs and CMCR, the incentive for a price increase is signalled if the calculated value of the indicator is non-negative, while in the case of GUPPI, the dividing line is some arbitrary threshold. In the case of quasi-simulations, the alarm condition is the reaching or exceeding of some positive threshold (i.e. the maximum tolerable price increase rate) that is to be adjusted to the properties of the relevant market.

[2] Epstein-Rubinfeld (2010) stated that UPP is essentially a special case of a merger simulation, while Farrell and Shapiro (2010c) refused it, claiming that their information requirement is quite different.

Table 1. Simple indicators in horizontal merger control (with my own notations).

		Asymmetry	Asymmetry (exception: $p_1 \coloneqq p_2$)	Full symmetry	Demand assumptions	Thold
Without pass-through factor	**UPP(FS)** Farrell-Shapiro (2010a)	$d_{12}(p_2 - c_2) - e_1 c_1$	$d_{12} m_2 - e_1(1 - m_1)$	$DM - E(1 - M)$	-	0
	UPP(Sch) Schmalensee (2009)	$d_{12}(p_2 - (1 - e_2)c_2) - e_1 c_1 = \mathrm{UPP}_1\ (\mathrm{FS}) + d_{12} e_2 c_2$	$d_{12}\big(m_2 + e_2(1 - m_2)\big) - e_1(1 - m_1)$	$DM - E(1 - D)(1 - M)$	-	0
	CMCR [a] Werden (1996)	$d_{12}(p_2 - c_2) + d_{12} d_{21}(p_1 - c_1) - e_1 c_1(1 - d_{12} d_{21}) = \mathrm{UPP}_1\ (\mathrm{FS}) + d_{12} d_{21}(p_1 - c_1(1 - e_1))$	$d_{12}(m_2 + d_{21} m_1) - e_1(1 - m_1)(1 - d_{12} d_{21})$	$DM - E(1 - D)(1 - M)$	-	0
	GUPPI Moresi (2010)	$\dfrac{d_{12}(p_2 - c_2)}{p_1}$	$d_{12} m_2$	DM	-	$\mathbb{R}^+$
With pass-through factor	**IPR(S)** Shapiro (2010)	$\dfrac{\left(2d_{12}\dfrac{p_2 - c_2}{p_2} - e_2\left(1 - \dfrac{p_2 - c_2}{p_2}\right)(d_{21} - d_{12})\right)\dfrac{p_2}{p_1} + d_{21}(d_{21} + d_{12})\dfrac{p_1 - c_1}{p_1}}{4 - (d_{21} + d_{12})^2} - \dfrac{e_1\left(1 - \dfrac{p_1 - c_1}{p_1}\right)\big(2 - d_{21}(d_{12} + d_{21})\big)}{4 - (d_{21} + d_{12})^2}$	$\dfrac{2d_{12} m_2 - e_2(1 - m_2)(d_{21} - d_{12}) + d_{21}(d_{21} + d_{12}) m_1}{4 - (d_{21} + d_{12})^2} - \dfrac{e_1(1 - m_1)\big(2 - d_{21}(d_{12} + d_{21})\big)}{4 - (d_{21} + d_{12})^2}$	$\dfrac{DM}{2(1 - D)} - \dfrac{E(1 - M)}{2}$	linear with equal slopes of -1 $\dfrac{\partial q_1}{\partial p_1} = \dfrac{\partial q_2}{\partial p_2} = -1$	$\mathbb{R}^+$
	IPR(S)i [b] Shapiro (2010)	-	-	$\dfrac{DM}{1 - D - M}$	isoelastic	$\mathbb{R}^+$
	IPR(H1) [c] Hausman et al. (2011)	$\dfrac{2d_{12}\dfrac{p_2 - c_2}{p_1} + d_{12} d_{21}\dfrac{p_1 - c_1}{p_1} + \dfrac{(p_1 - c_1)^2}{(p_2 - c_2) p_1} \cdot \dfrac{q_2}{q_1} d_{21}^2}{4 - 2d_{12} d_{21} - \dfrac{p_2 - c_2}{p_1 - c_1} \cdot \dfrac{q_1}{q_2} d_{12}^2 - \dfrac{p_1 - c_1}{p_2 - c_2} \cdot \dfrac{q_2}{q_1} d_{21}^2}$	$\dfrac{2d_{12} m_2 + d_{12} d_{21} m_1 + \dfrac{m_1^2}{m_2} \cdot d_{12}^2}{4 - 2d_{12} d_{21} - \dfrac{m_2}{m_1} d_{12}^2 - \dfrac{m_1}{m_2} d_{21}^2}$	$\dfrac{DM}{2(1 - D)}$	linear	$\mathbb{R}^+$
	IPR(H2) Hausman et al. (2011)	$\dfrac{d_{12}(p_2 - c_2) + d_{12} d_{21}(p_1 - c_1)}{2(1 - d_{12} d_{21}) p_1}$	$\dfrac{d_{12} m_2 + d_{12} d_{21} m_1}{2(1 - d_{12} d_{21})}$	$\dfrac{DM}{2(1 - D)}$	linear with Slutsky symmetry $\dfrac{\partial q_2}{\partial p_1} = \dfrac{\partial q_1}{\partial p_2}$	$\mathbb{R}^+$
	Gl Salop-Moresi (2009)	$\dfrac{\mathrm{GUPPI}_1}{2} \cdot \dfrac{p_2}{p_1} = \dfrac{d_{12}(p_2 - c_2)}{2p_1} \cdot \dfrac{p_2}{p_1}$	$\dfrac{d_{12} m_2}{2}$	$\dfrac{DM}{2}$	linear	$\mathbb{R}^+$
	Gi Salop-Moresi (2009)	$\dfrac{\mathrm{GUPPI}_1}{1 - m_1} \cdot \dfrac{p_2}{p_1} = \dfrac{d_{12}(p_2 - c_2)}{p_1(1 - m_1)} \cdot \dfrac{p_2}{p_1}$	$\dfrac{d_{12} m_2}{1 - m_1}$	$\dfrac{DM}{1 - M}$	isoelastic	$\mathbb{R}^+$

[a] In the case of full symmetry, the CMCR after some manipulation is $DM(1 + D) - E(1 - M)(1 - D)(1 + D)$, and so it can be divided by $1 + D$ since the threshold is zero.
[b] The formula is given for full symmetry only.
[c] In the cases of price symmetry and full symmetry, I also assumed quantity symmetry $(q_1 \coloneqq q_2)$.

3. Derivation of alarm probability

I call a test stricter if it indicates problem with greater probability when merging the ownership of two products randomly selected from the same product market.[3] In this experiment, if we have three variables, diversion ratio (D), margin (M), and efficiency (E), the sample space is a cuboid,

$$\Omega = \left\{ x = (D, M, E) \in \mathbb{R}^3 \;\middle|\; \begin{array}{c} 0 \le d_{min} \le D \le d_{max} < 1 \\ 0 \le m_{min} \le M \le m_{max} < 1 \\ 0 \le e_{min} \le E \le e_{max} < 1 \end{array} \right\},$$

where the minimum and maximum bounds are market specific. The indicator is a random variable, $I\colon \Omega \to \mathbb{R}$, and a test is a pair of an indicator and a threshold, i.e. (I, t). The set of outcomes that triggers alarm is then

$$A = \{x \in \Omega : I \ge t\}.$$

The probability measure of the alarm event is

$$\mathrm{P}(A) = \frac{A}{\Omega} = \frac{\iiint_A 1\, dE\, dM\, dD}{\iiint_\Omega 1\, dE\, dM\, dD} = \frac{\iiint_A 1\, dE\, dM\, dD}{(d_{max} - d_{min})(m_{max} - m_{min})(e_{max} - e_{min})},$$

where I assumed that the joint distribution of variables is uniform. In geometric terms, the ratio of the volumes of the homogenous objects shows the probability we are looking for.

Naturally, the concept can be generalised to higher dimensions (i.e. asymmetry of variables). In those cases, the sample space is an n-orthotope ("hypercuboid"), but otherwise the same applies as discussed above. However, over three dimensions or with non-uniform probability measure, it is cumbersome (if not impossible) to provide analytical solutions, either because of the hardship of finding the proper limits of integration or because of the integration procedure itself. In Section 5, using numerical methods, I analyse the robustness of the results to the assumptions on symmetry and distribution.

The derivation of the alarm set of each indicator can be found in the Appendix. The Excel calculator based on these results can be downloaded from: https://doi.org/10.5281/zenodo.10895535.

4. Discussion through an arbitrary example

For demonstrational purposes, I discuss the results of the previous section by choosing the domains of variables covering almost all of the products. In real-world applications, i.e. when considering a specific product market, the domains should be narrowed accordingly (although, in theory, it is not excluded that there are markets with similarly wide domains). With the provided Excel calculator, anyone can easily perform their customised calculations.

With regard to price-cost margins, there are some crutches we can rely on. Damodaran's (2018) collection of average gross profit margins by industry shows that they are mostly between 10 and 70

[3] If we know the nature of the demand, such a comparison only makes sense between indicators specialised for that particular demand.

percent in the USA.[4] Since they are averages, there must be some dispersion; thus, I expand this range into both directions with 10 percentage points.[5] As for diversion ratios, we can get a blurry picture from the publicly available information of past and recent merger cases. I assume a range of 0 to 50 percent, taking into account that the diversion ratios obtained from consumer surveys tend to overestimate the true values. Regarding efficiency gain, we have the least support. Röller et al. (2006) refutes the conventional wisdom that mergers and efficiency gain often go hand in hand. Moreover, the non-robust empirical examples they cited suggest that only 30 to 70 percent of the possible cost reduction is passed on to prices. It means that in order to offset a 10 percent price increase at a pass-through rate of 0.3, marginal cost should drop by 10/0.3=33.3 percent, which is, although a pretty high rate, not excluded by the authors from the realms of reality. Since the efficiency variable in the indicators expresses the effective rate, I assume a range of 0 to 20 percent for this variable.

First, let us get some impressions from the visualisation of the alarm sets (Figure 1).

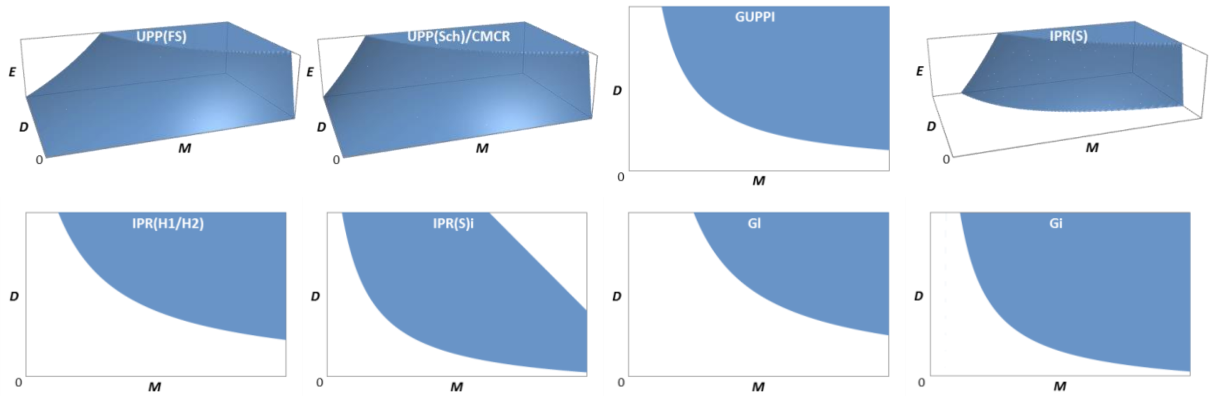

Figure 1. Alarm sets (threshold for GUPPI and quasi-simulations is 0.05).

Notes: Alarm sets expand to the southwest as threshold decreases. Recall that the floor of object IPR(S) is IPR(H1/H2).

The alarm set of UPP(Sch)/CMCR is somewhat greater than that of UPP(FS), which is to be expected due to the boosting feedback effect (see Table 1). With the given 5 percent threshold, the difference between the alarm sets of the quasi-simulations is also quite significant. It is not surprising if we compare indicators based on different demand structures, e.g. Gl and Gi, but a remarkable discrepancy is unfortunate if comparing two indicators from the same demand class, e.g. Gl and IPR(H1/H2) (or Gi and IPR(S)i). It logically follows that at least one of a pair is certainly flawed in terms of accuracy in this particular (arbitrary) market situation, but this issue is out of the scope of this paper. In the concrete example, the difference stems from the fact that, unlike IPR(H1/H2), Gl ignores feedback effects.

Notice the difference between UPPs and IPR(S) regarding their relationship with efficiency, which originates from their different foundations. Strictly speaking, UPPs consider efficiency gain as a must; without any, they would not give a green light to a merger. On the contrary, in the case of

[4] I excluded banks with profit margin of 100 percent. In Europe, the same range is approximately 15 to 85 percent.

[5] Note that Butler Consultants' (2018) database on average industry profit margins shows significantly greater values than Damodaran's (2018) in lots of industries.

IPR(S), the surface intersects the D-M plane in the examined domain. That is, there are green-light cases (i.e. the projected price rise is less than the threshold) even with some loss of efficiency.

Now, let us see the big picture on a reasonable spectrum of thresholds (Figure 2).

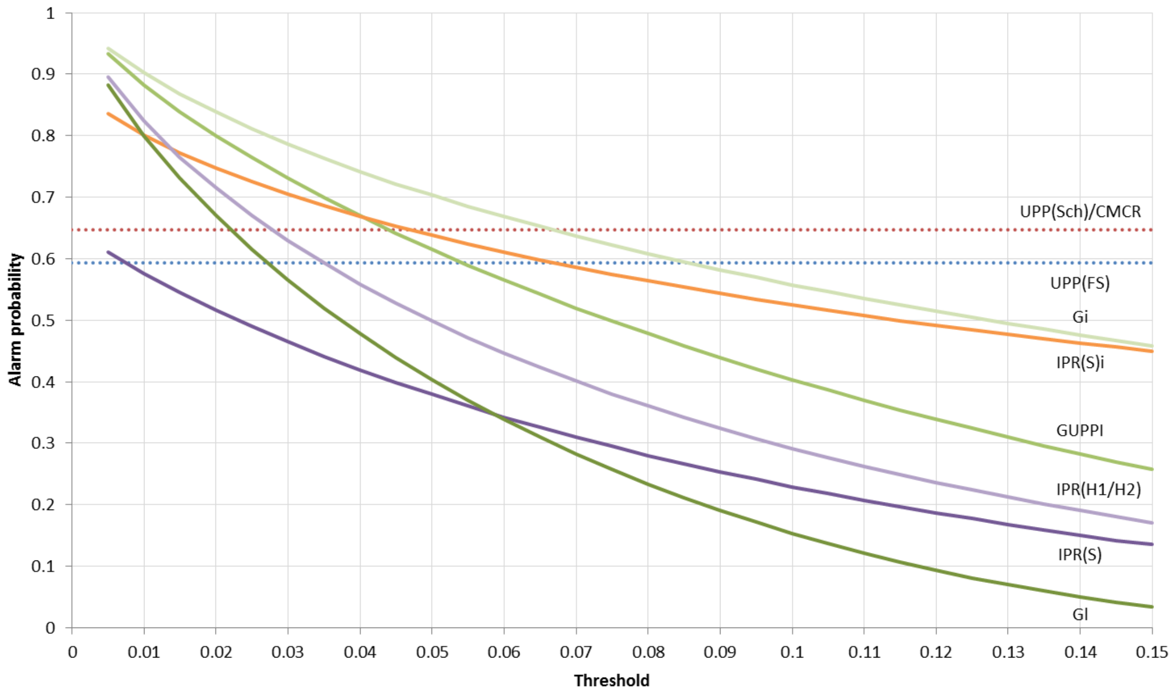

Figure 2. Alarm probabilities as a function of threshold.

Many things reappear from our previous evaluation of Figure 1. In the group of indicators without a pass-through factor, UPPs and CMCR are quite stringent, rejecting about 60–65 percent of the mergers. GUPPI is also not a very indulgent metric. Even with a threshold of 0.1, it flags about 40 percent of the mergers, and it becomes extremely harsh at lower thresholds. In the group of quasi-simulations, IPR(S) is lenient, which is perhaps not a surprise: it assumes linear demand and is the only one that incorporates efficiency gain. Gl is also mostly permissive, but also has a severe side, similar to IPR(H1/H2). Indicators assuming isoelastic demand, IPR(S)i and Gi, are naturally more relentless, even with very high thresholds.

With $d_{max} + m_{max} > 1$, it is easy to prove that the order of the indicated price-effects (without efficiency) is Gl < IPR(S)=IPR(H1/H2) < Gi < IPR(S)i if $1 - D - M > 0$, and IPR(S)i < Gl < IPR(S) = IPR(H1/H2) < Gi if $1 - D - M \leq 0$ (see Table 1). The peculiar position-switching of IPR(S)i comes from its "blank triangle" in the northeast (recall Figure 1). Note, however, that this phenomenon can only occur because we are exploring a sample space that includes the upper parts of the economically meaningful domain. That is, were $d_{max} + m_{max} < 1$, this issue would not arise at all.

4.1. Mapping indicator to indicator

Since the price-effects under linear and isoelastic demand systems are typically the lower and upper extremes, the effects under other formations (e.g. logit, almost ideal) tend to be located somewhere between them. On this basis, the above results can also be read in more complex ways if two indicators are paired and linked accordingly. I argue that it is a handhold when facing the chasm of an unknown demand structure.

For the sake of example, let us link Gl and Gi through a specific threshold or alarm probability. If the authority believes that the maximum tolerable price increase rate is 0.05 in the relevant market, but knows nothing about the true demand system, the portion of the problematic subpopulation can be estimated between about 40 to 70 percent (see Figure 2). Since (with $t_{\mathrm{Gl}} = t_{\mathrm{Gi}}$) Gl < Gi $\Leftrightarrow A_{\mathrm{Gl}} \subset A_{\mathrm{Gi}} \Leftrightarrow \mathrm{P}(A_{\mathrm{Gl}}) < \mathrm{P}(A_{\mathrm{Gi}})$, the problematic populations of the intermediate demand formations can be thought of as interpolations (see Figure 3, left side).

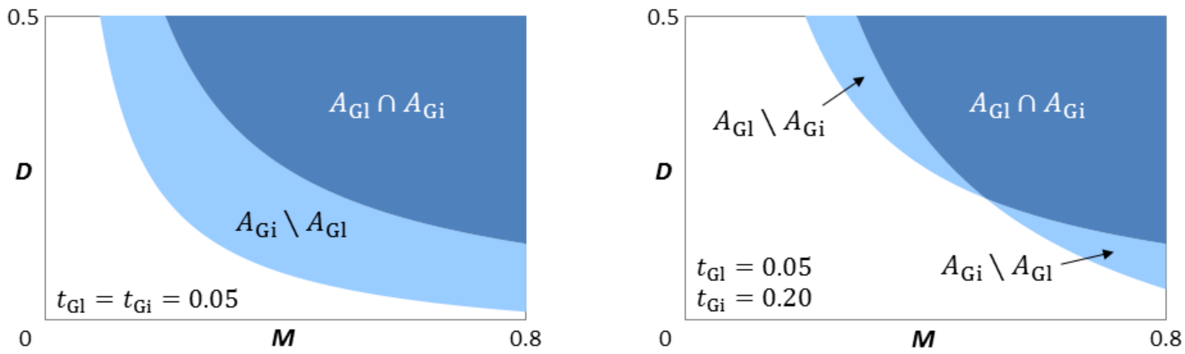

Figure 3. Overlap of alarm sets.

What about the reverse? If the authority seeks to flag no more than 40 percent of the population, it suggests that the implied threshold is between 0.05 and 0.2, depending on the (unknown) demand. If either Gl 0.05 or Gi 0.2 is used for testing, the rejection ratio will be 40 percent. However, $\mathrm{P}(A_{\mathrm{Gl}}) = \mathrm{P}(A_{\mathrm{Gi}}) \nRightarrow A_{\mathrm{Gl}} = A_{\mathrm{Gi}}$ (see Figure 3, right side)! $A_{\mathrm{Gl}} \setminus A_{\mathrm{Gi}}$ is the set of outcomes that indicates a price increase rate above 0.05 when the demand is linear, but below 0.2 when the demand is isoelastic. Consequently, here the upper bound of our interval estimate of the threshold is too high. $A_{\mathrm{Gi}} \setminus A_{\mathrm{Gl}}$ is the set of outcomes that indicates a price increase rate above 0.2 when the demand is isoelastic, but below 0.05 when it is linear. Thus, here the lower bound is the one that is overestimated. With a paired testing protocol of [Gl 0.05, Gi 0.2] with at least one rejection, the mergers associated with these two "outlier" subsets would also be blocked. It follows that the rejection ratio would be bigger than 40 percent, but not by much since $\frac{(A_{\mathrm{Gl}} \setminus A_{\mathrm{Gi}}) \cup (A_{\mathrm{Gi}} \setminus A_{\mathrm{Gl}})}{A_{\mathrm{Gl}} \cup A_{\mathrm{Gi}}} = 1 - \frac{(A_{\mathrm{Gl}} \cup A_{\mathrm{Gi}}) \setminus (A_{\mathrm{Gl}} \cap A_{\mathrm{Gi}})}{A_{\mathrm{Gl}} \cup A_{\mathrm{Gi}}}$ is moderately small.

Choosing IPR(S)i instead of Gi, the interval estimates would not be that favourable due to the peculiar feature of this indicator. Setting an equal threshold, Gl $\lesseqgtr$ IPR(S)i $\Longrightarrow A_{\mathrm{Gl}} \not\subset A_{\mathrm{IPR(S)i}}$, which implies that the size of the problematic subpopulation is not a monotonic function of the "curvature" of the demand system. Setting an equal probability, the corresponding alarm probability of the paired testing is significantly larger than the selected individual alarm probability, which indicates a weak overlap. However, recall that it is the matter of the selected sample space: markets with more moderate diversion ratios and margins are not affected by this issue.

In summary, mapping indicators along a specific threshold or alarm probability may come in handy if the nature of the underlying demand is unknown. The added value of such a procedure depends on two important caveats. First, by equal thresholds, the alarm set of a linear indicator is not necessarily a proper subset of the alarm set of an isoelastic indicator, which means that the isoelastic demand only tends to (i.e. not always) lead to higher price increases. For a reliable interval estimate of the problematic population (and so the alarm probability), these exceptions need to be modest. Second, the equality of the size of the two alarm objects (and so the equality of the two alarm probabilities) does not necessarily mean a perfect overlap of the alarm objects; therefore, their union according to the paired test can be significantly bigger than the individual object. For a reliable interval estimate of the threshold, this difference needs to be minimal. In a very limited information environment, I reckon that such mappings may be the best, undoubtedly rough, approaches.

Findings of this section:

- There can be huge differences between the alarm probabilities of various indicators with the same threshold, even within the same demand class.
- From the latter it logically follows that the accuracies can differ substantially.
- The respective trade-offs between threshold and alarm probability is provided for any product market with linear, isoelastic, or unknown demand system.
- If the demand system is unknown, a rough interval estimate of the problematic population ratio or the maximum tolerable price-effect can be given by linking the linear-demand and isoelastic-demand indicators.

5. Robustness analysis

It is out of question that the results depend on the basic assumptions – the point is, however, the degree of this dependence. In this section, I explore the "cost" of simplifications (i.e. uniformity of the joint distribution of variables and full symmetry of variables).

Unfortunately, there is no full-scale open-access database of the product variables, let alone available research on their joint probability distributions. Let us define then a non-uniform distribution, which closely reflects both theory and evidence. Werden and Froeb (2011) argued that in "Bertrand competition among differentiated products […] high degree of differentiation implies low diversion ratios causing high margins" (p. 166), and vice versa. But in practice, we might add, it is far from being a deterministic relationship. Higher than "usual" industry margins could not only be a sign of product uniqueness, but also the consequence of market inefficiency. Moreover, products with high market power could also be managed poorly, that is, with low margins. Therefore, suppose the following stochastic relationship between margin and diversion ratio:

$$M = -\frac{m_{max} - m_{min}}{d_{max} - d_{min}} D + m_{max} + \frac{m_{max} - m_{min}}{d_{max} - d_{min}} d_{min} + \xi \, .$$

The deterministic part gives the equation of an intersecting hyperplane of the sample space. Suppose that the intersection is the set of modes of the joint distribution, and that the occurrence density decreases in proportion to the orthogonal distance from this hyperplane (H),

$$f(D,M) = max\{dist^{\perp}(\hat{x},H): \forall \hat{x} \in \Omega\} - dist^{\perp}(x,H)$$
$$= \frac{(d_{max}-d_{min})\cdot(m_{max}-m_{min})}{\sqrt{(d_{max}-d_{min})^2+(m_{max}-m_{min})^2}}$$
$$- \frac{\left|\frac{m_{max}-m_{min}}{d_{max}-d_{min}}D + M - m_{max} - \frac{m_{max}-m_{min}}{d_{max}-d_{min}}d_{min}\right|}{\sqrt{\frac{(m_{max}-m_{min})^2}{(d_{max}-d_{min})^2}+1}},$$

where the maximum distance is naturally the distance from the edge(s) of the sample space opposite to the hyperplane. Figure 4 visualises this setup.

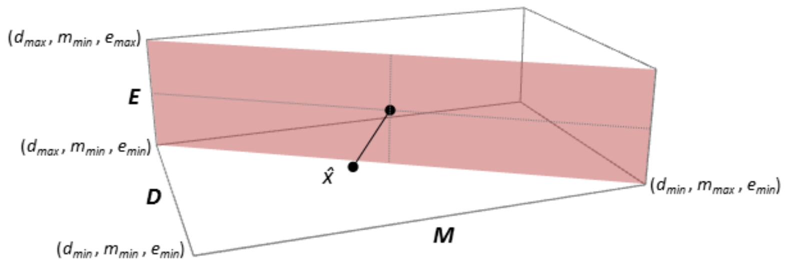

Figure 4. Sample space, set of modes, and an orthogonal distance.

It drives us to an "inverse" distance-weighted probability measure, which, in geometric terms, is the ratio of the masses of the non-homogeneous objects,

$$\mathrm{P}(A) = \frac{A}{\Omega} = \frac{\iiint_A f(D,M)\,dE\,dM\,dD}{\iiint_\Omega f(D,M)\,dE\,dM\,dD}.$$

Figure 5 shows the difference in probabilities between the baseline case and the alternative cases.

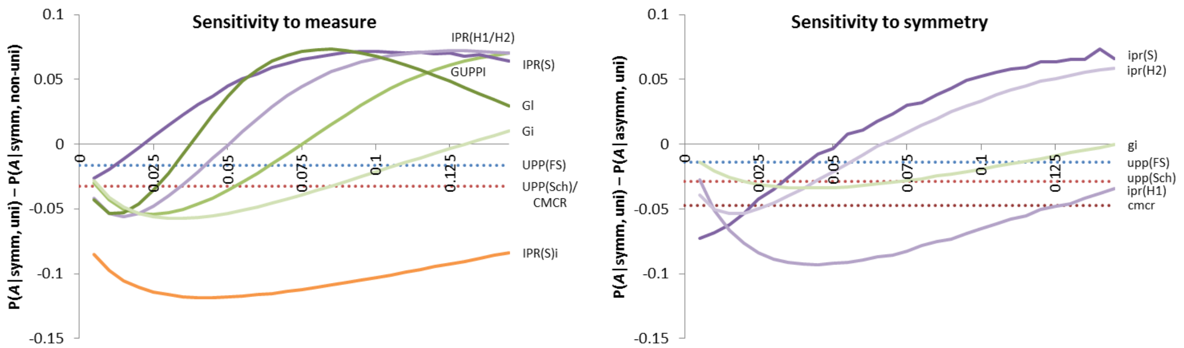

Figure 5. Difference between baseline and alternative probabilities.

Legend: Uppercase denotes full symmetry, lowercase denotes asymmetry.

Reminder: UPP(Sch)=CMCR, IPR(H1)=IPR(H2), P(A_{GUPPI})=P(A_{guppi}), P(A_{GI})=P(A_{gl}).

Switching to the non-uniform ("inverse" distance-weighted) measure did not affect probabilities greatly: almost all differences are between ±5 percentage points at lower thresholds. The odd-one-out is IPR(S)i, the reason for which is easy to understand from Figure 1. The alternative distribution puts more emphasis on the area close to the main diagonal, that is, the importance of the unflagged upper right triangle of IPR(S)i plummets, and so the rejection ratio springs up.

As for the issue of symmetry versus asymmetry, notice that (on the examined spectrum of threshold) full symmetry is usually, but not necessarily, more favourable to the merging parties; the exceptions are ipr(S) and ipr(H2) in higher ranges. Data collection challenges or short deadlines often impel competition authorities to assume symmetry, and thus it is good to be aware of the consequences of this decision. In our concrete case, the differences are mostly small, which advocates the use of the simpler approach (i.e. full symmetry).

Findings of this section (drawn from a large sample space):

- Changing the probability measure from uniform to non-uniform does not affect the alarm probabilities substantially, except for IPR(S)i.
- The alarm probabilities under full symmetry are typically lower than under asymmetry; nevertheless, the difference is not substantial.

6. Conclusions and outlook

Competition authorities have their visions for the tolerable post-merger price increase rates of the relevant markets. However, they do not know what portion of the relevant population is put under potential scrutiny as a result of this vision. I filled this gap by providing analytical solutions of alarm probabilities for each well-known indicator, and thus also how these indicators relate to each other. Based on these and the reverse approach (i.e. inferring the threshold from the population of interest), I also showed that the combination of two demand-specific indicators allows for more sophisticated screening tests than a "universal" indicator.

As a consequence, I argue that the exploration of the domain of diversion ratios, margins, and efficiency gains of the merger populations is inevitable for the proper use of indicators. I also showed that the exploration of the probability distribution or the use of asymmetric variables might be useful, but they are not necessarily that important.

From a practical point of view, the exploration of domains is incomparably easier and faster to perform than the exploration of the joint distribution. Note that competition authorities have the privilege of collecting precious data from a very broad spectrum of the economy. They could create their own dedicated database of the main variable values of their merger cases, with which they have the best chance to provide reliable estimates of the domains. Moreover, with such a database, they have the only chance of making robust estimates of the joint probability distributions. In a carefully anonymised form, this could also be made open access, classified by some product/activity standard (e.g. CPA/NACE, NAPCS/NAICS). In (sub)categories, where there are only a few items/economic actors, precaution is needed in order to prevent identifiability, but it can be easily remedied by setting the aggregation levels on a sector-specific basis. Thinking of cross-border mergers, it would be constructive if such national databases were implemented along a harmonised methodology. Not only competition authorities, but also the research community, and thus the average consumer could benefit from such "goldmines".

Finally, no one should forget that the results of analyses (e.g. accuracy studies) based on empirical merger data are only useful as long as the economic context is relatively stable. As soon as the domains and/or the underlying relationship of the variables change, the validity of these results expires. That is why customisable tools are indispensable, especially in turbulent times.

Use of AI tools declaration

The author declares that he did not use Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that there is no conflict of interest.

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Appendix: Derivations of alarm sets

Assumptions:
- full symmetry of variables,
- uniform distribution of variables.

For simplicity, let d, m, e denote the upper and δ, μ, ε denote the lower bounds of the variables. Since it is not confusing, I also use the conventional notation d for differential operator.

A. $\mathbf{UPP(FS)} = \boldsymbol{DM - E(1 - M) \geq 0}$

The alarm condition rearranged to efficiency is $E \leq \frac{DM}{1-M}$, which implies that $A \neq \emptyset$ if $\varepsilon < \frac{dm}{1-m}$. Since

$$\text{equality holds } \Rightarrow \begin{cases} \varepsilon = 0 \ \wedge \ (D = 0 \ \vee \ M = 0) \\ \varepsilon > 0 \ \wedge \ D > 0 \ \wedge \ M > 0 \end{cases},$$

the derivations are to be separated under different lower bound conditions.

$$\boxed{\text{If } \varepsilon < \frac{dm}{1-m} \leq e}$$

Lower bound condition 1: $\varepsilon = 0$

With $\delta = \mu = 0$, the volume is

$$\int_0^d \int_0^m \frac{DM}{1-M} dM \, dD .$$

Substituting $u = 1 - M$ $(du = -dM)$, the inner integral becomes

$$\int_0^m \frac{DM}{1-M} dM = D \int_1^{1-m} \frac{u-1}{u} du = D[u - \ln u]_1^{1-m} = D(-m - \ln(1-m)) . \qquad [1]$$

Thus, the outer integral is

$$\int_0^d D(-m - \ln(1-m)) \, dD = \frac{d^2(-m - \ln(1-m))}{2} .$$

With $\delta > 0$ or $\mu > 0$, the parts below these lower bounds is to be subtracted. Note that if $\delta > 0$ and $\mu > 0$, the part affected by double subtraction is to be compensated. Thus, the general formula for $\delta \geq 0$ and $\mu \geq 0$ is

$$\frac{d^2(-m-\ln(1-m))}{2} - \frac{\delta^2(-m-\ln(1-m))}{2} - \frac{d^2(-\mu-\ln(1-\mu))}{2} + \frac{\delta^2(-\mu-\ln(1-\mu))}{2}. \qquad [2]$$

Lower bound condition 2: $\varepsilon > 0$

In order to find the proper limits of integration, let us rearrange the alarm condition to D and M, which is $D > \frac{E(1-M)}{M}$ and $M > \frac{E}{E+D}$, and thus the alarm set is

$$\int_{\frac{\varepsilon(1-m)}{m}}^{d} \int_{\frac{\varepsilon}{\varepsilon+D}}^{m} \int_{\varepsilon}^{\frac{DM}{1-M}} 1 \, dE \, dM \, dD = \int_{\frac{\varepsilon(1-m)}{m}}^{d} \int_{\frac{\varepsilon}{\varepsilon+D}}^{m} \left(\frac{DM}{1-M} - \varepsilon\right) dM \, dD \, .$$

Using result from [1], the inner integral becomes

$$\int_{\frac{\varepsilon}{\varepsilon+D}}^{m} \left(\frac{DM}{1-M} - \varepsilon\right) dM = [D(-M-\ln(1-M)) - \varepsilon M]_{\frac{\varepsilon}{\varepsilon+D}}^{m}$$
$$= D\left(-m - \ln(1-m) + \frac{\varepsilon}{\varepsilon+D} + \ln\left(1 - \frac{\varepsilon}{\varepsilon+D}\right)\right) - \varepsilon m + \frac{\varepsilon^2}{\varepsilon+D},$$

and thus the outer integral is

$$\int_{\frac{\varepsilon(1-m)}{m}}^{d} \left((-m-\ln(1-m))D + \frac{\varepsilon D}{\varepsilon+D} + \ln\left(1-\frac{\varepsilon}{\varepsilon+D}\right) \cdot D - \varepsilon m + \frac{\varepsilon^2}{\varepsilon+D}\right) dD \, . \qquad [3]$$

First, I calculate the non-trivial indefinite integrals of the second and the third terms of [3]. Here and hereafter I do not use different notations for the constants of integration, i.e. I always stick to letter c.

The second term of [3] is (substituting $u = \varepsilon + D$)

$$\varepsilon \int \frac{D}{\varepsilon+D} dD = \varepsilon \int \frac{u-\varepsilon}{u} du = \varepsilon(u - \varepsilon \ln u) + c = \varepsilon(\varepsilon + D - \varepsilon \ln(\varepsilon+D)) + c$$
$$= \varepsilon(D - \varepsilon \ln(\varepsilon+D)) + c \, . \qquad [4]$$

For calculating the third term of [3], we need to integrate by parts ($\int fg' = fg - \int f'g$). Let

$$f = \ln\left(1 - \frac{\varepsilon}{\varepsilon+D}\right) \text{ and } g' = D \, .$$

Applying chain rule for getting f'

$$f' = \frac{d}{dD}\ln\left(1-\frac{\varepsilon}{\varepsilon+D}\right) = \frac{1}{1-\frac{\varepsilon}{\varepsilon+D}} \cdot \frac{d}{dD}\left(1-\frac{\varepsilon}{\varepsilon+D}\right) = \frac{1}{1-\frac{\varepsilon}{\varepsilon+D}} \cdot \frac{\varepsilon}{(\varepsilon+D)^2}$$
$$= \frac{\varepsilon}{(\varepsilon+D)^2 - \varepsilon(\varepsilon+D)} = \frac{\varepsilon}{D(\varepsilon+D)},$$

and thus

$$\int fg' = \int \ln\left(1 - \frac{\varepsilon}{\varepsilon + D}\right) D\, dD = \ln\left(1 - \frac{\varepsilon}{\varepsilon + D}\right)\frac{D^2}{2} - \int \frac{\varepsilon}{(\varepsilon + D)} \cdot \frac{D}{2} dD$$
$$= \frac{\ln\left(1 - \frac{\varepsilon}{\varepsilon + D}\right) D^2}{2} - \frac{\varepsilon(D - \varepsilon \ln(\varepsilon + D))}{2} + c\,,$$

where (for the last step) I used the result obtained at [4].

Remaining integrals of [3] come trivially, and thus the alarm set is

$$\left[\frac{(-m - \ln(1 - m))D^2}{2} + \varepsilon(D - \varepsilon \ln(\varepsilon + D)) + \frac{\ln\left(1 - \frac{\varepsilon}{\varepsilon + D}\right) D^2}{2} - \frac{\varepsilon(D - \varepsilon \ln(\varepsilon + D))}{2}\right.$$
$$\left. - \varepsilon m D + \varepsilon^2 \ln(\varepsilon + D)\right]_a^d$$
$$= \left[\frac{(-m - \ln(1 - m))D^2}{2} + \frac{\varepsilon(D - \varepsilon \ln(\varepsilon + D))}{2} + \frac{\ln\left(1 - \frac{\varepsilon}{\varepsilon + D}\right) D^2}{2} - \varepsilon m D\right.$$
$$\left. + \varepsilon^2 \ln(\varepsilon + D)\right]_a^d \quad [5]$$
$$= \left[\frac{(-m - \ln(1 - m))D^2}{2} + \frac{\varepsilon D}{2} + \frac{\varepsilon^2 \ln(\varepsilon + D)}{2} + \frac{\ln\left(1 - \frac{\varepsilon}{\varepsilon + D}\right) D^2}{2} - \varepsilon m D\right]_a^d$$
$$= \left[\frac{D}{2}\left((-m - \ln(1 - m))D + \varepsilon + \ln\left(1 - \frac{\varepsilon}{\varepsilon + D}\right) D - 2\varepsilon m\right) + \frac{\varepsilon^2 \ln(\varepsilon + D)}{2}\right]_a^d$$
$$= \left[\frac{D}{2}\left(D\left(\ln\left(1 - \frac{\varepsilon}{\varepsilon + D}\right) - m - \ln(1 - m)\right) + \varepsilon(1 - 2m)\right) + \frac{\varepsilon^2 \ln(\varepsilon + D)}{2}\right]_a^d,$$

where $a = \frac{\varepsilon(1-m)}{m}$. Since δ and μ may truncate this set, the general formula for $\delta \geq 0$ and $\mu \geq 0$ is

$$[5] - \left[5 \text{ with } d \coloneqq \delta \text{ if } \delta > \frac{\varepsilon(1 - m)}{m}\right] - \left[5 \text{ with } m \coloneqq \mu \text{ if } \mu > \frac{\varepsilon}{\varepsilon + d}\right]$$
$$+ \left[5 \text{ with } d \coloneqq \delta \text{ and } m \coloneqq \mu \text{ if } \delta > \frac{\varepsilon(1 - \mu)}{\mu} \text{ and } \mu > \frac{\varepsilon}{\varepsilon + \delta}\right]. \quad [6]$$

$$\boxed{\text{If } e < \frac{dm}{1 - m}}$$

In this case, the object part above e is to be subtracted. The alarm sets are then [2] – [6 with $\varepsilon \coloneqq e$] under Condition 1, and [6] – [6 with $\varepsilon \coloneqq e$] under Condition 2.

B. $\mathbf{UPP(Sch)/CMCR} = \boldsymbol{DM - E(1-D)(1-M)} \geq \mathbf{0}$

The alarm condition rearranged to efficiency is $E \leq \frac{DM}{(1-D)(1-M)}$, which implies that $A \neq \emptyset$ if $\varepsilon < \frac{dm}{(1-d)(1-m)}$. Since (just as with UPP(FS))

$$\text{equality holds} \Rightarrow \begin{cases} \varepsilon = 0 \ \wedge \ (D = 0 \ \vee \ M = 0) \\ \varepsilon > 0 \ \wedge \ D > 0 \ \wedge \ M > 0 \end{cases},$$

the derivation are to be separated under different lower bound conditions.

$$\boxed{\text{If } \varepsilon < \frac{dm}{(1-d)(1-m)} \leq e}$$

Lower bound condition 1: $\varepsilon = 0$

With $\delta = \mu = 0$, the volume is

$$\int_0^d \int_0^m \frac{DM}{(1-D)(1-M)} dM\, dD\, .$$

The inner integral becomes (like in [1])

$$\int_0^m \frac{DM}{(1-D)(1-M)} dM = \frac{D}{1-D}(-m - \ln(1-m))\, ,$$

then the outer integral is

$$\int_0^d \frac{D}{1-D}(-m - \ln(1-m))\, dD = (d + \ln(1-d))(m + \ln(1-m))\, .$$

With $\delta > 0$ or $\mu > 0$, the parts below these lower bounds is to be subtracted. If $\delta > 0$ and $\mu > 0$, the part affected by double subtraction is to be compensated. Thus, the general formula is

$$\begin{aligned}(d + \ln(1-d))(m + \ln(1-m)) - (\delta + \ln(1-\delta))(m + \ln(1-m)) \\ - (d + \ln(1-d))(\mu + \ln(1-\mu)) + (\delta + \ln(1-\delta))(\mu + \ln(1-\mu))\, .\end{aligned} \quad [7]$$

Lower bound condition 2: $\varepsilon > 0$

The alarm set is (using rearranged forms of alarm condition, $D \geq \frac{E(1-M)}{E(1-M)+M}, M \geq \frac{E(1-D)}{E(1-D)+D}$),

$$\int_{\frac{\varepsilon(1-m)}{\varepsilon(1-m)+m}}^{d} \int_{\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}}^{m} \int_{\varepsilon}^{\frac{DM}{(1-D)(1-M)}} 1\, dE\, dM\, dD$$

$$= \int_{\frac{\varepsilon(1-m)}{\varepsilon(1-m)+m}}^{d} \int_{\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}}^{m} \left(\frac{DM}{(1-D)(1-M)} - \varepsilon\right) dM\, dD\, .$$

The inner integral is

$$\int_{\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}}^{m}\left(\frac{DM}{(1-D)(1-M)}-\varepsilon\right)dM=\frac{D}{1-D}[-M-\ln(1-M)]_{\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}}^{m}-[\varepsilon M]_{\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}}^{m}$$
$$=\frac{D}{1-D}\left(-m-\ln(1-m)+\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}+\ln\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right)\right)-\varepsilon m$$
$$+\frac{\varepsilon^2(1-D)}{\varepsilon(1-D)+D},$$

so the outer integral is

$$\int_a^d\left(\frac{D}{1-D}(-m-\ln(1-m))+\frac{\varepsilon D}{\varepsilon(1-D)+D}+\ln\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right)\cdot\frac{D}{1-D}-\varepsilon m\right.$$
$$\left.+\frac{\varepsilon^2(1-D)}{\varepsilon(1-D)+D}\right)dD$$
$$=\int_a^d\left(\frac{D}{1-D}(-m-\ln(1-m))+\frac{\varepsilon(1-\varepsilon)D}{\varepsilon(1-D)+D}+\ln\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right)\right. \qquad [8]$$
$$\left.\cdot\frac{D}{1-D}-\varepsilon m+\frac{\varepsilon^2}{\varepsilon(1-D)+D}\right)dD\,,$$

where $a=\frac{\varepsilon(1-m)}{\varepsilon(1-m)+m}$.

I integrate [8] term by term as an indefinite integral. The first (combined) term is already known from previous calculations (see [1]):

$$\int\frac{D}{1-D}(-m-\ln(1-m))\,dD=(d+\ln(1-m))(m+\ln(1-m))+c \qquad [9]$$

The second term of [8] is (substituting $u=\varepsilon(1-D)+D=(1-\varepsilon)D+\varepsilon$ and $du=(1-\varepsilon)\,dD$)

$$\int\frac{\varepsilon(1-\varepsilon)D}{\varepsilon(1-D)+D}dD=\varepsilon(1-\varepsilon)\int\frac{\frac{u-\varepsilon}{1-\varepsilon}}{u}\cdot\frac{1}{1-\varepsilon}du=\frac{\varepsilon}{1-\varepsilon}\int\frac{u-\varepsilon}{u}du$$
$$=\frac{\varepsilon}{1-\varepsilon}(u-\varepsilon\ln(u))+c=\frac{\varepsilon\big((1-\varepsilon)D+\varepsilon\big)-\varepsilon^2\ln\big((1-\varepsilon)D+\varepsilon\big)}{1-\varepsilon}+c \qquad [10]$$
$$=\varepsilon D-\frac{\varepsilon^2\ln\big((1-\varepsilon)D+\varepsilon\big)}{1-\varepsilon}+c\,.$$

The third term of [8] is

$$\int\ln\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right)\cdot\frac{D}{1-D}dD\,.$$

Let $f=\ln\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right)$ and $g'=\frac{D}{1-D}$ and so

$$f' = \frac{d}{dD}\ln\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right) = \frac{1}{1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}}\cdot\frac{d}{dD}\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right). \qquad [11]$$

Applying chain rule and quotient rule

$$\begin{aligned}\frac{d}{dD}\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right) &= \varepsilon\left(-\frac{d}{dD}\frac{1}{\varepsilon(1-D)+D}+\frac{d}{dD}\frac{D}{\varepsilon(1-D)+D}\right)\\ &= \varepsilon\left(\frac{1-\varepsilon}{(\varepsilon(1-D)+D)^2}+\frac{\varepsilon(1-D)+D-D(1-\varepsilon)}{(\varepsilon(1-D)+D)^2}\right)\\ &= \varepsilon\left(\frac{1-\varepsilon}{(\varepsilon(1-D)+D)^2}+\frac{\varepsilon-\varepsilon D+D-D+\varepsilon D}{(\varepsilon(1-D)+D)^2}\right) = \frac{\varepsilon}{(\varepsilon(1-D)+D)^2}.\end{aligned} \qquad [12]$$

Writing [12] to [11]

$$\begin{aligned}f' &= \frac{1}{1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}}\cdot\frac{\varepsilon}{(\varepsilon(1-D)+D)^2} = \frac{\varepsilon}{(\varepsilon(1-D)+D)^2-\varepsilon(1-D)(\varepsilon(1-D)+D)}\\ &= \frac{\varepsilon}{D(\varepsilon(1-D)+D)}.\end{aligned}$$

Function g is obtained in a familiar way (see [1])

$$g = \int\frac{D}{1-D}dD = -D-\ln(1-D)+c\,.$$

Now, integrating by parts we get

$$\begin{aligned}\int f\,g' &= f\,g-\int f'g\\ &= \ln\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right)(-D-\ln(1-D))\\ &\quad -\int\frac{\varepsilon}{D(\varepsilon(1-D)+D)}(-D-\ln(1-D))\,dD\,.\end{aligned} \qquad [13]$$

First part of the integral of [13] is (substituting $u=\varepsilon(1-D)+D=(1-\varepsilon)D+\varepsilon$ and $du=(1-\varepsilon)\,dD$)

$$\int\frac{\varepsilon}{\varepsilon(1-D)+D}dD = \frac{\varepsilon}{1-\varepsilon}\int\frac{1}{u}du = \frac{\varepsilon\ln(\varepsilon(1-D)+D)}{1-\varepsilon}+c\,, \qquad [14]$$

and the second part is

$$\int\frac{\varepsilon\ln(1-D)}{D(\varepsilon(1-D)+D)}dD\,.$$

To solve the second part, the first move is partial fraction decomposition:

$$\frac{A}{D}+\frac{B}{\varepsilon(1-D)+D} = \frac{\varepsilon}{D(\varepsilon(1-D)+D)}$$

$$A(\varepsilon(1-D)+D)+BD=\varepsilon\,.$$

Choosing $D=0$ we get $A=1$. Using this value, by $D=1$ we get $B=\varepsilon-1$. Thus, the integral of the decomposed fraction is

$$\int\frac{\varepsilon\ln(1-D)}{D(\varepsilon(1-D)+D)}dD=\int\frac{\ln(1-D)}{D}dD+(\varepsilon-1)\int\frac{\ln(1-D)}{\varepsilon(1-D)+D}dD\,. \qquad [15]$$

Knowing that $\int_d^0\frac{\ln(1-D)}{D}dD=\mathrm{Li}_2(d)$, and considering that [8] (the outer integral) is on domain $[0,d]$, we can formalize the first integral of [15] as[6]

$$\int\frac{\ln(1-D)}{D}dD=-\mathrm{Li}_2(D)+c\,. \qquad [16]$$

Now, let us solve the second integral of [15]. Substituting $u=\ln(1-D)$ (which is $\exp(u)=1-D$, and so $\exp(u)\,du=-dD$) we get

$$\begin{aligned}(\varepsilon-1)\int\frac{\ln(1-D)}{\varepsilon(1-D)+D}dD&=-(\varepsilon-1)\int\frac{u}{\varepsilon\exp(u)+1-\exp(u)}\exp(u)\,du\\&=-(\varepsilon-1)\int u\frac{\exp(u)}{(\varepsilon-1)\exp(u)+1}\,.\end{aligned} \qquad [17]$$

Let $h=u$ and $i'=\frac{\exp(u)}{(\varepsilon-1)\exp(u)+1}$ and so (substituting $v=(\varepsilon-1)\exp(u)+1$ and $dv=(\varepsilon-1)\exp(u)\,du$)

$$i=\int\frac{\exp(u)}{(\varepsilon-1)\exp(u)+1}du=\frac{1}{\varepsilon-1}\int\frac{1}{v}dv=\frac{\ln v}{\varepsilon-1}+c=\frac{\ln\big((\varepsilon-1)\exp(u)+1\big)}{\varepsilon-1}+c\,.$$

Integrating by parts

$$\int h\,i'=h\,i-\int h'i=u\frac{\ln\big((\varepsilon-1)\exp(u)+1\big)}{\varepsilon-1}-\int\frac{\ln\big((\varepsilon-1)\exp(u)+1\big)}{\varepsilon-1}\,.$$

The latter integral term is (substituting $w=-(\varepsilon-1)\exp(u)$ and $dw=-(\varepsilon-1)\exp(u)\,du$)

$$\begin{aligned}\int h'i&=\int\frac{\ln\big((\varepsilon-1)\exp(u)+1\big)}{\varepsilon-1}du=\int\frac{\ln(1-w)}{\frac{w}{\exp(u)}}\cdot\frac{1}{(\varepsilon-1)\exp(u)}dw\\&=\frac{1}{\varepsilon-1}\int\frac{\ln(1-w)}{w}dw=-\frac{\mathrm{Li}_2(w)}{\varepsilon-1}+c=-\frac{\mathrm{Li}_2(-(\varepsilon-1)\exp(u))}{\varepsilon-1}+c\,.\end{aligned}$$

Putting together the pieces, $\int h\,i'$ becomes

$$\int h\,i'=u\frac{\ln\big((\varepsilon-1)\exp(u)+1\big)}{\varepsilon-1}+\frac{\mathrm{Li}_2(-(\varepsilon-1)\exp(u))}{\varepsilon-1}+c\,,$$

and so [17] is

[6] In the Excel calculator, the dilogarithm (Li_2) is approximated by trapezoidal rule with a step size of 10^{-5}.

$$
\begin{aligned}
(\varepsilon-1)\int \frac{\ln(1-D)}{\varepsilon(1-D)+D}\,dD &= -(\varepsilon-1)\int h\, i' \\
&= -u\ln\big((\varepsilon-1)\exp(u)+1\big) - \mathrm{Li}_2(-(\varepsilon-1)\exp(u)) + c \\
&= -\ln(1-D)\ln\big((\varepsilon-1)(1-D)+1\big) - \mathrm{Li}_2\big(-(\varepsilon-1)(1-D)\big) + c\,.
\end{aligned} \qquad [18]
$$

Substituting [16] and [18] to [15] it becomes

$$
\begin{aligned}
&\int \frac{\varepsilon\ln(1-D)}{D(\varepsilon(1-D)+D)}\,dD \\
&\quad = -\mathrm{Li}_2(D) - \ln(1-D)\ln\big(1-(1-\varepsilon)(1-D)\big) - \mathrm{Li}_2\big((1-\varepsilon)(1-D)\big) \\
&\quad + c\,.
\end{aligned} \qquad [19]
$$

And finally, writing [14] and [19] into [13], we earned the third term of [8], which is

$$
\begin{aligned}
&\int \ln\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right)\cdot\frac{D}{1-D}\,dD \\
&\quad = \ln\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right)(-D-\ln(1-D)) + \frac{\varepsilon\ln(\varepsilon(1-D)+D)}{1-\varepsilon} - \mathrm{Li}_2(D) \\
&\quad - \ln(1-D)\ln\big(1-(1-\varepsilon)(1-D)\big) - \mathrm{Li}_2\big((1-\varepsilon)(1-D)\big) + c\,.
\end{aligned} \qquad [20]
$$

The fourth term of [8] is trivial. The fifth term of [8] is (see [14])

$$
\int \frac{\varepsilon^2}{\varepsilon(1-D)+D}\,dD = \frac{\varepsilon^2\ln(\varepsilon(1-D)+D)}{1-\varepsilon} + c\,. \qquad [21]
$$

Gathering all terms from first to fifth (writing [9], [10], [20], the trivial fourth, and [21] into [8], respectively), the alarm set is

$$
\begin{aligned}
&\Bigg[(D+\ln(1-D))(m+\ln(1-m)) + \varepsilon D - \frac{\varepsilon^2\ln\big((1-\varepsilon)D+\varepsilon\big)}{1-\varepsilon} \\
&\quad - \ln\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right)(D+\ln(1-D)) + \frac{\varepsilon\ln(\varepsilon(1-D)+D)}{1-\varepsilon} - \mathrm{Li}_2(D) \\
&\quad - \ln(1-D)\ln\big(1-(1-\varepsilon)(1-D)\big) - \mathrm{Li}_2\big((1-\varepsilon)(1-D)\big) - \varepsilon m D \\
&\quad + \frac{\varepsilon^2\ln(\varepsilon(1-D)+D)}{1-\varepsilon}\Bigg]_a^d \\
&\quad = \Bigg[(D+\ln(1-D))(m+\ln(1-m)) + \varepsilon(1-m)D \\
&\quad - \ln\left(1-\frac{\varepsilon(1-D)}{\varepsilon(1-D)+D}\right)(D+\ln(1-D)) + \frac{\varepsilon\ln(\varepsilon(1-D)+D)}{1-\varepsilon} - \mathrm{Li}_2(D) \\
&\quad - \ln(1-D)\ln\big(1-(1-\varepsilon)(1-D)\big) - \mathrm{Li}_2\big((1-\varepsilon)(1-D)\big)\Bigg]_a^d ,
\end{aligned} \qquad [22]
$$

where $a = \frac{\varepsilon(1-m)}{\varepsilon(1-m)+m}$. Since δ and μ may truncate this set, the general formula for $\delta \geq 0$ and $\mu \geq 0$ is

$$[22] - \left[22 \text{ with } d \coloneqq \delta \text{ if } \delta > \frac{\varepsilon(1-m)}{\varepsilon(1-m)+m}\right] - \left[22 \text{ with } m \coloneqq \mu \text{ if } \mu > \frac{\varepsilon(1-d)}{\varepsilon(1-d)+d}\right] + \left[22 \text{ with } d \coloneqq \delta \text{ and } m \coloneqq \mu \text{ if } \delta > \frac{\varepsilon(1-\mu)}{\varepsilon(1-\mu)+\mu} \text{ and } \mu > \frac{\varepsilon(1-\delta)}{\varepsilon(1-\delta)+\delta}\right]. \quad [23]$$

$$\boxed{\text{If } e < \frac{dm}{(1-d)(1-m)}}$$

In this case, the object above e is to be subtracted. The alarm sets are then [7] – [23 with $\varepsilon \coloneqq e$] under Condition 1, and [23] – [23 with $\varepsilon \coloneqq e$] under Condition 2.

C. $\mathbf{IPR(S)} = \frac{\boldsymbol{DM-E(1-M)(1-D)}}{\boldsymbol{2(1-D)}} \geq \boldsymbol{t}$

The alarm condition rearranged to efficiency is $E \leq \frac{DM}{(1-D)(1-M)} - \frac{2t}{1-M}$, which implies that $A \neq \emptyset$ if $\varepsilon < \frac{dm}{(1-d)(1-m)} - \frac{2t}{1-m}$. Since $t > 0$, the equality holds $\Longrightarrow \varepsilon \geq 0 \wedge D > 0 \wedge M > 0$, and so it is sufficient to derive one triple integral with a general lower bound for E.

$$\boxed{\text{If } \varepsilon < \frac{dm - 2(1-d)t}{(1-d)(1-m)} \leq e}$$

The alarm set is (using rearranged forms of alarm condition, $D > \frac{E(1-M)+2t}{E(1-M)+M+2t}$, $M > \frac{(E+2t)(1-D)}{E(1-D)+D}$)

$$\int_{\frac{\varepsilon(1-m)+2t}{\varepsilon(1-m)+m+2t}}^{d} \int_{\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}}^{m} \int_{\varepsilon}^{\frac{DM}{(1-D)(1-M)}-\frac{2t}{1-M}} 1 \, dE \, dM \, dD = \int_{\frac{\varepsilon(1-m)+2t}{\varepsilon(1-m)+m+2t}}^{d} \int_{\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}}^{m} \left(\frac{DM}{(1-D)(1-M)} - \frac{2t}{1-M} - \varepsilon\right) dM \, dD .$$

The inner integral is

$$
\begin{aligned}
&\int_{\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}}^{m}\left(\frac{DM}{(1-D)(1-M)}-\frac{2t}{1-M}-\varepsilon\right)dM \\
&\qquad = \frac{D}{1-D}\left[-M-\ln(1-M)\right]_{\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}}^{m}+\left[2t\ln(1-M)\right]_{\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}}^{m} \\
&\qquad - \left[\varepsilon M\right]_{\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}}^{m} \\
&\qquad = \frac{D}{1-D}\left(-m-\ln(1-m)+\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}+\ln\left(1-\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)\right) \\
&\qquad + 2t\ln(1-m)-2t\ln\left(1-\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)-\varepsilon m+\frac{\varepsilon(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D},
\end{aligned}
$$

so the outer integral is

$$
\begin{aligned}
&\int_{a}^{d}\left(\frac{D}{1-D}(-m-ln(1-m))+\frac{(\varepsilon+2t)D}{\varepsilon(1-D)+D}+\frac{D}{1-D}\ln\left(1-\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)\right. \\
&\qquad + 2t\ln(1-m)-2t\ln\left(1-\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)-\varepsilon m \\
&\qquad \left.+\frac{\varepsilon(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)dD \\
&\qquad = \int_{a}^{d}\left(\frac{D}{1-D}(-m-\ln(1-m))+\frac{(1-\varepsilon)(\varepsilon+2t)D}{\varepsilon(1-D)+D}\right. \\
&\qquad + \frac{D}{1-D}\ln\left(1-\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)-2t\ln\left(1-\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right) \\
&\qquad \left.+ 2t\ln(1-m)-\varepsilon m+\frac{\varepsilon(\varepsilon+2t)}{\varepsilon(1-D)+D}\right)dD,
\end{aligned}
\qquad [24]
$$

where $a=\frac{\varepsilon(1-m)+2t}{\varepsilon(1-m)+m+2t}$.

I integrate [24] term by term as an indefinite integral. The first (combined) term is already known from previous calculations

$$
\int\frac{D}{1-D}(-m-\ln(1-m))\,dD=(D+\ln(1-D))(m+\ln(1-m))+c. \qquad [25]
$$

The second term of [24] comes easy from [10]

$$
\int\frac{(1-\varepsilon)(\varepsilon+2t)D}{\varepsilon(1-D)+D}dD=(\varepsilon+2t)D-\frac{\varepsilon(\varepsilon+2t)\ln\big((1-\varepsilon)D+\varepsilon\big)}{1-\varepsilon}+c. \qquad [26]
$$

The third term of [24] is

$$
\int\ln\left(1-\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)\cdot\frac{D}{1-D}dD.
$$

Let $f = \ln\left(1 - \frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)$ and $g' = \frac{D}{1-D}$ and so

$$f' = \frac{d}{dD}\ln\left(1 - \frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right) = \frac{1}{1 - \frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}} \cdot \frac{d}{dD}\left(1 - \frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right). \qquad [27]$$

The latter derivative comes easy from [12]

$$\frac{d}{dD}\left(1 - \frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right) = \frac{\varepsilon+2t}{(\varepsilon(1-D)+D)^2}. \qquad [28]$$

Writing [28] to [27]

$$\begin{aligned} f' &= \frac{1}{1 - \frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}} \cdot \frac{\varepsilon+2t}{(\varepsilon(1-D)+D)^2} \\ &= \frac{\varepsilon+2t}{(\varepsilon(1-D)+D)^2 - (\varepsilon+2t)(1-D)(\varepsilon(1-D)+D)} \\ &= \frac{\varepsilon+2t}{(\varepsilon(1-D)+D)\big(\varepsilon(1-D)+D-(\varepsilon+2t)(1-D)\big)} \\ &= \frac{\varepsilon+2t}{(\varepsilon(1-D)+D)\big((1+2t)D-2t\big)}. \end{aligned}$$

Function g comes like earlier (see [1])

$$g = \int \frac{D}{1-D}\, dD = -D - \ln(1-D) + c\,.$$

Now, integrating by parts we get

$$\begin{aligned} \int f\, g' &= f\, g - \int f' g \\ &= \ln\left(1 - \frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)(-D - \ln(1-D)) \\ &\quad - \int \frac{\varepsilon+2t}{(\varepsilon(1-D)+D)\big((1+2t)D-2t\big)}(-D - \ln(1-D))\, dD\,. \end{aligned} \qquad [29]$$

First part of the integral of [29] is

$$\int \frac{(\varepsilon+2t)D}{(\varepsilon(1-D)+D)\big((1+2t)D-2t\big)}\, dD\,.$$

For solving this integral, partial fraction decomposition is needed:

$$\frac{A}{(1+2t)D-2t} + \frac{B}{\varepsilon(1-D)+D} = \frac{(\varepsilon+2t)D}{\big((1+2t)D-2t\big)(\varepsilon(1-D)+D)}$$

$$A(\varepsilon(1-D)+D) + B\big((1+2t)D-2t\big) = (\varepsilon+2t)D\,.$$

Choosing $D=\frac{2t}{1+2t}$ we get $A=2t$, and choosing $D=-\frac{\varepsilon}{1-\varepsilon}$ we get $B=\varepsilon$. Thus, the integral of the decomposed fraction is

$$\int \frac{(\varepsilon+2t)D}{(\varepsilon(1-D)+D)\big((1+2t)D-2t\big)}dD = \int \frac{2t}{(1+2t)D-2t}dD + \int \frac{\varepsilon}{\varepsilon(1-D)+D}dD \\ = \frac{2t\ln\big((1+2t)D-2t\big)}{1+2t} + \frac{\varepsilon\ln\big((1-\varepsilon)D+\varepsilon\big)}{1-\varepsilon} + c\,. \tag{30}$$

The second part of the integral of [29] is

$$\int \frac{(\varepsilon+2t)\ln(1-D)}{(\varepsilon(1-D)+D)\big((1+2t)D-2t\big)}dD\,.$$

For the first step, we need partial fraction decomposition here, too:

$$\frac{A}{(1+2t)D-2t} + \frac{B}{\varepsilon(1-D)+D} = \frac{\varepsilon+2t}{(\varepsilon(1-D)+D)\big((1+2t)D-2t\big)}$$

$$A(\varepsilon(1-D)+D) + B\big((1+2t)D-2t\big) = \varepsilon+2t\,.$$

Choosing $D=\frac{2t}{1+2t}$ we get $A=1+2t$, and choosing $D=-\frac{\varepsilon}{1-\varepsilon}$ we get $B=\varepsilon-1$. Thus, the integral of this decomposed fraction is

$$\int \frac{(\varepsilon+2t)\ln(1-D)}{(\varepsilon(1-D)+D)\big((1+2t)D-2t\big)}dD \\ = \int \frac{(1+2t)\ln(1-D)}{(1+2t)D-2t}dD + \int \frac{(\varepsilon-1)\ln(1-D)}{\varepsilon(1-D)+D}dD\,. \tag{31}$$

Let us solve the first integral of [31]. Substituting $u=\ln(1-D)$ (which is $\exp(u)=1-D$, and so $\exp(u)\,du=-\,dD$) we get

$$(1+2t)\int \frac{\ln(1-D)}{(1+2t)D-2t}dD = -(1+2t)\int \frac{u}{(1+2t)(1-\exp(u))-2t}\exp(u)\,du \\ = -(1+2t)\int u\frac{\exp(u)}{1-(1+2t)\exp(u)}du\,. \tag{32}$$

Let $h=u$ and $i'=\frac{\exp(u)}{1-(1+2t)\exp(u)}$ and so (substituting $v=1-(1+2t)\exp(u)$ and $dv=-(1+2t)\exp(u)\,du$)

$$i = \int \frac{\exp(u)}{1-(1+2t)\exp(u)}du = -\frac{1}{1+2t}\int \frac{1}{v}dv = -\frac{\ln v}{1+2t} + c \\ = -\frac{\ln(1-(1+2t)\exp(u))}{1+2t} + c.$$

Integrating by parts

$$\int h\,i' = h\,i - \int h'i = -u\frac{\ln(1-(1+2t)\exp(u))}{1+2t} + \int \frac{\ln(1-(1+2t)\exp(u))}{1+2t}du\,.$$

The latter integral is (substituting $w = (1+2t)\exp(u)$ and $dw = (1+2t)\exp(u)\,du$)

$$\int h'i = \int \frac{\ln(1-(1+2t)\exp(u))}{1+2t}du = \int \frac{\ln(1-w)}{\frac{w}{\exp(u)}} \cdot \frac{1}{(1+2t)\exp(u)}dw$$
$$= \frac{1}{1+2t}\int \frac{\ln(1-w)}{w}dw = -\frac{\mathrm{Li}_2(w)}{1+2t} + c = -\frac{\mathrm{Li}_2\big((1+2t)\exp(u)\big)}{1+2t} + c\,.$$

Putting together the pieces, $\int h\,i'$ becomes

$$\int h\,i' = -u\frac{\ln(1-(1+2t)\exp(u))}{1+2t} - \frac{\mathrm{Li}_2\big((1+2t)\exp(u)\big)}{1+2t} + c\,,$$

and so [32] is

$$\begin{aligned}(1+2t)\int \frac{\ln(1-D)}{(1+2t)D-2t}dD &= -(1+2t)\int h\,i' \\ &= u\ln(1-(1+2t)\exp(u)) + \mathrm{Li}_2\big((1+2t)\exp(u)\big) + c \\ &= \ln(1-D)\ln\big(1-(1+2t)(1-D)\big) + \mathrm{Li}_2\big((1+2t)(1-D)\big) + c\,.\end{aligned} \quad [33]$$

The second integral of [31] is known from [18]:

$$\begin{aligned}&\int \frac{(\varepsilon-1)\ln(1-D)}{\varepsilon(1-D)+D}dD \\ &= -\ln(1-D)\ln\big(1-(1-\varepsilon)(1-D)\big) - \mathrm{Li}_2\big((1-\varepsilon)(1-D)\big) + c\,.\end{aligned} \quad [34]$$

Substituting [33] and [34] to [31], it becomes

$$\begin{aligned}&\int \frac{(\varepsilon+2t)\ln(1-D)}{(\varepsilon(1-D)+D)\big((1+2t)D-2t\big)}dD \\ &= \ln(1-D)\ln\big(1-(1+2t)(1-D)\big) + \mathrm{Li}_2\big((1+2t)(1-D)\big) \\ &\quad - \ln(1-D)\ln\big(1-(1-\varepsilon)(1-D)\big) - \mathrm{Li}_2\big((1-\varepsilon)(1-D)\big) + c\,.\end{aligned} \quad [35]$$

And finally, writing [30] and [35] into [29], we earned the third term of [24], which is

$$\begin{aligned}&\int \ln\left(1-\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)\cdot\frac{D}{1-D}dD \\ &= \ln\left(1-\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)(-D-\ln(1-D)) + \frac{2t\ln\big((1+2t)D-2t\big)}{1+2t} \\ &\quad + \frac{\varepsilon\ln\big((1-\varepsilon)D+\varepsilon\big)}{1-\varepsilon} + \ln(1-D)\ln\big(1-(1+2t)(1-D)\big) \\ &\quad + \mathrm{Li}_2\big((1+2t)(1-D)\big) - \ln(1-D)\ln\big(1-(1-\varepsilon)(1-D)\big) \\ &\quad - \mathrm{Li}_2\big((1-\varepsilon)(1-D)\big) + c\,.\end{aligned} \quad [36]$$

The fourth term of [24] is

$$\int -2t \ln\left(1 - \frac{(\varepsilon + 2t)(1 - D)}{\varepsilon(1 - D) + D}\right) dD \,. \qquad [37]$$

Let $f = \ln\left(1 - \frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)$ and $g' = 1$. From [27 and 28], we know that

$$f' = \frac{\varepsilon + 2t}{(\varepsilon(1 - D) + D)\big((1 + 2t)D - 2t\big)},$$

and so integrating by parts

$$\begin{aligned}\int f\, g' &= f\, g - \int f' g \\ &= \ln\left(1 - \frac{(\varepsilon + 2t)(1 - D)}{\varepsilon(1 - D) + D}\right) D - \int \frac{\varepsilon + 2t}{(\varepsilon(1 - D) + D)\big((1 + 2t)D - 2t\big)} D\, dD \\ &= \ln\left(1 - \frac{(\varepsilon + 2t)(1 - D)}{\varepsilon(1 - D) + D}\right) D - \frac{2t \ln\big((1 + 2t)D - 2t\big)}{1 + 2t} \\ &- \frac{\varepsilon \ln\big((1 - \varepsilon)D + \varepsilon\big)}{1 - \varepsilon} + c\,,\end{aligned} \qquad [38]$$

where, in the last step, I used the result of [30]. Finally, putting [38] to [37] we get the fourth term of [24]

$$\begin{aligned}-2t \int f\, g' = -2t \Bigg(\ln\left(1 - \frac{(\varepsilon + 2t)(1 - D)}{\varepsilon(1 - D) + D}\right) D - \frac{2t \ln\big((1 + 2t)D - 2t\big)}{1 + 2t} \\ - \frac{\varepsilon \ln\big((1 - \varepsilon)D + \varepsilon\big)}{1 - \varepsilon}\Bigg) + c\,.\end{aligned} \qquad [39]$$

The fifth (combined) term of [24] comes trivially. The sixth term of [24] is also obvious from [14]

$$\int \frac{\varepsilon(\varepsilon + 2t)}{\varepsilon(1 - D) + D} dD = \frac{\varepsilon(\varepsilon + 2t) \ln(\varepsilon(1 - D) + D)}{1 - \varepsilon} + c\,. \qquad [40]$$

Gathering all terms from first to sixth (writing [25], [26], [36], [39], the trivial fifth term, and [40] into [24], respectively), the outer definite integral becomes

$$\Big[(D+\ln(1-D))(m+\ln(1-m))+(\varepsilon+2t)D-\frac{\varepsilon(\varepsilon+2t)\ln\big((1-\varepsilon)D+\varepsilon\big)}{1-\varepsilon}$$
$$+\ln\left(1-\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)(-D-\ln(1-D))+\frac{2t\ln\big((1+2t)D-2t\big)}{1+2t}$$
$$+\frac{\varepsilon\ln\big((1-\varepsilon)D+\varepsilon\big)}{1-\varepsilon}+\ln(1-D)\ln\big(1-(1+2t)(1-D)\big)$$
$$+\mathrm{Li}_2\big((1+2t)(1-D)\big)-\ln(1-D)\ln\big(1-(1-\varepsilon)(1-D)\big)$$
$$-\mathrm{Li}_2\big((1-\varepsilon)(1-D)\big)$$
$$-2t\left(\ln\left(1-\frac{(\varepsilon+2t)(1-D)}{\varepsilon(1-D)+D}\right)D-\frac{2t\ln\big((1+2t)D-2t\big)}{1+2t}\right. \qquad [41]$$
$$\left.-\frac{\varepsilon\ln\big((1-\varepsilon)D+\varepsilon\big)}{1-\varepsilon}\right)+(2t\ln(1-m)-\varepsilon m)D$$
$$+\frac{\varepsilon(\varepsilon+2t)\ln(\varepsilon(1-D)+D)}{1-\varepsilon}\Big]_a^d,$$

where $a=\frac{\varepsilon(1-m)+2t}{\varepsilon(1-m)+m+2t}$. Since δ and μ may truncate this set, the general formula for $\delta \geq 0$ and $\mu \geq 0$ is

$$[41]-\left[41 \text{ with } d:=\delta \text{ if } \delta>\frac{\varepsilon(1-m)+2t}{\varepsilon(1-m)+m+2t}\right]$$
$$-\left[41 \text{ with } m:=\mu \text{ if } \mu>\frac{(\varepsilon+2t)(1-d)}{\varepsilon(1-d)+d}\right]$$
$$+\left[41 \text{ with } d:=\delta \text{ and } m:=\mu \text{ if } \delta>\frac{\varepsilon(1-\mu)+2t}{\varepsilon(1-\mu)+\mu+2t} \text{ and } \mu\right. \qquad [42]$$
$$\left.>\frac{(\varepsilon+2t)(1-\delta)}{\varepsilon(1-\delta)+\delta}\right].$$

$$\boxed{\text{If } e<\frac{dm-2(1-d)t}{(1-d)(1-m)}}$$

In this case, the object part above e is to be subtracted, and so the alarm set is [42] – [42 with $\varepsilon := e$].

D. $\mathbf{IPR(S)i}=\frac{\boldsymbol{DM}}{\mathbf{1}-\boldsymbol{D}-\boldsymbol{M}}\geq \boldsymbol{t}$

The alarm set is non-empty if $\frac{dm}{1-d-m}>t$ or $1-d-m<0$.

$\boxed{\text{If } \dfrac{dm}{1-d-m} > t}$

The alarm set is (using rearranged forms of alarm condition, $D > \frac{t(1-M)}{t+M}$, $M > \frac{t(1-D)}{t+D}$)

$$\int_{\frac{t(1-m)}{t+m}}^{d} \int_{\frac{t(1-D)}{t+D}}^{m} 1\, dM\, dD = \int_{\frac{t(1-m)}{t+m}}^{d} \left(m - \frac{t(1-D)}{t+D}\right) dD$$

$$= [mD - t\ln(t+D) + t(D - t\ln(t+D))]_{\frac{t(1-m)}{t+m}}^{d}$$

$$= [(m+t)D - (t+t^2)\ln(t+D)]_{\frac{t(1-m)}{t+m}}^{d}$$

$$= (m+t)d - (t+t^2)\ln(t+d) - t(1-m) + (t+t^2)\ln\left(t + \frac{t(1-m)}{t+m}\right) \qquad [43]$$

$$= (m+t)d - t(1-m) + (t+t^2)\ln\left(\frac{t+t^2}{(t+d)(t+m)}\right).$$

Since δ and μ may truncate this set, the general formula for $\delta \geq 0$ and $\mu \geq 0$ is

$$[43] - \left[43 \text{ with } d \coloneqq \delta \text{ if } \delta > \frac{t(1-m)}{t+m}\right] - \left[43 \text{ with } m \coloneqq \mu \text{ if } \mu > \frac{t(1-d)}{t+d}\right]$$
$$+ \left[43 \text{ with } d \coloneqq \delta \text{ and } m \coloneqq \mu \text{ if } \delta > \frac{t(1-\mu)}{t+\mu} \text{ and } \mu > \frac{t(1-\delta)}{t+\delta}\right]. \qquad [44]$$

$\boxed{\text{If } 1-d-m < 0}$

The overhanging part of the object is

$$\frac{\big(d-(1-m)\big)\big(m-(1-d)\big)}{2},$$

which is to be subtracted from [44] to get the alarm set.

E. IPR(H) $= \frac{\boldsymbol{DM}}{\boldsymbol{2(1-D)}} \geq \boldsymbol{t}$

The alarm set, which is non-empty if $\frac{dm}{2(1-d)} > t$, is (using rearranged forms of alarm condition, $D > \frac{2t}{2t+M}$, $M > \frac{2t(1-D)}{D}$)

$$\int_{\frac{2t}{2t+m}}^{d} \int_{\frac{2t(1-D)}{D}}^{m} 1\, dM\, dD = \int_{\frac{2t}{2t+m}}^{d} \left(m - \frac{2t(1-D)}{D}\right) dD = [mD - 2t\ln D + 2tD]_{\frac{2t}{2t+m}}^{d}$$

$$= md - 2t\ln d + 2td - \frac{2tm}{2t+m} + 2t\ln\frac{2t}{2t+m} - \frac{4t^2}{2t+m} \qquad [45]$$

$$= md + 2t\left(\ln\frac{2t}{d(2t+m)} + d - 1\right).$$

Since δ and μ may truncate this set, the general formula for $\delta \geq 0$ and $\mu \geq 0$ is

$$[45] - \left[45 \text{ with } d \coloneqq \delta \text{ if } \delta > \frac{2t}{2t+m}\right] - \left[45 \text{ with } m \coloneqq \mu \text{ if } \mu > \frac{2t(1-d)}{d}\right] + \left[45 \text{ with } d \coloneqq \delta \text{ and } m \coloneqq \mu \text{ if } \delta > \frac{2t}{2t+\mu} \text{ and } \mu > \frac{2t(1-\delta)}{\delta}\right].$$

F. GUPPI $= \boldsymbol{DM} \geq \boldsymbol{t}$

The alarm set, which is non-empty if $dm > t$, is (using rearranged forms of alarm condition, $D > \frac{t}{m}$, $M > \frac{t}{D}$)

$$\int_{\frac{t}{m}}^{d} \int_{\frac{t}{D}}^{m} 1 \, dM \, dD = \int_{\frac{t}{m}}^{d} \left(m - \frac{t}{D}\right) dD = [mD - t \ln D]_{\frac{t}{m}}^{d} = md - t \ln d - t + t \ln \frac{t}{m} = md + t\left(\ln \frac{t}{dm} - 1\right). \quad [46]$$

Since δ and μ may truncate this set, the general formula for $\delta \geq 0$ and $\mu \geq 0$ is

$$[46] - \left[46 \text{ with } d \coloneqq \delta \text{ if } \delta > \frac{t}{m}\right] - \left[46 \text{ with } m \coloneqq \mu \text{ if } \mu > \frac{t}{d}\right] + \left[46 \text{ with } d \coloneqq \delta \text{ and } m \coloneqq \mu \text{ if } \delta > \frac{t}{\mu} \text{ and } \mu > \frac{t}{\delta}\right].$$

G. Gl $= \frac{\textbf{GUPPI}}{\mathbf{2}} = \frac{\boldsymbol{DM}}{\mathbf{2}} \geq \boldsymbol{t}$

Using [46] obtained at GUPPI, the alarm set is

$$md + 2t\left(\ln \frac{2t}{dm} - 1\right) \quad [47]$$

if $dm > 2t$, else the set is empty. Since δ and μ may truncate this set, the general formula for $\delta \geq 0$ and $\mu \geq 0$ is

$$[47] - \left[47 \text{ with } d \coloneqq \delta \text{ if } \delta > \frac{2t}{m}\right] - \left[47 \text{ with } m \coloneqq \mu \text{ if } \mu > \frac{2t}{d}\right] + \left[47 \text{ with } d \coloneqq \delta \text{ and } m \coloneqq \mu \text{ if } \delta > \frac{2t}{\mu} \text{ and } \mu > \frac{2t}{\delta}\right].$$

H. Gi $= \frac{\textbf{GUPPI}}{\mathbf{1}-\boldsymbol{M}} = \frac{\boldsymbol{DM}}{\mathbf{1}-\boldsymbol{M}} \geq \boldsymbol{t}$

The alarm set, which is non-empty if $\frac{dm}{1-m} > t$, is (using rearranged forms of alarm condition, $D > \frac{t(1-M)}{M}$, $M > \frac{t}{t+D}$)

$$\int_{\frac{t(1-m)}{m}}^{d}\int_{\frac{t}{t+D}}^{m} 1\, dM\, dD = \int_{\frac{t(1-m)}{m}}^{d}\left(m - \frac{t}{t+D}\right) dD = [mD - t\ln(t+D)]_{\frac{t(1-m)}{m}}^{d}$$
$$= md - t\ln(t+d) - t(1-m) + t\ln\left(t + \frac{t(1-m)}{m}\right) \quad [48]$$
$$= md + t\left(\ln\frac{t}{m(t+d)} + m - 1\right).$$

Since δ and μ may truncate this set, the general formula for $\delta \geq 0$ and $\mu \geq 0$ is

$$[48] - \left[48 \text{ with } d \coloneqq \delta \text{ if } \delta > \frac{t(1-m)}{m}\right] - \left[48 \text{ with } m \coloneqq \mu \text{ if } \mu > \frac{t}{t+d}\right]$$
$$+ \left[48 \text{ with } d \coloneqq \delta \text{ and } m \coloneqq \mu \text{ if } \delta > \frac{t(1-\mu)}{\mu} \text{ and } \mu > \frac{t}{t+\delta}\right].$$