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Research article

Optimizing pricing and promotions for sustained profitability in declining markets: A Green-Centric inventory model

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Abstract: In the face of a competitive and ever-changing business landscape, companies often grapple with the challenge of sustaining their products in declining markets. To combat this issue, effective strategies such as promotional efforts play a pivotal role in boosting demand and maintaining market position. Additionally, businesses are increasingly focusing on ecological safety and greening efforts to minimize their environmental impact while ensuring the production of environmentally friendly products. These green initiatives not only contribute to environmental sustainability but can also enhance retailer profitability. This article presents an innovative inventory model tailored for perishable products within a stochastic environment. The model integrates elements such as linear pricing, time dynamics, promotional efforts, and a demand rate that depends non-linearly on the level of greening efforts. The model also considers partial backlogging of shortages, lost sales, time-dependent product deterioration, and investments in preservation technology to mitigate deterioration effects. The primary objective is to calculate the retailer's profit function, taking into account cycle time, selling price, promotional effort, and greening effort as key variables. To address this complex problem, the article introduces an algorithm for finding feasible solutions. Furthermore, the concavity of these solutions is demonstrated through graphical analysis. A numerical example is provided to illustrate the application of the model, and sensitivity analysis is conducted to elucidate how changes in inventory parameters impact decision variables. We will also depicted the short representation of proposed study in Figure 1.

Keywords: declining market; Price-Time-Promotional efforts and green Level-Dependent demand; pricing strategies; inventory management; stochastic demand; partial backlogging; lost sales; deterioration; preservation investments

JEL Codes: Q01, Q21

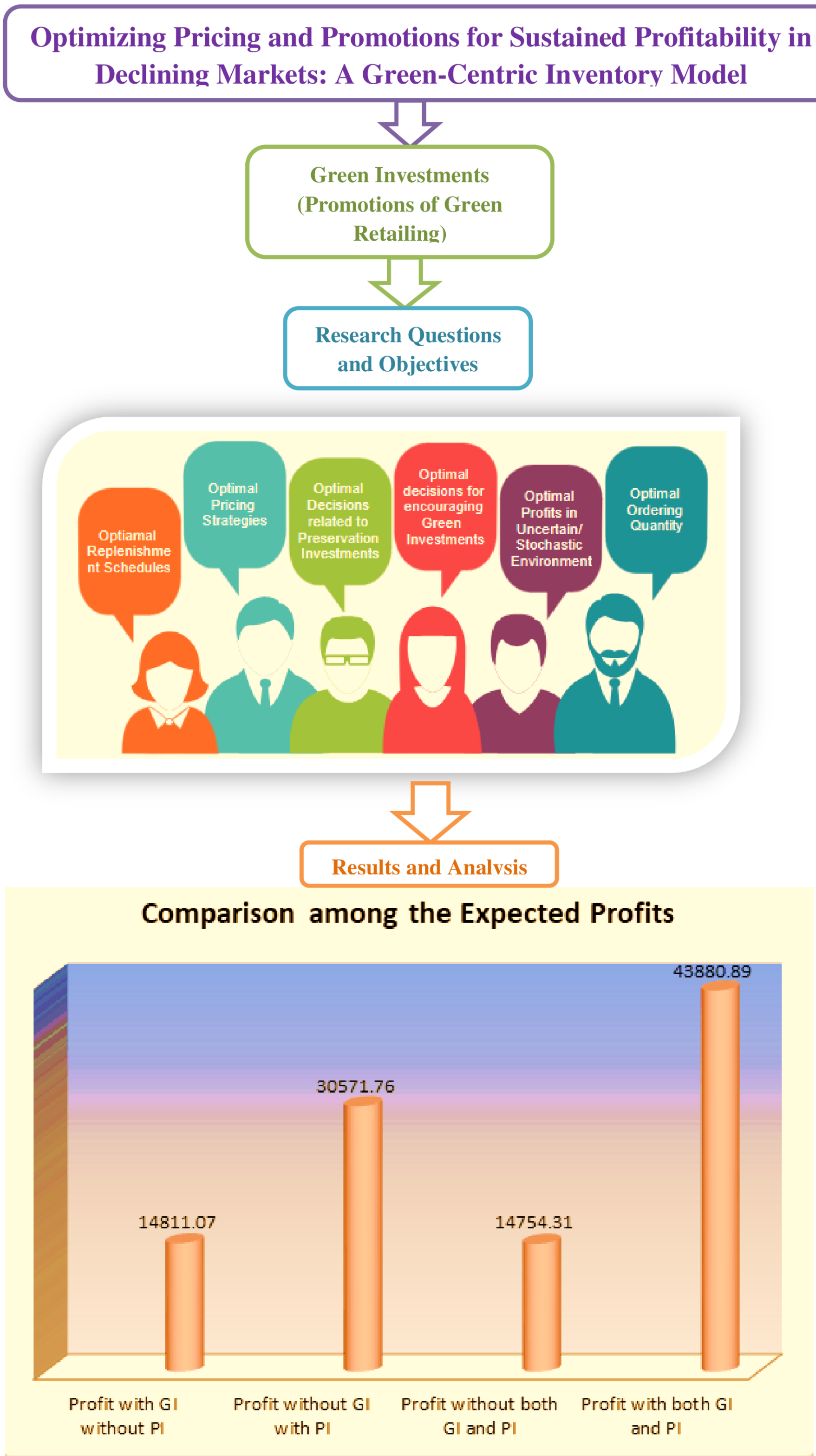


Figure 1. Graphical representation of abstract of the proposed study.

1. Introduction

1.1. Background of the proposed study

In the 21st century, environmental pollution has emerged as one of the most pressing global challenges facing humanity. Contamination of soil, water, and air is an ongoing issue, primarily driven by the unconscious actions of certain individuals and industries. This widespread pollution can be attributed, in part, to the prevalent use of harmful pesticides and chemical fertilizers, often employed to boost agricultural production swiftly. While these chemicals can lead to rapid and abundant yields, they also bring with them a host of health issues, affecting not only humans but also posing threats to various other species. This alarming trend has contributed to a concerning loss of biodiversity and environmental imbalance. In response to this environmental crisis, numerous nations are actively striving to raise public awareness about the detrimental effects of these pollution-inducing products. This awakening is gaining momentum, particularly in developed countries, where an increasing number of individuals are embracing environmental consciousness and sustainability as core principles. Consequently, the demand for green logistics, which aligns with eco-friendly practices and products, is on the rise, commensurate with the growth in environmental awareness. Many developing countries are placing a strong emphasis on conserving energy and safeguarding the environment. Notably, in India, major players in the manufacturing sector, such as Godrej, have set ambitious goals under their 'Good and Green Vision 2020 'project. Their aim is to reduce energy consumption by 25% through their green initiative, as detailed by Panja et al. (2019). Similarly, Kirloskar Brothers Limited has made significant strides, achieving a 30% reduction in specific energy consumption over the past five years, as documented by Panja et al. (2020). This highlights a growing dedication to sustainability within the manufacturing industry and Green retailing.

1.2. Significance of the proposed study

In today's business landscape, promotional effort plays a pivotal role in maintaining competition in competitive market. It serves as a highly effective tool for both product awareness and businesses visibility and bolsters product demand. Considering these factors, a model has been devised for a green retailing managers, greengrocer specializing in cultivating and selling organic vegetables. The demand for vegetables hinges on the selling price, advertising endeavours, and investments in eco-friendly practices.

This study endeavours to identify the optimal cycle duration, selling price, promotion expenditure, and eco-friendly investment amount that would maximize the green retailers overall profit. A mathematical model has been formulated and validated through a practical numerical example. Additionally, an investigation has been conducted to analyze how slight adjustments in fixed parameters linked to the model impact the decision variables crucial for maximizing total profit.

The main motivations of our paper are as follows:

1. Introducing an Economic Order Quantity model that optimizes profit by balancing ordering costs, holding costs, and shortage costs.
2. Recognizing that deterioration of inventory is time-dependent. This aspect is vital as it affects inventory management strategies and profit calculations.
3. Considering shortages in a more realistic manner, likely addressing how to manage customer

demand during shortage periods efficiently.

4. Allowing customers the option to partially backorder items during shortage periods in a standard way, providing flexibility while managing inventory.
5. Previously, most of the models were developed at constant or deterministic demand rates. However, the concept of stochastic demand has rarely been studied.
6. Many decision-making models are developed under partial backlogging. However, some of them considered partial backlogging and lost sales as shortages.
7. A lot of research models have developed with deterioration. But, how an inventory model help to obtain the maximum profit from deteriorating items through promotional efforts with partial backlogging and lost sale as shortages and declining demand (price and time-dependent) has rarely been investigated.
8. Also, the businesses should continue to invest in sustainable practices and green initiatives. The study underscores the positive impact of such investments on profitability, and as consumer demand for eco-conscious products grows, aligning with sustainability is likely to become a competitive advantage.
9. The integration of advanced technology, especially in preservation and inventory management systems, should remain a priority. The study highlights how technology can enhance product quality, reduce wastage, and contribute to overall profitability. Investing in cutting-edge solutions can offer long-term benefits.
10. The businesses should maintain flexibility in pricing strategies to adapt to changing market conditions and consumer behaviour. The pricing models that respond to price-sensitive and time-sensitive demand rates can optimize revenue while ensuring competitiveness.

Each of these elements is critical for a comprehensive inventory management model that can effectively optimize profit while considering real-world dynamics like demand fluctuations, inventory decay, and shortage handling.

2. Literature survey

2.1. Literature survey of green inventory models

In recent years, there has been a significant global focus on environmental sustainability and corporate responsibility. Businesses are increasingly integrating green initiatives into their operations to reduce environmental impact and meet the demands of eco-conscious consumers. This shift highlights the importance of green level investments (g) in supporting sustainability objectives. These investments not only promote environmental goals but also affect a company's competitiveness, brand image, and long-term viability. The practices of global retailers have undergone a substantial transformation, with green retailing (GR) emerging as a crucial element of their business strategies. In the retail sector, major players like Tesco, Carrefour, Wal-Mart, and Zara are taking proactive steps towards integrating environmental protection into their operations. Tesco, for example, places a strong emphasis on reducing carbon emissions and has set an ambitious goal of becoming a zero-carbon business by 2050 (Saha et al., 2017).

Green retailing embodies an eco-conscious management approach, prioritizing environmental safety in delivering goods and services to consumers. This encompasses a range of eco-friendly practices, from recyclable packaging to the adoption of sustainable technologies like solar panels and energy-

Figure 2. Some ways of green retailing (green initiatives).

efficient lighting. Not only does this enhance efficiency in production and distribution, but it also leads to significant waste reduction, illustrates in Figure 2. The benefits of green retailing are far-reaching. They encompass not only cost savings and health benefits but also recycling advantages and streamlined management. Raw materials for green packaging are readily accessible and tend to be more environmentally friendly than their factory-produced counterparts. They bypass many expensive factory processes and are easy to maintain, recyclable, and less harmful to the environment. This growing consumer awareness underscores the rising significance of sustainable practices in the retail industry.

This transformation is driven by the need to balance economic profitability with environmental responsibility, driven by pressures from various stakeholders, including customers, regulators, non-governmental organizations, and others. GR practices encompass a range of environmentally friendly initiatives such as energy conservation, eco-efficient environments, responsible resource utilization, employee satisfaction enhancement, waste reduction, and innovative store layouts (Lai et al., 2010; Jones et al., 2017; Chkanikova, 2015; Tang et al., 2016). Vinish et al. (2015) offered a detailed exploration of green retailing within the context of the Indian market. Their research aimed to provide a thorough understanding of the various initiatives and practices adopted by the Indian retail sector as it moves towards more environmentally conscious operations. Taking a forward-looking approach, Paul et al. (2023) introduced an innovative inventory model. This model specifically takes into account retail investments in green operations, incorporating elements such as variable holding costs. The concept of green investments has gained prominence in recent studies (Mondal et al., 2020; Hakin et al., 2022; Li et al., 2023; Paul et al., 2023; Shah et al., 2023), and these studies take into account influential market factors. The valuable contributions and research gaps in this area have been systematically summarized in tabular form.

2.2. Literature survey of deteriorating products

The management of inventory is a technique for maintaining stock items at the desired level. It encompasses various aspects, including product demand, storage capacity and maintenance, inventory decisions, strategies for remaining competitive in the market, promotional policies, quality maintenance, business reputation, financial growth, and so forth. Inventory managers strive to boost demand and sustain profitability even in a declining market. To address this challenge, we require mathematical models that address how much and when to order, while also accounting for constraints on facilities like production, storage, time, and finances. Inventory modeling is the art of formulating optimal policies while considering various realistic parameters and outputs. Numerous research efforts have been made to extend the EOQ model to account for real-world situations. Empirical observations led researchers to coin the term "deterioration".

Deterioration can result in damage, spoilage, dryness, vaporization, and so forth. The issue of deteriorating inventory has garnered significant attention in recent decades, particularly for items such as medicine, dairy products, chemicals, electronic components, radioactive substances, and so on, which begin to deteriorate once they are manufactured. Wagner et al. (1958) were the first to discuss an inventory model for deteriorating items. They empirically observed the lifespan of various items, including food commodities like fruits and vegetables, as well as some volatile liquids that deplete due to evaporation. The concept of non-instantaneous deterioration (NID) pertains to items that do not degrade immediately upon receipt by a retailer from a supplier. Instead, they maintain their freshness for a limited period, during which spoilage is minimal. This phenomenon is exemplified by perishable goods like fresh fruits and vegetables. Wu et al. (2006) introduced the NID concept, emphasizing that inventory reduction during this period is primarily driven by customer demand. Subsequent studies by Chang et al. (2010) and Chang et al. (2015) delved into optimal replenishment policies, considering stock-dependent demand and order-size-dependent delay in payment. Rabbani et al. (2017) integrated dynamic pricing and inventory control for NID items. Other researchers, including Maihami et al. (2017), Jaggi et al. (2015), and Li et al. (2019), further refined inventory models for these items, contributing significantly to the field of inventory management. These studies collectively offer valuable insights for retailers to effectively manage NID items, minimizing losses and optimizing customer satisfaction.

2.3. Literature survey in shortages (partially backlogged and lost sales)

Shortages are one of the major concerns for researchers. It refers to demands that cannot be fulfilled immediately. It is the stage or situation in which necessary items are either insufficiently available or unavailable. Some needy or impatient customers are unwilling to wait and prefer to shop elsewhere. This could lead to order cancellations and massive losses in sales. According to the literature, most inventory models were anticipated with the assumption that "shortages are permitted and completely backlogged ". For shortages, two types of inventory policies are developed: IFS (inventory followed by shortages) and SFI (shortages followed by inventory). But, many previous studies proved that the system cost of the SFI policy would be less than that of the IFS policy. After that, all the inventory modellers considered the SFI policy of replenishment in their models. The problem of Hollier et al. (1983) reconsidered by applying the SFI policy in which an exponentially declining demand is taken and it is assumed that a constant fraction θ where $0 < \theta < 1$ of the on-hand inventory deteriorates per unit of time. Sana (2010) and Soni et al. (2018) also developed their inventory models by following the SFI policy.

In the real world, backlogging is only incorporated for trendy commodities and high-tech products with short life expectancies that decrease with waiting time. Thus, the term "partial backlogging"is introduced, which is a more efficient and realistic assumption to strengthen business performance in the present market scenario. It is a process that offers an opportunity for the customer to wait for the desired product for a specific time to stay in that inventory system. It is highly dependent on the length of the delivery interval, while complete backlogging is applicable to a monopolistic market. A general and practical model was proposed for deteriorating items with time-varying demand and an appropriate time-dependent partial backlogging rate, introducing the opportunity cost of lost sales. This demonstrated the effects of changes in backlogging parameters and unit opportunity cost on total cost and the optimal number of replenishments (Wang, 2002). Abad (1996) and Abad (2001) developed pricing and ordering policies for a variable rate of deterioration and partial backlogging. The partial backlogging was assumed to be an exponential function of the time until the next replenishment. He did not add the backlogging costs and lost sales into the total profit since these costs were difficult to estimate. The obtained result was not relevant for dealing with actual market conditions.

After that, Dye (2007) obtained an optimal selling price and lot size with a varying rate of deterioration and exponential partial backlogging. They incorporated both the backorder costs and lost sales into the total profit to make the model more relevant and applicable in practice. However, in the real world, these assumptions of exponential backlogging were impractical. Shah and Shukla (2009) reconsidered Abad's model by assuming that backlogged units are proportional to waiting time, making it more realistic and applicable in practice where demand and deterioration rates were constant. In most deteriorating inventory models, the holding cost and demand rate were treated as constant functions. But, in reality, the holding cost varies with time and also the demand cannot be treated as constant. Mishra et al. (2011) developed a deterministic deteriorating inventory model in which the demand rate and holding cost were both linear functions of time, with a constant rate of deterioration, variable backlogging rate, and dependence on the next replenishment schedule. Shortages were allowed and were partially backlogged.

2.4. Literature survey of emerging marketing factors of demand rate in declining environment

Declining markets pose another significant challenge in the inventory system. As a market transitions from a period of stability, where sales remain steady or may even experience occasional boosts, to multiple periods, there is a decrease in sales. The market is a complex system influenced by numerous factors that can lead to either success or failure in selling a product. Both internal and external factors collectively contribute to the declining demand in the market. The concept of replenishment policies for deteriorating items in a declining market was pioneered by Hollier et al. (1983). They proposed two mathematical models for an inventory system in which units deteriorate at a constant rate and demand experiences exponential decline. The first model assumed that replenishment orders were placed at regular intervals, while the second model treated replenishment times as variables, eliminating the need for orders to be placed at fixed intervals. Optimal replenishment policies were determined for both cases. The demand for an item is a crucial factor that motivates producers to increase production and sellers to place more orders. Thus, demand plays a pivotal role in the success of any business Rastogi et al. (2019). It can be constant, deterministic, stochastic, or influenced by a variety of factors such as selling price, stock levels, time, various offers, advertising of new or existing goods with new approaches, seasonal needs, and so on. Several studies have recognized that demand varies with time and price independently. In recent decades, researchers like You (2005), Maihami et al. (2012), Panda et al. (2013), San-Jose L et al. (2018), and Saha et al. (2019) have investigated the combined effect of price and time on demand. Traditional inventory models assume that customer demand is predictable and remains constant over time. However, in real-world systems, demand for a product is uncertain and must be treated as a random variable. The "newsboy"or "newsvendor"problem is one of the most well-known inventory problems that assumes random demand for the product. The uncertainty in demand is determined based on the quantity of stock and the selling price.

Commonly, uncertainty in demand is considered as probabilistic or stochastic demand. The stochastic demand is defined as $D(p, \varepsilon) = d(p) + \varepsilon$ in the additive case and $D(p, \varepsilon) = d(p) \cdot \varepsilon$ in the multiplicative case, where $d(p)$ is a deterministic decreasing/increasing function that captures the dependency between demand and price, and ε is a random variable. For example, Dada et al. (2007) developed the stochastic-demand in newsvendor model to account for many unrealistic assumptions.

In inventory management, a common and effective strategy that motivates customers to buy more is the promotional effort or initiatives taken by retailers. Examples of promotional policies include gifts, price discounts, displays, allowable payment delays, special services, and advertising. The sales team in an oligopoly marketing system faces significant pressure to increase the sales of their products. Promoting the product on a global scale is beneficial as it raises product awareness and encourages more people to purchase it. A well-designed promotional strategy and its proper execution can boost product sales to new heights. It is a significant factor, so the effect of promotional efforts in inventory literature cannot be underestimated. Numerous research studies have considered promotional efforts. Decisions regarding promotional efforts and retail prices were adjusted either upward or downward in response to changes in market demand. The problem is formulated as a bivariate optimization model, and it is solved through an iterative search process [52]. They used this method to maximize the total profit, the optimal retail price, promotional effort, and replenishment quantities throughout a multi-cycle planning horizon.

Several researchers, such as Zand et al. (2019), He et al. (2009), De et al. (2013), and Dash et al. (2014), have developed models for joint decision-making on pricing, promotion, and inventory control. Maihami et al. (2014) devised an inventory model for determining the optimal pricing policy for non-instantaneous deteriorating items with promotional efforts, considering price-sensitive demand, which is stochastic in nature, and shortages were assumed to be partially backlogged. The outcomes showed a significant effect on total profit. Following that, Singh et al. (2015) presented a model for deteriorating items with production reliability and considered situations both with and without shortages. They assumed stochastic demand and used the rectangular distribution density function. They suggested that their model was applicable in various industries, such as garment businesses, leather businesses, and others. Chen et al. (2017) addressed the optimization problem for inventory replenishment, production, and promotion effects with risks of production disruption and stochastic demand for a production–retail system. Based on decentralized and centralized decision models, they obtained the optimal inventory replenishment, the optimal promotional effort of the retailer, and the production quantity of the manufacturer. Soni et al. (2018) formulated a profit maximization inventory model for deteriorating items with stochastic price-sensitive demand and promotional efforts. They

Table 1. Valuable contributions of previous studies related to present inventory model.

Authors (Year)	Weibull Deterioration	Stochastic Demand	Promotional Efforts	Preservation Investments	Green Investments	Demand Rate
Rajeswari *et al.* (2012)	+					Time
Mishra *et al.* (2011)	+					Time
Saha *et al.* (2017)				+	+	Price, Green, Reference price
Saha *et al.* (2019)				+		Price and Time
Soni *et al.* (2018)		+	+	+		Price
Soni *et al.* (2019)		+	+			Price, Promotional Efforts
Li *et al.* (2019)				+		Price
Hakim *et al.* (2022)						Price, Time
Nath *et al.* (2021)					+	Price, Time
Paul *et al.* (2023)		+		+		Price and Green Level
Manna *et al.* (2022)					+	Price, Green Level
Jauhari *et al.* (2023)		+			+	Fuzzy
Khedlekar *et al.* (2023)		+	+	+		Price
Shah *et al.* (2023)		+	+	+		Price, Stock, Advertisement
Zhang *et al.* (2008)		+				Price
Zand *et al.* (2019)		+				Price and Green level
Li *et al.* (2023)	+					Green level
This Model	**+**	**+**	**+**	**+**	**+**	**Price, Time, Green Level Promotional Efforts**

generalized time-proportional deterioration and the partial backlogging rate. Following that, Soni et al. (2019) devised an inventory model for non-instantaneous deteriorating items that allowed for price and promotional effort-sensitive stochastic demand, time and promotional effort-sensitive demand during shortages, and a time-proportional deterioration rate. The proposed pricing and inventory management decisions aim to maximize profits. Promotional efforts, stock levels, and pricing all influence market demand. From this perspective, a mathematical inventory model was formulated for deteriorating items with a constant rate of deterioration. It assumed that the supplier provides the retailer with successive discounts on goods purchases if the order size exceeds predefined quantity levels (Shah et al. (2021)).

Within supply chain models, market demand is frequently modeled to be influenced by both price and environmental considerations. Noteworthy studies on demand sensitivity to these dual factors include Panja's (2020) formulation of a two-layer green supply chain model accounting for green- and price-sensitive demand. Similarly, Ghosh *et al.* (2012) examined a green supply chain model incorporating demand sensitivity to both price and environmental considerations. Additionally, Zand *et al.* (2019) discussed a supply chain model where demand is contingent on both greening level and price sensitivity. In this study, demand is intricately tied to both the selling price and the environmental sustainability of the product, creating a dynamic influenced by consumer consciousness of green products and pricing considerations. This multifaceted demand behaviour reflects an evolving consumer awareness of the dual impact of environmental and pricing factors on purchasing decisions.

In the table (Table 1), various authors have made valuable contributions to inventory models, addressing different aspects such as stochastic demand, promotional efforts, preservation investments, and green initiatives. However, there are noteworthy research gaps that our proposed study aims to address more effectively. These gaps include the lack of models incorporating weibull deterioration, the absence of integrated approaches considering both weibull deterioration and stochastic demand simultaneously, the limited consideration of combined promotional efforts and preservation investments, and the absence of comprehensive models that encompass multiple factors like time, green level, and promotional efforts. Our proposed study seeks to provide a more comprehensive and integrated inventory model that bridges these gaps, offering a more effective and holistic approach to inventory management.

3. Materials and method

3.1. Problem statement, notations and hypotheses

3.1.1. Problem statement

In today's fiercely competitive and ever-evolving business environment, companies operating in declining markets face the formidable task of sustaining their products. To overcome this challenge, businesses must deploy effective strategies, including promotional campaigns and environmentally responsible practices, known as "greening"efforts. Greening initiatives are crucial for minimizing the environmental impact of business activities while offering eco-friendly products. Balancing promotional efforts, greening initiatives, and optimizing inventory management within a stochastic setting presents a multifaceted problem that necessitates exploration.

The central problem to be addressed is as follows:

How can companies optimize their inventory management, pricing strategies, replenishment schedules, promotional efforts, and greening initiatives for perishable products in declining markets, considering factors such as time-dependent deterioration, partial backlogging of shortages, lost sales and technology investments to control deterioration, with the ultimate goal of maximizing retailer profitability?

This problem statement underscores the complex nature of managing perishable inventory in the context of environmental responsibility and competitive markets. It underscores the necessity for a comprehensive model that integrates these variables to guide decision-making and strategy formulation for companies navigating such challenging market dynamics.

3.1.2. Notations

Following notations are used throughout this paper.

Parameters

A : The cost of ordering per order
C_o : The cost of purchasing per unit
C_h : The cost of retaining a unit for a given period of time
C_s : The cost of a backorder per unit of time
C_l : The per-unit cost of lost sales
C_d : The cost of deterioration per unit for per unit time
$I(t)$: Inventory level during the scheduling period t
ε : Non-negative and continuous random variable with $E(\varepsilon)=\mu$
w : The maximum amount of money can be put into preservation technology investments
$*$: Represents the optimal value
a : Market potential $a > b$
b : Price sensitive component of demand rate
c : Time sensitive component of demand rate
d : Greenness sensitive component of demand rate
e : Promotional efforts sensitive parameter
f : constant, $f \geq 0$
θ : Deterioration rate ($0 \leq \theta < 1$)
δ : Backlogging parameter.

Variables in Decision-Making

t_1 : The time frame for which shortages are permitted, $0 \leq t \leq t_1$
t_2 : The time when the inventory is depleted, after replenishment
ρ : The public relations effort/ promotional efforts, $\rho \geq 1$
p : Selling price of the product per unit
ϕ : Preservation Technology Investment
g : Cost of green efforts $g > 0$

Variables

Π : Total profit per unit time for the inventory system
Π_A : The optimal total average profit per unit time for the inventory system

$m(\phi)$: fraction of reduced deterioration rate of product $0 \leq m(\phi) \leq 1$

$I_s(t)$: Shortage inventory level at time t, where $0 \leq t \leq t_1$

$I_h(t)$: On hand inventory level at time t, where $t_1 \leq t \leq t_1 + t_2$

$R(p,t,g,\rho)$: Demand function (it should be noted that price, time, promotional efforts and green level-dependent function)

Q : The total ordering quantity per cycle $Q = Q_1 + Q_2$, where Q_1 is backorder quantity and Q_2 is replenishment quantity

3.1.3. Hypotheses

The following hypotheses are assist to model the proposed inventory problem:

H1: Lead time is assumed to be zero. This implies that the time between placing an order and receiving it is negligible.

H2: The order size is finite, and replenishment is assumed to be instantaneous. This assumption implies that the inventory is replenished immediately after an order is placed.

H3: Greenness is a paramount concern in today's business landscape, with our hypothesis positing that demand for a suboptimal green product relies on both price and non-linear variations in its green attributes, expressed mathematically as an intricate model capturing these dynamics and it is mathematically represented the stochastic demand function $(R(p,t,g,\rho) + \varepsilon)$ is continuous and dependent on both price, time, promotional effort and green effort. The deterministic demand function is represented as $R(p,t,g,\rho)$, where $a > 0$, $b > 0$, $c > 0$, $p > 0$, $g > 0$, $\rho > 0$ and ε is a non-negative continuous random variable with $E(\varepsilon) = \mu$. This hypothesis considers the potential demand of the product (a), the price component of demand (b) and for (e), is the parameters so chosen to best fit the demand function, d parameter to check green investment sensitivity and the time component of demand (c). For analysis of this non-linear green level model, we may refer from the previous works of Hakim *et al.* (2022)

H4: The expense of promotional efforts is modeled as an increasing function of the promotional effort. It is represented as $PC = K\rho^{\eta}$, where $K > 0$ and η is a constant. This hypothesis considers the impact of promotional efforts on the total cost and assumes that the cost increases with higher promotional efforts and demand variability.

H5: The deterioration rate in this scenario is time-dependent and follows a two-parameter Weibull distribution represented by $\theta(t) = \alpha\theta_0 t^{\theta_0 - 1}$, with $\alpha > 0$ as the scale parameter, $\theta_0 > 0$ as the shape parameter, and $t > 0$ denoting the time of deterioration. These assumptions are widely applied in various time-varying recent inventory models like Cholodowicz *et al.* (2021), Barman *et al.* (2023) and many more.

H6: In the context of preserving products, the rate of deterioration can be mitigated by allocating resources to preservation technology, denoted as ϕ. It is assumed that the function $m(\phi)$ represents the proportion of reduced deterioration rate, and this function is continuous, steadily increasing, and has a concave shape with respect to the level of preservation technology investment.

H7: During the shortage period, a portion of the demand is back-ordered, while the remaining demand is a combination of lost sales and partial backlogging. The backlogging rate is given by:

$$\beta(t) = \begin{cases} e^{-\delta t} & \text{if } \delta > 0 \\ 1 & \text{if } \delta = 0 \end{cases} \tag{3.1}$$

where $0 < \delta < 1$ is the backlogging parameter (cf. Papachristos and Skouri (2000); Dye *et al.* (2007); Sana (2010); Soni and Chauhan (2018)).

H8: During the specified time frame, no replacements or repairs are considered for perishable products.

H9: Shortages are allowed and are treated as a combination of partial backorders and lost sales in the inventory model.

These hypotheses lay the foundation for the development of the inventory model and serve as the basis for analysing the impact of various factors on inventory management decisions. By incorporating these hypotheses, the proposed study aims to provide effective and influential insights into optimizing inventory control strategies.

Impact of variables on demand

In this section, we analyze the impact of various variables on the demand rate, $R(p, t, g, \rho)$, using mathematical derivatives. We consider the variables p (selling price), t (time), g (green level investment), and ρ (promotional effort investment).

Lemma 3.1. ***Impact of*** g ***on demand***

- *First Derivative*
  The first derivative of the demand rate $R(p, t, g, \rho)$ with respect to green level investment g is given by:

$$\frac{dR(p, t, g, \rho)}{dg} = d\lambda g^{\lambda-1}$$

  As $\lambda > 0$, $\frac{dR(p,t,g,\rho)}{dg} > 0$. This implies that an increase in green level investment (g) leads to a positive increase in the demand rate (R).
- *Second Derivative*
  The second derivative of the demand rate R with respect to g is:

$$\frac{d^2R(p, t, g, \rho)}{dg^2} = d\lambda(\lambda - 1)g^{\lambda-2}$$

  a) If $\lambda > 1$, $\frac{d^2R(p,t,g,\rho)}{dg^2} > 0$. This means that for $\lambda > 1$, a further increase in g leads to an acceleration in demand rate growth.
  b) If $0 < \lambda < 1$, $\frac{d^2R(p,t,g,\rho)}{dg^2} < 0$. For $0 < \lambda < 1$, an increase in g leads to a deceleration in demand rate growth.

We also illustrate the graphical representation of Impact of green investments on demand rate in Figure 3 and Figure 4.

Lemma 3.2. ***Impact of*** ρ ***on demand***

- *First derivative*
  The impact of promotional effort investment (ρ) on the demand rate is as follows:

$$\frac{dR(p, t, g, \rho)}{d\rho} = \frac{e}{(1 + \rho)^2}$$

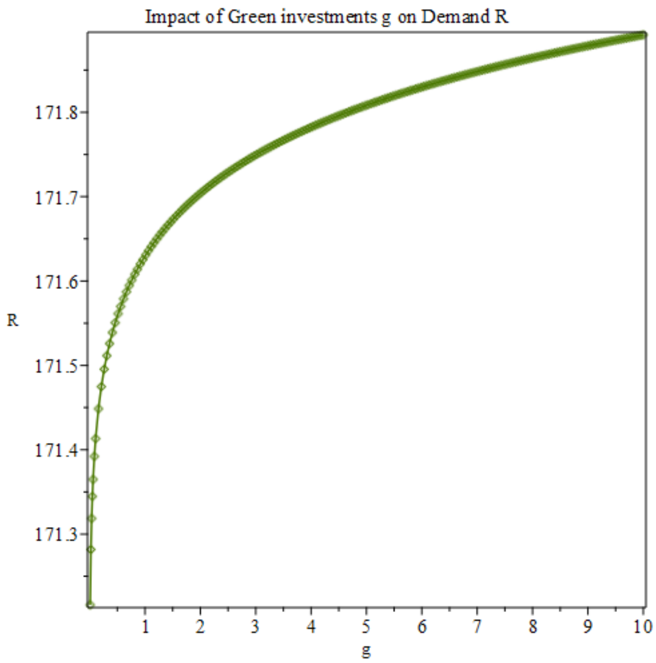


Figure 3. Graphical representation of impact of green investments g on demand rate R.

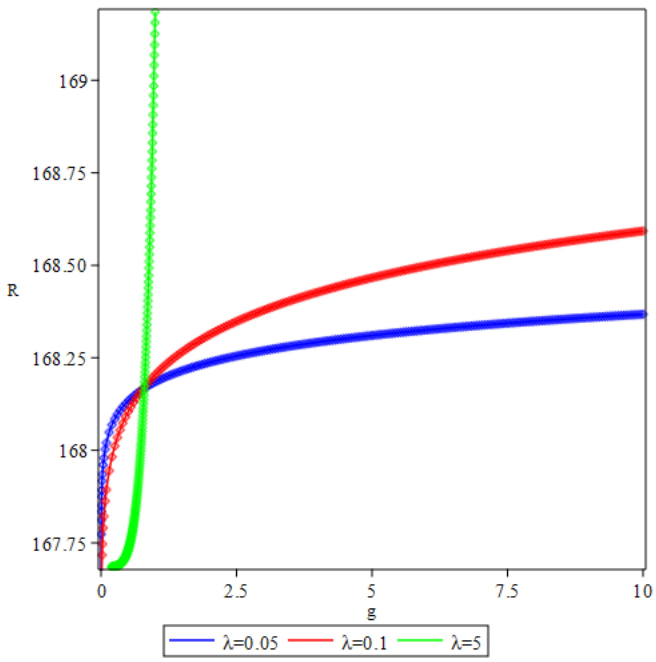


Figure 4. Graphical representation of impact of green investments parameter λ on demand rate R.

- *Second derivative*
  The second derivative of the demand rate $R(p, t, g, \rho)$ with respect to ρ is:

$$\frac{d^2R(p, t, g, \rho)}{d\rho^2} = -\frac{2e}{(1 + \rho)^3}$$

  a) *An increase in promotional effort investment (ρ) leads to a diminishing increase in the demand rate $R(p, t, g, \rho)$.*
  b) *The rate of increase in demand with respect to ρ is decreasing.*

We also illustrate the graphical representation of Impact of promotional efforts on demand rate in Figure 5.

Lemma 3.3. ***Impact of*** p ***on demand***
The impact of selling price (p) on the demand rate is as follows:

$$\frac{dR(p, t, g, \rho)}{dp} = -b$$

An increase in selling price (p) leads to a linear decrease in the demand rate $R(p, t, g, \rho)$.

Lemma 3.4. ***Impact of*** t ***on demand***
The impact of time (t) on the demand rate is as follows:

$$\frac{dR(p, t, g, \rho)}{dt} = -c$$

An increase in time (t) leads to a linear decrease in the demand rate (R).

Efficiency of the demand rate function

The demand rate function $R(p, t, g, \rho) = a - bp - ct + dg^\lambda + \frac{e\rho}{1+\rho}$ is an essential tool for optimizing business efficiency. It allows for a comprehensive analysis of how changes in pricing (p), time (t), green level investment (g), and promotional effort investment (ρ) impact revenue and resource allocation. By examining the derivatives with respect to each variable, businesses can make informed decisions on pricing strategies, time-sensitive planning, sustainability initiatives, and promotional efforts. This data-driven approach not only enhances operational efficiency but also provides a competitive advantage in adapting to dynamic market conditions and fostering long-term profitability.

Lemma 3.5. *The backlog function $\beta(t) = e^{-\delta t}$ exhibits the following behavior:*

1. *It starts at its maximum value of 1 when $t = 0$.*
2. *It is monotonically decreasing for $t > 0$.*
3. *The rate of decrease is determined by the positive constant δ.*
4. *The second derivative is positive, indicating an accelerating rate of decrease.*
5. *As t approaches infinity, $\beta(t)$ approaches zero, but never actually reaches it.*

Proof. Here, the backlog function is $e^{-\delta t}$ with $\beta(0) = 1$.

1. When $t = 0$, the backlog function starts at its maximum value of 1.

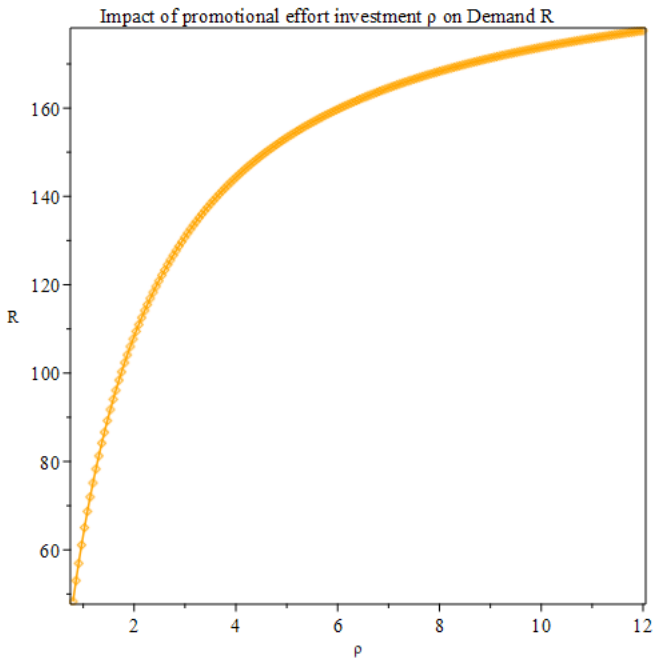


Figure 5. Graphical representation of impact of promotional efforts ρ on demand rate R.

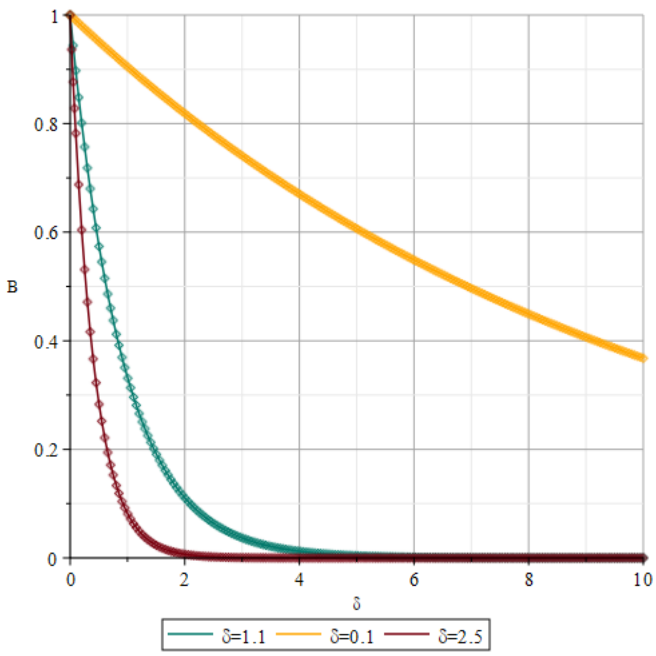


Figure 6. Graphical representation of impact of backlog parameter δ on backlog function $\beta(t)$.

2. The first derivative is $\frac{d\beta(t)}{dt} = -\delta e^{-\delta t} < 0 \;\forall\; \delta,\ t > 0$, indicating that the backlog function is monotonically decreasing for positive values of t.
3. The rate of decrease is determined by the positive constant δ. A higher δ leads to a faster decrease, while a lower δ results in a slower decrease.
4. The second derivative is $\frac{d^2\beta(t)}{dt^2} = \delta^2 e^{-\delta t} > 0 \;\forall\; \delta,\ t > 0$, indicating that the rate of decrease is accelerating over time.
5. As t approaches infinity, the backlog function approaches zero, but never actually reaches it.

Thus, the backlog function $\beta(t) = e^{-\delta t}$ exhibits the stated behaviour. We also illustrate the graphical representation of impact of backlog parameter δ on backlog function $\beta(t)$ in Figure 6. □

Lemma 3.6. *Consider a deterioration rate in this scenario, which is time-dependent and follows a two-parameter Weibull distribution represented by* $\theta(t) = \alpha\theta_0 t^{\theta_0 - 1}$, *with* $\alpha > 0$ *as the scale parameter,* $\theta_0 > 0$ *as the shape parameter, and* $t > 0$ *denoting the time of deterioration.*

Proof. We can discuss the following cases:

a) Case 1: $\theta_0 < 1$
In this case, the deterioration rate decreases over time. The system becomes more reliable as time progresses, making it suitable for applications where early reliability is critical.

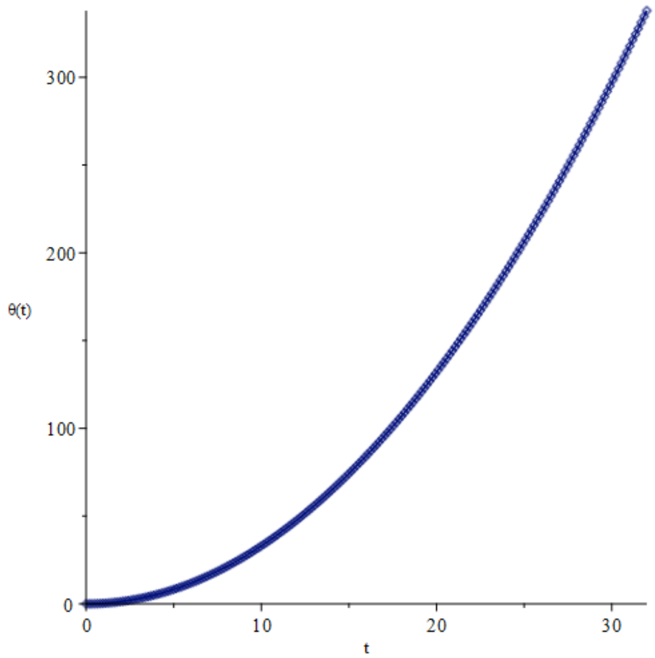


Figure 7. Graphical representation of deterioration function $\theta(t)$ with time (t).

b) Case 2: $\theta_0 = 1$
When $\theta_0 = 1$, the deterioration rate is constant over time. This represents a scenario where the system's reliability remains consistent throughout its lifetime.

c) Case 3: $\theta_0 > 1$
In this case, the deterioration rate increases over time. The system becomes less reliable as time

progresses, making it suitable for applications where long-term reliability is less critical, or where planned replacements or maintenance can be scheduled and we also illustrate graphically in Figure 7.

d) Case 4: Special Cases
When α takes specific values (e.g., $\alpha = 1$), it can lead to special cases of interest. Further analysis is needed to understand the implications of these specific values.

In this context, based on our proposed inventory problem, the optimal scenario is when $\theta_0 > 1$ and $\alpha < 1$. This implies an accelerating deterioration process with a scale parameter (α) less than 1. It is recommended to consider this scenario for our specific inventory problem. □

Lemma 3.7. *Let $PC = K\rho^{\eta}$ represent the promotional cost (PC) as a function of promotional effort (ρ), where $K > 0$ and η is a constant.*

The relationship between promotional effort and cost varies based on different values of η:
Case 1: *If $\eta > 1$, the promotional cost increases rapidly with an increase in effort.*
Case 2: *For $0 < \eta < 1$, the cost still increases but at a decreasing rate as effort rises.*
Case 3: *When $\eta = 1$, the cost increases proportionally to the effort.*
Case 4: *If $\eta < 0$, the cost decreases as effort increases, representing an inverse relationship.*
Case 5: *When $\eta = 0$, the cost remains constant regardless of promotional effort.*

To examine the rate of change in cost concerning effort, let's find the first and second derivatives:

- First Derivative
  The first derivative of PC with respect to ρ is:

$$\frac{dPC}{d\rho} = K\eta\rho^{\eta-1}$$

- Second Derivative
  The second derivative of PC with respect to ρ is:

$$\frac{d^2PC}{d\rho^2} = K\eta(\eta-1)\rho^{\eta-2}$$

Considering the need for a controlled cost increment, $\eta = 2$ or another value slightly greater than 2 could be relatively more suitable. But there is need to make balance between promotional cost and promotional effort both. We also illustrated graphically the impact of changing η on PC in Figure 8.

Lemma 3.8. *Let $m(\phi) = 1 - e^{-f\phi}$ be a preservation function, where $\phi \geq 0$ represents time, and $f > 0$ is a constant. The first derivative of $m(\phi)$ with respect to ϕ is given by*

$$\frac{dm}{d\phi} = fe^{-f\phi},$$

and the second derivative is

$$\frac{d^2m}{d\phi^2} = -f^2e^{-f\phi}.$$

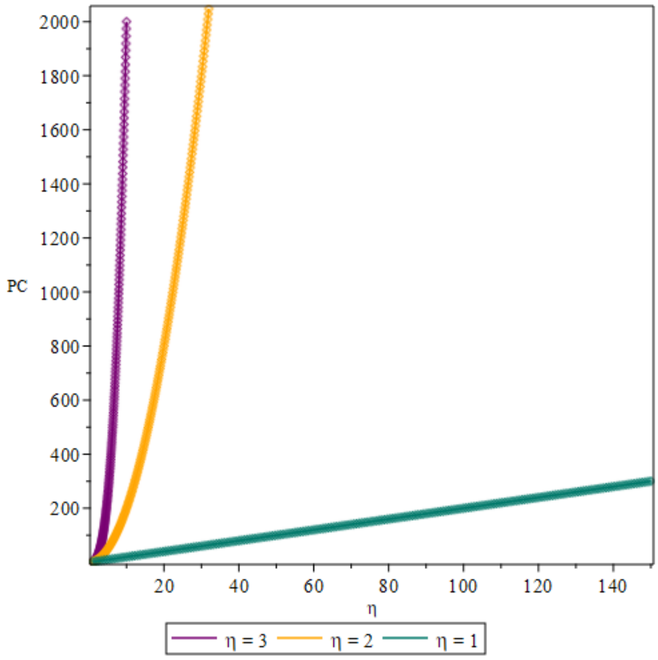


Figure 8. Graphical representation of promotional effort cost PC with promotional effort parameter η.

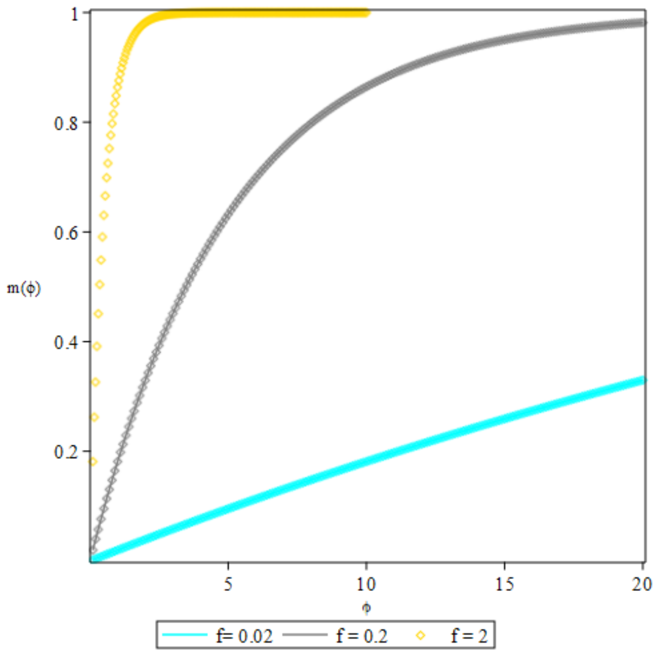


Figure 9. Graphical representation of the preservation function $m(\phi)$ with respect to f.

Proof. We begin by computing the derivatives of the preservation function $m(\phi)$ with respect to f.

- First derivative
$$\frac{dm}{d\phi} = \frac{d}{d\phi}(1 - e^{-f\phi}) = fe^{-f\phi}.$$
The first derivative $\frac{dm}{d\phi}$ represents the rate of change in preservation effectiveness, which is positive for all $\phi \geq 0$, indicating a continuous increase over time.
- Second derivative
$$\frac{d^2m}{d\phi^2} = \frac{d}{d\phi}(fe^{-f\phi}) = -f^2e^{-f\phi}.$$
The second derivative $\frac{d^2m}{d\phi^2}$ shows a negative trend for all $\phi \geq 0$, implying a decreasing rate of increase in preservation effectiveness.

We discuss following cases:

1. For f values close to zero, the preservation effectiveness increases slowly over time.
2. Moderate f values lead to a balanced increase in preservation effectiveness, neither too slow nor too fast.
3. Larger f values cause a rapid initial increase in preservation effectiveness, which gradually slows down over time. We also illustrated graphically the impact f on preservation technology function $m(\phi)$ in Figure 9.

□

3.2. Mathematical modelling and analysis

This paper studies an inventory control system for a single non-instantaneous deteriorating item with partial backlogging. We consider a scenario where the business initially faces a shortage, due to this some demand are partially backlogged which accumulates during the time interval $[0, t_1]$ and some are lost sales in the same time interval. At time t_1, an instantaneous replenishment Q is made to fulfill a portion of the backlogged demand Q_1, while the remaining demand Q_2 is fulfilled during the interval $[t_1, t_2]$, leading to the complete depletion of inventory by time t_2. Due to the non-instantaneous deteriorating characteristic of the product, there is no deterioration during a fixed time t_1. After that, the product undergoes continuous deterioration at a variable deterioration rate, where the deterioration rate $\theta(t)$ is affected by the preservation technology investment ϕ. The units of the product that have already deteriorated cannot be repaired or replaced. The demand for the product is stochastic in nature and depends on selling price, time, green level and promotional effort. The primary objective of the proposed study is to find the optimal holding period, the optimal shortage period, the optimal price, optimal investments in promotional efforts and optimal green efforts to maximize the expected total profit. We also illustrate the proposed study in Figure 10. Thus, the proposed inventory model satisfied the governed differential equation:

Inventory Dynamics

i) Shortage Inventory Dynamics
$$\frac{dI_s(t)}{dt} = -(R(p, t, g, \rho) + \varepsilon)\beta(t_1 - t), \quad 0 \leq t \leq t_1 \tag{3.2}$$

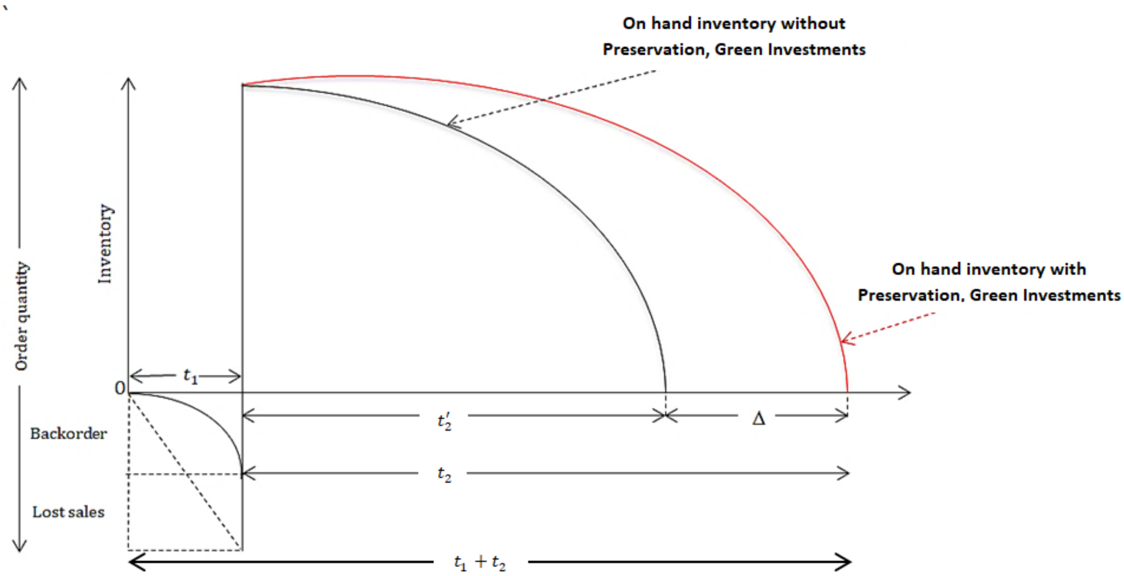


Figure 10. Graphical representation of the proposed study.

with $I_s(0) = 0$
This equation represents the rate of change of shortage inventory over time. It captures the effect of promotional efforts, demand rate, and the backlog function. The negative sign indicates that shortage inventory decreases over time. Understanding this dynamics is essential for evaluating the impact of inventory shortages and managing the back-ordering process effectively.

Shortage Inventory Level: The differential equation (3.2) yields

$$I_s(t) = -\int_0^t \left[(R(p,t,g,\rho) + \varepsilon)\beta(t_1 - t)\right] dt \tag{3.3}$$

This expression calculates the level of shortage inventory at any given time within the range $[0, t_1]$. It integrates the demand rate, promotional efforts, and backlog function. Analyzing this level helps in understanding the magnitude of inventory shortages and their implications on customer satisfaction and potential lost sales.

ii) On-Hand Inventory Dynamics:

$$\frac{dI_h(t)}{dt} = -\theta(t)(1 - m(\phi))I_h(t) - (R(p,t,g,\rho) + \varepsilon), \quad 0 \le t \le t_2 \tag{3.4}$$

with $I_h(t_2) = 0$ This equation describes the rate of change of on-hand inventory over time. It considers factors such as the promotional efforts, deterioration rate, and demand rate. The negative sign indicates that on-hand inventory depletes over time. Understanding this dynamics is crucial for managing inventory replenishment and avoiding stock-outs.

On-Hand Inventory Level

$$I_h(t) = e^{-g(t)(1-m(\phi)}\left(\int_0^{t_2}\left[(R(p,t,g,\rho)+\varepsilon)e^{g(x)(1-m(\phi)}\right]dx\right) \tag{3.5}$$

where,

$$g(t) = \int_0^t \theta(x)\,dx$$

This expression calculates the level of on-hand inventory at any given time within the range $[0, t_2]$. It incorporates the effect of promotional efforts, time-dependent demand, and deterioration. Analyzing this level helps in optimizing inventory levels, minimizing holding costs, and ensuring product availability.

Cost Analysis

i) Total Size of Replenishment

$$Q = I_h(0) - I_s(t_1) = \left[e^{-g(0)(1-m(\phi)}\left(\int_0^{t_2}(R(p,t,g,\rho)+\varepsilon)e^{g(x)(1-m(\phi)}dx\right)\right] + \left[\int_0^{t}(R(p,t,g,\rho)+\varepsilon)\beta(t_1-t)\,dt\right]$$

This expression calculates the total size of replenishment required to fulfill both on-hand and backlog inventory. It considers the demand rate, promotional efforts, and the time intervals $[0, t_1]$ and $[0, t_2]$. Analyzing the replenishment size aids in effective inventory management and maintaining optimal stock levels.

ii) Total Cost of Lost Sales

$$TLC = E\left(C_l\int_0^{t_1} I_l(t)dt\right) = C_1\int_0^{t_1}(R(p,t,g,\rho)+\mu)\,(1-\beta(t_1-t))\,dt$$

where, the amount of money lost on sales at time t is

$$I_l(t) = (R(p,t,g,\rho)+\varepsilon)(1-\beta(t_1-t)), \qquad 0 \le t \le t_1$$

This expression calculates the expected cost of lost sales during the time interval $[0, t_1]$. It considers the cost per lost sale, the demand rate, and the backlog function. Analyzing this cost helps in understanding the financial impact of stockouts and optimizing inventory levels to minimize lost sales.

iii) Total Cost of Stock-Out Backlogs

$$TSC = E\left(C_s\int_0^{t_1}[-I_s(t)]dt\right) = C_s\left[\int_0^{t_1}\left\{\int_0^{t}(R(p,x,g,\rho)+\mu)\beta(t_1-x)dx\right\}dt\right]$$

This expression calculates the expected cost associated with stock-out backlogs during the time interval $[0, t_1]$. It considers the cost per unit of backlog and the backlog fulfillment process.

Analyzing this cost helps in evaluating the financial implications of backorders and optimizing order fulfillment strategies.

iv) Total Inventory Holding Cost

$$THC = E\left(C_h \int_0^{t_2} I_h(t)dt\right) = C_h \left[\int_0^{t_2} e^{g(x)(1-m(\phi)} \left\{\int_t^{t_2} (R(p,x,g,\rho) + \mu)e^{g(x)(1-m(\phi)}dx\right\} dt\right]$$

This expression calculates the expected cost of holding inventory over the time interval $[0, t_2]$. It considers the cost per unit of inventory, the deterioration rate, and the on-hand inventory level. Analyzing this cost helps in optimizing inventory levels and balancing holding costs with the risk of stockouts.

v) Overall Cost of Purchase

$$TPC = E(C_0 * Q) = C_0 e^{g(0)(1-m(\phi)} \left\{\int_0^{t_2} (R(p,x,g,\rho) + \mu)e^{g(x)(1-m(\phi)}dx + \int_0^{t_1} \beta(t_1 - t)(R(p,x,g,\rho) + \mu)dt\right\}$$

This expression calculates the overall cost of purchasing inventory, including both on-hand and backlog replenishment. It considers the cost per unit of purchase, the replenishment quantity, and the time intervals $[0, t_1]$ and $[0, t_2]$. Analyzing this cost provides insights into the overall procurement expenses and aids in optimizing purchasing decisions.

vi) Total Revenue from Sales

$$\begin{aligned} TRV = \quad & E\left(p\int_0^{t_2} (R(p,x,g,\rho) + \mu)dx + (-I_s(t_1)\right) \\ & = p\left\{\int_0^{t_2} (R(p,x,g,\rho) + \mu)dx \right. \\ & \left. + \int_0^{t_1} (R(p,x,g,\rho) + \mu)\beta(t_1 - x)dx\right\} \end{aligned}$$

This expression calculates the expected revenue generated from sales during the time intervals $[0, t_1]$ and $[0, t_2]$. It considers the unit selling price, the demand rate, and the impact of stockouts. Analyzing the revenue helps in assessing the financial performance of the inventory system and optimizing pricing and sales strategies.

vii) Cost of Promotional Efforts

$$PC = K\rho^{\eta}$$

This expression calculates the expected cost of promotional efforts over the time interval $[0, t_2]$. Analysing this cost helps in evaluating the effectiveness and financial impact of promotional activities.

viii) Total Expected Deterioration Cost

$$TDC = E\left(C_d \int_0^{t_2} \theta(t) I_h(t) dt\right) = C_d \int_0^{t_2} \theta(t) e^{g(t)(1-m(\phi)} \left(\int_t^{t_2} (R(p, x, g, \rho) + \mu) e^{(1-m(\phi))g(x)} dx\right) dt$$

This expression calculates the expected cost associated with the deterioration of inventory during the time interval $[0, t_2]$. It considers the cost per unit of deterioration, the deterioration rate, and the on-hand inventory level. Analyzing this cost helps in managing inventory quality, minimizing losses due to spoilage or damage, and optimizing inventory replenishment strategies.

ix) Total Preservation Cost (TPTC)

$$TPTC = \phi(t_1 + t_2)$$

Total Preservation Technology Cost ($TPTC$) represents the cumulative cost associated with preserving the product throughout its shelf life, which includes both the shortage period (t_1) and the replenishment period (t_2). It takes into account the Preservation Technology Investment (ϕ) made to extend the product's shelf life and the total time period ($t_1 + t_2$) for which preservation is required. This cost component is essential for inventory managers to consider, as it directly impacts the overall cost structure of the inventory system. Effective preservation technology investments can lead to reduced TPTC and, consequently, lower total inventory costs.

x) Total Green Technology Investment (TGTI) The formula for TGTI is as follows:

$$TGTI = \alpha g(t_1 + t_2)$$

Total Green Technology Investment ($TGTI$) represents the cumulative investment made in green or environmentally sustainable technologies throughout the entire inventory cycle. It is calculated by multiplying the Green Investment Sensitivity Parameter (α), the level of green efforts or technology adoption (g), and the total time period ($t_1 + t_2$) for which these green efforts are in place.

This component reflects the commitment of the organization to environmentally friendly practices in its inventory management. Higher values of $TGTI$ indicate a greater investment in green technologies and sustainability initiatives. Inventory managers should assess the trade-offs between green investments and potential cost savings to determine the optimal level of green technology adoption.

3.3. Total profit function

The total profit function, denoted as $\Pi(p, t_1, t_2, g, \rho, \phi)$ represents the overall profitability of the inventory system. It is calculated by subtracting the sum of various costs from the total expected revenue from sales (TRV). The costs include the cost of stock-out backlogs (TSC), the cost of lost sales (TLC), the inventory holding cost (TIC), the cost of purchase (TPC), the cost of promotional efforts (PC), the cost of deterioration (TDC), the cost of total preservation investments ($TPTC$) and the cost of total green technology investments ($TGTI$).

$$\Pi(p, t_1, t_2, g, \rho, \phi) = TRV - (A + TSC + TLC + TIC + TPC + PC + TDC + TPTC + TGTI)$$

$$
\begin{aligned}
\Pi(p,t_1,t_2,g,\rho,\phi) = &\left[p\left(\int_0^{t_2}(R(p,x,g,\rho)+\mu)\,dx+\int_0^{t_1}(R(p,x,g,\rho)+\mu)\beta(t_1-x)\,dx\right)\right]\\
&-\left[A+C_1\int_0^{t_1}(R(p,t,g,\rho)+\mu)\,dt\right]\left[1-\int_0^{t_1}\beta(t_1-x)\,dx\right]\\
&+C_h\left[\int_0^{t_2}e^{g(x)(1-m(\phi))}\left(\int_t^{t_2}(R(p,x,g,\rho)+\mu)e^{g(x)(1-m(\phi))}\,dx\right)dt\right]\\
&+C_d\int_0^{t_2}\theta(t)e^{g(t)(1-m(\phi))}\left(\int_t^{t_2}(R(p,x,g,\rho)+\mu)e^{(1-m(\phi))g(x)}\,dx\right)dt\\
&+C_s\left[\int_0^{t_1}\left(\int_0^{t}(R(p,x,g,\rho)+\mu)\beta(t_1-x)\,dx\right)dt\right]\\
&+C_0e^{g(0)(1-m(\phi))}\left[\int_0^{t_2}(R(p,x,g,\rho)+\mu)e^{g(x)(1-m(\phi))}\,dx\right.\\
&\quad\left.+\int_0^{t_1}\beta(t_1-t)(R(p,x,g,\rho)+\mu)\,dt\right]\\
&+\phi(t_1+t_2)+\alpha g(t_1+t_2)
\end{aligned}
$$

Therefore, the total average profit is

$$
\Pi_A(p,t_1,t_2,g,\rho,\phi)=\frac{\Pi(p,t_1,t_2,g,\rho,\phi)}{(t_1+t_2)} \tag{3.6}
$$

where, $\Pi(t_1,t_2,p,g,\rho)$ represents the total profit of the inventory system over the time period (t_1+t_2).

This equation quantifies the average profitability of the inventory system by dividing the total profit by the total time duration. It provides a normalized measure that reflects the average financial performance of the system over the entire time period.

3.4. Solution procedure

To solve the mathematical model presented in this study, a classical optimization method is applied using analytical techniques. The problem is to maximize the total average profit $\Pi_A(t_1,t_2,p,g,\rho,\phi)$ with respect to the decision variables t_1,t_2,p,g,ρ and ϕ. subject to the constraints $C \leq p$, and $t_1,t_2,p,g,\rho,\phi \geq 0$

The objective function $\Pi_A(t_1,t_2,p,g,\rho,\phi)$ is maximized with respect to the decision variables t_1 and t_2 , while p is fixed. The problem can be expressed as:

$$
Maximum\ \ \Pi_A(t_1,t_2,p,g,\rho,\phi) \tag{3.7}
$$

The aim is to find the optimal value of p that maximizes the total average profit over all possible values of t_1 and t_2. Throughout the optimization process, rigorous mathematical techniques are employed, including the analysis of derivatives, critical points, and the Hessian matrix, to ensure a comprehensive exploration of the objective function and the determination of globally optimal solutions. By following this standard solution procedure, the study aims to obtain robust and reliable results that optimize the total average profit and meet the specified constraints. Problem (2.10) is notably characterized by its highly nonlinear nature involving four decision variables. Analytically demonstrating its optimality proves to be a complex challenge. Therefore, the proposed model is effectively tackled using the MAPLE software.

Optimality

The primary objective of the study is to maximize profit by jointly optimizing the order replenishment cycle time ($t_1 + t_2$), the allocation of investment in preservation technology (ϕ), the consideration of environmental impact (green efforts, g), promotional efforts (ρ), and the selling price (p).

For simplification, we expand $e^{(1-m(\phi))g(t)}$ and ignoring the second and higher powers of $(1-m(\phi))g(t)$.

$$
\begin{aligned}
\Pi_A(p,t_1,t_2,g,\rho,\phi) = \ & \Big[p\Big(\int_0^{t_2}(R(p,x,g,\rho)+\mu)\,dx+\int_0^{t_1}(R(p,x,g,\rho)+\mu)\beta(t_1-x)\,dx\Big)\Big] \\
& -\Big[A+C_1\int_0^{t_1}(R(p,t,g,\rho)+\mu)\,dt\Big]\Big[1-\int_0^{t_1}\beta(t_1-x)\,dx\Big] \\
& +C_h\Big[\int_0^{t_2}e^{g(x)(1-m(\phi))}\Big(\int_t^{t_2}(R(p,x,g,\rho)+\mu)e^{g(x)(1-m(\phi))}\,dx\Big)\,dt\Big] \\
& +C_d\int_0^{t_2}\theta(t)e^{g(t)(1-m(\phi))}\Big(\int_t^{t_2}(R(p,x,g,\rho)+\mu)e^{(1-m(\phi))g(x)}\,dx\Big)\,dt \\
& +C_s\Big[\int_0^{t_1}\Big(\int_0^{t}(R(p,x,g,\rho)+\mu)\beta(t_1-x)\,dx\Big)\,dt\Big] \\
& +C_0e^{g(0)(1-m(\phi))}\Big[\int_0^{t_2}(R(p,x,g,\rho)+\mu)e^{g(x)(1-m(\phi))}\,dx \\
& +\int_0^{t_1}\beta(t_1-t)(R(p,x,g,\rho)+\mu)\,dt\Big] \\
& +\phi(t_1+t_2)+\alpha g(t_1+t_2)
\end{aligned} \tag{3.8}
$$

Computational Steps	Description
Step 1	Allocate values of the parameters other than decision variables in equation (3.8).
Step 2	Differentiate equation (3.8) with respect to t_1, t_2, p, g, ρ, and ϕ to obtain first-order partial derivatives.
Step 3	Set the first-order partial derivatives to zero and solve for $(t_1, t_2, p, g, \rho, \phi)$ equation (3.8):

1. To optimize the shortage inventory cycle time t_1:

$$\frac{\partial \Pi_A(p, t_1, t_2, g, \rho, \phi)}{\partial t_1} = 0 \tag{3.9}$$

2. To optimize the holding inventory cycle time t_2:

$$\frac{\partial \Pi_A(p, t_1, t_2, g, \rho, \phi)}{\partial t_2} = 0 \tag{3.10}$$

3. To optimize the selling price p:

$$\frac{\partial \Pi_A(p, t_1, t_2, g, \rho, \phi)}{\partial p} = 0 \tag{3.11}$$

4. To optimize the environmental impact (green effect) g:

$$\frac{\partial \Pi_A(p, t_1, t_2, g, \rho, \phi)}{\partial g} = 0 \tag{3.12}$$

5. To optimize promotional efforts ρ:

$$\frac{\partial \Pi_A(p, t_1, t_2, g, \rho, \phi)}{\partial \rho} = 0 \tag{3.13}$$

6. To optimize the investment in preservation technology ϕ:

$$\frac{\partial \Pi_A(p, t_1, t_2, g, \rho, \phi)}{\partial \phi} = 0 \tag{3.14}$$

Step 4	Use the obtained values in equation (3.8) to find the total profit.
Step 5	To assess the concavity of the profit function, we examined the Hessian matrix, as depicted below.It is sufficient condition for the profit function:

$$\begin{bmatrix}
\frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial p^2} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial p t_1} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial p t_2} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial p g} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial p \rho} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial p \phi} \\
\frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial t_1 p} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial t_1^2} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial t_1 t_2} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial t_1 g} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial t_1 \rho} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial t_1 \phi} \\
\frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial t_2 p} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial t_2 t_1} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial t_2^2} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial t_2 g} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial t_2 \rho} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial t_2 \phi} \\
\frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial g p} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial g t_1} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial g t_2} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial g^2} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial g \rho} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial g \phi} \\
\frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial \rho p} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial \rho t_1} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial \rho t_2} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial \rho g} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial \rho^2} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial \rho \phi} \\
\frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial \phi p} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial \phi t_1} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial \phi t_2} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial \phi g} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial \phi \rho} & \frac{\partial^2\Phi(p,t_1,t_2,g,\rho,\phi)}{\partial \phi^2}
\end{bmatrix}$$

For this matrix, we computed the principal minors, denoted as H_1, H_2, H_3, H_4, H_5 and H_6. It must satisfy the conditions $H_1 < 0$, $H_2 > 0$, $H_3 < 0$, $H_4 > 0$, $H_5 < 0$ and $H_6 > 0$ to check the concavity of proposed expected profit function.

4. Numerical illustration

To demonstrate the applicability of the proposed model and the solution procedure developed in this article, let's consider a practical numerical example.

Example 1: Managing perishable goods in an online grocery store

Consider an online grocery store that specializes in providing fresh produce, dairy products, and perishable items to its customers. This store prides itself on delivering high-quality, organic products to environmentally conscious consumers. However, due to the nature of perishable goods, efficient inventory management is difficult task to minimize waste and meet customer demand.

The perishable goods, which include organic fruits and vegetables, have a limited shelf life and begin to deteriorate over time. Their deterioration rate follows a two-parameter Weibull distribution with $\alpha = 0.1$ and $\theta_0 = 3$. To control the deterioration process, we employed preservation technology $m(\phi) = (1 - e^{-f\phi})$ and limitation for investment in preservation is predefined i.e. $w = \$50$. Given the competitive online grocery market, pricing and promotional efforts significantly influence customer buying behavior. To accurately model customer demand, the store uses a stochastic demand equation:

$$(R(p, t, g, \rho) + \varepsilon) = 620 - 40p - 2.5t + 1.5g^{\lambda} + 270\left(\frac{\rho}{1+\rho}\right) + \varepsilon$$

Here, $\varepsilon \sim N(20, 1)$ represents the random component of demand when a product is temporarily out of stock.

To effectively manage the inventory of perishable goods, the store must take decisions carefully that about when to reorder, how much to order, and at what price to offer these products. The cost structure includes a \$4 cost per unit for ordering perishable goods, \$100 per order as the ordering cost, and \$2 per unit per year for holding costs. Additionally, the estimated opportunity cost of lost sales for perishable goods is \$12 per unit, and the shortage cost is \$8 per unit per year. The promotional cost function is described with constants $K = 2$ and $\eta = 3$.

Applying the proposed computational method to these parameters and constraints, the optimized solution is as follows:

$$\begin{aligned} t_1^* &= 4.53\ days \\ t_2^* &= 30.84\ days \\ g^* &= \$5.99\ (approx\ 6) \\ \rho^* &= 9.20\ (approx\ 9) \\ \phi^* &= \$19.62 \end{aligned}$$

Consequently, the optimal selling price for perishable goods is $p^* = \$15.21$. With this pricing and inventory management strategy tailored for perishable items in the online grocery store, the store can expect to achieve an expected total profit of $\Pi_A(p, t_1, t_2, g, \rho, \phi) = \$43880.89$ per unit of time. The optimal order quantity per cycle for perishable goods is $Q^* = 7452.31$ units.

For the concavity of the profit function, the required conditions are:

$$H_1 < 0, \quad H_2 > 0, \quad H_3 < 0, \quad H_4 > 0, \quad H_5 < 0, \quad \text{and} \quad H_6 > 0.$$

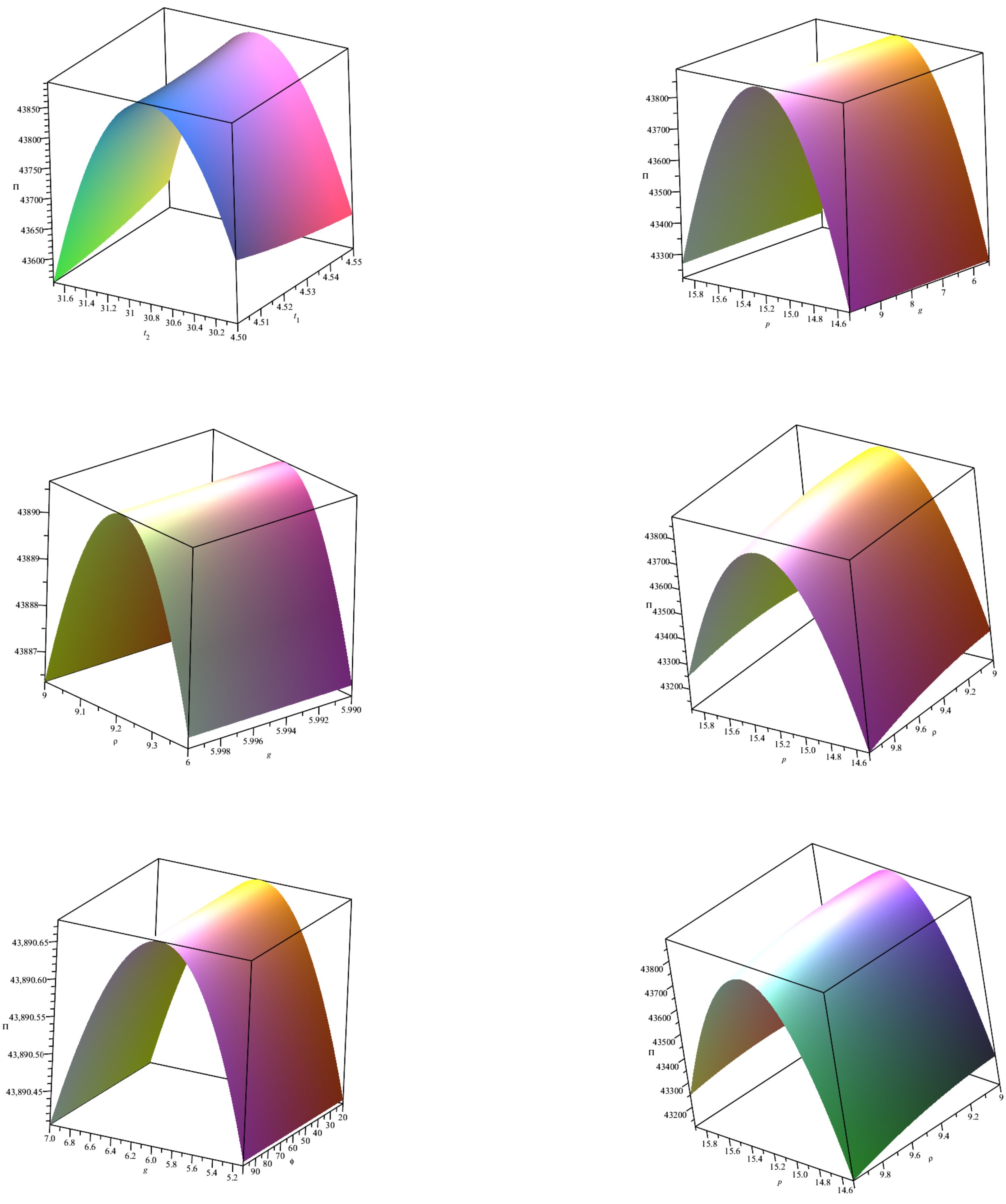


Figure 11. Concavity of profit $\Pi_A(p, t_1, t_2, g, \rho, \phi)$ with different key parameters combinations.

Table 2. Computational results for different distributions scenario of expected random demand.

$E(\varepsilon) = \mu$	p	t_1	t_2	g	ρ	ϕ	$\Pi_A(p, t_1, t_2, g, \rho, \phi)$	Q
$N \sim (2, 1)$	14.95	4.72	30.03	5.77	9.07	17.48	40442.71	7048.89
$N \sim (6, 1)$	15.01	4.67	30.21	5.82	9.10	17.97	41191.42	7137.41
$N \sim (9, 1)$	15.05	4.64	30.34	5.86	9.12	18.33	41758.67	7204.23
$U \sim (12, 8)$	15.06	4.63	30.39	5.87	9.13	18.45	41948.85	7226.58
$exp(12)$	15.09	4.61	30.47	5.89	9.14	18.69	42330.84	7271.41

Due to the complexity of the calculation, we used Maple 2021 software to obtain values of the minors. Thus, by substituting the values of the decision variables and solving the above minors, we obtain:

$$
\begin{aligned}
H_1 &= -4505.12 < 0,\\
H_2 &= 8.49 \times 10^7 > 0,\\
H_3 &= -2.44 \times 10^{11} < 0,\\
H_4 &= 8.46 \times 10^{11} > 0,\\
H_5 &= -1.17 \times 10^8 < 0,\\
H_6 &= 1.49 \times 10^{16} > 0.
\end{aligned}
$$

Therefore, based on these determinants and their signs, we can conclude that the profit function is concave with respect to the obtained values of the decision variables. We also illustrated the concave behaviour of profit function with respect to all the decision variables in Figure 11.

The table (Table 2) presents a set of scenarios related to expected random demand, denoted as $E(\varepsilon) = \mu$, for a deteriorating product. Each row represents a distinct scenario with specific parameter values (p, t_1, t_2, g, ρ, ϕ) and calculates two outcomes, $\Pi_A(p, t_1, t_2, g, \rho, \phi)$ and Q. These outcomes likely hold significance for decision-making or analysis in the context of the deteriorating product. The table demonstrates how varying the parameters across scenarios affects both expected demand and the resulting calculated values. For instance, as the mean demand (μ) changes, it influences the calculated outcomes, showing how different demand scenarios impact the system's performance. Similarly, variations in other parameters such as p, t_1, t_2, g, ρ, and ϕ also play a role in determining the final results. The calculated values $\Pi_A(p, t_1, t_2, g, \rho, \phi)$ and Q represent important metrics or measures of performance in each scenario. Analyzing these variations and their sensitivity to parameter changes can provide valuable insights into how to manage and make decisions regarding the deteriorating product under different conditions. Overall, this table serves as a comprehensive tool for understanding the relationships between parameters, expected demand, and system outcomes in various scenarios.

Proposition 1. *For given feasible $p^*, t_1^*, t_2^*, \rho^*, \phi^*$, $\Pi_A(g|p^*, t_1^*, t_2^*, \rho^*, \phi^*)$ is strictly concave in g.*

Proof. We have, from eq^n.(3.8), taking second derivative for $\Pi_A(p, t_1, t_2, g, \rho, \phi)$ with respect to g, it is found that:

$$\frac{\partial \Pi_A^2}{\partial g^2} = -\lambda d\,(-1+\lambda)\,g^{\lambda-2}\Bigg[-\frac{\alpha^2}{3}\left(\left(\beta-\frac{5}{8}\right)(f\phi-1)\,\alpha+\frac{\beta}{8}-\frac{1}{8}\right)\beta^2\,(f\phi-1)\,C_d\delta^2 t_2^4\,(-t_1)^{\beta-1}$$
$$-\frac{C_d\alpha^3\beta^3\delta^2 t_2^5\,(f\phi-1)^2\,(\beta-1)\,(-t_1)^{\beta-2}}{15}-\frac{C_d\alpha\beta\delta^2 t_2^2}{2}\,(\alpha f\phi-\alpha+1)\,(\alpha f\phi-\alpha-1)\,(-t_1)^{\beta+1}$$
$$-\frac{2\alpha}{3}\left(\left(\beta-\frac{1}{4}\right)(f\phi-1)^2\,\alpha^2+\frac{\beta}{4}\,(f\phi-1)\,\alpha-\frac{\beta}{4}+\frac{1}{4}\right)\beta C_d\delta^2 t_2^3\,(-t1)^{\beta}$$
$$+\,t_1^2\,(((p-C_s)\,t_1-C_1-C_o)\,\delta-C_s)\,e^{-\delta t_1}\Bigg]<0 \tag{4.1}$$

Therefore, $\Pi_A(p,t_1,t_2,g,\rho,\phi)$ is concave function of p,t_1,t_2,g,ρ,ϕ where p,t_1,t_2,g,ρ,ϕ can be solved by following Computational Steps. Numerically we also found that $\frac{\partial \Pi_A^2}{\partial g^2}=-0.60<0$ This completes the proof. We also show the graphical representation (Figure 12) of the concavity of profit function with respect to g. □

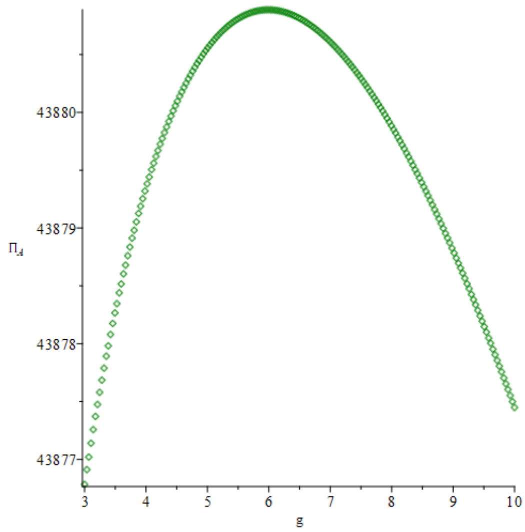


Figure 12. Concavity of the profit function $\Pi_A(p,t_1,t_2,g,\rho,\phi)$ with green investments g.

Proposition 2. *For given feasible* $g^*,t_1^*,t_2^*,\rho^*,\phi^*$, $\Pi_A(p|g^*,t_1^*,t_2^*,\rho^*,\phi^*)$ *is strictly concave in* p.

Proof. We have, from eq^n(3.8), taking second derivative for $\Pi_A(p,t_1,t_2,g,\rho,\phi)$ with respect to p, it is found that:

$$\frac{\partial \Pi_A^2}{\partial p^2} = -\left[2\,(\exp(-\delta t_1)t_1-t_2\delta-t_1)\,b/\delta\right]<0 \tag{4.2}$$

Therefore, $\Pi_A(p,t_1,t_2,g,\rho,\phi)$ is concave function of p,t_1,t_2,g,ρ,ϕ where p,t_1,t_2,g,ρ,ϕ can be solved by following Computational Steps. Numerically we also found that $\frac{\partial \Pi_A^2}{\partial p^2}=-2612.27<0$. This completes the proof. We also show the graphical representation (Figure 13) of the concavity of profit function with respect to p. □

Proposition 3. *For given feasible* $g^*,t_1^*,t_2^*,p^*,\phi^*$, $\Pi_A(\rho|g^*,t_1^*,t_2^*,p^*,\phi^*)$ *is strictly concave in* ρ.

Proof. We have, from $eq^n(3.8)$, taking second derivative for $\Pi_A(p, t_1, t_2, g, \rho, \phi)$ with respect to ρ, it is found that:

$$\begin{aligned}
\frac{\partial \Pi_A^2}{\partial \rho^2} = & \frac{-1}{60\delta^2(1+\rho)^3 t_1^2}\Big[40\delta^2\beta^2 C_d t_2^4 e(f\phi - 1)\alpha^2(-t_1)^{\beta-1} \\
& + 8\delta^2 e t_2^5 (f\phi - 1)^2\alpha^3\beta^3(-t_1)^{\beta-2} \\
& + 60\delta^2 e t_2^2(-t_1)^{1+\beta}\alpha\beta(\alpha f\phi - \alpha + 1)(\alpha f\phi - \alpha - 1) \\
& - 120\delta t_1^2 e \exp(-\delta t_1)((p - Cs)t_1 - C1 - Co) \\
& - 80\delta^2\beta C_d t_2^3 e\alpha(-t_1)^{\beta}\Big[(f\phi - 1)^2\alpha^2(\beta - \frac{1}{4}) \\
& + \frac{\beta}{4}(f\phi - 1)\alpha - \frac{\beta}{4} + \frac{1}{4}\Big] \\
& - 60K\delta^2 t_1^2\eta(\eta - 1)\Big[\rho^{\eta-2} + 3\rho^{\eta-1} + \rho^{\eta+1} + 3\rho^{\eta}\Big] \\
& - \frac{1}{2}\delta^2\Big[\frac{t_1^2\beta^2\alpha^2 C_h(f\phi - 1)^2 t_2^3}{4} \\
& - t_1^2\alpha t_1\beta(f\phi - 1)(1 + (f\phi - 1)\alpha)C_h t_2^2 \\
& - t_1^2(p - C_o)t_2\Big]\Big] < 0
\end{aligned} \tag{4.3}$$

Therefore, $\Pi_A(p, t_1, t_2, g, \rho, \phi)$ is concave function of $p, t_1, t_2, g, \rho, \phi$ where $p, t_1, t_2, g, \rho, \phi$ can be solved by following Computational Steps. Numerically we also found that $\frac{\partial \Pi_A^2}{\partial \rho^2} = -210.06 < 0$. This completes the proof. We also show the graphical representation (Figure 14) of the concavity of profit function with respect to ρ. ◻

Similarly, we can observe that the concavity of the profit function significantly depends on the remaining two decision variables, t_2 and ϕ, as illustrated in Figure 15 and Figure 16.

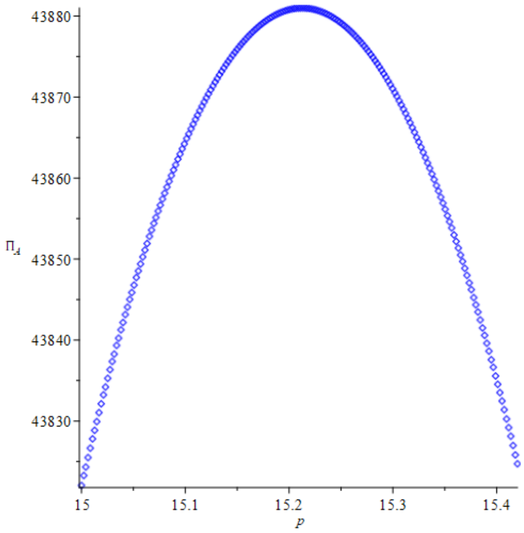


Figure 13. Concavity of the profit function $\Pi_A(p, t_1, t_2, g, \rho, \phi)$ with selling price p.

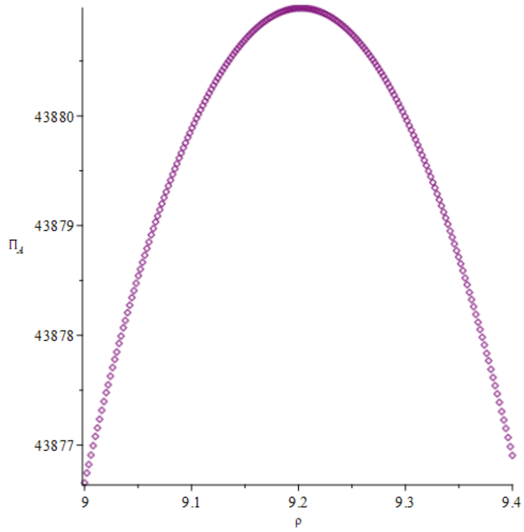


Figure 14. Concavity of the profit function $\Pi_A(p, t_1, t_2, g, \rho, \phi)$ with promotional efforts ρ.

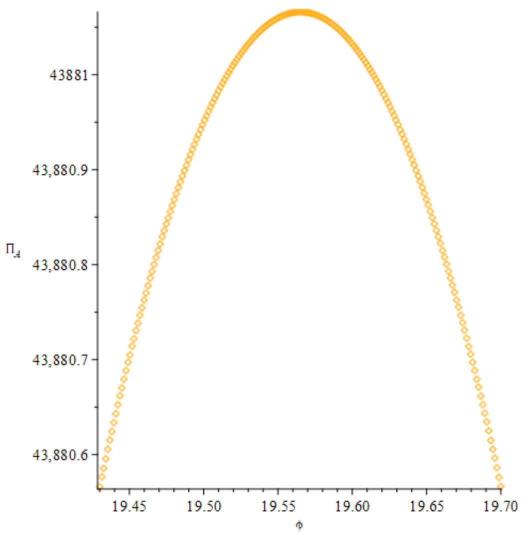


Figure 15. Concavity of the profit function $\Pi_A(p, t_1, t_2, g, \rho, \phi)$ with preservation technology ϕ.

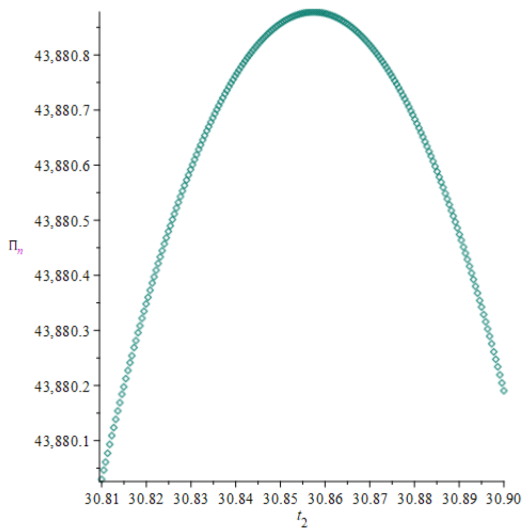


Figure 16. Concavity of the profit function $\Pi_A(p, t_1, t_2, g, \rho, \phi)$ with replenishment time t_2.

Sensitivity analysis and managerial insights

We use same dataset as in above example to perform the sensitivity analysis of optimal decision making parameters and associated the expected total profit to change the values of other inventory decision parameters which associated with proposed model. We illustrates the sensitivity analysis of the key parameters graphically in Figure 17 and also shows the changes % in Table 3.

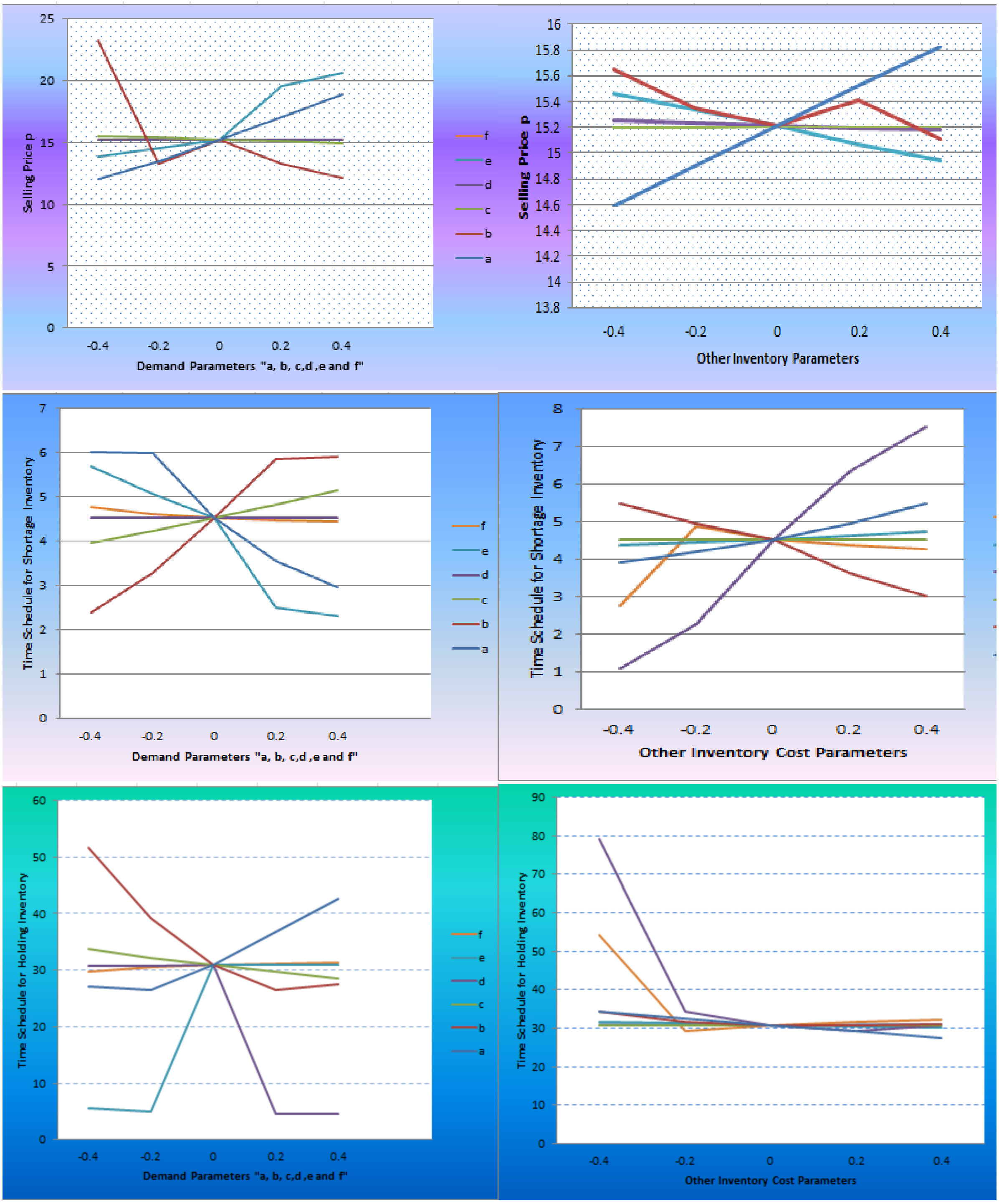
4.1. Sensitivity analysis of t_1 with respect to the all assumed inventory parameters

The sensitivity analysis examines how various parameters influence the permissible time frame for shortages (t_1). Market potential (a) plays a pivotal role, with a decrease from 0.4 to -0.4 resulting in an increase in t_1. This underscores the importance of recognizing market potential in effectively managing shortages. Price sensitivity (b) also significantly impacts t_1, as a decrease from 2.96 to 2.4 leads to a decrease in the permissible time frame. Understanding this influence is crucial for making informed decisions regarding pricing strategies and their impact on shortages. The time sensitivity component (c) exhibits a relatively minor effect on t_1 across scenarios, indicating that variations in time sensitivity have minimal impact on shortage management. Similarly, the greenness sensitivity component (d) and generic parameter (f) demonstrate limited influence on t_1.

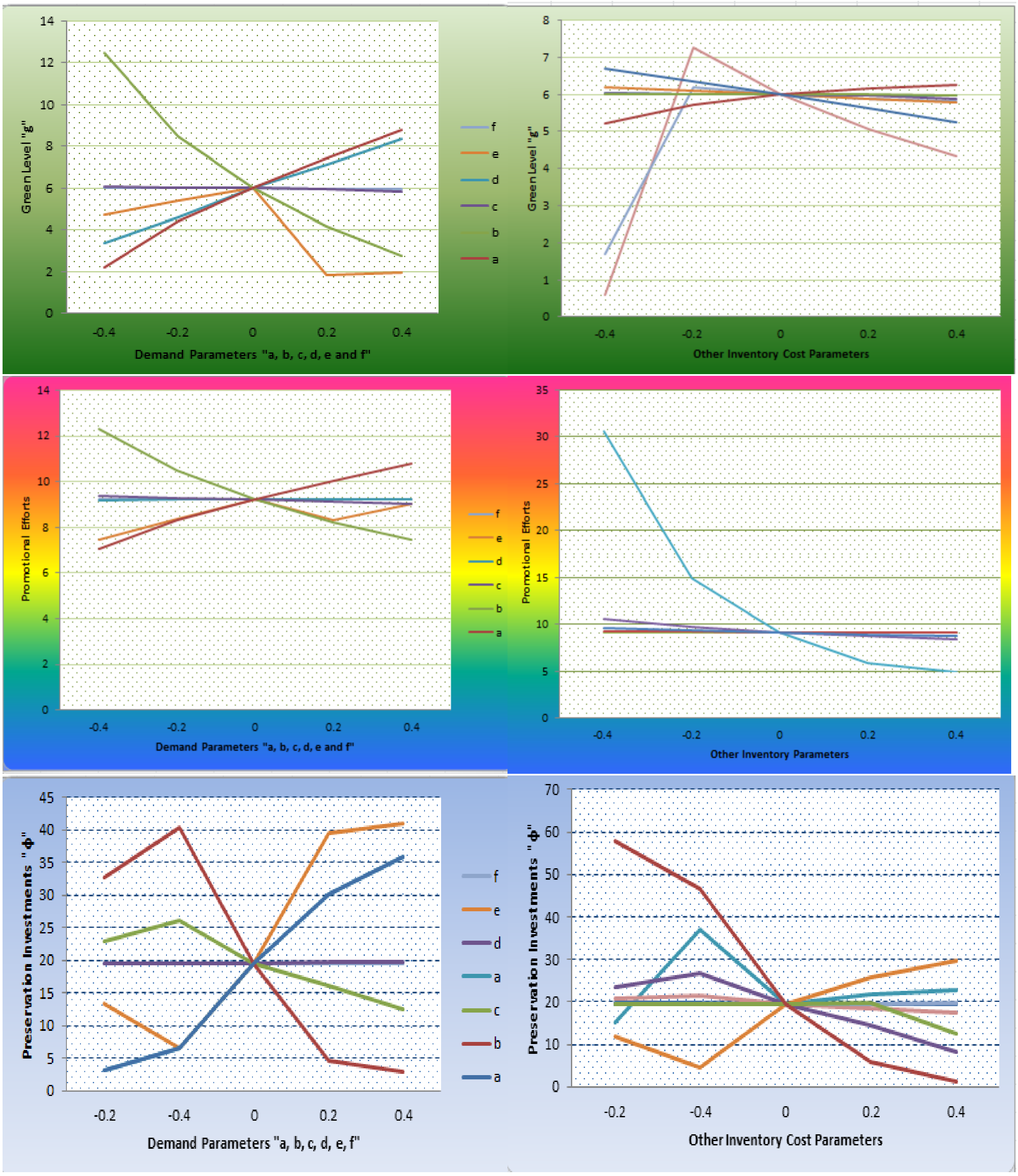
Moving on to the managerial insights, the sensitivity analysis provides valuable guidance for optimizing shortage management. The retailer should order as early as possible to avoid backlogging costs and lost sales. These shortage costs reduce the desired profit. The other opportunity provided by model is that to increase the replenishment size or order earlier than existing, since both shortage period (t_1) and replenishment period (t_2) are reduced continuously.

4.2. Sensitivity analysis of t_2 with respect to the all assumed inventory parameters

The behaviour of the replenishment time (t_2) in inventory management is closely tied to various parameters, each exerting its influence. Ordering Costs (A) and Purchasing Costs (C_o) directly affect t_2—higher costs lead to shorter replenishment cycles to minimize expenses. Retention Costs (C_h) similarly influence t_2, with higher holding costs promoting shorter replenishment times. Market Potential (a) plays a role in demand dynamics. A higher market potential often leads to shorter t_2 as businesses strive to meet increased demand promptly. Price-Sensitive Demand Rate (b) and Time-Sensitive Demand Rate (c) both push for shorter t_2 when demand sensitivity is high. Greenness-Sensitive Demand Rate (d) introduces environmental considerations into inventory decisions, potentially affecting t_2 based on green product preferences. Balancing Backorder Costs (C_s) and Lost Sales Costs (C_l) is a pivotal challenge. High backorder costs can lead to shorter t_2 to minimize stockouts, while high lost sales costs may prompt shorter cycles to avoid revenue losses. Deterioration Costs (C_d) impact t_2 as well. Higher deterioration costs may encourage shorter cycles to reduce product wastage. Promotional efforts (ρ) and preservation technology constant (f) can introduce fluctuations in t_2 as promotional campaigns and technology investments influence demand and product shelf life. Lastly, the backlogging parameter (δ) affects how shortages and backorders are managed during high-demand periods, potentially influencing t_2 decisions. Understanding these dynamics is crucial for inventory managers as they fine-tune replenishment strategies to strike a balance between cost optimization and meeting customer demands effectively.



Selling Price p
Demand Parameters "a, b, c,d ,e and f"
f
e
d
c
b
a
Selling Price p
Other Inventory Parameters
Time Schedule for Shortage Inventory
Demand Parameters "a, b, c,d ,e and f"
f
e
d
c
b
a
Time Schedule for Shortage Inventory
Other Inventory Cost Parameters
Time Schedule for Holding Inventory
Demand Parameters "a, b, c,d ,e and f"
f
e
d
c
b
a
Time Schedule for Holding Inventory
Other Inventory Cost Parameters


Green Level "g"
Demand Parameters "a, b, c, d, e and f"
Other Inventory Cost Parameters
Promotional Efforts
Demand Parameters "a, b, c, d, e and f"
Other Inventory Parameters
Preservation Investments "φ"
Demand Parameters "a, b, c, d, e, f"
Other Inventory Cost Parameters
f
e
d
c
b
a

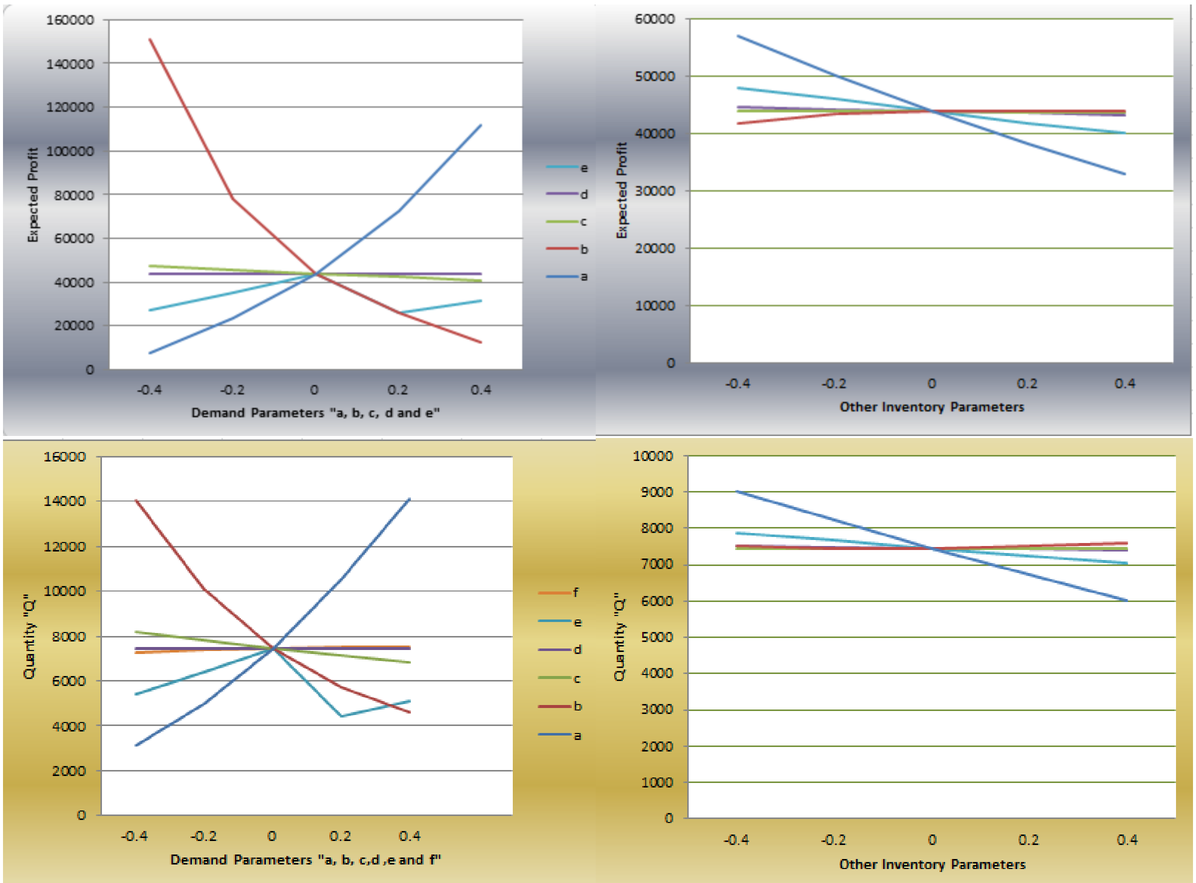


Figure 17. Sensitivity analysis of profitability and quantity with respect to changes in parameters.

It reveals that above analysis requires synchronized t_2 adjustments to maximize the effectiveness of marketing strategies. Retailer needs strategic investments in constant parameter of preservation technology (f) offer opportunities for cost savings and improved inventory control. They can extend product shelf life, potentially leading to longer t_2 periods. Lastly, skilfully managing backlogging and shortages during high-demand periods, influenced by the backlogging parameter (δ), ensures customer satisfaction and minimizes potential revenue losses. These insights empower inventory managers to refine strategies, achieving cost-efficiency, improved customer service, and operational excellence.

4.3. Sensitivity analysis of selling price (p) with respect to the all assumed inventory parameters

In this sensitivity analysis, we examine how the selling price (p) responds to changes in various parameters, shedding light on its behavior in different scenarios. Firstly, market potential (a) significantly influences the selling price. When market potential is high, businesses can often command higher selling prices (p) to capitalize on strong demand. Conversely, in less favorable market conditions, adjusting the selling price downwards may be necessary to stimulate sales. Price-sensitive demand rate (b) and time-sensitive demand rate (c) play a vital role in pricing decisions. An increase in price sensitivity or time sensitivity may prompt businesses to carefully evaluate and, if needed, revise their selling prices to remain competitive. Greenness-sensitive demand rate (d) reflects consumer preferences for environmentally friendly products. A higher greenness sensitivity can justify premium selling prices (p) for eco-friendly goods, aligning with the growing demand for sustainability. Promotional efforts (e) and preservation technology investments constant (f) can also impact selling prices. Effective marketing campaigns and investments in product quality and preservation technology can support higher selling prices. Moreover, the costs associated with inventory management and operations are essential considerations. Ordering costs (K), purchasing costs (λ), and other operational expenses may necessitate adjustments in selling prices (p) to ensure profitability.

Managers must continuously monitor these parameters to strike a balance between maximizing revenue and meeting customer expectations. Greenness-sensitive demand rate (d) provides an opportunity for environmentally conscious product positioning. Understanding this sensitivity empowers businesses to justify higher selling prices (p) for eco-friendly offerings, appealing to environmentally conscious consumers. Promotional efforts (e) remain a powerful tool for influencing demand and pricing. Effective marketing campaigns can support higher selling prices, provided they align with customer perceptions of value. Additionally, preservation technology investments (f), being constant, underline the importance of long-term product quality and shelf life. This investment can support higher prices for products with extended freshness or durability.

4.4. Sensitivity analysis of green level investment (g) with respect to the all assumed inventory parameters

In this sensitivity analysis, we delve into how the green level investment (g) responds to changes in various parameters, shedding light on its behavior across different scenarios. Market potential (a) holds a significant sway over green level investments. In markets with a high potential for eco-friendly products, businesses may allocate more resources to green initiatives, aiming to meet consumer demand for sustainability. Conversely, in markets where green products have limited appeal, a more conservative approach to green level investments may be prudent. Price-sensitive demand rate (b) and

Table 3. Sensitivity analysis of proposed study.

Parameters		**Change (%) in Decision Variables**				**Parameters**		**Change (%) in Decision Variables**			
		-0.4	**-0.2**	**0.2**	**0.4**			**-0.4**	**-0.2**	**0.2**	**0.4**
a	$\Delta p/p\%$	-20.77	-11.24	12.03	24.26	δ	$\Delta p/p\%$	22.49	-0.53	0.26	0.46
	$\Delta g/g\%$	-63.10	-26.21	23.70	46.41		$\Delta g/g\%$	-71.95	3.17	-1.84	-3.17
	$\Delta\rho/\rho\%$	-23.47	-09.34	8.91	17.28		$\Delta\rho/\rho\%$	-71.95	3.17	-1.83	-3.17
	$\Delta\phi/\phi\%$	-66.56	-32.82	53.46	82.72		$\Delta\phi/\phi\%$	89.34	-22.53	10.19	16.10
	$\Delta t_1/t_1\%$	32.89	32.00	-21.41	-34.65		$\Delta t_1/t_1\%$	-38.63	7.50	-3.53	-5.52
	$\Delta t_2/t_2\%$	-12.02	-14.39	19.06	38.39		$\Delta t_2/t_2\%$	75.77	-4.89	-90.06	4.24
	$\Delta Q/Q\%$	-57.88	-33.13	41.44	89.39		$\Delta Q/Q\%$	-49.98	-2.43	1.11	1.76
	$\Delta\Pi_A/\Pi_A\%$	-59.52	-46.38	65.8	154.85		$\Delta\Pi_A/\Pi_A\%$	-50.24	3.03	-1.91	-3.25
b	$\Delta p/p\%$	52.59	19.98	-12.75	-20.18	f	$\Delta p/p\%$	-0.26	-0.13	0.06	0.06
	$\Delta g/g\%$	108.68	41.4	-29.89	-54.09		$\Delta g/g\%$	0.50	0.16	-0.16	-0.33
	$\Delta\rho/\rho\%$	33.58	14.13	-10.43	-18.80		$\Delta\rho/\rho\%$	-0.43	-0.21	0.10	0.20
	$\Delta\phi/\phi\%$	106.32	66.51	-76.5	-85.42		$\Delta\phi/\phi\%$	41.08	18.45	-14.01	-24.71
	$\Delta t_1/t_1\%$	-47.01	-27.15	29.35	30.46		$\Delta t_1/t_1\%$	5.51	1.98	-1.32	-1.98
	$\Delta t_2/t_2\%$	67.5	26.84	-13.78	-16.99		$\Delta t_2/t_2\%$	-3.40	-1.23	0.77	1.32
	$\Delta Q/Q\%$	88.67	35.56	-23.09	-37.83		$\Delta Q/Q\%$	-2.24	-0.79	0.49	0.84
	$\Delta\Pi_A/\Pi_A\%$	244.23	77.27	-41.29	-71.13		$\Delta\Pi_A/\Pi_A\%$	-0.96	-0.38	0.26	0.46
c	$\Delta p/p\%$	-8.79	-4.50	4.63	9.38	η	$\Delta p/p\%$	1.64	0.85	-0.92	-1.77
	$\Delta g/g\%$	108.68	41.4	-29.89	-54.09		$\Delta g/g\%$	3.50	1.83	-1.83	3.67
	$\Delta\rho/\rho\%$	33.58	14.13	-10.43	-18.80		$\Delta\rho/\rho\%$	232.82	62.93	-29.78	-46.08
	$\Delta\phi/\phi\%$	106.32	66.51	-76.5	-85.42		$\Delta\phi/\phi\%$	9.83	5.35	-5.65	-11.26
	$\Delta t_1/t_1\%$	-47.01	-27.15	29.35	30.46		$\Delta t_1/t_1\%$	-3.75	-1.98	2.20	-4.19
	$\Delta t_2/t_2\%$	67.5	26.84	-13.78	-16.99		$\Delta t_2/t_2\%$	2.59	1.39	-1.39	-2.69
	$\Delta Q/Q\%$	88.67	35.56	-23.09	-37.83		$\Delta Q/Q\%$	5.45	2.89	-2.88	-5.56
	$\Delta\Pi_A/\Pi_A\%$	244.23	77.27	-41.29	-71.13		$\Delta\Pi_A/\Pi_A\%$	9.60	4.86	-4.49	-8.44
d	$\Delta p/p\%$	-0.06	-0.06	0	0.07	K	$\Delta p/p\%$	0.26	0.13	-0.13	-0.19
	$\Delta g/g\%$	-42.40	-21.36	-8.23	0.17		$\Delta g/g\%$	0.66	0.33	-0.16	-0.50
	$\Delta\rho/\rho\%$	-0.10	0.00	8.65	16.97		$\Delta\rho/\rho\%$	14.56	6.08	-4.78	-8.69
	$\Delta\phi/\phi\%$	0.35	-0.20	13.60	25.38		$\Delta\phi/\phi\%$	1.73	0.81	-0.66	-1.27
	$\Delta t_1/t_1\%$	0.00	0.00	-0.22	-1.32		$\Delta t_1/t_1\%$	-0.66	-0.44	0.22	0.44
	$\Delta t_2/t_2\%$	-0.06	-0.06	0.03	0.78		$\Delta t_2/t_2\%$	0.42	0.19	-0.19	-0.32
	$\Delta Q/Q\%$	-0.21	-0.10	15.09	31.47		$\Delta Q/Q\%$	0.92	0.41	-0.35	-0.66
	$\Delta\Pi_A/\Pi_A\%$	244.23	77.27	-41.29	-71.13		$\Delta\Pi_A/\Pi_A\%$	1.72	0.77	-0.66	-1.23
e	$\Delta p/p\%$	-8.79	-4.5	4.63	9.38	λ	$\Delta p/p\%$	0.00	0.00	0.00	0.00
	$\Delta g/g\%$	-34.31	-25.39	-8.23	0.17		$\Delta g/g\%$	-44.40	-23.20	25.37	53.58
	$\Delta\rho/\rho\%$	-19.14	-9.19	8.65	16.97		$\Delta\rho/\rho\%$	0.00	0.00	0.00	0.00
	$\Delta\phi/\phi\%$	-30.80	-15.09	13.60	25.38		$\Delta\phi/\phi\%$	0.35	-0.20	0.20	0.40
	$\Delta t_1/t_1\%$	17.07	8.41	-7.69	-14.42		$\Delta t_1/t_1\%$	0.00	0.00	0.00	0.00
	$\Delta t_2/t_2\%$	-9.59	-5.21	5.88	12.21		$\Delta t_2/t_2\%$	-0.03	-0.03	0.00	0.00
	$\Delta Q/Q\%$	-26.36	-13.82	15.09	31.47		$\Delta Q/Q\%$	-0.03	-0.01	0.01	0.04
	$\Delta\Pi_A/\Pi_A\%$	-39.47	-21.28	24.60	52.86		$\Delta\Pi_A/\Pi_A\%$	-0.03	-0.01	0.01	0.04
C_o	$\Delta p/p\%$	-4.07	-2.03	2.03	4.07	C_h	$\Delta p/p\%$	-8.79	-4.5	4.63	9.38
	$\Delta g/g\%$	11.51	5.84	-6.01	-12.18		$\Delta g/g\%$	-34.31	-25.39	-8.23	0.17
	$\Delta\rho/\rho\%$	4.78	2.39	-2.50	-4.89		$\Delta\rho/\rho\%$	-19.14	-9.19	8.65	16.97
	$\Delta\phi/\phi\%$	35.67	19.92	-25.53	-57.44		$\Delta\phi/\phi\%$	-30.80	-15.09	13.60	25.38
	$\Delta t_1/t_1\%$	-13.90	-7.72	9.27	20.97		$\Delta t_1/t_1\%$	17.07	8.41	-7.69	-14.42
	$\Delta t_2/t_2\%$	11.12	5.60	-5.64	-10.95		$\Delta t_2/t_2\%$	-9.59	-5.21	5.88	12.21
	$\Delta Q/Q\%$	20.99	10.29	-9.84	-19.09		$\Delta Q/Q\%$	-26.36	-13.82	15.09	31.47
	$\Delta\Pi_A/\Pi_A\%$	29.98	14.28	-12.91	-24.52		$\Delta\Pi_A/\Pi_A\%$	-39.47	-21.28	24.60	52.86
C_l	$\Delta p/p\%$	-20.77	-11.24	12.03	24.26	C_s	$\Delta p/p\%$	-0.30	-0.15	0.15	0.30
	$\Delta g/g\%$	-63.10	-26.21	23.70	46.41		$\Delta g/g\%$	0.50	0.33	-0.33	-0.50
	$\Delta\rho/\rho\%$	-23.47	-09.34	8.91	17.28		$\Delta\rho/\rho\%$	0.10	0.00	-0.10	-0.10
	$\Delta\phi/\phi\%$	-66.56	-32.82	53.46	82.72		$\Delta\phi/\phi\%$	-0.30	-0.15	0.15	0.30
	$\Delta t_1/t_1\%$	32.89	32.00	-21.41	-34.65		$\Delta t_1/t_1\%$	0.00	0.00	0.00	-0.22
	$\Delta t_2/t_2\%$	-12.02	-14.39	19.06	38.39		$\Delta t_2/t_2\%$	-0.09	-0.06	0.03	0.06
	$\Delta Q/Q\%$	-57.88	-33.13	41.44	89.39		$\Delta Q/Q\%$	3.65	3.15	2.40	2.10
	$\Delta\Pi_A/\Pi_A\%$	-59.52	-46.38	65.8	154.85		$\Delta\Pi_A/\Pi_A\%$	-1.27	-2.23	-3.69	-4.28
C_d	$\Delta p/p\%$	-0.19	-0.13	0.06	0.19	$E(\epsilon) = \mu$	$\Delta p/p\%$	-0.78	-0.39	0.39	0.72
	$\Delta g/g\%$	0.16	0.00	-0.16	-0.50		$\Delta g/g\%$	-1.66	-0.83	0.83	1.50
	$\Delta\rho/\rho\%$	0.10	0	-0.10	-0.10		$\Delta\rho/\rho\%$	-0.65	-0.32	0.32	0.54
	$\Delta\phi/\phi\%$	-0.10	-0.05	0.05	0.10		$\Delta\phi/\phi\%$	-4.74	-2.34	2.29	4.58
	$\Delta t_1/t_1\%$	0.00	0.00	0.00	-0.22		$\Delta t_1/t_1\%$	1.76	0.88	-0.88	-1.76
	$\Delta t_2/t_2\%$	-0.09	-0.06	0.03	0.06		$\Delta t_2/t_2\%$	-1.66	-0.83	0.83	1.50
	$\Delta Q/Q\%$	0.11	0.05	-0.05	-0.11		$\Delta Q/Q\%$	-2.42	-1.21	1.22	2.46
	$\Delta\Pi_A/\Pi_A\%$	0.31	0.15	-0.15	-0.31		$\Delta\Pi_A/\Pi_A\%$	-3.53	-1.77	1.79	3.61

time-sensitive demand rate (c) play pivotal roles in green level investment decisions. As these rates increase, businesses may be more inclined to invest in green initiatives to capture a broader customer base driven by price and time considerations. Promotional efforts parameter (e) can also influence green level investments. Effective marketing campaigns that highlight a product's eco-friendliness may justify higher green investments to align with consumer preferences. Preservation technology, represented by the constant parameter (f), interacts with green level investments. Businesses may consider preservation technology as a complementary aspect of green initiatives, investing in both to ensure product quality and sustainability. Additionally, the costs associated with green level investments (g) must be balanced with potential benefits. Ordering costs (A), purchasing costs (C_o), and other operational expenses should be factored into investment decisions to optimize eco-friendly practices effectively.

Managers should assess market conditions carefully and align their green initiatives with market potential. In high-potential markets, increasing green investments can capitalize on consumer demand for environmentally friendly products, enhancing brand reputation. In contrast, cautious green investments may be prudent in markets with lower potential. Price-sensitive demand rate (b) and time-sensitive demand rate (c) underscore the importance of pricing and timing in sustainability efforts. Higher sensitivity to these factors signifies greater consumer willingness to pay for green products and time-related considerations. Businesses should leverage these insights to justify increased green investments and capture value from sustainability. Promotional efforts (e) offer a lever for driving green investments. Effective marketing campaigns that highlight a product's eco-friendliness can justify higher green expenditures. Businesses should integrate promotional strategies with sustainability goals to boost the perceived value of green products. Operational costs, including ordering costs (A), purchasing costs (C_o), and inventory-related expenses (λ, C_d, C_s, K, C_h, C_l), necessitate careful management. Balancing these costs with green investments ensures that sustainability practices do not compromise profitability. Efficient resource allocation is key to maintaining a sustainable and profitable business. The backlogging parameter (δ) and the preservation technology constant (f) introduce nuances to green investment decisions. A higher backlogging parameter may motivate businesses to invest more in green practices to minimize shortages and enhance customer satisfaction.

4.5. Sensitivity analysis of promotional efforts (ρ) with respect to the all assumed inventory parameters

In this sensitivity analysis, we explore how promotional efforts (ρ) respond to changes in various parameters, shedding light on the dynamics of marketing strategies and their impact on business outcomes. Market potential (a) is a central factor influencing promotional efforts. In markets with substantial potential for a product, higher promotional investments are often justified to capture a larger share of the market. Conversely, in markets with lower potential, businesses may adopt a more cost-effective approach to promotional activities. Price-sensitive demand rate (b) and time-sensitive demand rate (c) significantly affect promotional strategies. Higher sensitivity to price and time factors signals that consumers are responsive to price fluctuations and seasonal trends. Businesses can tailor their promotional efforts to leverage these sensitivities, optimizing their marketing campaigns for maximum impact.Operational costs, including ordering costs (K), purchasing costs (λ), and various inventory-related expenses (C_o, C_d, C_s, C_h, C_l), need to be managed efficiently alongside promotional efforts. Striking a balance between marketing investments and operational costs ensures that promotional strategies contribute positively to the bottom line. The backlogging parameter (δ) and the preservation technology constant (f) introduce nuances to promotional decisions. A higher backlogging parameter may necessitate more aggressive promotional efforts to reduce backorders, while the preservation technology constant affects the availability of products for promotion.

4.6. *Sensitivity analysis of preservation technology investment (ϕ) with respect to the all assumed inventory parameters*

The sensitivity analysis of preservation technology investment (ϕ) sheds light on its responsiveness to changes in various parameters, providing crucial insights for decision-makers in managing technological advancements. Market potential (a) plays a pivotal role in determining the level of preservation technology investment. In high-potential markets, businesses may be inclined to allocate more resources towards technological advancements to cater to growing consumer demand and maintain a competitive edge. Conversely, in markets with lower potential, cost-effective preservation solutions may be prioritized to ensure profitability. The price-sensitive demand rate (b) and time-sensitive demand rate (c) significantly impact preservation technology decisions. Markets with heightened price and time sensitivity may require more advanced preservation technologies to extend product shelf life and meet demand fluctuations effectively. Businesses should align their investment in preservation technology with these consumer sensitivities. Operational costs, including ordering costs (K), purchasing costs (λ), and various inventory-related expenses (C_o, C_d, C_s, C_h, C_l), must be considered alongside preservation technology investments. Striking a balance between technological advancements and operational efficiency ensures that investments in preservation technology contribute positively to overall cost management. The deterioration rate parameter (α) and backlogging parameter (δ) introduce intricacies into preservation technology decisions. Higher deterioration rates may necessitate more significant investments in preservation to combat product deterioration effectively. Additionally, the backlogging parameter impacts the urgency of maintaining adequate stock levels to prevent backorders.

The sensitivity analysis of promotional efforts (ρ) offers valuable managerial insights for crafting effective marketing strategies and optimizing business performance. Market potential (a) emerges as a guiding factor in promotional decisions. Businesses operating in markets with high potential should consider allocating more resources to promotional efforts. These investments can help capitalize on consumer demand, expand market share, and enhance brand recognition. In contrast, markets with lower potential may require a more strategic and cost-effective approach to promotional activities. Price-sensitive demand rate (b) and time-sensitive demand rate (c) underscore the importance of pricing and timing in marketing campaigns. Higher sensitivity to price fluctuations and seasonal trends necessitates dynamic pricing strategies and timely promotions. Managers should align promotional efforts with these consumer sensitivities to maximize sales and profitability. Operational costs, including ordering costs (K), purchasing costs (λ), and inventory-related expenses (C_o, C_d, C_s, C_h, C_l), need to be carefully managed alongside promotional activities. Balancing these costs with promotional investments ensures that marketing strategies contribute positively to the company's financial performance. The backlogging parameter (δ) and the preservation technology constant (f) introduce additional considerations into promotional decisions. A higher backlogging parameter may prompt businesses to intensify promotional efforts to minimize backorders and meet customer demand promptly. The preservation technology constant affects product availability for promotions, emphasizing the need for alignment between inventory management and marketing strategies.

4.7. Sensitivity analysis of expected total profit (Π_A)

The sensitivity analysis of expected total profit (Π_A) provides valuable insights into the impact of different parameters on overall profitability. Market conditions, represented by parameters such as market potential (a) and price sensitivity (b), significantly influence Π_A. In markets with higher potential and price sensitivity, Π_A tends to be more substantial, highlighting the importance of identifying and targeting lucrative market segments to maximize profits. Operational costs, including ordering costs (K), purchasing costs (λ), and various inventory-related expenses (C_o, C_d, C_s, C_h, C_l)), play a pivotal role in profit generation. Effective cost management strategies can enhance Π_A by optimizing resource allocation and minimizing overheads. The deterioration rate parameter (α) and backlogging parameter (δ) exhibit contrasting effects on Π_A. Higher deterioration rates tend to reduce Π_A due to increased product losses, necessitating investments in preservation technology and efficient inventory management. Conversely, a well-tuned backlogging parameter can positively impact Π_A by allowing partial backorders during peak demand periods, preventing lost sales. Investments in preservation technology (f) significantly influence Π_A. Optimal allocation of resources to enhance product preservation can lead to higher Π_A by extending the shelf life of products and reducing waste.

The sensitivity analysis of total expected profit (Π_A) reveals essential considerations for inventory managers. Market segmentation based on market potential (a) and price sensitivity (b) allows managers to focus on high-potential segments, optimizing resource allocation. Cost control, including minimizing ordering costs (K) and other inventory-related expenses, remains critical. Investments in preservation technology (f) can enhance profitability by extending product shelf life. Managing demand variability through the backlogging parameter (δ) is key to balancing customer satisfaction and cost control. Moreover, addressing higher deterioration rates (α) necessitates efficient inventory turnover strategies. By embracing these insights, inventory managers can make informed decisions to maximize profits and maintain a competitive edge.

4.8. Sensitivity analysis of ordering quantity (Q)

The sensitivity analysis of the ordering quantity (Q) provides valuable insights for inventory management decisions. It is evident that variations in market potential (a), price sensitivity (b), time sensitivity (c), and other parameters significantly affect the optimal ordering quantity. Market segmentation based on these parameters can help managers tailor their ordering strategies to specific customer segments, ensuring efficient resource allocation. Additionally, the impact of promotional efforts (ρ) on ordering quantity underscores the importance of marketing strategies in influencing demand. Managers must strike a balance between demand variability, backlogging parameters (δ), and inventory holding costs (C_h) to optimize the ordering quantity. By carefully considering these insights, inventory managers can fine-tune their ordering decisions to enhance profitability and customer satisfaction.

Managers should closely monitor these factors and adjust their order quantities accordingly to prevent overstocking or under-stocking. The analysis underscores the impact of promotional efforts (ρ) on order quantities. Managers should align marketing campaigns with inventory planning to leverage promotions for increasing demand while minimizing inventory holding costs. This highlights the importance of cross-functional collaboration between marketing and supply chain teams. The sensitivity to backlogging parameters (δ) and inventory holding costs (C_h) indicates that inventory managers need to carefully balance between minimizing costs and fulfilling customer demand.

Implementing efficient inventory replenishment strategies and utilizing technology for real-time demand forecasting can help strike this balance effectively.

5. Conclusions

In conclusion, this study delves into the complex interplay of inventory management, pricing strategies, and sustainability considerations within the context of declining markets. Through rigorous analysis, we have addressed critical research questions and provided valuable insights for businesses navigating challenging market conditions. Our findings underscore the significance of strategic decision-making in optimizing inventory management and pricing policies. Moreover, the integration of sustainability practices, particularly Green Investment (GI) and Preservation Investment (PI), emerges as a compelling avenue for businesses. These investments not only enhance profitability but also align with the growing consumer demand for eco-conscious products. The holistic approach of combining GI and PI proves to be the most beneficial, offering the potential for maximum profitability and sustainability. This study equips businesses with actionable strategies to thrive in competitive, environmentally conscious markets, fostering long-term success and responsible business practices. In essence, the study underscores the pivotal role of innovation, adaptability, and sustainability in modern business strategies, guiding companies towards greater profitability, consumer satisfaction, and a greener future. In comparing the profitability scenarios, it becomes evident that Green Investment (GI)

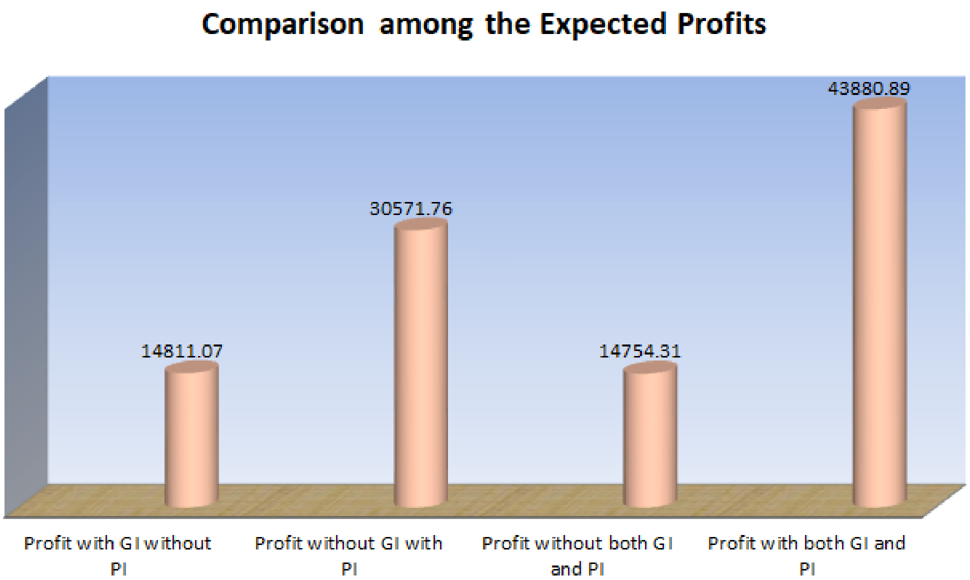


Figure 18. Comparison among the expected profits.

alone, without Preservation Investment (PI), results in a profit of \$14,811.07. Conversely, when Preservation Investment (PI) is implemented without Green Investment (GI), the profit significantly increases to \$30,571.76. Interestingly, forgoing both Green Investment (GI) and Preservation Investment (PI) yields a profit of \$14,754.31, which is comparable to the profit generated solely by Green Investment (GI). However, the combined implementation of both Green Investment (GI) and Preservation Investment (PI) leads to the highest profit, totaling \$43,880.89. The differences in profits are illustrated in Figure 18. This analysis underscores the substantial impact of Preservation Investment (PI) on profitability, especially when complemented by Green Investment (GI).

In conclusion, this study provides businesses with valuable insights to navigate declining markets effectively. By optimizing inventory management and pricing strategies while integrating sustainability practices like Green Investment (GI), Promotional Efforts (PE) and Preservation Investment (PI), companies can enhance profitability, align with consumer preferences for eco-conscious products, and ensure long-term viability in competitive marketplaces. The combined approach of GI and PI stands out as particularly beneficial, offering the potential for maximum profitability and sustainability, reinforcing the importance of embracing environmentally conscious practices in today's business landscape. The proposed study opens up several avenues for future research in the field of pricing and inventory management in a declining market. Several future recommendations emerge from this study's insights into inventory management and pricing strategies in the context of shifting consumer. Furthermore, continued research into consumer preferences and sustainability trends is essential. Understanding how these factors evolve will enable businesses to stay ahead of market shifts and tailor their strategies accordingly. Next, the collaboration with regulatory bodies and industry peers can foster sustainable practices and ensure compliance with emerging environmental standards. By sharing best practices and advocating for environmentally responsible policies, businesses can collectively contribute to a greener future.

In summary, embracing sustainability, investing in technology, maintaining pricing flexibility, staying attuned to consumer trends, and fostering collaboration are key future recommendations for businesses seeking to thrive in an ever-changing business landscape. These strategies will not only enhance profitability but also position companies as responsible stewards of the environment, ensuring long-term success.

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

Availability of data and material

For numerical validation of different cases are obtained by optimizing respective functions through MAPLE 21 software based on classical optimization method.

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