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Research article

On the time-optimal control problem for a fourth order parabolic equation in the three-dimensional space

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Abstract: In this paper, we consider the problem of optimal time control for a fourth-order parabolic-type equation describing the growth process of a thin film in a bounded three-dimensional space. The control function is defined on a certain part of a boundary. It is proved that the optimal time depends on the parameters of the growth process when the average value of the growth interface height of the thin film in the domain is close to the critical value.

Keywords: fourth order parabolic equation; Volterra integral equation; admissible control; initial-boundary problem; critical value; minimal time
Mathematics Subject Classification: Primary: 35K15, 35K35; Secondary: 58J35

1. Introduction

Let $R > 0$, and let $\Omega = B_R = \{x \in \mathbb{R}^3 : |x| < R\}$ be given. Moreover, let Γ with $mes\Gamma > 0$ (i.e., the surface measure of Γ, distinct from Lebesgue measure $|\Omega|$) be some subset of $\partial\Omega$, which is the boundary of Ω. Suppose that the given functions $h_i(x)$ $(i = 1, 2)$ and $a(x)$ are not exactly zero for $x \in \partial\Omega$ and are piecewise smooth non-negative functions, and it is regarded that the following relations hold for these functions:

$$h_i(x) = 0, \quad x \in \Gamma \quad \text{and} \quad a(x) = 0, \quad x \notin \Gamma.$$

In this paper, we investigate the fourth-order parabolic equation

$$u_t(x,t) + \Delta^2 u(x,t) = 0, \quad x \in \Omega, \quad t > 0, \tag{1.1}$$

with boundary conditions

$$\frac{\partial u(x,t)}{\partial n} + h_1(x)\, u(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0, \tag{1.2}$$

and

$$\frac{\partial \Delta u(x,t)}{\partial n} + h_2\,(x)\Delta u(x,t) = -\,a(x)\,v(t), \quad x \in \partial\Omega, \quad t > 0, \tag{1.3}$$

and initial condition

$$u(x,0) = 0, \tag{1.4}$$

where n is the outward normal on $\partial\Omega$ and $v(t)$ is the control function.

In what follows, $\overline{\mathbb{R}}_+$ is the non-negative half-line, $\overline{\mathbb{R}}_+ = \{t \in \mathbb{R} : t \geq 0\}$. Assume $M > 0$ is a given number. The function $v(t)$ is considered an *admissible control* if it is measurable on the half-line $\overline{\mathbb{R}}_+$ and meets the following requirement:

$$|v(t)| \leq M, \quad t \geq 0.$$

This paper is devoted to the study of the time-optimal control problem for a fourth-order parabolic equation in a bounded domain with a piecewise smooth boundary in three-dimensional space. It is known that the mathematical model of the epitaxial growth process of nanoscale thin films is represented by fourth-order parabolic equations [1]. The main objective of the work is to find the optimal estimate of the time required to obtain a thin film of a given average thickness. In Eq. (1.1), the function $u(x,t)$ describes the height of the growth interface at the spatial point $x \in \Omega$ at time $t \geq 0$, and the term $\Delta^2 u$ is used to describe a random adatom distribution that attempts to minimize the chemical potential of the system.

It is well known that publication [2] examined preliminary findings on optimal time control problems for parabolic type partial differential equations (PDEs). The control problem for a linear parabolic type equation in a one-dimensional domain with Robin boundary condition was studied by Fattorini and Russell [3]. The control problem of parabolic type PDE in an infinite dimensional domain was first examined in [4]. Many details regarding optimal control problems can be found in the monographs [5,6] by Lions and Fursikov.

In the three-dimensional domain, control problems for the second-order parabolic equation are examined in [7]. The boundary control problem for a second order parabolic type equation with a piecewise smooth boundary in an $n-$dimensional domain was studied in [8] and an estimate for the minimum time required to reach a given average temperature was found. In [9], a mathematical model of thermocontrol processes for a second-order parabolic equation was studied.

In [10], a specific time optimum control problem with a closed ball centered at zero as the target was studied. The problem was governed by the internal controlled second order parabolic equation. Some practical problems of the control problem with different boundary conditions for the linear second order parabolic equation were studied in [11].

In [12], the global existence and blow-up of solutions to the initial-boundary value problem for a pseudo-parabolic equation with a single potential were considered. Additionally, Wang and Xu [13] investigated the initial-boundary value problems for the nonlocal semilinear pseudo-parabolic equation. The control problems associated with a pseudo-parabolic equation in a one-dimensional domain in [14,15] were considered, and it was shown that there is admissible control by solving the Volterra integral equation of the second kind using the Laplace transform method.

The fourth-order semilinear parabolic equation in a bounded domain in $\mathbb{R}^N$ was studied by Xu et al. [16] and, by using the potential well method, showed that the solutions exist globally or blow-up in finite time, depending on whether or not the initial data are in the potential well. In [17], the global dynamical properties of solutions to a class of finite degenerate fourth-order parabolic equations with mean curvature nonlinearity were examined through the initial-boundary value problem.

The boundary value control problem for a fourth-order parabolic equation was studied in [18], and an optimal estimate for the control function was found. In [19], the optimal time problem for a fourth-order parabolic equation in a square domain was studied, and an optimal time estimate for the minimum time was found. In [1], the epitaxial growth of nanoscale thin films whose mathematical model is represented by a fourth-order parabolic equation is studied, and the existence, uniqueness, and regularity of solutions in the corresponding functional domain are shown.

In [20], Winkler examined the equation

$$u_t + \Delta^2 u + \mu_1 \Delta u + \mu_2 \Delta \left(|\nabla u|^2\right) = f(x),$$

when Ω is a bounded convex domain in $\mathbb{R}^N$, under the conditions $\frac{\partial u}{\partial n} = \frac{\partial \Delta u}{\partial n} = 0$ on the boundary $\partial\Omega$ with bounded initial date, and proved the existence of global weak solutions with spatial dimensional $N \leq 3$, under suitable conditions on date.

The behavior of the cost of the null controllability of a fourth-order parabolic equation using boundary controls with a transport term and vanishing diffusion coefficient was examined by the authors in the recent work [21]. Although a Carleman inequality was used in all of the earlier research, [22] presented the first null controllability result pertaining to a fourth-order parabolic equation without establishing a Carleman inequality. In [23], an initial-boundary-value problem for a class of fourth-order nonlinear parabolic equations modeling the epitaxial growth of thin films was studied, and the global existence, asymptotic behavior, and finite-time explosion of weak solutions are obtained using potential well theory. In [24], the studies of a fourth-order parabolic equation modeling thin film growth proved the existence, uniqueness, and regularity of solutions. Numerical simulations illustrate the model's ability to capture grain coarsening, island formation, and thickness growth observed in experiments. There are still a lot of efforts in the fourth-order parabolic equation [16] and second-order version [25–29].

We now consider the following eigenvalue problem corresponding to the initial boundary value problem (1.1)-(1.4):

$$\Delta^2 w_k(x) = \lambda_k \, w_k(x), \quad x \in \Omega, \tag{1.5}$$

$$\frac{\partial w_k(x)}{\partial n} + h_1(x) \, w_k(x) = 0, \quad x \in \partial\Omega, \tag{1.6}$$

and

$$\frac{\partial \Delta w_k(x)}{\partial n} + h_2\,(x) \Delta w_k(x) = 0, \quad x \in \partial\Omega. \tag{1.7}$$

The asymptotic behavior of the fourth-order parabolic equation's solution is known to be mostly dependent on the corresponding selfadjoint extension's first eigenvalue. By using the properties of the first eigenfunction w_1, we are able to estimate the minimal time required to reach film height. Therefore, we must first show that the first eigenfunction of the eigenvalue problem (1.5)-(1.7) above is positive.

It is known that the property on positivity conserving of the first eigenfunction for the Laplace operator belongs to every domain in $\mathbb{R}^n$. However, an original assumption of Boggio and Hadamard [30] asserted that such a result would be valid for the biharmonic operator on optional strictly convex domains, and after this approach, a number of counterexamples were provided. For numerous domains, the biharmonic Drichlet problem does not have a order-conserving property, and its first eigenfunction does not keep the positivity property. See the counterexamples to the Boggio-Hadamard assumption in [31] and [32].

The sign-preserving property for (1.5)-(1.7) is equivalent to having a positive Green function. By an applications of Jentzsch's theorem [33], or the Krein-Rutman theorem [34], it follows that a strictly positive Green function implies that the first eigenfunction of the problem (1.5)-(1.7) is simple and that the corresponding eigenfunction is positive (see [35]).

According to the above considerations, the spectral problem (1.5)-(1.7) is self-adjoint in $L_2(\Omega)$ and there is a sequence of eigenvalues $\{\lambda_k\}$. Hence,

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_k \to +\infty, \quad k \to \infty.$$

It is known that the eigenfunctions $\{w_k\}_{k\in\mathbb{N}}$ form a complete orthonormal system in $L_2(\Omega)$ and these functions belong to $C(\overline{\Omega})$. From the orthogonality of the eigenfunctions $w_1(x)$ and $w_2(x)$ and the non-negative of the first eigenfunction, we may write

$$\lambda_1 < \lambda_2.$$

Denote by

$$W := \left\{ \rho(x) \in H^4(\Omega) \left| \int\limits_\Omega \rho(x)\, dx = 1, \quad \rho(x) \geq 0, \quad x \in \Omega \right. \right\},$$

the set whose functions $\rho(x)$, which satisfy the boundary conditions (1.6)-(1.7), where $H^4(\Omega)$ is the Hilbert space.

For any $\theta \geq 0$ we consider the following equality:

$$\theta = \int\limits_\Omega \rho(x)\, u(x,t)\, dx, \tag{1.8}$$

where the solution $u(x,t)$ of the problem (1.1)-(1.4) depends on the control function $v(t)$.

The value of (1.8) can be the average value of the height of the growth interface over a subdomain of the domain Ω.

The minimum time required to reach θ, which is determined by the value of the average growth height of the thin film, is denoted by the symbol $T(\theta)$. In other words, the equation (1.8) is valid for $t = T(\theta)$ and not valid for $t < T(\theta)$. We show the critical value θ^* such that, for each $\theta < \theta^*$, the equality (1.8) is impossible, and for $\theta \geq \theta^*$ there exists the requisite admissible control $v(t)$ and matching value of $T(\theta) < +\infty$. This work aims to determine the dependence of $T(\theta)$ on the parameters of the thin film growth process as θ approaches the critical value θ^*.

We consider the following function:

$$L(t) = \int\limits_\Omega \rho(x)\, u(x,t)\, dx, \quad t > 0. \tag{1.9}$$

We denote by $(\Delta^2)^{-1}$ the inverse operator of the biharmonic operator Δ^2. Also, σ is the surface measure induced by the Lebesgue measure on Γ. In particular,

$$\int\limits_\Gamma d\sigma(x) = mes\Gamma.$$

We set

$$\theta^* = M \int_\Gamma ((\Delta^2)^{-1}\rho(x))a(x)\, d\sigma(x), \tag{1.10}$$

and

$$\beta = \frac{M}{\lambda_1}(\rho, w_1) \int_\Gamma w_1(y)a(y)d\sigma(y), \tag{1.11}$$

where

$$(\rho, w_1) = \int_\Omega \rho(x)\, w_1(x)\, dx.$$

We present the main theorem in this work.

Theorem 1. *Let* $\rho \in W$, *and let* $\theta^* > 0$ *be defined by equality* (1.10). *Then:*

1) *for each* θ *from the interval* $0 < \theta < \theta^*$ *there is* $T(\theta)$ *such that*

$$L(t) < \theta, \quad 0 < t < T(\theta),$$

and

$$L(T(\theta)) = \theta.$$

2) *for each* $\theta \to \theta^*$ *the following estimate holds:*

$$T(\theta) = \frac{1}{\lambda_1} \ln \frac{1}{\varepsilon(\theta)} + \frac{1}{\lambda_1} \ln \beta + O(\varepsilon(\theta)^{(\lambda_2-\lambda_1)/\lambda_1}),$$

where $\varepsilon(\theta) = |\theta^* - \theta|$.

3) *for each* $\theta \geq \theta^*$ *there is no* $T(\theta)$.

2. Main integral equation

In this section, we reduce the optimal time control problem to a Volterra integral equation of the first kind. To find a solution to a mixed problem, we first need the Green function and its properties.

We consider the following Green function:

$$G(x, y, t) = \sum_{k=1}^{\infty} e^{-\lambda_k t}\, w_k(x)\, w_k(y), \quad x, y \in \Omega, \quad t > 0.$$

It is known that this function is the solution of the following initial-boundary value problem:

$$G_t(x, y, t) + \Delta^2 G(x, y, t) = 0, \quad x, y \in \Omega, \quad t > 0,$$

with boundary conditions

$$\frac{\partial G(x, y, t)}{\partial n} + h_1(x)G(x, y, t) = 0, \quad x, y \in \partial\Omega, \quad t > 0,$$

and

$$\frac{\partial \Delta G(x, y, t)}{\partial n} + h_2(x)\, \Delta G(x, y, t) = 0, \quad x, y \in \partial\Omega, \quad t > 0,$$

and initial condition

$$G(x, y, 0) = \delta(x - y),$$

where $\delta(x - y)$ is the Dirac delta function,

$$\delta(x - y) = \begin{cases} 0, & x \neq y, \\ \infty, & x = y, \end{cases} \qquad \int_{-\infty}^{+\infty} \delta(x - y)\, dx = 1.$$

Let $u(x, t)$ be the solution of the initial-boundary value problem (1.1)-(1.4). Then, it is defined as

$$u(x, t) = \int_0^t \nu(\tau)\, d\tau \int_\Gamma G(x, y, t - \tau)\, a(y)\, d\sigma(y). \tag{2.1}$$

Indeed, we write the solution of problem (1.1)-(1.4) in the following form (see [36]):

$$u(x, t) = \sum_{k=1}^{\infty} \phi_k(t)\, w_k(x), \tag{2.2}$$

where

$$\phi_k(t) \;=\; \int_\Omega w_k(x)\, u(x, t)\, dx.$$

Then,

$$\begin{aligned}
\phi_k{}'(t) &= \int_\Omega w_k(x)\, u_t(x, t)\, dx \;=\; -\int_\Omega w_k(x)\, \Delta^2 u(x, t)\, dx \\
&= -\int_\Omega \Delta u(x, t)\, \Delta w_k(x)\, dx \\
&\quad - \int_{\partial\Omega} \left(w_k(x) \left(\frac{\partial \Delta u}{\partial n} + h_2 \Delta u \right) - \Delta u(x, t) \left(\frac{\partial w_k}{\partial n} + h_1\, w_k \right) \right) d\sigma(x) \\
&= -\int_\Omega u(x, t) \Delta^2 w_k(x)\, dx \;+ \nu(t) \int_{\partial\Omega} w_k(x)\, a(x)\, d\sigma(x) \\
&\quad - \int_{\partial\Omega} \left(\Delta w_k(x) \left(\frac{\partial u}{\partial n} + h_1 u \right) - u(x, t) \left(\frac{\partial \Delta w_k}{\partial n} + h_2\, \Delta w_k \right) \right) d\sigma(x) \\
&= -\lambda_k \int_\Omega u(x, t)\, w_k(x)\, dx \;+\; \nu(t) \int_\Gamma w_k(x) a(x)\, d\sigma(x) \\
&= -\lambda_k\, \phi_k\,(t) + \Lambda_k\, \nu(t),
\end{aligned}$$

where Λ_k is

$$\Lambda_k \;=\; \int_\Gamma w_k(x)\, a(x)\, d\sigma(x).$$

Since $\phi_k(0) = 0$, we have

$$\phi_k(t) \;=\; \Lambda_k \int_0^t e^{-\lambda_k (t-\tau)}\, \nu(\tau)\, d\tau, \quad t > 0.$$

Thus, by (2.2), we have the solution

$$\begin{aligned} u(x,t) &= \sum_{k=1}^{\infty} \phi_k(t)\, w_k(x) = \sum_{k=1}^{\infty} \Lambda_k\, w_k(x) \int_0^t e^{-\lambda_k (t-\tau)}\, \nu(\tau)\, d\tau \\ &= \int_0^t \nu(\tau)\, d\tau \int_\Gamma G(x,y,t-\tau)\, a(y)\, d\sigma(y). \end{aligned}$$

Denote

$$F(x,t) = \int_\Omega \rho(y)\, G(x,y,t) dy, \quad x \in \Omega, \quad t > 0. \tag{2.3}$$

Clearly, the function (2.3) is a solution to the fourth-order equation

$$F_t(x,t) + \Delta^2 F(x,t) = 0, \quad x \in \Omega, \quad t > 0,$$

with boundary conditions

$$\frac{\partial F(x,t)}{\partial n} + h_1(x)\, F(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0,$$

and

$$\frac{\partial \Delta F(x,t)}{\partial n} + h_2(x)\, \Delta F(x,t) = 0, \quad x \in \partial\Omega, \quad t > 0,$$

and initial condition

$$F(x,0) = \rho(x), \quad x \in \Omega.$$

The function $F(x,t)$ can be written as follows using Green function

$$\begin{aligned} F(x,t) &= \sum_{k=1}^{\infty} e^{-\lambda_k t}\, w_k(x) \int_\Omega \rho(y)\, w_k(y)\, dy \\ &= \sum_{k=1}^{\infty} (\rho, w_k)\, e^{-\lambda_k t}\, w_k(x) \\ &= (\rho, w_1) e^{-\lambda_1 t} w_1(x) \;+\; F_1(x,t), \end{aligned} \tag{2.4}$$

where

$$F_1(x,t) \;=\; \sum_{k=2}^{\infty} (\rho, w_k) e^{-\lambda_k t} w_k(x), \quad t \geq 0. \tag{2.5}$$

Lemma 1. *The following estimate is valid:*

$$\Lambda_1 \;=\; \int_\Gamma w_1(y)\, a(y)\, d\sigma(y) > 0. \tag{2.6}$$

Proof. It is known that $a(y) > 0$ for $y \in \Gamma$. Let us assume that Λ_1 is zero. Then, w_1 must be equal to 0 on a certain part Γ_1 of the surface Γ. That is,

$$w_1(y) = 0, \quad y \in \Gamma_1.$$

By the conditions $h_i(x) = 0$ $(i = 1, 2)$ for $x \in \Gamma$ and from boundary conditions (1.6), (1.7), we obtain the following conditions:

$$\frac{\partial w_1(y)}{\partial n} = 0, \quad \frac{\partial \Delta w_1(y)}{\partial n} = 0, \quad y \in \Gamma_1.$$

Therefore, $w_1(y)$ is a solution to homogeneous Cauchy problem and from the uniqueness of the solution $w_1(y) \equiv 0$, and this contradicts the assumption that $w_1(y)$ is an eigenfunction. It follows that the integral defined by (2.6) is positive. □

We set

$$G_2(x, y) = \sum_{k=2}^{\infty} \frac{w_k(x)w_k(y)}{\lambda_k^2}. \tag{2.7}$$

Lemma 2. *Assume that* $\rho \in W$. *Then, the following estimate*

$$\left|F_1(x, t)\right| \leq C \sqrt{G_2(x, x)} e^{-\lambda_2 t}, \quad t \geq 0, \quad x \in \overline{\Omega},$$

is valid, where C *is a positive constant.*

Proof. By (2.5), we can write

$$\begin{aligned} \left|F_1(x, t)\right|^2 &= \left|\sum_{k=2}^{\infty} (\rho, w_k) e^{-\lambda_k t} w_k(x)\right|^2 \\ &\leq \left(\sum_{k=2}^{\infty} |(\rho, w_k)|^2 \lambda_k^2\right)\left(\sum_{k=2}^{\infty} e^{-2\lambda_k t} |w_k(x)|^2 \lambda_k^{-2}\right), \quad t \geq 0. \end{aligned}$$

Since $\rho(x) \in W$ and using equality (1.5), we have

$$\sum_{k=1}^{\infty} |(\rho, w_k)|^2 \lambda_k^2 = \|\Delta^2 \rho\|^2.$$

Therefore, for $k \geq 2$, we can write:

$$\sum_{k=2}^{\infty} |(\rho, w_k)|^2 \lambda_k^2 = \|\Delta^2 \rho\|^2 - |(\rho, w_1)|^2 \lambda_1^2 = C^2,$$

where C is a non-negative constant.

We now consider the following two cases:

Case 1. If $C = 0$, then $\rho(x) = A \cdot w_1(x)$, where A is a constant. In this case, $F_1(x, t) = 0$ because $(\rho, w_k) = 0$ for all $k \geq 2$. Thus, the inequality in Lemma 2 holds trivially.

Case 2. If $C > 0$, then we have

$$\left|F_1(x, t)\right| \leq C \sqrt{G_2(x, x)}\, e^{-\lambda_2 t}.$$

In both cases, the inequality in Lemma 2 holds. This completes the proof. □

Using the solution (2.1) and the condition (1.9), we can write

$$L(t) = \int_{\Omega} \rho(x)\, u(x,t)\, dx$$
$$= \int_{\Omega} dx \int_0^t \nu(\tau)\, d\tau \int_{\Gamma} G(x,y,t-\tau)\, a(y)\, d\sigma(y).$$

By (2.3), we may write

$$L(t) = \int_0^t \nu(\tau)\, d\tau \int_{\Gamma} F(y,t-\tau)\, a(y)\, d\sigma(y).$$

Now, we introduce the function

$$B(t) = \int_{\Gamma} F(y,t)\, a(y)\, d\sigma(y). \tag{2.8}$$

As a result, we get the following main Volterra integral equation:

$$\int_0^t B(t-\tau)\, \nu(\tau)\, d\tau = L(t), \quad t > 0. \tag{2.9}$$

From (2.5) and Lemma 2, we get

$$\left| \int_{\Gamma} F_1(y,t)\, a(y)\, d\sigma(y) \right| \le C_1\, e^{-\lambda_2 t}, \quad t > 0,$$

where

$$C_1 = C \int_{\Gamma} \sqrt{G_2(y,y)}\, a(y)\, d\sigma(y).$$

Thus, by equality (2.4), we can write the following assumption for the function $B(t)$:

$$B(t) = (\rho, w_1) e^{-\lambda_1 t} \int_{\Gamma} w_1(y)\, a(y) d\sigma(y) + \int_{\Gamma} F_1(y,t)\, a(y)\, d\sigma(y)$$
$$= \Lambda_1\, (\rho, w_1) e^{-\lambda_1 t} + O(e^{-\lambda_2 t}). \tag{2.10}$$

3. Proof of Theorem 1

In this section, we give the proof of the Theorem 1, which is the main result.
Denote

$$H(x,t) = \int_0^t F(x,\tau)\, d\tau. \tag{3.1}$$

Then, using (2.3) we can write

$$
\begin{aligned}
H(x,t) &= \sum_{k=1}^{\infty}(\rho, w_k)\, w_k(x) \int_0^t e^{-\lambda_k \tau}\, d\tau \\
&= \sum_{k=1}^{\infty} \frac{1-e^{-\lambda_k t}}{\lambda_k}(\rho, w_k)\, w_k(x) \\
&= (\Delta^2)^{-1}\rho(x) \; - \; \frac{e^{-\lambda_1 t}}{\lambda_1}(\rho, w_1)\, w_1(x) \; - \; H_1(x,t),
\end{aligned}
$$

where

$$
H_1(x,t) \;=\; \sum_{k=2}^{\infty} \frac{e^{-\lambda_k t}}{\lambda_k}(\rho, w_k) w_k(x). \tag{3.2}
$$

Set

$$
\|\rho\| = \left(\sum_{k=2}^{\infty} |(\rho, w_k)|^2 \right)^{1/2}.
$$

Lemma 3. *The following estimate is valid:*

$$
|H_1(x,t)| \;\le\; e^{-\lambda_2 t}\, \sqrt{G_2(x,x)}\, \|\rho\|.
$$

Proof. From (2.7), (3.2) and using the Cauchy-Bunyakovsky inequality, we can write

$$
\begin{aligned}
|H_1(x,t)| &= \left| \sum_{k=2}^{\infty} \frac{e^{-\lambda_k t}}{\lambda_k}(\rho, w_k) w_k(x) \right| \\
&\le \; e^{-\lambda_2 t} \left(\sum_{k=2}^{\infty} |(\rho, w_k)|^2 \right)^{1/2} \cdot \left(\sum_{k=2}^{\infty} \frac{|w_k(x)|^2}{\lambda_k^2} \right)^{1/2}.
\end{aligned}
$$

Thus, we get the required estimate

$$
|H_1(x,t)| \;\le\; e^{-\lambda_2 t}\, \sqrt{G_2(x,x)}\, \|\rho\|.
$$

□

We introduce the function

$$
Q(t) = \int_0^t B(t-\tau)\, d\tau = \int_0^t B(\tau)\, d\tau. \tag{3.3}
$$

The physical meaning of this function is that $Q(t)$ is the average thickness of the thin film in the Ω. According to Lemma 1 and (2.10), (3.3), we may write $Q(0) = 0$ and $Q'(t) = B(t) \ge 0$.

Based on (2.8) and (3.1), we get

$$
\int_\Gamma H(x,t)\, a(x)\, d\sigma(x) = \int_0^t d\tau \int_\Gamma F(x,\tau)\, a(x) d\sigma(x)
$$

$$= \int_0^t B(\tau)\,d\tau = Q(t). \tag{3.4}$$

On the other hand, we can also write the following equation:

$$\int_\Gamma H(x,t)a(x)\,d\sigma(x) = \int_\Gamma [(\Delta^2)^{-1}\rho(x)]a(x)\,d\sigma(x) - \frac{\Lambda_1}{\lambda_1}(\rho, w_1)e^{-\lambda_1 t} - \int_\Gamma H_1(x,t)a(x)\,d\sigma(x). \tag{3.5}$$

Set

$$Q^* = \lim_{t\to\infty} Q(t) = \int_0^\infty B(\tau)\,d\tau. \tag{3.6}$$

We can see that Q^* is positive and finite using (2.10). In conclusion, we can say that the average thickness of a thin film in the Ω domain cannot be greater than Q^*.

From (3.4) and (3.5), we get

$$Q(t) = \int_\Gamma [(\Delta^2)^{-1}\rho(x)]a(x)\,d\sigma(x) - \frac{\Lambda_1}{\lambda_1}(\rho, w_1)e^{-\lambda_1 t} - \int_\Gamma H_1(x,t)a(x)\,d\sigma(x).$$

As a result, by Lemma 3, we can write

$$\theta(t) = M\,Q(t) = \theta^* - \beta\, e^{-\lambda_1 t} + O(e^{-\lambda_2 t}), \tag{3.7}$$

where θ^* and β are defined by (1.10), (1.11), respectively.

We can write the following equation from (3.7):

$$\theta^* = MQ^*. \tag{3.8}$$

Lemma 4. *Assume that a function $f(r)$ increases in the interval $(0, 1]$ and is defined, for some $\alpha > 0$,*

$$f(r) = \beta\, r + O(r^{1+\alpha}). \tag{3.9}$$

Then, for inverse function $r = f^{-1}(s)$ the following estimate holds:

$$\ln\frac{1}{r} = \ln\frac{1}{s} + \ln\beta + O(s^\alpha),$$

where β is defined by (1.11).

Proof. We can write from (3.9),

$$s = \beta r[1 + \gamma(r)], \tag{3.10}$$

where $\gamma(r)$ is

$$\gamma(r) = O(r^{\alpha}).$$

It is clear that $f(r) > 0$ in the interval $(0, 1]$. Consequently,

$$s \geq C_2 r, \quad r \in (0, 1], \tag{3.11}$$

where constant $C_2 > 0$.

By (3.11), we may write

$$r(s) = f^{-1}(s) \leq \frac{1}{C_2} \cdot s,$$

and

$$r(s) = O(s).$$

Hence, we get

$$\gamma(r(s)) = O(s^{\alpha}).$$

Thus, according to (3.10),

$$\begin{aligned} \ln\frac{1}{s} &= \ln\frac{1}{\beta r} + \ln\frac{1}{1+\gamma(r)} = \ln\frac{1}{\beta r} - \ln[1 + \gamma(r)] \\ &= \ln\frac{1}{r} + \ln\frac{1}{\beta} + O(|\gamma(r)|) = \ln\frac{1}{r} - \ln\beta + O(s^{\alpha}), \end{aligned}$$

where β is defined by (1.11). □

Corollary 1. *The following equality is true:*

$$t = \frac{1}{\lambda_1}\ln\frac{1}{|\theta^* - \theta(t)|} + \frac{1}{\lambda_1}\ln\beta + O\left(|\theta^* - \theta(t)|^{(\lambda_2-\lambda_1)/\lambda_1}\right).$$

Proof. Indeed, according to (3.7),

$$\theta^* - \theta(t) = \beta e^{-\lambda_1 t} + O(e^{-\lambda_2 t}).$$

Set

$$r = e^{-\lambda_1 t}, \quad s = \theta^* - \theta(t), \quad \xi = \frac{\lambda_2}{\lambda_1} - 1.$$

Then, we can write

$$e^{-\lambda_2 t} = e^{-\lambda_1 t(1+\xi)} = r^{1+\xi}.$$

Using Lemma 4, we can obtain

$$t = \frac{1}{\lambda_1}\ln\frac{1}{|\theta^* - \theta(t)|} + \frac{1}{\lambda_1}\ln\beta + O\left(|\theta^* - \theta(t)|^{\xi}\right).$$

□

Lemma 5. *A real-valued measurable function $v(t)$ and $T(\theta) > 0$ exist such that $|v(t)| \leq M$ and the following equality holds:*

$$L(T) = \int_0^T B(T-\tau)\, v(\tau)\, d\tau. \tag{3.12}$$

Proof. The proof of this lemma is derived from the properties of the function $Q(t)$. By setting $v(t) = M$, we get the following equation:

$$\int_0^t B(t-\tau)\, v(\tau)\, d\tau \;=\; M \int_0^t B(t-\tau)\, d\tau \;=\; M\, Q(t).$$

Due to the equation (3.12), there exists $T(\theta) > 0$. Hence, the following equation is valid:

$$M\, Q(T) = L(T).$$

It is known that the equality (1.8) would be valid at $t = T(\theta)$. Therefore, it is not difficult to understand that the value $T(\theta)$ is a root of the following equation

$$Q(T) = \frac{T(\theta)}{M} = \frac{\theta}{M}. \tag{3.13}$$

□

Proof of Theorem 1. Using (3.7), (3.8), we can say that for every θ from the interval $(0, \theta^*)$ there exists $T(\theta)$

$$L(t) \;<\; \theta, \quad t \in (0, T(\theta)),$$

and using (3.13), we may write

$$L(T(\theta)) \;=\; \theta.$$

Then, using Corollary 1 for $\theta \to \theta^*$, we have the following estimate:

$$T(\theta) \;=\; \frac{1}{\lambda_1} \ln \frac{1}{\varepsilon(\theta)} + \frac{1}{\lambda_1} \ln \beta + O(\varepsilon(\theta)^{(\lambda_2 - \lambda_1)/\lambda_1}),$$

where $\varepsilon(\theta)$ is

$$\varepsilon(\theta) = |\theta^* - \theta|.$$

Theorem 1 is proved.

4. Conclusions

The initial boundary value problem is solved using the separation of variables method, and the control problem is reduced to a Volterra integral equation of the first kind using an additional integral condition. Using the properties of the first eigenfunction, an asymptotic estimate for the kernel of the Volterra integral equation is found. Using this estimate, it is proven that the average value of the growth interface height of a thin film in a domain is dependent on the parameters of the optimal time growth process when it is close to a critical value. The optimal time estimate derived from the study can be applied to improve thin film growth processes in the design of semi-conductors. The achieving of time-optimal control is beneficial to enhance efficiency, reduce production costs, and improve the quality of thin films.

Author contributions

Farrukh Dekhkonov: Writing-original draft, Writing-review & editing, Methodology, Formal Analysis, Investigation; Wenke Li: Writing-review & editing, Methodology, Formal Analysis, Validation; Weipeng Wu: Writing-review & editing, Methodology.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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