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Research article

Boundedness and stabilization in a quasilinear chemotaxis model with nonlocal growth term and indirect signal production

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Abstract: In this paper, we study the following fully parabolic chemotaxis system with nonlocal growth and indirect signal production:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (u\nabla v) + f(u), & x \in \Omega,\ t > 0, \\ v_t = \Delta v + w - v, & x \in \Omega,\ t > 0, \\ w_t = \Delta w + u - w, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in \Omega, \end{cases}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 3)$, where $D(u) \geq D_1 u^\gamma$, $f(u) = u(a_0 - a_1 u^\sigma + a_2 \int_\Omega u^\sigma dx)$, D_1, a_1, a_2 are positive constants, $a_0, \gamma \in \mathbb{R}$, $\sigma \geq \max\{1, -\gamma\}$ and $a_1 - a_2|\Omega| > 0$. It is shown that the above system admits a globally bounded classical solution if $\frac{4}{n} + \frac{\gamma}{\sigma} > \frac{3-\sigma}{1+\sigma}$. Furthermore, by the method of Lyapunov functionals, the global stability of steady states with convergence rates is established.

Keywords: chemotaxis system; global existence; boundedness; nonlocal term; asymptotic stability
Mathematics Subject Classification: 35A01, 65L10, 65L12, 65L20, 65L70

1. Introduction

Chemotaxis describes the directional movement of cells or organisms in the direction of the concentration gradient of chemical signals. In order to simulate the phenomenon that cells are attracted to the high concentrations of chemical signals secreted by themselves. In 1970, Keller and Segel proposed a classical biological chemotaxis model [1] as follows

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(S(u)\nabla v) + f(u), & x\in\Omega,\ t>0,\\ \tau v_t = \Delta v - v + u, & x\in\Omega, t>0,\\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), \tau v(x,0) = \tau v_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^n, \tau\in\{0,1\}$, ν denotes the outward unit normal vector on $\partial\Omega$, $u(x,t)$ represents the density of cells and $v(x,t)$ denotes the density of a chemical signal. $\nabla\cdot(D(u)\nabla u)$ and $-\nabla\cdot(S(u)\nabla v)$ represent self-diffusion and cross-diffusion, respectively. The function $f(u)$ describes cell proliferation and death. As we all know, chemotaxis research has important applications in both biology and medicine, so it has been one of the hottest research focuses in applied mathematics nowadays. The system (1.1) with $\tau = 0$ or $\tau = 1$ has been investigated extensively in the past few decades. For $\tau = 0, D(u)\equiv 1, S(u) = \chi u$ and $f\in C^1([0,\infty))$ is assumed to satisfy $f(u)\le a-\mu u^2$ for all $u\ge 0$ with some positive constants a,μ. The solutions to (1.1) are global and bounded for arbitrarily small $\mu>0$ with $n\le 2$ [2], or $n\ge 3$ and $\mu > \frac{(n-2)\chi}{n}$ sufficiently large [3]. When $f(u) = \lambda u - \mu u^\alpha$ with $\alpha>1, \lambda\ge 0$ and $\mu>0$, Winkler [4] introduced a concept of very weak solutions and proved global existence of such solutions for any nonnegative initial data $u_0\in L^1(\Omega)$ under the assumption that $\alpha > 2-\frac{1}{n}$. In the case of $D,S\in C^2([0,\infty))$ and $D(u)\ge c_0u^\rho, c_1u^q\le S(u)\le c_2u^q$, and $f(u)$ is a smooth function fulfilling $f(0)\ge 0$ and $f(u)\le au-\mu u^2$ for all $u>0$ with constants $a\ge 0$ and $\mu\ge 0$, Cao in [5] showed that there exists a unique global bounded classical solution. For $\tau = 1, D(u) = 1, S(u) = \chi u$ and $f(u) = 0$ with $\chi>0$, the system (1.1) has the global solutions with $n = 1$ [6]; when $n = 2, \int_\Omega u_0 > \frac{8\pi}{\chi}$, the solution of the system (1.1) will blow up in finite time [7], if $\int_\Omega u_0 < \frac{8\pi}{\chi}$, the system (1.1) possesses a globally bounded classical solution [8]. In the case of $n\ge 3$, if Ω is a ball, then for arbitrarily small mass $m := \int_\Omega u_0 > 0$, there exists the finite-time blow-up solutions [9] with proper initial coditions. Besides, for $f(u) = 0, S(u)/D(u)\le K(u+\varepsilon)^\alpha$ with $u>0, \alpha<\frac{2}{N}(N\in\mathbb{N}), K>0, \varepsilon\ge 0$, Ishida et al. [10] ruled out convexity of Ω, then established global-in-time existence and uniform-in-time boundedness of solutions. For $f(u) = au-\mu u^2, a\in\mathbb{R}$, Cao [11] used an approach based on maximal Sobolev regularity and proved that if the ratio $\frac{\mu}{\chi}$ is sufficiently large, then the unique nontrivial spatially homogeneous equilibrium given by $(\frac{a_+}{\mu},\frac{a_+}{\mu})$ is globally asymptotically stable without the restrictions $\tau = 1$ and the convexity of Ω. For the case $f(u) = u(1-u^\gamma)$, $D(0)>0$, $D(u)\ge K_1u^{m_1}$ and $S(u)\le K_2u^{m_2}$, $\forall u\ge 0$, $K_i\in\mathbb{R}^+$, $m_i\in\mathbb{R}$, $i = 1,2$, Wang et al. [12] showed that the system admits global classical solutions and they are uniformly bounded in time with the parameter pair (m_1,m_2) lies in some specific regions and $N\ge 2$. When $D(u)\simeq a_0(u+1)^{-\alpha}, S(u)\simeq b_0u(u+1)^{\beta-1}$ and $f(u) = ru-\mu u^{1+\sigma}$, Zheng [13] showed the globally bounded classical solutions of the system (1.1) if $0<\alpha+\beta<\max\{\sigma+\alpha,\frac{2}{n}\}$, or $\beta = \sigma$ with μ sufficiently large. In addition, the signal generation may be in a nonlinear form. Zhuang [14] established that (1.1) admits a globally bounded classical solution under $\beta<\sigma-1$ or $\beta = \sigma-1$ with $r = 1, \mu>0$ sufficiently large, and the second equation replaced by $v_t = \Delta v - v + u(u+1^{\beta-1})$. When $D(u)\ge a_0(u+1)^{-\alpha}, 0\le S(u)\le b_0u(u+1)^{\beta-1}$ with $a_0,b_0>0$ and $\alpha,\beta\in\mathbb{R}$, Ding [15] provided a boundedness result under $\alpha+\beta+\gamma<\frac{2}{n}$, or $\beta+\gamma<1+\sigma$, or $\beta+\gamma = 1+\sigma$ with μ large enough and the bounded classical solution $(u,v)\to((\frac{r}{\mu})^{\frac{1}{\sigma}},(\frac{r}{\mu})^{\frac{1}{\sigma}})$ in $L^\infty(\Omega)$ exponentially under the condition of $b>b_0$. More relevant works of system (1.2) can refer to ([16–18]). Furthermore, many scholars consider the situation which the growth or death of cells is influenced by external factors, that is, the logistic sources contains nonlocal growth term. Negreanu and Tello [19] proposed the following model

$$\begin{cases} u_t = \Delta u - \chi \nabla \cdot (u^m \nabla v) + u(a_0 - a_1 u - a_2 \int_\Omega u dx), & x \in \Omega,\ t > 0, \\ -\Delta v + \lambda v = f + u, & x \in \Omega,\ t > 0, \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^n$ is a smooth bounded domain, if $m = 1$ and $a_1 > 2\chi + |a_2|$, it is shown that the solution of (1.2) satisfies $\lim_{t\to\infty} \|u - \frac{a_0}{a_1 + a_2}\|_{L^\infty(\Omega)} = 0$. when the second is replaced by $v_t = \Delta v - v + u^\gamma$ and the logistic source term is $u^\alpha(1 - \frac{\sigma}{|\Omega|}\int_\Omega u^\beta dx)$ with $\alpha \geq 1, \beta > 1$ and $m = \gamma = \frac{\sigma}{|\Omega|} = 1$, Bian [20] proved that the system (1.2) possesses a unique global strong solution which is uniformly bounded in whole space under either $\gamma + 1 \leq \sigma < 1 + \frac{2\beta}{n}$ or $\sigma < \gamma + 1 < \frac{2(\sigma+\beta)}{n+2} + \frac{n}{n+2}$. In reference [21], when $\gamma + m \leq \alpha < 1 + \frac{2\beta}{n}$ or $\frac{n+4}{2} - \beta < \alpha < \gamma + m$, the system (1.2) admits a uniformly bounded global classical solution. For the logistic source term is replaced by $u(a_0 - a_1 u^\alpha + a_2 \int_\Omega u^\alpha dx)$, Negreanu [22] obtained the global-in-time existence of classical solutions and the convergence to steady state $(a_0^{\frac{1}{\alpha}}(a_1 - a_2|\Omega|)^{-\frac{1}{\alpha}}, (a_0^{\frac{1}{\alpha}}(a_1 - a_2|\Omega|)^{-\frac{1}{\alpha}})^\gamma)$ under assumptions that $\alpha, \gamma \geq 1, m > 1, \alpha + 1 > m + \gamma, a_1 > 0$ and $a_1 - a_2|\Omega| > 0$. Moreover, when the logistic source term is replaced by $u^\sigma(a_0 - a_1 u - a_2 \int_\Omega u^\beta dx)$, Ren [23] proved that system (1.2) possesses a unique global classical solution in three different cases, namely parabolic-elliptic, fully parabolic, and parabolic-parabolic-elliptic. For more chemotaxis systems with nonlocal terms, we can find the literature works ([24–27]). The chemotactic signal is produced directly by cells in the classical Keller–Segel system, yet the signal generation undergoes intermediate stages in some realistic biological processes [28]. The related models can be described as follows:

$$\begin{cases} u_t = \nabla \cdot (D(u)\nabla u) - \nabla \cdot (S(u)\nabla v) + \mu(u - u^\gamma), & x \in \Omega,\ t > 0, \\ \tau v_t = \Delta v - v + w, & x \in \Omega,\ t > 0, \\ \tau w_t = \Delta w - w + u, & x \in \Omega,\ t > 0, \\ \partial_\nu u = \partial_\nu v = \partial_\nu w = 0, & x \in \partial\Omega,\ t > 0, \\ u(x,0) = u_0(x), \tau v(x,0) = \tau v_0(x), \tau w(x,0) = \tau w_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where u, v, w represent the density of cells, the density of chemical substances and the concentration of indirect signal, respectively. For $\tau = 0, D(s) \geq a_0(s+1)^\alpha, |S(s)| \leq b_0(1+s)^{\beta-1}$ for all $s \geq 0$ with $a_0, b_0 > 0, \alpha, \beta \in \mathbb{R}$, Li in [29] have obtained the nonnegative classical solution (u, v, w) is global in time and bounded for $\beta \leq \gamma - 1$. Moreover, if μ satisfies some suitable conditions, the solution (u, v, w) converges to $(1,1,1)$ in L^∞-norm as $t \to \infty$. When the signal generation is in a nonlinear form, it also has been shown that the boundedness and large time behaviors of classical solutions in [30]. For $\tau = 1, D(u) = 1, S(u) = u$, Zhang in [28] proved that if $\gamma > \frac{n}{4} + \frac{1}{2}$, the solution is globally bounded; if $\mu > 0$ is sufficiently large, (u, v, w) satisfies $\|u(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|v(\cdot,t) - 1\|_{L^\infty(\Omega)} + \|w(\cdot,t) - 1\|_{L^\infty(\Omega)} \to 0$ as $t \to \infty$. For $D(u) \geq D_1 u^\alpha, S(u) = u$ with $D_1 > 0, \alpha \in \mathbb{R}, \gamma \geq 2$ and $\mu > 0$, Wu in [31] proved the global existence and boundedness of solutions if the assumption $\frac{\alpha}{n} + \frac{\gamma}{2} > \frac{1}{2}$ holds with $n \geq 3$. When $D, S \in C^2([0,\infty))$ satisfying $D(s) \geq a_0(s+1)^{-\alpha}, 0 \leq S(s) \leq b_0(s+1)^\beta$ for $a_0, b_0 > 0$ and logistic source term is replaced by $b - \mu s^\gamma$ for all $s, b \geq 0, \gamma \geq 1$, Wang [32] obtained the global boundedness of solutions in four cases: the self-diffusion dominates the cross-diffusion; the logistic source suppresses the cross-diffusion for $\mu > 0$ sufficiently large; the logistic dampening balances the cross-diffusion; the self-diffusion and the logistic source both balance the cross-diffusion with $\mu > 0$ suitably large. If D, S

are smooth functions satisfying $D(s) \geq a_0(s+1)^{\alpha}$, $|S(s)| \leq b_0 s(s+1)^{\beta-1}$ for all $s > 0$ and the logistic source term is $\mu(s - s^{\gamma})$ with $s \geq 0, \gamma > 1$, Zhang in [33] showed that the system (1.3) possesses a globally bounded classical solution (u, v, w).

The above systems only discuss the forms of direct or indirect generation of chemical signals, but do not discuss systems with nonlocal source terms and indirect signal production.Thereafter, inspired by reference [22], considering the growth or death of cells is influenced by external factors, this article added the production of chemical signals goes through intermediate stages and studied the following fully parabolic chemotaxis system with a nonlocal growth term and indirect signal production

$$\begin{cases} u_t = \nabla\cdot(D(u)\nabla u) - \nabla\cdot(u\nabla v) + f(u), & x\in\Omega,\ t>0,\\ v_t = \Delta v + w - v, & x\in\Omega,\ t>0,\\ w_t = \Delta w + u - w, & x\in\Omega,\ t>0,\\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = \frac{\partial w}{\partial n} = 0, & x\in\partial\Omega,\ t>0,\\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x\in\Omega, \end{cases} \tag{1.4}$$

in a bounded domain $\Omega \subset \mathbb{R}^n (n \geq 3)$ with smooth boundary $\partial\Omega$. The diffusion function $D(u)$ satisfies

$$D \in C^2([0,\infty)) \text{ and } D(u) \geq D_1 u^{\gamma}, u > 0, \tag{1.5}$$

where D_1 is a positive constant and $\gamma \in \mathbb{R}$. And $f(u)$ is the logistic function, which satisfies

$$f(u) = u(a_0 - a_1 u^{\sigma} + a_2 \int_{\Omega} u^{\sigma} dx) \tag{1.6}$$

with

$$\sigma \geq \max\{1, -\gamma\} \text{ and } a_0 \in \mathbb{R}, a_i(i=1,2) > 0, a_1 - a_2|\Omega| > 0, \tag{1.7}$$

the initial data fulfill

$$\begin{cases} u_0 \in C^0(\bar{\Omega}),\ u_0 \geq 0,\ u_0 \not\equiv 0,\\ v_0 \in W^{1,\infty}(\bar{\Omega}),\ v_0 \geq 0,\\ w_0 \in W^{1,\infty}(\bar{\Omega}),\ w_0 \geq 0. \end{cases} \tag{1.8}$$

Under these assumptions, our main results on the global boundedness and large time behavior of solutions to system (1.4) are as follows:

Theorem 1. *Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded domain with smooth boundary. Suppose that functions $D(u), f(u)$ and parameters $\sigma, a_i (i = 0, 1, 2)$ satisfy (1.5)-(1.7) with $\frac{4}{n} + \frac{\gamma}{\sigma} > \frac{3-\sigma}{1+\sigma}$. For any nonnegative initial data (u_0, v_0, w_0) evolve from (1.8), the system (1.4) possesses a global classical solution*

$$(u, v, w) \in (C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})))^3, \tag{1.9}$$

which is bounded in $\Omega \times (0, \infty)$.

Remark 1. *When $a_2 = 0$, the result in Theorem 1 is consistent with that in [31] with $S(u) = u$ in system (1.3), which needs $\mu > 0$. If $a_2 > 0$, system (1.4) does not require any restrictions on a_0. These findings suggest that the nonlocal term $a_2 \int_{\Omega} u^{\sigma} dx$ could play a crucial role in ensuring the global existence and boundedness of solutions in (1.4).*

Remark 2. *This work extends the study in [22] to the system with nonlinear diffusion and indirect signal production. Our results show that the nonlinear diffusion mechanism and nonlocal logistic sources have an inhibitory effect on the blow-up solution.*

Theorem 2. *Assume the conditions in Theorem 1 hold. Let (u, v, w) be the solution of (1.4) obtained in Theorem 1, then there exist some $\kappa > 0$ and $C = C(a_0, a_1, a_2, \sigma, \gamma, |\Omega|) > 0$ such that*

$$\|u(\cdot, t) - u_*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v_*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w_*\|_{L^\infty(\Omega)} \leq Ce^{-\kappa t}$$

for all $t > 0$, where $\kappa = \frac{1}{4(n+2)}$, $u_* = v_* = w_* = (\frac{a_0}{a_1 - a_2|\Omega|})^{\frac{1}{\sigma}}$.

The framework structure of this article is as follows. In Sect.2, we give the local existence of the solution in (1.4) and show several related inequalities. In Sect.3, we consider the global boundedness of solutions for problem (1.4) under some suitable conditions and prove Theorem 1. In Sect.4, we give a lower bound for u via comparison [34]. In Sect.5, we obtain the asymptotic behavior of (1.4) by constructing Lyapunov functions, thus completing the proof of Theorem 2.

2. Local existence and preliminaries

In this section, we present several important lemmas that will be used in the following sections. First, we will state the local-in-time existence result of solutions for problem (1.4), which can be proved by adapting well-established approaches for parabolic-parabolic chemotaxis models (see [35]).

Lemma 1. *Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded domain with smooth boundary, $D(u)$, $f(u)$, and initial data (u_0, v_0, w_0) satisfy (1.5), (1.6), and (1.8), respectively. Then there exist $T_{max} \in (0, \infty]$ and a tripe (u, v, w) of nonnegative functions*

$$(u, v, w) \in (C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})))^3$$

which solves (1.4) in the classical sense. Moreover, if $T_{max} < \infty$, we can see that

$$\lim_{t \nearrow T_{max}} \sup \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

We next state a lemma which guarantees L^1-boundedness of u, v and w.

Lemma 2. *Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded domain with smooth boundary. The functions $D(u)$, $f(u)$ and the parameters $\sigma, a_i (i = 0, 1, 2)$ satisfy (1.5)-(1.7). Assume that (u, v, w) is the solution of (1.4). Then there exist constants $M_i (i = 1, 2, 3, 4)$ such that*

$$\|u(\cdot, t)\|_{L^1(\Omega)} \leq M_1 := \max\Big\{\int_\Omega u_0 dx,\ (\frac{a_0}{a_1 - a_2|\Omega|})^{\frac{1}{\sigma}}|\Omega|\Big\}, \tag{2.2}$$

$$\|v(\cdot, t)\|_{L^1(\Omega)} \leq M_2 := \max\Big\{\int_\Omega v_0,\ M_3\Big\}, \tag{2.3}$$

$$\|w(\cdot, t)\|_{L^1(\Omega)} \leq M_3 := \max\Big\{\int_\Omega w_0,\ M_1\Big\}, \tag{2.4}$$

for all $t \in (0, T_{max})$. Moreover,

$$\int_t^{t+\tau} \int_\Omega u^{\sigma+1} dx dt \leq M_4 = (a_1 - a_2|\Omega|)^{-1} M_1 (1 + a_0 \tau), \tag{2.5}$$

for all $\tau \in (0, T_{\max})$ and $t \in (0, T_{max} - \tau)$.

Proof. Integrating the first equation of the system (1.4) over Ω, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega udx =& \int_\Omega u(a_0 - a_1 u^\sigma + a_2 \int_\Omega u^\sigma dx)dx \\ =& a_0 \int_\Omega udx - a_1 \int_\Omega u^{\sigma+1}dx + a_2 \int_\Omega u(\int_\Omega u^\sigma dx)dx.\end{aligned} \tag{2.6}$$

By applying Hölder's inequality, we derive

$$\int_\Omega u(\int_\Omega u^\sigma dx)dx \leq (\int_\Omega u^{\sigma+1}dx)^{\frac{1}{\sigma+1}}|\Omega|^{\frac{\sigma}{\sigma+1}}(\int_\Omega u^{\sigma\cdot\frac{\sigma+1}{\sigma}}dx)^{\frac{\sigma}{\sigma+1}}|\Omega|^{\frac{1}{\sigma+1}} = |\Omega|\int_\Omega u^{\sigma+1}dx, \tag{2.7}$$

therefore

$$a_2\int_\Omega u(\int_\Omega u^\sigma dx)dx \leq a_2|\Omega|\int_\Omega u^{\sigma+1}dx.$$

Assuming (1.7) holds, applying the Hölder's inequality again yields

$$\begin{aligned}\frac{d}{dt}\int_\Omega udx \leq& a_0\int_\Omega udx - (a_1 - a_2|\Omega|)\int_\Omega u^{\sigma+1}dx \\ \leq& a_0\int_\Omega udx - \frac{a_1 - a_2|\Omega|}{|\Omega|^\sigma}(\int_\Omega udx)^{\sigma+1},\end{aligned} \tag{2.8}$$

for all $t \in (0, T_{max})$.
On an ordinary differential equation(ODE) comparison, this implies that

$$\int_\Omega udx \leq M_1, \tag{2.9}$$

where $M_1 := \max\left\{\int_\Omega u_0 dx, (\frac{a_0}{a_1 - a_2|\Omega|})^{\frac{1}{\sigma}}|\Omega|\right\}$. Integrating the third equation in (1.4) over Ω yields

$$\frac{d}{dt}\int_\Omega wdx = \int_\Omega udx - \int_\Omega wdx, \text{ for all } t \in (0, T_{max}), \tag{2.10}$$

then

$$\frac{d}{dt}\int_\Omega wdx + \int_\Omega wdx = \int_\Omega udx \leq M_1, \text{ for all } t \in (0, T_{max}). \tag{2.11}$$

So (2.2) and (2.11) imply (2.4). By the same method, using the second equation of the system (1.4) and (2.4), we obtain (2.3). Moreover, integrating (2.8) upon $(t, t+\tau)$ for $t \in (0, T_{max} - \tau)$, and using (2.9), we have

$$\int_t^{t+\tau}\int_\Omega u^{\sigma+1}dxdt \leq M_4, \text{ for all } t \in (0, T_{max} - \tau),$$

where $M_4 = (a_1 - a_2|\Omega|)^{-1}M_1(1 + a_0\tau)$. □

We provide the well-known Neumann heat semigroup theory without proof for our subsequent work (see Ref. [36, 37]).

Lemma 3. *Assume that $\lambda \in \{0,1\}$, $p \in [1,\infty]$, $q \in (1,\infty)$ and $\theta \in (0,1)$. Then there exists positive constant c_1 such that for all $u \in D(A^\theta)$, $A := -\Delta + \lambda$,*

$$\|u\|_{W^{m,p}(\Omega)} \leq c_1\|(-\Delta+1)^\theta u\|_{L^q(\Omega)}, \tag{2.12}$$

if

$$m - \frac{n}{p} \leq 2\theta - \frac{n}{q}.$$

If in addition $q \geq p$, then there exist some constants $c_2 > 0$ and $\alpha > 0$ such that for all $u \in L^p(\Omega)$,

$$\|A^\theta e^{-tA}u\|_{L^q(\Omega)} \leq c_2 t^{-\theta-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})} e^{-\alpha t}\|u\|_{L^p(\Omega)}, \tag{2.13}$$

where the associated diffusion semigroup $\{e^{-tA}\}_{t\geq 0}$ maps $L^p(\Omega)$ into $D(A^\theta)$. Moreover, for any $p \in (1,\infty)$ and $\varepsilon > 0$, there exist $c_3 > 0$ and $\mu > 0$ such that

$$\|A^\theta e^{t\Delta}\nabla\cdot u\|_{L^p(\Omega)} \leq c_3 t^{-\theta-\frac{1}{2}-\varepsilon} e^{-\mu t}\|u\|_{L^p(\Omega)} \tag{2.14}$$

is valid for all R^n-Valued $u \in L^p(\Omega)$.

We recall the Gagliardo–Nirenberg interpolation inequality (see Ref. [38] for detail), which will be used frequently in the proof of our main results.

Lemma 4. *Let $u \in W^{1,q}(\Omega) \cap L^p(\Omega)$ and $h \geq 1$, $p \in (0,h)$, where $\Omega \subset \mathbb{R}^n (n \geq 3)$ is a smooth bounded domain. Then there exists a constant $c_4 > 0$ such that*

$$\|u\|_{L^h(\Omega)} \leq c_4(\|\nabla u\|^\lambda_{L^q(\Omega)}\|u\|^{1-\lambda}_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}), \tag{2.15}$$

where

$$\frac{1}{h} = \lambda(\frac{1}{q} - \frac{1}{n}) + (1-\lambda)\frac{1}{p},$$

and $\lambda \in (0,1)$ satisfies

$$\lambda = \frac{\frac{n}{p} - \frac{n}{h}}{1 - \frac{n}{q} + \frac{n}{p}}.$$

Finally, we give the following lemma from [39], which is also important for our proof.

Lemma 5. *Let $T > 0$, $c_5 > 0$, $c_6 > 0$ and $\tau \in (0,T)$. Assume that the function $z : [0,T) \to [0,\infty)$ is absolutely continuous and such that the following inequality holds:*

$$z'(t) + c_5 z(t) \leq c(t), \tag{2.16}$$

for a.e $t \in (0,T)$, where $c \in L^1_{loc}([0,T))$ is a nonnegative function satisfying

$$\int_t^{t+\tau} c(s)ds \leq c_6, \tag{2.17}$$

for all $t \in [0, T-\tau)$.

Then

$$z(t) \leq \max\{z(0) + c_6, \frac{c_6}{c_5\tau} + 2c_6\}, \tag{2.18}$$

for a.e $t \in (0,T)$.

3. Glocal existence and boundedness

In this section, we will give the proof of global existence and boundedness of solutions to system (1.4). Some necessary estimations are needed. We first give an inequality involving $\int_\Omega w^{p+1}dx$ and then use this inequality to establish the L^p-estimate for w.

Lemma 6. *Assume that* (u, v, w) *is the solution of (1.4) on* $[0, T_{\max})$ *as in Lemma 2.1 and* $p > 1$. *Then we have the following inequality*

$$\frac{d}{dt}\int_\Omega w^{p+1}dx + p\int_\Omega w^{p+1}dx \le p^p \int_\Omega u^{p+1}dx, \tag{3.1}$$

for all $t \in (0, T_{max})$.

Proof. Multiplying the third equation of (1.4) by w^p, integrating over Ω and with the help of Young's inequality, we obtain

$$\begin{aligned}\frac{1}{p+1}\frac{d}{dt}\int_\Omega w^{p+1}dx =& \int_\Omega w^p(\Delta w + u - w)dx\\ =& -p\int_\Omega w^{p-1}|\nabla w|^2dx + \int_\Omega w^p u dx - \int_\Omega w^{p+1}dx\\ \le& \frac{1}{p+1}\int_\Omega w^{p+1}dx + \frac{p^p}{p+1}\int_\Omega u^{p+1}dx - \int_\Omega w^{p+1}dx\\ =& -\frac{p}{p+1}\int_\Omega w^{p+1}dx + \frac{p^p}{p+1}\int_\Omega u^{p+1}dx.\end{aligned}$$

□

In addition, in order to obtain the proof of Theorem 1, a key step is to derive the upper bounds of $\|w(\cdot,t)\|_{L^{\sigma+1}(\Omega)}$ and $\|\nabla v(\cdot,t)\|_{L^\beta(\Omega)}$. We will expand the proof from the following lemmas.

Lemma 7. *Let* $\Omega \subset \mathbb{R}^n (n \ge 3)$ *be a bounded domain with smooth boundary. The functions* $D(u), f(u)$ *and parameters* $\sigma, a_i (i = 1, 2)$ *satisfy (1.5)-(1.7). Assume that* (u, v, w) *is the solution of (1.4). Then for each* $\beta \in (1, \frac{n(\sigma+1)}{n-(\sigma+1)})$, *there exist positive constants* M_5 *and* M_6 *such that*

$$\|w(\cdot,t)\|_{L^{\sigma+1}(\Omega)} \le M_5, \|\nabla v(\cdot,t)\|_{L^\beta(\Omega)} \le M_6, \text{ for all } t \in (0, T_{max}). \tag{3.2}$$

Proof. Using Lemma 6 for $p = \sigma$, we have

$$\frac{1}{\sigma+1}\frac{d}{dt}\int_\Omega w^{\sigma+1}dx \le -\frac{\sigma}{\sigma+1}\int w^{\sigma+1}dx + \frac{\sigma^\sigma}{\sigma+1}\int_\Omega u^{\sigma+1}dx. \tag{3.3}$$

Taking $z(t) := \int_\Omega w^{\sigma+1}dx, t \in (0, T_{max})$, and $c(t) := \sigma^\sigma \int_\Omega u^{\sigma+1}(\cdot,t)dx, t \in (0, T_{max})$, we obtain

$$z'(t) + \sigma z(t) \le c(t), \quad \text{for all } t \in (0, T_{max}). \tag{3.4}$$

According to (2.5), one implies

$$\int_t^{t+\tau} c(t)dt = \sigma^\sigma \int_t^{t+\tau}\int_\Omega u^{\sigma+1}dxdt \le C_1, \quad \text{for all } t \in (0, T_{max} - \tau), \tag{3.5}$$

where $C_1 = \sigma^\sigma M_4 > 0$.
In view of (3.5) and Lemma 5, this yields that

$$\int_\Omega w^{\sigma+1} dx \le \max\Big\{ \int_\Omega w_0^{\sigma+1} + C_1, \frac{C_1}{\sigma\tau} + 2C_1 \Big\},$$

thus, we have

$$\|w(\cdot,t)\|_{L^{\sigma+1}(\Omega)} \le M_5, \tag{3.6}$$

where $M_5 = \max\Big\{ \int_\Omega w_0^{\sigma+1} + C_1, \frac{C_1}{\sigma\tau} + 2C_1 \Big\}^{\frac{1}{\sigma+1}}$.
Next, we obtain the L^β-bound of ∇v by applying semigroup arguments (see, for example, [36, 37]). First, using the variation of constants formula for the second equation of system (1.4) indicates

$$v(\cdot,t) = e^{t(\Delta-1)} v_0 + \int_0^t e^{(t-s)(\Delta-1)} w(\cdot,s) ds. \tag{3.7}$$

Choosing $\tau < 1$ and setting $A := -\Delta + 1, \lambda = 1, m = 1, q = \sigma + 1, \beta \in (1, \frac{n(\sigma+1)}{n-(\sigma+1)})$ in Lemma 3, which makes $\theta \in (\frac{1}{2} + \frac{n}{2(\sigma+1)} - \frac{n}{2\beta}, 1)$. Then there exist positive constants ε, γ such that

$$\begin{aligned}
\|v(\cdot,t)\|_{W^{1,\beta}(\Omega)} \le& C_2 \|A^\theta v(\cdot,t)\|_{L^{\sigma+1}(\Omega)} \\
\le& C_2 \|A^\theta e^{-tA} v_0\|_{L^{\sigma+1}(\Omega)} + C_2 \int_0^t \|A^\theta e^{-(t-s)A}\| \|w(\cdot,s)\|_{L^{\sigma+1}(\Omega)} ds \\
\le& C_2 t^{-\theta - \frac{n}{2}(\frac{1}{\sigma+1} - \frac{1}{\sigma+1})} e^{-\varepsilon t} \|v_0\|_{L^{\sigma+1}(\Omega)} \\
&+ C_2 \int_0^t (t-s)^{-\theta} e^{-\gamma(t-s)} \|w(\cdot,s)\|_{L^{\sigma+1}(\Omega)} ds \\
\le& C_2 t^{-\theta} \|v_0\|_{L^{\sigma+1}(\Omega)} + C_2 M_5 \int_0^t (t-s)^{-\theta} e^{-\gamma(t-s)} ds,
\end{aligned} \tag{3.8}$$

for all $t \in (\tau, T_{max})$. So

$$\|v(\cdot,t)\|_{W^{1,\beta}(\Omega)} \le C_2 \tau^{-\theta} \|v_0\|_{L^{\sigma+1}(\Omega)} + C_2 \Gamma(1-\theta) := C_3, \tag{3.9}$$

where C_2, C_3 represent different positive constants, and when $(1-\theta) > 0, \Gamma(1-\theta) > 0$. Therefore, we obtain that $\|\nabla v(\cdot,t)\|_{L^\beta(\Omega)}$ is bounded for all $t \in (0, T_{max})$. □

Next we will calculate the first equation of system (1.4) to obtain an inequality for $\int_\Omega u^p dx$ by applying the classical Green's formula, Young's inequality, and Hölder's inequality.

Lemma 8. *Let $\Omega \subset \mathbb{R}^n (n \ge 3)$ be a bounded domain with smooth boundary. The functions $D(u), f(u)$ and parameters $\sigma, a_i (i = 1, 2)$ satisfy (1.5)-(1.7). Assume that (u, v, w) is the solution of (1.4). Then for any $p > \max\{2, \gamma\}, \sigma > -\gamma$ there exist $C_4, C_5 > 0$ depend on p, such that the following inequality*

$$\frac{d}{dt}\int_\Omega u^p dx + \int_\Omega u^p dx \le -(a_1 - a_2|\Omega|) \int_\Omega u^{p+\sigma} dx + C_4 \int_\Omega |\nabla v|^{\frac{2(p+\sigma)}{\sigma+\gamma}} dx + C_5 \tag{3.10}$$

holds, for all $t \in (0, T_{max})$.

Proof. Multiplying the first equation in (1.4) by u^{p-1} and integrating over Ω by parts, we have

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx =& \int_\Omega u^{p-1}\nabla\cdot(D(u)\nabla u)dx - \int_\Omega u^{p-1}\nabla\cdot(u\nabla v)dx \\
&+ \int_\Omega u^p(a_0 - a_1u^\sigma + a_2\int_\Omega u^\sigma dx)dx \\
\leq& -(p-1)\int_\Omega u^{p-2}D(u)|\nabla u|^2dx + (p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v dx \\
&+ a_0\int_\Omega u^p dx - a_1\int_\Omega u^{p+\sigma}dx + a_2\int_\Omega u^p dx\int_\Omega u^\sigma dx \\
\leq& -(p-1)\int_\Omega D_1u^{p+\gamma-2}|\nabla u|^2dx + (p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v dx \\
&+ a_0\int_\Omega u^p dx - a_1\int_\Omega u^{p+\sigma}dx + a_2\int_\Omega u^p dx\int_\Omega u^\sigma dx,
\end{aligned}\tag{3.11}$$

for all $t \in (0, T_{max})$.
Using Young's inequality to the second term on the right side of (3.11) yields

$$\begin{aligned}
(p-1)\int_\Omega u^{p-1}\nabla u\cdot\nabla v dx \leq& \frac{(p-1)D_1}{2}\int_\Omega u^{p+\gamma-2}|\nabla u|^2dx + \frac{(p-1)}{2D_1}\int_\Omega u^{p-\gamma}|\nabla v|^2dx \\
\leq& \frac{(p-1)D_1}{2}\int_\Omega u^{p+\gamma-2}|\nabla u|^2dx + \frac{C_4}{p}\int_\Omega |\nabla v|^{\frac{2(p+\sigma)}{\sigma+\gamma}}dx \\
&+ \frac{a_1 - a_2|\Omega|}{2p}\int_\Omega u^{p+\sigma}dx, \text{ for all } t \in (0, T_{max}).
\end{aligned}\tag{3.12}$$

According to Young's inequality, the combination of (3.11) and (3.12) leads to

$$\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx + \frac{1}{p}\int_\Omega u^p dx \\
\leq& -\frac{(p-1)D_1}{2}\int_\Omega u^{p+\gamma-2}|\nabla u|^2dx + (a_0 + \frac{1}{p})\int_\Omega u^p dx + \frac{C_4}{p}\int_\Omega |\nabla v|^{\frac{2(p+\sigma)}{\sigma+\gamma}}dx \\
&+ (\frac{a_1 - a_2|\Omega|}{2p} - a_1)\int_\Omega u^{p+\sigma}dx + a_2\int_\Omega u^p dx\int_\Omega u^\sigma dx \\
\leq& -\frac{(p-1)D_1}{2}\int_\Omega u^{p+\gamma-2}|\nabla u|^2dx + (a_0 + \frac{1}{p})\int_\Omega u^p dx + \frac{C_4}{p}\int_\Omega |\nabla v|^{\frac{2(p+\sigma)}{\sigma+\gamma}}dx \\
&+ (\frac{a_1 - a_2|\Omega|}{2p} - a_1)\int_\Omega u^{p+\sigma}dx + a_2|\Omega|\int_\Omega u^{p+\sigma}dx \\
\leq& -\frac{(p-1)D_1}{2}\int_\Omega u^{p+\gamma-2}|\nabla u|^2dx + \frac{C_4}{p}\int_\Omega |\nabla v|^{\frac{2(p+\sigma)}{\sigma+\gamma}}dx \\
&- (\frac{a_1 - a_2|\Omega|}{p})\int_\Omega u^{p+\sigma}dx + \frac{C_5}{p},
\end{aligned}\tag{3.13}$$

where we note that our assumption $a_1 - a_2|\Omega| > 0$ and $p > 2$ warrant that

$$a_1 - a_2|\Omega| - \frac{3(a_1 - a_2|\Omega|)}{2p} > 0.$$

Therefore, the following inequality

$$(a_0 + \frac{1}{p}) \int_\Omega u^p dx \le (a_1 - a_2|\Omega| - \frac{3(a_1 - a_2|\Omega|)}{2p}) \int_\Omega u^{p+\sigma} dx + \frac{C_5}{p}$$

holds by Young's inequality. Then (3.10) is obtained. □

In order to obtain the L^p boundedness of u , we need to estimate the second term on the right side of (3.10), thus an inequality for $\int_\Omega |\nabla v|^{2q} dx$ has to be required.

Lemma 9. *Let (u, v, w) be a solution of (1.4), then for any $q > 2$ there exist positive constants C_6 and C_7 such that*

$$\begin{aligned}&\frac{d}{dt}\int_\Omega |\nabla v|^{2q} dx + 2q \int_\Omega |\nabla v|^{2q} dx + \frac{2q-2}{q} \int_\Omega |\nabla|\nabla v|^q|^2 dx \\ \le&(p-1) \int_\Omega w^{p+1} dx + C_6 \int_\Omega |\nabla v|^{\frac{(2q-2)(p+1)}{p-1}} dx + C_7, \quad \textit{for all } t \in (0, T_{max}).\end{aligned} \tag{3.14}$$

Proof. According to the second equation of (1.4), we can obtain

$$\begin{aligned}\frac{1}{2q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q} dx &= \int_\Omega |\nabla v|^{2q-2} \nabla v \cdot \nabla v_t dx \\ &= \int_\Omega |\nabla v|^{2q-2} \nabla v \cdot \nabla(\Delta v + w - v) dx \\ &= \int_\Omega |\nabla v|^{2q-2} \nabla v \cdot \nabla w dx - \int_\Omega |\nabla v|^{2q} dx \\ &\quad + \int_\Omega |\nabla v|^{2q-2} \nabla v \cdot \nabla(\Delta v) dx,\end{aligned} \tag{3.15}$$

for all $t \in (0, T_{max})$.

Now we estimate the first and third terms of (3.15), where an important inequality (3.10) in reference [10] as follows is needed

$$\frac{1}{2}\int_{\partial\Omega} |\nabla v|^{2q-2} \frac{\partial |\nabla v|^2}{\partial n} dx \le \frac{(q-1)}{2q^2} \int_\Omega |\nabla|\nabla v|^q|^2 dx + C_8, \tag{3.16}$$

with some $C_8 > 0$.

Using (3.16), we can infer that

$$\begin{aligned}&\int_\Omega |\nabla v|^{2q-2} \nabla v \cdot \nabla(\Delta v) dx \\ =&\frac{1}{2}\int_\Omega |\nabla v|^{2q-2} \Delta |\nabla v|^2 dx - \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 dx \\ =&-\frac{1}{2}\int_\Omega \nabla |\nabla v|^{2q-2} \cdot \nabla |\nabla v|^2 dx + \frac{1}{2}\int_{\partial\Omega} |\nabla v|^{2q-2} \cdot \frac{\partial |\nabla v|^2}{\partial n} dx - \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 dx \\ =&-\frac{q-1}{2}\int_\Omega |\nabla v|^{2q-4} \cdot |\nabla |\nabla v|^2|^2 dx + \frac{1}{2}\int_{\partial\Omega} |\nabla v|^{2q-2} \cdot \frac{\partial |\nabla v|^2}{\partial n} dx - \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 dx \\ \le&-\frac{2(q-1)}{q^2} |\nabla|\nabla v|^q|^2 dx + \frac{q-1}{2q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx - \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 dx + C_8 \\ \le&-\frac{3(q-1)}{2q^2} |\nabla|\nabla v|^q|^2 dx - \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 dx + C_8,\end{aligned} \tag{3.17}$$

for all $t \in (0, T_{max})$, where we have used $\Delta|\nabla v|^2 = 2\nabla v \cdot \nabla(\Delta v) + 2|D^2 v|^2$. Applying Young's inequality and $|\Delta v|^2 \leq n|D^2 v|^2$, we get

$$\begin{aligned}\int_\Omega |\nabla v|^{2q-2}\nabla v \cdot \nabla w dx =& -(q-1)\int_\Omega w|\nabla v|^{2(q-2)}\nabla|\nabla v|^2 \cdot \nabla v dx - \int_\Omega w|\nabla v|^{2q-2}\Delta v dx \\ \leq& \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx + (q-1)\int_\Omega w^2|\nabla v|^{2q-2}dx \\ &+ \sqrt{n}\int_\Omega w|\nabla v|^{2q-2}|D^2 v|dx \\ \leq& \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx + (q-1)\int_\Omega w^2|\nabla v|^{2q-2}dx \\ &+ \int_\Omega |\nabla v|^{2q-2}|D^2 v|^2 dx + \frac{n}{4}\int_\Omega w^2|\nabla v|^{2q-2}dx \\ \leq& \frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx + ((q-1) + \frac{n}{4})\int_\Omega w^2|\nabla v|^{2q-2}dx \\ &+ \int_\Omega |\nabla v|^{2q-2}|D^2 v|^2 dx,\end{aligned} \tag{3.18}$$

for all $t \in (0, T_{max})$.

And we also have

$$((q-1) + \frac{n}{4})\int_\Omega w^2|\nabla v|^{2q-2}dx \leq \frac{p-1}{2q}\int_\Omega w^{p+1}dx + \frac{C_6}{2q}\int_\Omega |\nabla v|^{\frac{(2q-2)(p+1)}{p-1}}dx, \tag{3.19}$$

for all $t \in (0, T_{max})$. Substituting (3.17), (3.18), and (3.19) into (3.15), we have

$$\begin{aligned}\frac{1}{2q}\frac{d}{dt}\int_\Omega |\nabla v|^{2q}dx \leq& -\frac{q-1}{q^2}\int_\Omega |\nabla|\nabla v|^q|^2 dx - \int_\Omega \nabla v|^{2q}dx + \frac{p-1}{2q}\int_\Omega w^{p+1}dx \\ &+ \frac{C_6}{2q}\int_\Omega |\nabla v|^{\frac{(2q-2)(p+1)}{p-1}}dx + C_8,\end{aligned} \tag{3.20}$$

for all $t \in (0, T_{max})$. Then we readily derive (3.14), where $C_7 := 2qC_8$. □

Combining Lemma 6 and Lemma 8 with Lemma 9 and using Young's inequality, the following inequality will be gained. And next we will establish the boundedness of $\|u(\cdot,t)\|_{L^p(\Omega)}$, $\|\nabla v(\cdot,t)\|_{L^{2p}(\Omega)}$ and $\|w(\cdot,t)\|_{L^{p+1}(\Omega)}$.

Lemma 10. *Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded domain with smooth boundary. The functions $D(u), f(u)$ and the parameters $\sigma, a_i (i = 1, 2)$ satisfy (1.5)-(1.7). Assume that (u, v, w) is the solution of (1.4). Then for any $p > max\{2, \gamma\}, q > 2$, there exist positive constants M_7, M_8 and M_9 such that*

$$\|u(\cdot,t)\|_{L^p(\Omega)} \leq M_7, \|\nabla v(\cdot,t)\|_{L^{2q}(\Omega)} \leq M_8, \|w(\cdot,t)\|_{L^{p+1}(\Omega)} \leq M_9, \tag{3.21}$$

for all $t \in (0, T_{max})$.

Proof. If $\sigma > 1$ or $\sigma \geq -\gamma$, we combine (3.1), (3.10) with (3.14) and apply Young's inequality to obtain

$$\begin{aligned}
&\frac{d}{dt}\Big(\int_\Omega u^p dx + \int_\Omega |\nabla v|^{2q} dx + \int_\Omega w^{p+1} dx\Big) + \int_\Omega u^p dx \\
&+ 2q\int_\Omega |\nabla v|^{2q} dx + \int_\Omega w^{p+1} dx + \frac{2q-2}{q}\int_\Omega |\nabla|\nabla v|^q|^2 dx \\
\leq& p^p\int_\Omega u^{p+1} dx - (a_1 - a_2|\Omega|)\int_\Omega u^{p+\sigma} dx + C_4\int_\Omega |\nabla v|^{\frac{2(p+\sigma)}{\sigma+\gamma}} dx + C_6\int_\Omega |\nabla v|^{\frac{(2q-2)(p+1)}{p-1}} dx + C_5 + C_7 \\
\leq& C_4\int_\Omega |\nabla v|^{\frac{2(p+\sigma)}{\sigma+\gamma}} dx + C_6\int_\Omega |\nabla v|^{\frac{(2q-2)(p+1)}{p-1}} dx + C_9, \quad \text{for all } t \in (0, T_{max}).
\end{aligned} \tag{3.22}$$

If $\sigma = 1$, we have the similar inequality and omit it here. In order to obtain the boundedness above, we need to estimate the right two terms of (3.22). By employing reference [38] and Gagliardo–Nirenberg inequality, we deduce that

$$\begin{aligned}
C_4\int_\Omega |\nabla v|^{\frac{2(p+\sigma)}{\sigma+\gamma}} dx =& C_4\Big\||\nabla v|^q\Big\|^{\frac{2(p+\sigma)}{q(\sigma+\gamma)}}_{L^{\frac{2(p+\sigma)}{q(\sigma+\gamma)}}(\Omega)} = C_4\Big\||\nabla v|^q\Big\|^{\frac{\alpha_1}{q}}_{L^{\frac{\alpha_1}{q}}(\Omega)} \\
\leq& C_{10}\Big(\Big\|\nabla|\nabla v|^q\Big\|^{\lambda_1}_{L^2(\Omega)}\Big\||\nabla v|^q\Big\|^{1-\lambda_1}_{L^{\frac{\beta}{q}}(\Omega)} + \Big\||\nabla v|^q\Big\|_{L^{\frac{\beta}{q}}(\Omega)}\Big)^{\frac{\alpha_1}{q}} \\
=& C_{10}\Big(\Big\|\nabla|\nabla v|^q\Big\|^{\lambda_1}_{L^2(\Omega)}\Big\|\nabla v\Big\|^{q(1-\lambda_1)}_{L^\beta(\Omega)} + \Big\|\nabla v\Big\|^q_{L^\beta(\Omega)}\Big)^{\frac{\alpha_1}{q}} \\
\leq& C_{10}\Big(\Big\|\nabla|\nabla v|^q\Big\|^{\lambda_1}_{L^2(\Omega)} M_6^{q(1-\lambda_1)} + M_6^q\Big)^{\frac{\alpha_1}{q}} \\
\leq& C_{10}\max\Big\{M_6^{q(1-\lambda_1)}, M_6^q\Big\}^{\frac{\alpha_1}{q}} \cdot \Big(\Big\|\nabla|\nabla v|^q\Big\|^{\lambda_1}_{L^2(\Omega)} + 1\Big)^{\frac{\alpha_1}{q}} \\
\leq& C_{10}\max\Big\{M_6^{q(1-\lambda_1)}, M_6^q\Big\}^{\frac{\alpha_1}{q}} \cdot 2^{\frac{\alpha_1}{q}-1}\Big(\Big\|\nabla|\nabla v|^q\Big\|^{\frac{\alpha_1\lambda_1}{q}}_{L^2(\Omega)} + 1\Big) \\
\leq& \widetilde{C}_{10}\Big(\Big\|\nabla|\nabla v|^q\Big\|^{h_1}_{L^2(\Omega)} + 1\Big) \\
\leq& \frac{q-1}{q}\Big\|\nabla|\nabla v|^q\Big\|^2_{L^2(\Omega)} + C_{11}, \quad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
C_6\int_\Omega |\nabla v|^{\frac{(2q-2)(p+1)}{p-1}} dx =& C_6\Big\||\nabla v|^q\Big\|^{\frac{(2q-2)(p+1)}{q(p-1)}}_{L^{\frac{(2q-2)(p+1)}{q(p-1)}}(\Omega)} = C_6\Big\||\nabla v|^q\Big\|^{\frac{\alpha_2}{q}}_{L^{\frac{\alpha_2}{q}}(\Omega)} \\
\leq& C_{12}\Big(\Big\|\nabla|\nabla v|^q\Big\|^{\lambda_2}_{L^2(\Omega)}\Big\||\nabla v|^q\Big\|^{1-\lambda_2}_{L^{\frac{\beta}{q}}(\Omega)} + \Big\||\nabla v|^q\Big\|_{L^{\frac{\beta}{q}}(\Omega)}\Big)^{\frac{\alpha_2}{q}} \\
=& C_{12}\Big(\Big\|\nabla|\nabla v|^q\Big\|^{\lambda_2}_{L^2(\Omega)}\Big\|\nabla v\Big\|^{q(1-\lambda_2)}_{L^\beta(\Omega)} + \Big\|\nabla v\Big\|^q_{L^\beta(\Omega)}\Big)^{\frac{\alpha_2}{q}} \\
\leq& \widetilde{C}_{12}\Big(\Big\|\nabla|\nabla v|^q\Big\|^{h_2}_{L^2(\Omega)} + 1\Big) \\
\leq& \frac{q-1}{q}\Big\|\nabla|\nabla v|^q\Big\|^2_{L^2(\Omega)} + C_{13}, \quad \text{for all } t \in (0, T_{max}),
\end{aligned} \tag{3.24}$$

where

$$\alpha_1 = \frac{2(p+\sigma)}{\sigma+\gamma},\ \alpha_2 = \frac{(2q-2)(p+1)}{p-1},$$
$$\lambda_1 = \frac{\frac{q}{\beta} - \frac{q}{\alpha_1}}{\frac{q}{\beta} - (\frac{1}{2} - \frac{1}{n})},\ \lambda_2 = \frac{\frac{q}{\beta} - \frac{q}{\alpha_2}}{\frac{q}{\beta} - (\frac{1}{2} - \frac{1}{n})}, \tag{3.25}$$
$$h_1 = \frac{\frac{\alpha_1}{\beta} - 1}{\frac{q}{\beta} - (\frac{1}{2} - \frac{1}{n})},\ h_2 = \frac{\frac{\alpha_2}{\beta} - 1}{\frac{q}{\beta} - (\frac{1}{2} - \frac{1}{n})}.$$

And for large $p > 1$, it will be shown that there exists constant $q > 1$ such that $\lambda_1, \lambda_2 \in (0,1)$ and $h_1, h_2 < 2$ hold in (3.25).

To ensure that $\lambda_1, \lambda_2 \in (0,1)$ and $h_1, h_2 < 2$, we can choose suitable parameters such that

$$\alpha_1, \alpha_2 > \beta \text{ and } q > \frac{\alpha_1}{2} - \frac{\beta}{n}, q > \frac{\alpha_2}{2} - \frac{\beta}{n}. \tag{3.26}$$

According to continuity argument, we discuss the case $\beta \in (1, \frac{n(\sigma+1)}{n-(\sigma+1)})$, which is inserted into (3.26) to have

$$p > \frac{n(\sigma+1)}{2(n-(\sigma+1))}(\sigma+\gamma) - \sigma, q > \frac{n(\sigma+1)}{2(n-(\sigma+1))} + 1, \tag{3.27}$$

and

$$\frac{p+\sigma}{\sigma+\gamma} - \frac{\sigma+1}{n-(\sigma+1)} < q < \frac{p+1}{2} + \frac{(\sigma+1)(p-1)}{2(n-(\sigma+1))}. \tag{3.28}$$

Thus if

$$\frac{p+\sigma}{\sigma+\gamma} - \frac{\sigma+1}{n-(\sigma+1)} < \frac{p+1}{2} + \frac{(\sigma+1)(p-1)}{2(n-(\sigma+1))}, \tag{3.29}$$

holds, then (3.28) is true. And $\frac{4}{n} + \frac{\gamma}{\sigma} > \frac{3-\sigma}{1+\sigma}$ implies (3.29). Then for any $p > p_0$, we can choose suitable q such that (3.27) and (3.28) are fulfilled, where $p_0 := \max\{2, \gamma, \frac{n(\sigma+1)}{2(n-(\sigma+1))}(\sigma+\gamma) - \sigma\}$.

Hence, a combination of (3.22)-(3.24) entails that

$$\frac{d}{dt}\Big(\int_\Omega u^p + \int_\Omega |\nabla v|^{2q} + \int_\Omega w^{p+1}\Big)dx + \int_\Omega u^p dx + 2q\int_\Omega |\nabla v|^{2q} dx + \int_\Omega w^{p+1} dx \le C_{14}, \tag{3.30}$$

for all $t \in (0, T_{max})$. By employing the Grönwall's inequality for (3.30), the desired result (3.21) is gained. □

Lemma 11. *Let $\Omega \subset \mathbb{R}^n (n \ge 3)$ be a bounded domain with smooth boundary. The functions $D(u), f(u)$ and parameters $\sigma, a_i (i = 1, 2)$ satisfy (1.5)-(1.7). Assume (u, v, w) is the solution of (1.4), then there exists a positive constant M_{10} such that*

$$\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \le M_{10}, \ \textit{for all } t \in (0, T_{max}). \tag{3.31}$$

Proof. (3.31) can be deduced by applying the semigroup arguments (see [36, 37] for details) to the third and second equations of (1.4). □

Next, by using Lemma 11 and the standard Alikakos–Moser iteration, we establish the L^∞ bound of u.

Lemma 12. *Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded domain with smooth boundary. The functions $D(u), f(u)$ and parameters $\sigma, a_i (i = 1, 2)$ satisfy (1.5)-(1.7). Assume that (u, v, w) is the solution of (1.4). Then for any $p > \max\{2, \gamma\}, \sigma > -\gamma$, there exists a positive constant $M_{11} > 0$ such that*

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq M_{11}, \ \textit{for all } t \in (0, T_{max}). \tag{3.32}$$

Proof. For any $p > \max\{2, \gamma\}$, multiplying the first equation of (1.4) by u^{p-1}, integrating over Ω, and using Hölder's inequality, one obtains

$$\begin{aligned}
\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx =& \int_\Omega u^{p-1}\nabla \cdot (D(u)\nabla u)dx - \int_\Omega u^{p-1}\nabla \cdot (u\nabla v)dx + \int_\Omega u^{p-1} f(u)dx \\
\leq& -(p-1)\int_\Omega u^{p-2}D(u)|\nabla u|^2 dx + (p-1)\int_\Omega u^{p-1}\nabla u \cdot \nabla v dx \\
&+ a_0\int_\Omega u^p dx - a_1\int_\Omega u^{p+\sigma}dx + a_2\int_\Omega u^p\int_\Omega u^\sigma dx dx \\
\leq& -(p-1)D_1\int_\Omega u^{p+\gamma-2}|\nabla u|^2 dx + (p-1)\int_\Omega u^{p-1}\nabla u \cdot \nabla v dx \\
&+ a_0\int_\Omega u^p dx - (a_1 - a_2|\Omega|)\int_\Omega u^{p+\sigma}dx
\end{aligned} \tag{3.33}$$

Once more employing Young's inequality, we have

$$\frac{d}{dt}\int_\Omega u^p dx + \int_\Omega u^p dx \leq C_{15}. \tag{3.34}$$

Integrating (3.34) yields that

$$\int_\Omega u^p dx \leq C_{16}, C_{16} := \max\Big\{\int_\Omega u_0^p dx, C_{15}\Big\}. \tag{3.35}$$

Then, we can prove the following inequality by using Alikakos–Moser iteration (see [16,40] for details)

$$\|u(\cdot, t)\|_{L^\infty(\Omega)} \leq M_{11}, \ \text{ for all } t \in (0, T_{max}). \tag{3.36}$$

Now, we complete the proof of Theorem 1. □

Proof of Theorem 1. Along with Lemma 1 part 2, this proves that $T_{max} = \infty$ and the standard parabolic regularity makes sure that (u, v, w) is bounded for $(x, t) \in \Omega \times (0, \infty)$. Hence the desired result of Theorem 1 is obtained. □

4. A lower bound for u

To further prove the asymptotic behavior of the solution, we are going to estimate the lower bound for u and provide some lemmas as follows.

Lemma 13. *Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded domain with smooth boundary. The functions $D(u), f(u)$ and parameters $\sigma, a_i (i = 1, 2)$ satisfy (1.5)-(1.7). If (u, v, w) is the solution of (1.4) and(u_0, v_0, w_0) satisfy (1.8), then there exists a constant positive C_{17} such that*

$$\limsup_{t\to\infty} \|\Delta v(\cdot, t)\|_{L^\infty(\Omega)} \leq M_{12}. \tag{4.1}$$

Proof. We split the proof into two steps:

Step 1. Fix $\theta_0 \in (0,1)$ and set $m = 2, p = \infty$ in Lemma 3, we verify $\lim_{t\to\infty}\sup \|A^{\theta_0} w(\cdot,t)\|_{L^p(\Omega)} \le C_{17}$ for any $p > \max\{2, \gamma, \frac{n(\sigma+1)}{2(n-(\sigma+1))}(\sigma+\gamma)-\sigma\}$ with some $C_{17} = C_{17}(\theta_0, p, \lambda, \gamma, \sigma) > 0$.

Review of (3.21), we know that there exists suitably large $t_0 > 0$ satisfying

$$\|u(\cdot,t)\|_{L^p(\Omega)} \le M_7, \quad \text{for all } t \ge t_0. \tag{4.2}$$

In view of the variation-of-constants representation, we get

$$\begin{aligned} w(\cdot,t) =& e^{(t-t_0)(\Delta-1)} w(\cdot,t_0) + \int_{t_0}^{t} e^{(t-s)(\Delta-1)} u(\cdot,s) ds \\ =& e^{-(1-\lambda)(t-t_0)} e^{-(t-t_0)A} w(\cdot,t_0) + \int_{t_0}^{t} e^{-(1-\lambda)(t-s)} e^{-(t-s)A} u(\cdot,s) ds \end{aligned}$$

for all $t \ge t_0$. Recalling that $A = -\Delta + \lambda$ in reference [36] and applying the (4.2), then there exist constants C_{18}, C_{19}, we can derive

$$\begin{aligned} \|A^{\theta_0} w(\cdot,t)\|_{L^p(\Omega)} \le& e^{-(1-\lambda)(t-t_0)} \|A^{\theta_0} e^{-(t-t_0)A} w(\cdot,t_0)\|_{L^p(\Omega)} \\ &+ \int_{t_0}^{t} e^{-(1-\lambda)(t-s)} \|A^{\theta_0} e^{-(t-s)A} u(\cdot,s)\|_{L^p(\Omega)} ds \\ \le& C_{18} e^{-(1-\lambda)(t-t_0)} (t-t_0)^{-\theta_0} \|w(\cdot,t_0)\|_{L^p(\Omega)} \\ &+ C_{19} \int_{t_0}^{t} e^{-(1-\lambda)(t-s)} (t-s)^{-\theta_0} \|u(\cdot,s)\|_{L^p(\Omega)} ds \\ \le& C_{18} e^{-(1-\lambda)(t-t_0)} (t-t_0)^{-\theta_0} \|w(\cdot,t_0)\|_{L^p(\Omega)} + C_{20} \end{aligned}$$

for all $t \ge t_0$, where $C_{20} := M_7 C_{19} \int_0^\infty e^{-(1-\lambda)\tau} \tau^{-\theta_0} d\tau < \infty$ with $\theta_0 \in (0,1)$. Thereafter

$$\lim_{t\to\infty} \sup \|A^{\theta_0} w(\cdot,t)\|_{L^\infty(\Omega)} \le C_{21},$$

and there exists $t_0 > 0$ large enough fulfilling

$$\|A^{\theta_0} w(\cdot,t)\|_{L^\infty(\Omega)} \le 2C_{21}, \quad \text{for all } t \ge t_0. \tag{4.3}$$

Step 2. The solution v satisfies (4.1).

Let us fix any $\theta_1 \in (1,2)$ on the condition that $\theta_1 - 1 < \theta_0 < 1$ and choose $p > \max\{2, \gamma, \frac{n}{n-2}(\sigma+\gamma)-\sigma\}$ satisfying $p > \frac{n}{2(\theta_1-1)}$, thus

$$2\theta_1 - \frac{n}{p} > 2\theta_1 - 2(\theta_1 - 1) = 2. \tag{4.4}$$

Then following the same procedure as Step 1 and invoking (2.11), (4.3), and reference [36], we estimate

$$\begin{aligned} \|v(\cdot,t)\|_{W^{2,\infty}} \le& C_{22} \|A^{\theta_1} v(\cdot,t)\|_{L^p(\Omega)} \\ \le& C_{22} e^{-(1-\lambda)(t-t_0)} \|A^{\theta_1} e^{-(t-t_0)A} v(\cdot,t_0)\|_{L^p(\Omega)} \\ &+ C_{22} \int_{t_0}^{t} e^{-(1-\lambda)(t-s)} \|A^{\theta_1} e^{-(t-s)A} w(\cdot,s)\|_{L^p(\Omega)} ds \\ \le& C_{23} e^{-(1-\lambda)(t-t_0)} (t-t_0)^{-\theta_1} \|v(\cdot,t_0)\|_{L^p(\Omega)} + C_{24} \end{aligned}$$

for all $t \geq t_0$, where $C_{24} := 2C_{21}C'_{22}\int_0^\infty e^{-(1-\lambda)\tau}\tau^{-(\theta_1-\theta_0)}d\tau < \infty$ with $\theta_1 - 1 < \theta_0 < 1$. This implies (4.1) for some $M_{12} > 0$. In accordance with (4.1), we can pick $t_0 > 0$ suitably large satisfying

$$\|\Delta v(\cdot, t)\|_{L^\infty(\Omega)} \leq 2M_{12}, \quad \text{for all } t \geq t_0. \tag{4.5}$$

With the help of the comparison principle [34], a positive lower bound for u is hereafter addressed after some suitable waiting time. □

Lemma 14. *Let $\Omega \subset \mathbb{R}^n (n \geq 3)$ be a bounded domain with smooth boundary. The functions $D(u)$, $f(u)$, and parameters $\sigma, a_i (i = 1, 2)$ satisfy (1.5)-(1.7) with the positive number p being as defined in (4.4). Assume that (u, v, w) is the solution of (1.4) and the initial data fulfills (1.8). Then there exists a constant M_{13} such that*

$$\liminf_{t\to\infty} \big(\inf_{x\in\Omega} u(x,t)\big) \geq \Big(1 - \frac{a_0 - a_1 - 2M_{12}}{a_1}\Big)^{\frac{1}{\sigma}} := M_{13}, \tag{4.6}$$

Proof. Dealing with the first equation in (1.4), a combination of (3.34), and (4.5) yields

$$\begin{aligned}
u_t =& \nabla \cdot (D(u)\nabla u) - \nabla \cdot (u\nabla v) + u(a_0 - a_1 u^\sigma + a_2 \int_\Omega u^\sigma dx) \\
=& \nabla \cdot (D(u)\nabla u) - \nabla u \cdot \nabla v - u\Delta v + a_0 u - a_1 u^{\sigma+1} + a_2 u \int_\Omega u^\sigma dx \\
\geq& \nabla \cdot (D(u)\nabla u) - \nabla u \cdot \nabla v - 2M_{12}u + a_0 u - a_1 u^{\sigma+1} + a_2 u \int_\Omega u^\sigma dx \\
\geq& \nabla \cdot (D(u)\nabla u) - \nabla u \cdot \nabla v + (a_0 - 2M_{12})u - a_1 u^{\sigma+1},
\end{aligned}$$

for all $x \in \Omega$ and $t \geq t_0$. Moreover, let the function $y(t) \in C^1([t_0, \infty))$ be defined by

$$\begin{cases} y'(t) = (a_0 - 2M_{12}) \cdot y(t) - a_1 y^{\sigma+1}(t), t \geq t_0, \\ y(t_0) = \inf\limits_{x\in\Omega} u(x, t_0). \end{cases}$$

Applying the comparison argument, we gain

$$u(x,t) \geq y(t), y(t) = \frac{C_{25}}{e^{\sigma(a_0 - 2M_{12})t}} + C_{26}, \quad \text{for all } x \in \Omega \text{ and } t \geq t_0,$$

and

$$y(t) \to \Big(1 - \frac{a_0 - a_1 - 2M_{12}}{a_1}\Big)^{\frac{1}{\sigma}}, \quad \text{as } t \to \infty.$$

Which implies that

$$\liminf_{t\to\infty} \big(\inf_{x\in\Omega} u(x,t)\big) \geq \liminf_{t\to\infty} \inf y(t) \geq \Big(1 - \frac{a_0 - a_1 - 2M_{12}}{a_1}\Big)^{\frac{1}{\sigma}}.$$

□

5. Asymptotic behavior

The key to proving Theorem 2 relies on seeking so-called Lyapunov functions. Thus in this section, we will construct the appropriate Lyapunov functions in the following lemmas to obtain the large-time behavior of the solution (u_*, v_*, w_*) of the system (1.4).

Lemma 15. *(Lemma 3.1 in [41]) Let $f : (1, \infty) \to [0, \infty)$ be uniformly continuous such that $\int_1^\infty f(t)dt < \infty$, then*

$$f(t) \to 0, \quad as\ t \to \infty. \tag{5.1}$$

Lemma 16. *Let functions $D(u)$ and $f(u)$ and parameters σ and $a_i (i = 1, 2)$ satisfy (1.5)-(1.7). Then for any classical solution (u, v, w) of (1.4) in $\Omega \times (t_0, \infty)$ verifying*

$$\sup_{t\in[t_0,\infty)} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)}) \le M_{14},$$

with some $t_0 \ge 0$ and $M_{14} > 0$.
Then there exist $\bar{\theta} \in (0, 1)$ and $C_{27} = C_{27}(M_{14}, D_1, \sigma, \gamma, |\Omega|, \|\nabla u(\cdot, t_0 + 2)\|_{C(\bar{\Omega})}) > 0$ such that

$$\|\nabla u(\cdot, t)\|_{C(\bar{\Omega})} \le C_{27}((t - t_0)^{\bar{\theta}} + 1), t > t_0 + 2.$$

Proof. First, using the Neumann heat semigroup estimate [34] to the third and the second equations in system (1.4) we can derive that $\|\nabla v\|_{L^\infty(\Omega)}$ and $\|\nabla w\|_{L^\infty(\Omega)}$ is bounded in $t \in (0, T_{max})$. According to Theorem 1, the system (1.4) possesses a global bounded classical solution. Thus, we can find $t \in [t_0, \infty)$ such that

$$\sup_{t\in[t_0,\infty)} (\|u(\cdot, t)\|_{L^\infty(\Omega)} + \|v(\cdot, t)\|_{W^{1,\infty}(\Omega)}) \le M_{14}.$$

And because of $(u, v, w) \in (C^0(\bar{\Omega} \times [0, T_{max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{max})))^3$, applying a proof method similar to proposition 3.4 in Reference [15], we can obtain that there exists $\bar{\theta} \in (0, 1)$ such that $\|\nabla u(\cdot, t)\|_{C(\bar{\Omega})} \le C_{27}((t - t_0)^{\bar{\theta}} + 1)$ holds. □

Lemma 17. *Assume functions $D(u)$ and $f(u)$ and parameters σ and $a_i (i = 1, 2)$ satisfy (1.5)-(1.7) and (u, v, w) is the solution of (1.4). The initial data (u_0, v_0, w_0) evolves from (1.8). Then there exists a positive constant $H_0 \in [M_{13}, M_{11}]$ such that functions E and F are defined by*

$$E(t) := \int_\Omega (u - u_* - u_* \ln \frac{u}{u_*})dx + \frac{H_0 u_*}{4} \int_\Omega (v - v_*)^2 dx + \frac{H_0 u_*}{2} \int_\Omega (w - w_*)^2 dx, \tag{5.2}$$

$$F(t) := \int_\Omega (u - u_*)^2 dx + \frac{H_0 u_*}{4} \int_\Omega (v - v_*)^2 dx + \frac{H_0 u_*}{4} \int_\Omega (w - w_*)^2 dx, \tag{5.3}$$

where $t > 0, u_* = v_* = w_* = (\frac{a_1}{a_1 - a_2|\Omega|})^{\frac{1}{\sigma}}$, and we have

$$\frac{d}{dt} E(t) \le -F(t), \quad for\ all\ t > 0. \tag{5.4}$$

Proof. Note that

$$E(t) = \int_\Omega (u - u_* - u_* \ln \frac{u}{u_*})dx + \frac{H_0 u_*}{4}\int_\Omega (v - v_*)^2 dx + \frac{H_0 u_*}{2}\int_\Omega (w - w_*)^2 dx \\ :=E_1(t) + E_2(t) + E_3(t). \tag{5.5}$$

By system (1.4) and the assumption in Theorem 2, applying Young's inequality, we deduce that

$$\begin{aligned} E_1^{'}(t) =& \int_\Omega \frac{u-u_*}{u} u_t dx \\ =& \int_\Omega \frac{u-u_*}{u}(\nabla\cdot(D(u)\nabla u) - \nabla\cdot(u\nabla v) + u(a_0 - a_1 u^\sigma + a_2\int_\Omega u^\sigma dx))dx \\ =& - u_*\int_\Omega \frac{|\nabla u|^2}{u^2}D(u)dx + u_*\int_\Omega \frac{\nabla u\cdot\nabla v}{u}dx \\ &+ \int_\Omega (u-u_*)(a_0 - a_1 u^\sigma + a_2\int_\Omega u^\sigma dx)dx \\ \leq& - \frac{u_*}{2}\int_\Omega D_1 u^\gamma \frac{|\nabla u|^2}{u^2}dx + \frac{u_*}{2}\int_\Omega \frac{1}{D_1 u^\gamma}|\nabla v|^2 dx \\ &- a_1\int_\Omega (u-u_*)(u^\sigma - u_*^\sigma)dx + a_2\int_\Omega (u-u_*)dx\int_\Omega (u^\sigma - u_*^\sigma)dx \\ \leq& \frac{H_0 u_*}{2}\int_\Omega |\nabla v|^2 dx - a_1\int_\Omega (u-u_*)(u^\sigma - u_*^\sigma)dx \\ &+ a_2\int_\Omega (u-u_*)dx\int_\Omega (u^\sigma - u_*^\sigma)dx, \end{aligned} \tag{5.6}$$

where $a_0 = a_1 u_*^\sigma - a_2\int_\Omega u_*^\sigma dx$, H_0 is obtained by u with a lower bound.
A simple calculation yields

$$(u - u_*)(u^\sigma - u_*^\sigma) \geq u_*^{\sigma-1}(u - u_*)^2,$$

and hence

$$- a_1\int_\Omega (u-u_*)(u^\sigma - u_*^\sigma)dx \leq -a_1 u_*^{\sigma-1}\int_\Omega (u-u_*)^2 dx. \tag{5.7}$$

We treat the last integral in (5.6) by estimating the integrated function in a pointwise way. Suppose (1.5) is valid, according to reference [15], and the Mean value theorem ensure

$$\begin{aligned} a_2\int_\Omega (u-u_*)dx\int_\Omega (u^\sigma - u_*^\sigma)dx \leq& a_2(\int_\Omega (u-u_*)^2 dx)^{\frac{1}{2}}|\Omega|^{\frac{1}{2}}(\int_\Omega (u^\sigma - u_*^\sigma)^2 dx)^{\frac{1}{2}}|\Omega|^{\frac{1}{2}} \\ \leq& a_2|\Omega|\int_\Omega (u-u_*)^2 dx + a_2|\Omega|\int_\Omega (u^\sigma - u_*^\sigma)^2 dx \\ \leq& a_2|\Omega|\int_\Omega (u-u_*)^2 dx + a_2|\Omega|\int_\Omega (\sigma(u+u_*)^{\sigma-1}|u-u_*|)^2 dx \\ \leq& a_2|\Omega|\int_\Omega (u-u_*)^2 dx + a_2|\Omega|\sigma^2(M_1 + lu_*)^{2\sigma-2}\int_\Omega (u-u_*)^2 dx, \end{aligned} \tag{5.8}$$

where $l > 0$, together with (5.7)-(5.9), we can conclude that

$$\begin{aligned} E_1'(t) \le & \frac{H_0 u_*}{2}\int_\Omega |\nabla v|^2 dx - a_1 u_*^{\sigma-1}\int_\Omega (u-u_*)^2 dx + a_2|\Omega|\int_\Omega (u-u_*)^2 dx \\ & + a_2|\Omega|\sigma^2(M_1 + lu_*)^{2\sigma-2}\int_\Omega (u-u_*)^2 dx. \end{aligned} \tag{5.9}$$

Then we use Young's equality to estimate $E_2(t)$ and $E_3(t)$,

$$\begin{aligned} E_2'(t) = & \frac{H_0 u_*}{2}\int_\Omega v_t(v-v_*)dx \\ = & \frac{H_0 u_*}{2}\int_\Omega (v-v_*)\Delta v dx + \frac{H_0 u_*}{2}\int_\Omega w(v-v_*)dx - \frac{H_0 u_*}{2}\int_\Omega v(v-v_*)dx \\ = & -\frac{H_0 u_*}{2}\int_\Omega |\nabla v|^2 dx + \frac{H_0 u_*}{2}\int_\Omega (v-v_*)(w-w_*)dx - \frac{H_0 u_*}{2}\int_\Omega (v-v_*)^2 dx \\ \le & -\frac{H_0 u_*}{2}\int_\Omega |\nabla v|^2 dx + \frac{H_0 u_*}{4}\int_\Omega (v-v_*)^2 dx + \frac{H_0 u_*}{4}\int_\Omega (w-v_*)^2 dx \\ & -\frac{H_0 u_*}{2}\int_\Omega (v-v_*)^2 dx \\ = & -\frac{H_0 u_*}{2}\int_\Omega |\nabla v|^2 dx - \frac{H_0 u_*}{4}\int_\Omega (v-v_*)^2 dx + \frac{H_0 u_*}{4}\int_\Omega (w-v_*)^2 dx \\ = & -\frac{H_0 u_*}{2}\int_\Omega |\nabla v|^2 dx - \frac{H_0 u_*}{4}\int_\Omega (v-v_*)^2 dx + \frac{H_0 u_*}{4}\int_\Omega (w-w_*)^2 dx \end{aligned} \tag{5.10}$$

and

$$\begin{aligned} E_3'(t) = & H_0 u_*\int_\Omega w_t(w-w_*)dx \\ = & H_0 u_*\int_\Omega (w-w_*)\Delta w dx + H_0 u_*\int_\Omega u(w-w_*)dx - H_0 u_*\int_\Omega w(w-w_*)dx \\ \le & -H_0 u_*\int_\Omega |\nabla w|^2 dx - \frac{H_0 u_*}{2}\int_\Omega (w-w_*)^2 dx + \frac{H_0 u_*}{2}\int_\Omega (u-w_*)^2 dx \\ = & -H_0 u_*\int_\Omega |\nabla w|^2 dx - \frac{H_0 u_*}{2}\int_\Omega (w-w_*)^2 dx + \frac{H_0 u_*}{2}\int_\Omega (u-u_*)^2 dx. \end{aligned} \tag{5.11}$$

Combing $E_1'(t)$, $E_2'(t)$ and $E_3'(t)$, we obtain

$$\begin{aligned} & \frac{d}{dt}E(t) + a_1 u_*^{\sigma-1}\int_\Omega (u-u_*)^2 dx \\ & + \frac{H_0 u_*}{4}\int_\Omega (v-v_*)^2 dx + \frac{H_0 u_*}{4}\int_\Omega (w-w_*)^2 dx \\ \le & (a_2|\Omega| + a_2|\Omega|\sigma^2(M_1 + lu_*)^{2\sigma-2} + \frac{H_0 u_*}{2})\int_\Omega (u-u_*)dx. \end{aligned} \tag{5.12}$$

Since

$$\begin{aligned} & \frac{a_2|\Omega| + a_2|\Omega|\sigma^2(M_1 + lu_*)^{2\sigma-2} + \frac{H_0 u_*}{2}}{a_1 u_*^{\sigma-1}} \le \frac{a_2|\Omega| + a_2|\Omega|\sigma^2(M_1 + lu_*)^{2\sigma-2} + \frac{H_0 u_*}{2}}{a_1 u_*^{\sigma-1} - a_2|\Omega|u_*^{\sigma-1}} \\ = & \frac{a_2|\Omega| + a_2|\Omega|\sigma^2(M_1 + l(\frac{a_0}{a_1 - a_2|\Omega|})^{\frac{1}{\sigma}})^{2\sigma-2} + \frac{H_0}{2}(\frac{a_0}{a_1 - a_2|\Omega|})^{\frac{1}{\sigma}}}{a_0\cdot(\frac{a_1 - a_2|\Omega|}{a_0})^{\frac{1}{\sigma}}} \to 0 \text{ as } a_1 \to \infty, \end{aligned}$$

there is $\widetilde{a_1} > 0$ such that

$$a_1 u_*^{\sigma-1} > a_2|\Omega| + a_2|\Omega|\sigma^2(M_1 + lu_*)^{2\sigma-2} + \frac{H_0 u_*}{2} + 1$$

provided $a_1 > \widetilde{a_1}$. This in conjunction with (5.13) entails that

$$\frac{d}{dt}E(t) + \int_\Omega (u-u_*)^2 dx + \frac{H_0 u_*}{4}\int_\Omega (v-v_*)^2 dx + \frac{H_0 u_*}{4}\int_\Omega (w-w_*)^2 dx \le 0,$$

therefore

$$\begin{aligned}\frac{d}{dt}E(t) \le & -\int_\Omega (u-u_*)^2 dx - \frac{H_0 u_*}{4}\int_\Omega (v-v_*)^2 dx - \frac{H_0 u_*}{4}\int_\Omega (w-w_*)^2 dx \\ \le & -F(t),\end{aligned} \tag{5.13}$$

where $a_1 - a_2|\Omega| > 0$. Lemma 5.3 is proved. □

We are now in a positive to prove our main result.

Proof of Theorem 2. Integrating (5.13) from t_0 to ∞, we have

$$\int_{t_0}^\infty \int_\Omega (u-u_*)^2 dxdt + \int_{t_0}^\infty \int_\Omega (v-v_*)^2 dxdt + \int_{t_0}^\infty \int_\Omega (w-w_*)^2 dxdt \le \infty,$$

According to the standard parabolic regularity theory [42], with the global boundedness of (u, v, w), we can see that there exists $\vartheta \in (0,1)$ and $C_{28} > 0$ such that

$$\|u\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\bar{\Omega}\times[t,t+1])} + \|v\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\bar{\Omega}\times[t,t+1])} + \|w\|_{C^{2+\vartheta,1+\frac{\vartheta}{2}}(\bar{\Omega}\times[t,t+1])} \le C_{28}, \tag{5.14}$$

for all $t > 1$. This clearly implies that $\int_\Omega (u(\cdot,t)-u_*)^2 dx + \int_\Omega (v(\cdot,t)-v_*)^2 dx + \int_\Omega (w(\cdot,t)-w_*)^2 dx$ is uniformly continuous with respect to $t \ge t_0$ provided $t_0 > 1$. Therefore, we infer from Lemma 15 that the following inequality

$$\int_\Omega (u(\cdot,t)-u_*)^2 dx + \int_\Omega (v(\cdot,t)-v_*)^2 dx + \int_\Omega (w(\cdot,t)-w_*)^2 dx \to 0 \ \text{ as } t \to \infty. \tag{5.15}$$

holds.

The Gagliardo–Nirenberg inequality says

$$\|\varpi\|_{L^\infty(\Omega)} \le C_{GN}\|\varpi\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|\varpi\|_{L^2(\Omega)}^{\frac{2}{n+2}} \ \text{ for all } \varpi \in W^{1,\infty}(\Omega). \tag{5.16}$$

Using ϖ as $u-u_*, v-v_*$ and $w-w_*$ in (5.16), respectively, we have from (5.14) and (5.15) that

$$\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)} + \|v(\cdot,t)-v_*\|_{L^\infty(\Omega)} + \|w(\cdot,t)-w_*\|_{L^\infty(\Omega)} \to 0, \ \text{ as } t \to \infty. \tag{5.17}$$

We next defined $\varphi : (0,\infty) \to \mathbb{R}$ and

$$\varphi(s) := s - u_* - u_* \ln\frac{s}{u_*}, \ s > 0.$$

By the Taylor expansion with $s \in (0, \infty)$, there is $\xi(s) \in (0, 1)$ such that

$$\begin{aligned}\varphi(s) :&= \varphi(u_*) + \varphi'(u_*)(s - u_*) + \frac{\varphi''(\xi s + (1-\xi)u_*)}{2}(s - u_*)^2 \\ &= \frac{u_*}{2(\xi s + (1-\xi)u_*)^2}(s - u_*)^2, \ s > 0.\end{aligned}$$

Obviously, $\varphi(s) \geq 0$, and use L'Hôpital's rule to see

$$\lim_{s \to u_*} \frac{\varphi(s)}{(s - u_*)^2} = \lim_{s \to u_*} \frac{u_*}{2(\xi s + (1-\xi)u_*)^2} = \frac{1}{2u_*}.$$

Furthermore, we can pick $t_1 > 0$ such that

$$\frac{1}{4u_*}(u - u_*)^2 \leq \varphi(u(\cdot, t)) \leq \frac{1}{u_*}(u - u_*)^2, \text{ for all } t \geq t_1, \tag{5.18}$$

namely,

$$\frac{1}{4u_*}\int_\Omega (u(\cdot, t) - u_*)^2 dx \leq E_1(t) \leq \frac{1}{u_*}\int_\Omega (u(\cdot, t) - u_*)^2 dx, \text{ for all } t \geq t_1, \tag{5.19}$$

In view of (5.13) and (5.19), we have

$$\frac{d}{dt}E(t) \leq -F(t) \leq -\frac{1}{2}E(t), \text{ for all } t \geq t_1,$$

thus, we get by the Gronwall inequality that

$$E(t) \leq E(t_1)e^{-\frac{1}{2}(t - t_1)}, \text{ for all } t \geq t_1,$$

Together with (5.18), we obtain the estimate

$$\|u(\cdot, t) - u_*\|^2_{L^2(\Omega)} + \|v(\cdot, t) - v_*\|^2_{L^2(\Omega)} + \|w(\cdot, t) - w_*\|^2_{L^2(\Omega)} \leq C_{29}e^{-\frac{1}{2}(t - t_1)}, \tag{5.20}$$

for all $t \geq t_1$, with $C_{29} = C_{29}(a_0, a_1, a_2, \sigma, \gamma, |\Omega|) > 0$. By using the Gagliardo–Nirenberg inequality with Lemma 16 and (5.14)-(5.15),

$$\begin{aligned}\|u(\cdot, t) - u_*\|_{L^\infty(\Omega)} &\leq C_{30}\|\nabla u(\cdot, t)\|^{\frac{n}{n+2}}_{L^\infty(\Omega)}\|u(\cdot, t) - u_*\|^{\frac{2}{n+2}}_{L^2(\Omega)} \\ &\leq C_{31}((t - t_1)^{\bar{\theta}} + 1)e^{-\frac{(t-t_1)}{2(n+2)}} \\ &\leq C_{32}e^{-\frac{(t-t_1)}{4(n+2)}}, \ t > t_1 + 2\end{aligned} \tag{5.21}$$

with $C_{30} = C_{30}(|\Omega|), C_{31} = C_{31}(a_0, a_1, a_2, \sigma, \gamma, |\Omega|, \|\nabla u(\cdot, t_1)\|_{L^\infty(\Omega)}) > 0, C_{32} = C_{32}(C_{31}, \bar{\theta})$. Similarly, an application of the Gagliardo–Nirenberg inequality with combining (3.31) and (5.20) indicates that

$$\|v(\cdot, t) - v_*\|_{L^\infty(\Omega)} \leq C_{33}e^{-\frac{(t-t_1)}{4(n+2)}}, \ t > t_1 + 2 \tag{5.22}$$

$$\|w(\cdot, t) - w_*\|_{L^\infty(\Omega)} \leq C_{34}e^{-\frac{(t-t_1)}{4(n+2)}}, \ t > t_1 + 2 \tag{5.23}$$

for some $C_{33} = C_{33}(a_0, a_1, a_2, \sigma, \gamma, |\Omega|, \|\nabla v(\cdot, t_1)\|_{L^\infty(\Omega)}) > 0$, and $C_{34} = C_{34}(a_0, a_1,$ $a_2, \sigma, \gamma, |\Omega|, \|\nabla w(\cdot, t_1)\|_{L^\infty(\Omega)}) > 0$.
A combination of (5.21)-(5.23), gives us

$$\|u(\cdot, t) - u_*\|_{L^\infty(\Omega)} + \|v(\cdot, t) - v_*\|_{L^\infty(\Omega)} + \|w(\cdot, t) - w_*\|_{L^\infty(\Omega)} \leq Ce^{-\frac{(t-t_1)}{4(n+2)}},$$

with $t \geq t_1$ and $C = C(a_0, a_1, a_2, \sigma, \gamma, |\Omega|) > 0$, which completes the proof of Theorem 2. □

6. Conclusion

In this paper, we considered that the growth or death of cells is influenced by external factors and the production of chemical signals goes through intermediate stages, and thus we investigate a fully parabolic chemotaxis system with a nonlocal growth term and indirect signal production. The work is carried out under the condition of spatial dimension $n \geq 3$, when the initial data, the diffusion function, the logistic source term and related parameters satisfy certain conditions, the global boundedness of solutions to system (1.4) is proved by applying the maximum principle, variation-of constants formula, Neumann heat semigroup estimation, Young's inequality, Gagliardo–Nirenberg inequality and so on. In addition, by constructing appropriate Lyapunov functions, we obtained the asymptotic behavior of solutions.

Author contributions

Min Jiang, Dandan Liu and Rengang Huang: Methodology; Min Jiang and Dandan Liu: Writing-original draft; Dandan Liu and Rengang Huang: Writing-review and editing.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

References

1. E. F. Keller, L. A. Segel, Initiation of slime mold aggregation viewed as an instability, *J. Theor. Biol.*, **26** (1970), 399–415.
2. K. Osaki, T. Tsujikawa, A. Yagi, M. Mimura, Exponential attractor for a chemotaxis-growth system of equations, *Nonlinear Anal.*, **51** (2002), 119–144. https://doi.org/10.1016/s0362-546x(01)00815-x
3. J. I. Tello, M. Winkler, A chemotaxis system with logistic source, *Commun. Partial Differ. Equ.*, **32** (2007), 849–877. https://doi.org/10.1080/03605300701319003

4. M. Winkler, Chemotaxis with logistic source: very weak global solutions and their boundedness properties, *J. Math. Anal. Appl.*, **348** (2008), 708–729. https://doi.org/10.1016/j.jmaa.2008.07.071
5. X. Cao, Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with logistic source, *J. Math. Anal. Appl.*, **412**(2014), 181-188. https://doi.org/10.1016/j.jmaa.2013.10.061
6. K. Osaki, A. Yagi, Finite dimensional attractor for one-dimensional Keller-Segel equations, *Funkcial. Ekvac.*, **44** (2001), 441–469.
7. M. A. Herrero, J. J. Velázquez, A blow-up mechanism for a chemotaxis model, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, **24** (1997), 633–683.
8. T. Nagai, T. Senba, K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, *Funkcial. Ekvac.*, **40** (1997), 411–433.
9. M. Winkler, Finite-time blow-up in the higher-dimensional parabolic-parabolic Keller-Segel system, *J. Math. Pure. Appl.*, **100** (2013), 748–767. https://doi.org/10.1016/j.matpur.2013.01.020
10. S. Ishida, K. Seki, T. Yokota, Boundedness in quasilinear Keller-Segel systems of parabolic-parabolic type on non-convex bounded domains, *J. Differ. Equ.*, **256** (2014), 2993–3010. https://doi.org/10.1016/j.jde.2014.01.028
11. X. Cao, Large time behavior in the logistic Keller-Segel model via maximal Sobolev regularity, *Discrete Contin. Dyn. Syst. Ser. B.*, **22** (2017), 3369–3378. https://doi.org/10.3934/dcdsb.2017141
12. Q. Wang, J. Yang, F. Yu, Boundedness in logistic keller-segel models with nonlinear diffusion and sensitivity functions, *Discrete Contin. Dyn. Syst.*, **37** (2017), 5021-5036. https://doi.org/10.3934/dcds.2017216
13. J. Zheng, Boundedness of solutions to a quasilinear parabolic-parabolic Keller-Segel system with a logistic source, *J. Math. Anal. Appl.*, **431** (2015), 867–888. https://doi.org/10.1016/j.jmaa.2015.05.071
14. M. Zhuang, W. Wang, S. Zheng, Boundedness in a fully parabolic chemotaxis system with logistic-type source and nonlinear production, *Nonlinear Anal-Real.*, **47** (2019), 473–483. https://doi.org/10.1016/j.nonrwa.2018.12.001
15. M. Ding, W. Wang, S. Zhou, S. Zheng, Asymptotic stability in a fully parabolic quasilinear chemotaxis model with general logistic source and signal production, *J. Differ. Equ.*, **268** (2020), 6729–6777. https://doi.org/10.1016/j.jde.2019.11.052
16. Y. Tao, M. Winkler, Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity, *J. Differ. Equ.*, **252** (2012), 692–715. https://doi.org/10.1016/j.jde.2011.08.019
17. D. Liu, Y. Tao, Boundedness in a chemotaxis system with nonlinear signal production, *Appl. Math. J. Chin. Univ. Ser. B.*, **31** (2016), 379–388. https://doi.org/10.1007/s11766-016-3386-z
18. M. Winkler, A critical blow-up exponent in a chemotaxis system with nonlinear signal production, *Nonlinearity*, **31** (2018), 2031–2056. https://doi.org/10.1088/1361-6544/aaaa0e
19. M. Negreanu, J. I. Tello, On a competitive system under chemotactic effects with non-local terms, *Nonlinearity*, **26** (2013), 1083–1103. https://doi.org/10.1088/0951-7715/26/4/1083
20. S. Bian, L. Chen, E. A. Latos, Chemotaxis model with nonlocal nonlinear reaction in the whole space, *Discrete Contin. Dyn. Syst.*, **38** (2018), 5067–5083. https://doi.org/10.3934/dcds.2018222

21. E. A. Latos, Nonlocal reaction preventing blow-up in the supercritical case of chemotaxis, arXiv preprint arXiv: 2011.10764, 2020. https://doi.org/10.48550/arXiv.2011.10764

22. M. Negreanu, J. I. Tello, A. M. Vargas, On a fully parabolic chemotaxis system with nonlocal growth term, *Nonlinear Anal.*, **213** (2021), 112518. https://doi.org/10.1016/j.na.2021.112518

23. G. Ren, Global boundedness and asymptotic behavior in an attraction-repulsion chemotaxis system with nonlocal terms, *Z. Angew. Math. Phys.*, **73** (2022), 200. https://doi.org/10.1007/s00033-022-01832-7

24. T. B. Issa, R. B. Salako, Asymptotic dynamics in a two-species chemotaxis model with non-local terms, *Discrete Contin. Dyn. Syst. Ser. B.*, **22** (2017), 3839–3874. https://doi.org/10.3934/dcdsb.2017193

25. T. B. Issa, W. Shen, Persistence, coexistence and extinction in two species chemotaxis models on bounded heterogeneous environments, *J. Dyn. Differ. Equ.*, **31** (2019), 1839–1871. https://doi.org/10.1007/s10884-018-9686-7

26. P. Zheng, On a parabolic-elliptic Keller-Segel system with nonlinear signal production and nonlocal growth term, *Dynam. Part. Differ. Eq.*, **21** (2024), 61–76. https://doi.org/10.4310/DPDE.2024.v21.n1.a3

27. Y. Chiyo, F. G. Düzgün, S. Frassu, G. Viglialoro, Boundedness through nonlocal dampening effects in a fully parabolic chemotaxis model with sub and superquadratic growth, *Appl. Math. Opt.*, **89** (2024), 9. https://doi.org/10.1007/s00245-023-10077-3

28. W. Zhang, P. Niu, S. Liu, Large time behavior in a chemotaxis model with logistic growth and indirect signal production, *Nonlinear Anal. Real Word Appl.*, **50** (2019), 484–497. https://doi.org/10.1016/j.nonrwa.2019.05.002

29. D. Li, Z. Li, Asymptotic behavior of a quasilinear parabolic-elliptic-elliptic chemotaxis system with logistic source, *Z. Angew. Math. Phys.*, **73** (2022), 1–17. https://doi.org/10.1007/s00033-021-01655-y

30. C. Wang, Y. Zhu, X. Zhu, Long time behavior of the solution to a chemotaxis system with nonlinear indirect signal production and logistic source, *Electron. J. Qual. Theo.*, **2023** (2023), 1–21. https://doi.org/10.14232/ejqtde.2023.1.11

31. S. Wu, Boundedness in a quasilinear chemotaxis model with logistic growth and indirect signal production, *Acta. Appl. Math.*, **176** (2021), 1–14. https://doi.org/10.1007/s10440-021-00454-x

32. W. Wang, A quasilinear fully parabolic chemotaxis system with indirect signal production and logistic source, *J. Math. Anal. Appl.*, **477** (2019), 488–522. https://doi.org/10.1016/j.jmaa.2019.04.043

33. W. Zhang, S. Liu, P. Niu, Asymptotic behavior in a quasilinear chemotaxis-growth system with indirect signal production, *J. Math. Anal. Appl.*, **486** (2020), 123855. https://doi.org/10.1016/j.jmaa.2020.123855

34. M. Winkler, Global asymptotic stability of constant equilibria in a fully parabolic chemotaxis system with strong logistic dampening, *J. Differ. Equ.*, **257** (2014), 1056–1077. https://doi.org/10.1016/j.jde.2014.04.023

35. Y. Tao, M. Winkler, A chemotaxis-haptotaxis model: the roles of nonlinear diffusion and logistic source, *SIAM J. Math. Anal.*, **43** (2011), 685–704. https://doi.org/10.1137/100802943

36. D. Horstmann, M. Winkler, Boundedness vs. blow-up in a chemotaxis system, *J. Differ. Equ.*, **215** (2005), 52–107. https://doi.org/10.1016/j.jde.2004.10.022

37. M. Winkler, Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model, *J. Differ. Equ.*, **248** (2010), 2889–2905. https://doi.org/10.1016/j.jde.2010.02.008

38. L. Nirenberg, On elliptic partial differential equations, *Ann. Scuola. Norm-Sci.*, **13** (1959), 115–162.

39. C. Stinner, C. Surulescu, M. Winkler, Global weak solutions in a PDE-ODE system modeling multiscale cancer cell invasion, *SIAM. J. Comput.*, **46** (2014), 1969–2007. https://doi.org/10.1137/13094058X

40. C. Mu, L. Wang, P. Zheng, Q. Zhang, Global existence and boundedness of classical solutions to a parabolic-parabolic chemotaxis system, *Nonlinear Anal-Real.*, **14** (2013), 1634–1642. https://doi.org/10.1016/j.nonrwa.2012.10.022

41. X. Bai, M. Winkler, Equilibration in a fully parabolic two-species chemotaxis system with competitive kinetics, *Indiana. U. Math. J.*, **65** (2016), 553–583. https://doi.org/10.1512/iumj.2016.65.5776

42. O. A. Ladyzhenskaia, V. A. Solonnikov, N. N. Ural'tseva, *Linear and quasi-linear equations of parabolic type*, AMS, 1968.