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Research article

Asymptotic symmetry of solutions for reaction-diffusion equations via elliptic geometry

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Abstract: In this paper, we investigated the asymptotic symmetry and monotonicity of positive solutions to a reaction-diffusion equation in the unit ball, utilizing techniques from elliptic geometry. First, we discussed the properties of solutions in the elliptic space. Then, we established crucial principles, including the asymptotic narrow region principle. Finally, we employed the method of moving planes to demonstrate the asymptotic symmetry of the solutions.

Keywords: parabolic equation; asymptotic symmetry; monotonicity; elliptic geometry
Mathematics Subject Classification: 35R11, 35B07

1. Introduction

In this paper, we investigate the asymptotic symmetry and monotonicity of positive solutions to the following reaction-diffusion equation:

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - \Delta u(x,t) = f(|x|, u(x,t), t), & (x,t) \in B_1(0) \times (0,\infty), \\ u(x,t) = 0, & (x,t) \in \partial B_1(0) \times (0,\infty), \end{cases} \tag{1.1}$$

where $B_1(0)$ is the unit ball in $\mathbb{R}^N$, $N \geqslant 2$.

Reaction-diffusion equations are fundamental mathematical models that describe how the concentration of substances evolves over time under the influence of both chemical reactions and diffusion. They are widely applied in fields such as biology for pattern formation, chemistry for reaction waves, ecology for population dynamics, neuroscience for modeling brain activity, and physics for phase transitions and material science [1, 2]. These equations provide critical insights into self-organization in complex systems and form a theoretical basis for understanding various spatial structures and dynamic behaviors in nature. In terms of the application to the asymptotic symmetry of solutions, they also have been used in the proofs of convergence results for some autonomous and time-periodic equations [3].

For elliptic equations, the method of moving planes, initially introduced by Alexandrov [4] and Serrin [5], and later developed by Berestycki and Nirenberg [6], Gidas, Ni, and Nirenberg [7], and Chen and Li [8], among others, is a powerful tool for investigating the symmetry and monotonicity of solutions. In [7], Gidas, Ni, and Nirenberg proved that if for each $u \in (0, \infty)$, the function $r \mapsto f(r, u) : (0, 1) \to \mathbb{R}$ is non-increasing, then any positive $C^2(\bar{B}_R(0))$ solution of

$$\begin{cases} \Delta u + f(|x|, u) = 0, & x \in B_R(0), \\ u = 0, & x \in \partial B_R(0), \end{cases} \tag{1.2}$$

is radially symmetric and decreasing in r. In recent years, several systematic approaches have emerged for studying symmetry and monotonicity in both local and nonlocal elliptic equations. These include the method of moving planes in integral form [9–12], the direct method of moving planes [13–18], the method of moving spheres [19–23], and sliding methods [6,24,25]. For further details on these methods, we refer to [26–29] and the references therein.

Notably, by combining hyperbolic geometry with the spirit of the moving plane method, a completely new monotonicity result can be achieved. Under the condition that $(1 - r^2)^{(N+2)/2} f(r, (1 - r^2)^{-(N-2)/2} u)$ is decreasing in $r \in (0, 1)$, for a fixed $u \in (0, \infty)$, Naito, Nishimoto, and Suzuki have sequentially achieved that each positive solution of (1.2) is radially symmetric and $(1 - r^2)^{-(N-2)/2} u$ is decreasing in $r \in (0, 1)$ in the cases of $N = 2$ [30] and $N \geqslant 2$ [31]. In [32], using not only hyperbolic geometry but also elliptic geometry, Shioji and Watanabe established the symmetry and monotonicity properties of a wide class of strong solutions of (1.2).

As to the parabolic equations, the situation becomes significantly more intricate. There have been some preliminary studies of symmetric solutions of the periodic-parabolic problem [33–35] and further results concerning entire solutions where the time variable $t \in \mathbb{R}$ [36,37]. In a different type of symmetry considering the Cauchy-Dirichlet problem of reaction-diffusion equations, asymptotic symmetry shows a tendency of positive solutions to improve their symmetry as time variable $t \in (0, \infty)$ increases, becoming "symmetric and monotone in the limit" as $t \to \infty$. In this context, Li [38] obtained symmetry for positive solutions in the case that the initial solutions are symmetric. Without the symmetric initial solutions, in bounded, symmetric, and strictly convex domain Ω, Hess and Poláčik [39] showed the asymptotic symmetry and monotonicity of positive solutions to the following problem:

$$\begin{cases} \partial_t u = \Delta u(x, t) + f(u(x, t), t), & (x, t) \in \Omega \times (0, \infty), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty). \end{cases} \tag{1.3}$$

It was assumed in [39] that f is uniformly Lipschitz-continuous in u and Hölder-continuous of exponent $(\alpha, (\alpha/2))$ with respect to (u, t). Subsequently, in both bounded and unbounded domains, Poláčik [40,41] extended the result to the following generalized fully nonlinear parabolic equation:

$$\begin{cases} u_t = F(t, x, u, Du, D^2 u), & (x, t) \in \Omega \times (0, \infty), \\ u(x, t) = 0, & (x, t) \in \partial\Omega \times (0, \infty). \end{cases} \tag{1.4}$$

In an independent work, Babin and Sell [42] gave a similar result. With the symmetry conclusion of entire solutions of (1.4), Poláčik [37] gave a survey that summarized the following limitations of existing asymptotic symmetry results: the regularity requirements of the time-dependence of the nonlinearities and domain, the compactness requirements of spatial derivatives, and the strong positivity requirements on the nonlinearities in the case of nonsmooth domains [43]. Moreover, Saldaña and Weth [44] established the asymptotic foliated Schwarz symmetry which indicates that all positive solutions of (1.1) become axially symmetric with respect to a common axis passing through the origin as $t \to \infty$.

The motivation for this paper is to extend the results of [30–32] to the framework of reaction-diffusion equations. Our approach, which integrates elliptic geometry with the method of moving planes, is inspired by the work of [45] and [32].

In order to clarify the theorem, we first introduce the ω-limit-set of u:

$$\omega(u) := \left\{ \varphi \in C_0(\overline{B_1(0)}) \,|\, \exists\, t_k \to \infty \ \ \textit{such that}\ \varphi = \lim_{k\to\infty} u(\cdot, t_k) \right\}. \tag{1.5}$$

By the discussion of u in the Appendix, the orbit $\{u(\cdot,t) : t > 0\}$ is relatively compact in $C(\overline{B_1(0)})$ and $\omega(u)$ is a nonempty compact subset of $C(\overline{B_1(0)})$.

From now on, we will assume the nonlinearity $f : (0,1) \times (0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfies

(F1) for each $M > 0$, $f(r,u,t)$ is Lipschitz-continuous in u uniformly with respect to t and r in the region $(0,1) \times [-M, M] \times [0,\infty)$;

(F2) for each $r \in (0,1)$ and $\tau > 0$ there exist a constant H and a small constant ε_0 both independent of τ such that
$$|f(r_1,u,t_1) - f(r_2,u,t_2)| \leqslant H(|r_1 - r_2|^{\alpha} + |t_1 - t_2|^{\frac{\alpha}{2}}),$$
for all $u \in (0,\infty)$ and (r_1,t_1), $(r_2,t_2) \in (0,1) \times [\tau - \varepsilon_0, \tau + \varepsilon_0]$;

(F3) for all $t \in [0,\infty)$ and each fixed $u \in (0,\infty)$, $(1+r^2)^{\frac{N+2}{2}} f(r, (1+r^2)^{-\frac{N-2}{2}} u, t)$ is decreasing in $r \in (0,1)$.

Theorem 1.1. *Let f satisfy conditions (F1), (F2), and (F3). Assume $u \in C^{2,1}(B_1(0) \times (0,\infty)) \cap C(\overline{B_1(0)} \times [0,\infty))$ is a positive bounded solution of* (1.1) *satisfies that $\partial u/\partial t$ is non-negative, bounded, and $\nabla u(x,t)$ is bounded for all $(x,t) \in B_1(0) \times (0,\infty)$. Then for each $\varphi(x) \in \omega(u)$, the following holds:*

- *Either $\varphi(x) \equiv 0$;*
- *Or $\varphi(x)$ is radially symmetric about the origin and satisfies $\partial_r \left((1+|r|^2)^{\frac{N-2}{2}} \varphi(r)\right) < 0$ for $r = |x| \in (0,1)$, where $x \in B_1(0)$.*

The structure of the paper is as follows: Section 2 presents the preliminary results that form the foundation for our main findings. In Section 3, we introduce the asymptotic narrow region principle, which plays a crucial role in the proof of our main theorem. The proof of Theorem 1.1 is provided in detail in Section 4. Finally, the Appendix includes additional properties of the solutions.

2. Preliminaries

In this section, we will establish some essential preliminaries for applying the moving plane method in the space $(B_1(0), g)$, where the metric tensor g is defined by

$$\frac{4|dx|^2}{(1+|x|^2)^2}. \tag{2.1}$$

Here, $|\cdot|$ denotes the standard Euclidean norm, consistent with the notation used in other sections. For each $\lambda \in (0,1)$, let $T_\lambda \subset B_1(0)$ be a totally geodesic plane intersecting the x_1-axis orthogonally at the point $(\lambda, 0, \ldots, 0)$. It follows that

$$T_\lambda = \left\{x \in B_1(0) : |x - e_\lambda| = \left(\frac{1+\lambda^2}{2\lambda}\right)\right\}, \tag{2.2}$$

where

$$e_\lambda = \left(\left(\frac{1-\lambda^2}{-2\lambda}\right), 0, \cdots, 0\right). \tag{2.3}$$

Define

$$\Sigma_\lambda = \left\{x \in B_1(0) \,\middle|\, |x - e_\lambda| > \left(\frac{1+\lambda^2}{2\lambda}\right)\right\}. \tag{2.4}$$

For each $x \in \Sigma_\lambda$, let x^λ denote the reflection of x with respect to T_λ in the space $(B_1(0), g)$. This reflection can be expressed as

$$x^\lambda = e_\lambda + \left(\frac{1+\lambda^2}{2\lambda}\right)^2 \frac{x - e_\lambda}{|x - e_\lambda|^2}. \tag{2.5}$$

We remark that

$$|x|^2 > |x^\lambda|^2. \tag{2.6}$$

The proof is presented in Lemma 5.1.

The Laplace-Beltrami operator $\Delta_{(g,x)}$ in the space $(B_1(0), g)$ at $x \in B_1(0)$ is defined by

$$\Delta_{(g,x)} = \left(\frac{1+|x|^2}{2}\right)^2 \left(\Delta - \frac{2(N-2)}{1+|x|^2} \sum_{i=1}^N x_i \frac{\partial}{\partial x_i}\right), \tag{2.7}$$

where $\Delta = \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2}$.

Let $u(x,t)$ be a solution to the parabolic problem given by equation (1.1). For each $\lambda \in (0,1)$, we introduce new functions $v(x,t)$, $w_\lambda(x,t)$, and $z_\lambda(x,t)$ to compare the value of $u(x,t)$ with $u(x^\lambda, t)$ and to simplify the analysis of the gradient's impact. These functions are defined as follows:

$$v(x,t) = (1+|x|^2)^{\frac{N-2}{2}} u(x,t), \qquad (x,t) \in B_1(0) \times (0,\infty), \tag{2.8}$$

$$w_\lambda(x,t) = v(x^\lambda, t) - v(x,t), \qquad (x,t) \in \Sigma_\lambda \times (0,\infty), \tag{2.9}$$

$$z_\lambda(x,t) = (1+|x|^2)^{-\frac{N-2}{2}} w_\lambda(x,t), \qquad (x,t) \in \Sigma_\lambda \times (0,\infty). \tag{2.10}$$

From (2.8)–(2.10), for $(x,t) \in \Sigma_\lambda \times (0,\infty)$, we obtain

$$\begin{aligned} z_\lambda(x,t) &= (1+|x|^2)^{-\frac{N-2}{2}} \left((1+|x^\lambda|^2)^{\frac{N-2}{2}} u(x^\lambda,t) - (1+|x|^2)^{\frac{N-2}{2}} u(x,t)\right) \\ &= \left(\frac{1+|x^\lambda|^2}{1+|x|^2}\right)^{\frac{N-2}{2}} u(x^\lambda,t) - u(x,t). \end{aligned} \tag{2.11}$$

Clearly, $z_\lambda \in C^{2,1}(\Sigma_\lambda \times (0,\infty)) \cap C(\overline{\Sigma_\lambda} \times [0,\infty))$ and $z_\lambda = 0$ on T_λ. For each $\varphi(x) \in \omega(u)$, denote

$$\begin{aligned} \psi_\lambda(x) &= (1+|x|^2)^{-\frac{N-2}{2}} \left((1+|x^\lambda|^2)^{\frac{N-2}{2}} \varphi(x^\lambda) - (1+|x|^2)^{\frac{N-2}{2}} \varphi(x)\right) \\ &= \left(\frac{1+|x^\lambda|^2}{1+|x|^2}\right)^{\frac{N-2}{2}} \varphi(x^\lambda) - \varphi(x), \end{aligned} \tag{2.12}$$

which is an ω-limit of $z_\lambda(x,t)$.

By virtue of the definitions given in (2.7) and (2.8), we observe that for $x \in B_1(0)$, the function v satisfies

$$\begin{aligned}\Delta_{(g,x)}v(x,t) &= \left(\frac{1+|x|^2}{2}\right)^2\left(\Delta v(x,t) - \frac{2(N-2)}{1+|x|^2}\sum_{i=1}^{N} x_i\frac{\partial v}{\partial x_i}\right)\\ &= \frac{(1+|x|^2)^{\frac{N+2}{2}}}{4}\Delta u(x,t) + \frac{N(N-2)}{4}(1+|x|^2)^{\frac{N-2}{2}}u(x,t)\\ &= \frac{(1+|x|^2)^{\frac{N+2}{2}}}{4}\left(\frac{\partial u}{\partial t} - f(|x|,u(x,t),t)\right) + \frac{N(N-2)}{4}v(x,t)\\ &= \left(\frac{1+|x|^2}{2}\right)^2\frac{\partial v}{\partial t} - \frac{(1+|x|^2)^{\frac{N+2}{2}}}{4}f(|x|,(1+|x|^2)^{-\frac{N-2}{2}}v(x,t),t) + \frac{N(N-2)}{4}v(x,t).\end{aligned} \tag{2.13}$$

In addition to the above notations, we now present the following lemmas to establish the properties of $z_\lambda(x,t)$.

Lemma 2.1. *Let $u \in C^{2,1}(B_1(0)\times(0,\infty))\cap C(\overline{B_1(0)}\times[0,\infty))$ be a positive bounded solution of* (1.1) *that satisfies $\frac{\partial u}{\partial t}(x,t) \geqslant 0$ for all $(x,t) \in B_1(0)\times(0,\infty)$. Assume that the function $(1+r^2)^{\frac{N+2}{2}}f(r,(1+r^2)^{-\frac{N-2}{2}}s,t)$ is nonincreasing in $r \in (0,1)$ for each fixed $s \in (0,\infty)$ and $t \in (0,\infty)$. Under these conditions, z_λ satisfies*

$$\frac{\partial z_\lambda}{\partial t} - \Delta z_\lambda \geqslant c_\lambda(x,t)z_\lambda \tag{2.14}$$

in $\Sigma_\lambda\times(0,\infty)$, where

$$c_\lambda(x,t) = \frac{f(|x|,(1+|x|^2)^{-\frac{N-2}{2}}v(x^\lambda,t),t) - f(|x|,(1+|x|^2)^{-\frac{N-2}{2}}v(x,t),t)}{(1+|x|^2)^{-\frac{N-2}{2}}v(x^\lambda,t) - (1+|x|^2)^{-\frac{N-2}{2}}v(x,t)}. \tag{2.15}$$

Proof. Let $\lambda \in (0,1)$, $x \in \Sigma_\lambda$, and set $y = x^\lambda$. Since the Laplace-Beltrami operator is invariant under the isometry, as shown in Lemma 5.2 in the Appendix, we have

$$\Delta_{(g,y)}v(y,t) = \Delta_{(g,x)}v(x^\lambda,t). \tag{2.16}$$

Using this equality, together with (2.6) and the monotonicity assumption (F3) of $(1+r^2)^{\frac{N+2}{2}}f(r,(1+r^2)^{-\frac{N-2}{2}}s,t)$, we deduce that

$$\begin{aligned}0 &= \Delta_{(g,y)}v(y,t) - \frac{N(N-2)}{4}v(y,t) + \frac{(1+|y|^2)^{\frac{N+2}{2}}}{4}f(|y|,(1+|y|^2)^{-\frac{N-2}{2}}v(y,t),t) - \left(\frac{1+|y|^2}{2}\right)^2\frac{\partial v(y,t)}{\partial t}\\ &\quad -\Delta_{(g,x)}v(x,t) + \frac{N(N-2)}{4}v(x,t) - \frac{(1+|x|^2)^{\frac{N+2}{2}}}{4}f(|x|,(1+|x|^2)^{-\frac{N-2}{2}}v(x,t),t) + \left(\frac{1+|x|^2}{2}\right)^2\frac{\partial v(x,t)}{\partial t}\\ &= \Delta_{(g,x)}w_\lambda(x,t) - \frac{N(N-2)}{4}w_\lambda(x,t)\\ &\quad +\frac{(1+|x^\lambda|^2)^{\frac{N+2}{2}}}{4}f(|x^\lambda|,(1+|x^\lambda|^2)^{-\frac{N-2}{2}}v(x^\lambda,t),t) - \frac{(1+|x|^2)^{\frac{N+2}{2}}}{4}f(|x|,(1+|x|^2)^{-\frac{N-2}{2}}v(x,t),t)\\ &\quad -\left(\frac{1+|x|^2}{2}\right)^2\frac{\partial w_\lambda(x,t)}{\partial t} + \frac{\partial v(y,t)}{\partial t}\left(\left(\frac{1+|x|^2}{2}\right) - \left(\frac{1+|x^\lambda|^2}{2}\right)\right)\end{aligned}$$

$$\geqslant \Delta_{(g,x)}w_\lambda(x,t) - \frac{N(N-2)}{4}w_\lambda(x,t) - \left(\frac{1+|x|^2}{2}\right)^2\frac{\partial w_\lambda(x,t)}{\partial t}$$
$$+\frac{(1+|x|^2)^{\frac{N+2}{2}}}{4}f(|x|,(1+|x|^2)^{-\frac{N-2}{2}}v(x^\lambda,t),t) - \frac{(1+|x|^2)^{\frac{N+2}{2}}}{4}f(|x|,(1+|x|^2)^{-\frac{N-2}{2}}v(x,t),t)$$
$$= \Delta_{(g,x)}w_\lambda(x,t) - \frac{N(N-2)}{4}w_\lambda(x,t) + \left(\frac{1+|x|^2}{2}\right)^2 c_\lambda(x,t)w_\lambda(x,t) - \left(\frac{1+|x|^2}{2}\right)^2\frac{\partial w_\lambda(x,t)}{\partial t}$$
$$= \left(\frac{1+|x|^2}{2}\right)^2\left(-\frac{\partial w_\lambda(x,t)}{\partial t} + \Delta w_\lambda - \frac{2(N-2)}{1+|x|^2}\sum_{i=1}^N x_i\frac{\partial w_\lambda}{\partial x_i} - \frac{N(N-2)}{(1+|x|^2)^2}w_\lambda(x,t) + c_\lambda(x,t)w_\lambda(x,t)\right).$$

Thus we obtain

$$\frac{\partial w_\lambda(x,t)}{\partial t} - \Delta w_\lambda + \frac{2(N-2)}{1+|x|^2}\sum_{i=1}^N x_i\frac{\partial w_\lambda}{\partial x_i} + \frac{N(N-2)}{(1+|x|^2)^2}w_\lambda(x,t) \geqslant c_\lambda(x,t)w_\lambda(x,t). \tag{2.17}$$

From (2.10), an elementary computation shows that

$$\begin{aligned}
\frac{\partial w_\lambda(x,t)}{\partial t} &= (1+|x|^2)^{\frac{N-2}{2}}\frac{\partial z_\lambda}{\partial t},\\
\Delta z_\lambda(x,t) &= (\Delta(1+|x|^2)^{-\frac{N-2}{2}})w_\lambda + 2\nabla(1+|x|^2)^{-\frac{N-2}{2}})\cdot\nabla w_\lambda + (1+|x|^2)^{-\frac{N-2}{2}})\Delta w_\lambda\\
&= (1+|x|^2)^{-\frac{N-2}{2}}\left(\Delta w_\lambda - \frac{2(N-2)}{1+|x|^2}\sum_{i=1}^N x_i\frac{\partial w_\lambda}{\partial x_i} - \frac{N(N-2)}{(1+|x|^2)^2}w_\lambda(x,t)\right).
\end{aligned}$$

Therefore, z_λ, as defined in (2.10), satisfies the inequality given in (2.14). □

Lemma 2.2. *Assume that for some* $(x_0,t_0)\in T_\lambda\times(0,\infty)$*, there holds*

$$\frac{\partial z_\lambda}{\partial n}(x_0,t_0) < 0,$$

where n denotes the unit outer normal to $\partial\Sigma_\lambda$. Then,

$$\frac{\partial v}{\partial n}(x_0,t_0) > 0. \tag{2.18}$$

Proof. Since $z_\lambda(x,t)=0$ on $T_\lambda\times(0,\infty)$, we have

$$\frac{\partial w_\lambda}{\partial n}(x_0,t_0) = (1+|x_0|^2)^{\frac{N-2}{2}}\frac{\partial z_\lambda}{\partial n}(x_0,t_0) < 0. \tag{2.19}$$

We define x_p as $x_p = x_0 - pn(x_0)$, $p>0$, and $n(x_0)$ is the unit outer normal to $\partial\Sigma_\lambda$ at point x_0. Specifically, $n(x_0) = -(x_0-e_\lambda)/|x_0-e_\lambda|$.

For $x_p\in\Sigma_\lambda$, using the definition in (2.5), we find that

$$x_p^\lambda = x_q = x_0 + qn(x_0),$$

where

$$q = \frac{(1+\lambda^2)p}{1+\lambda^2+2p\lambda} > 0.$$

This result follows from the property given in (2.5) that

$$|x_p - e_\lambda| = \frac{1+\lambda^2}{2\lambda} + p, \quad |x_q - e_\lambda| = \frac{1+\lambda^2}{2\lambda} - q, \quad |x_p - e_\lambda||x_q - e_\lambda| = \left(\frac{1+\lambda^2}{2\lambda}\right)^2.$$

Then, it follows that

$$\begin{aligned}
\frac{\partial w_\lambda(x_0,t)}{\partial n} &= \lim_{p\to 0^+} \frac{w_\lambda(x_p,t) - w_\lambda(x_0,t)}{-p} \\
&= \lim_{p\to 0^+} \left(\frac{v(x_q,t) - v(x_0,t)}{-p} + \frac{v(x_p,t) - v(x_0,t)}{p}\right) \\
&= \lim_{q\to 0^+} \frac{v(x_q,t) - v(x_0,t)}{-q}\left(\frac{q}{p}\right) + \lim_{p\to 0^+} \frac{v(x_p,t) - v(x_0,t)}{p} \\
&= -2\frac{\partial v(x_0,t)}{\partial n}.
\end{aligned}$$

From (2.19), we conclude that inequality (2.18) holds. □

3. Asymptotic maximum principles

In this section, we present the following asymptotic narrow region principle, which will play a crucial role in establishing Theorem 1.1.

Lemma 3.1. *(Asymptotic narrow region principle) Assume that Ω is a bounded narrow region with respect to e_λ contained within the annulus defined by*

$$\left\{x \in B_1(0) \,\middle|\, \frac{1+\lambda^2}{2\lambda} < |x - e_\lambda| < \frac{1+\lambda^2}{2\lambda} + \delta\right\}, \tag{3.1}$$

for some small $\delta > 0$, where e_λ is defined by (2.3).

For sufficiently large $\bar{t}$, assume that $z_\lambda(x,t) \in C^2(\Omega) \times C^1([\bar{t},\infty])$ is bounded and lower semi-continuous in x on $\overline{\Omega}$, and satisfies

$$\begin{cases}
\dfrac{\partial z_\lambda(x,t)}{\partial t} - \Delta z_\lambda(x,t) \geqslant c_\lambda(x,t) z_\lambda(x,t), & (x,t) \in \Omega \times [\bar{t},\infty), \\
z_\lambda(x,t) \geqslant 0, & (x,t) \in \partial\Omega \times [\bar{t},\infty),
\end{cases} \tag{3.2}$$

where $c_\lambda(x,t)$ is bounded from above. Then for sufficiently small δ the following statement holds:

$$\varliminf_{t\to\infty} z_\lambda(x,t) \geqslant 0, \quad \forall x \in \Omega. \tag{3.3}$$

Proof. Let m be a fixed positive constant that will be determined later. Define

$$\tilde{z}_\lambda(x,t) = \frac{e^{mt} z_\lambda(x,t)}{\phi(x)}.$$

Then $\tilde{z}_\lambda(x,t)$ satisfies

$$\frac{\partial \tilde{z}_\lambda(x,t)}{\partial t} = m\frac{e^{mt} z_\lambda(x,t)}{\phi(x)} + \frac{e^{mt}}{\phi(x)}\frac{\partial z_\lambda(x,t)}{\partial t},$$

and

$$\Delta\tilde{z}_\lambda(x,t) \quad = \quad e^{mt}\left(\frac{\Delta z_\lambda(x,t)}{\phi(x)} - 2\frac{\nabla\tilde{z}_\lambda(x,t)\nabla\phi(x)}{\phi(x)} - \frac{\Delta\phi(x)}{\phi(x)}\frac{z_\lambda(x,t)}{\phi(x)}\right).$$

Thus, we find that

$$\frac{\partial\tilde{z}_\lambda(x,t)}{\partial t} - \Delta\tilde{z}_\lambda(x,t) - 2e^{mt}\frac{\nabla\tilde{z}_\lambda(x,t)\nabla\phi(x)}{\phi(x)} \geqslant \left(c_\lambda(x,t) + \frac{\Delta\phi(x)}{\phi(x)} + m\right)\tilde{z}_\lambda(x,t). \tag{3.4}$$

Let $\phi(x)$ be defined as

$$\phi(x) = \sin\left(\frac{|x-e_\lambda| - \frac{1+\lambda^2}{2\lambda}}{\delta} + \frac{\pi}{2}\right) = \sin\left(\frac{\sqrt{\left|x_1 - \frac{1-\lambda^2}{-2\lambda}\right|^2 + |x_2|^2 + \cdots + |x_N|^2} - \frac{1+\lambda^2}{2\lambda}}{\delta} + \frac{\pi}{2}\right).$$

For each $x \in \Omega$, we have

$$\frac{\pi}{2} < \frac{|x-e_\lambda| - \frac{1+\lambda^2}{2\lambda}}{\delta} + \frac{\pi}{2} < 1 + \frac{\pi}{2} < \pi.$$

Direct calculation shows

$$\begin{aligned}\Delta\phi(x) \quad &= \quad -\frac{1}{\delta^2}\sin\left(\frac{|x-e_\lambda| - \frac{1+\lambda^2}{2\lambda}}{\delta} + \frac{\pi}{2}\right) + \cos\left(\frac{|x-e_\lambda| - \frac{1+\lambda^2}{2\lambda}}{\delta} + \frac{\pi}{2}\right)\frac{1}{|x-e_\lambda|\delta}(n-1) \\ &< \quad -\frac{1}{\delta^2}\sin\left(\frac{|x-e_\lambda| - \frac{1+\lambda^2}{2\lambda}}{\delta} + \frac{\pi}{2}\right).\end{aligned}$$

Thus,

$$\frac{\Delta\phi(x)}{\phi(x)} < -\frac{1}{\delta^2}. \tag{3.5}$$

We claim that for any $T > \bar{t}$,

$$\tilde{z}_\lambda(x,t) \geqslant \min\left\{0, \inf_\Omega \tilde{z}_\lambda(x,\bar{t})\right\}, \quad (x,t) \in \Omega \times [\bar{t}, T]. \tag{3.6}$$

If (3.6) is not true, by (3.2) and the lower semi-continuity of z_λ on $\overline{\Omega} \times [\bar{t}, T]$, there exists (x_0, t_0) in $\Omega \times (\bar{t}, T]$ such that

$$\tilde{z}_\lambda(x_0,t_0) = \min_{\overline{\Omega}\times(\bar{t},T]} \tilde{z}_\lambda(x,t) < \min\{0, \inf_\Omega \tilde{z}_\lambda(x,\bar{t})\}. \tag{3.7}$$

Since

$$\tilde{z}_\lambda(x,t) = 0, \quad (x,t) \in \overline{\Omega} \cap T_\lambda,$$

and

$$\tilde{z}_\lambda(x,t) > 0, \quad (x,t) \in \overline{\Omega} \cap \partial B_1(0),$$

where T_λ is defined in (2.2), the minimum point x_0 is an interior point of Ω. Therefore,

$$\frac{\partial\tilde{z}_\lambda(x_0,t_0)}{\partial t} \quad \leqslant \quad 0, \tag{3.8}$$

$$\Delta \tilde{z}_\lambda(x_0, t_0) \ \geqslant \ 0, \tag{3.9}$$

$$\nabla \tilde{z}_\lambda(x_0, t_0) \ = \ 0. \tag{3.10}$$

From (3.4), (3.5), and (3.8)–(3.10), we have

$$0 \geqslant \frac{\partial \tilde{z}_\lambda(x_0, t_0)}{\partial t} - \Delta \tilde{z}_\lambda(x_0, t_0) > \left(c_\lambda(x_0, t_0) - \frac{1}{\delta^2} + m \right) \tilde{z}_\lambda(x_0, t_0). \tag{3.11}$$

Since $c_\lambda(x, t)$ is bounded from above for all $(x, t) \in B_1(0) \times (0, \infty)$, we can choose δ small enough such that

$$c_\lambda(x_0, t_0) - \frac{1}{\delta^2} < -\frac{1}{2\delta^2}.$$

Taking $m = 1/2\delta^2$, we derive that the right-hand side of the second inequality of (3.11) is strictly greater than 0, since $\tilde{z}_\lambda(x_0, t_0) < 0$. This contradicts with (3.11).

Therefore, given the boundedness of z_λ, there exists a constant $C > 0$ such that

$$\tilde{z}_\lambda(x, t) \geqslant \min\{0, \inf_\Omega \tilde{z}_\lambda(x, \bar{t})\} \geqslant -C, \quad (x, t) \in \Omega \times [\bar{t}, T].$$

Thus, we have

$$z_\lambda(x, t) \geqslant e^{-mt}(-C), \quad \forall t > \bar{t}.$$

Taking the limit $t \to \infty$, we obtain

$$\varliminf_{t \to \infty} z_\lambda(x, t) \geqslant 0, \quad \forall x \in \Omega.$$

This completes the proof of Lemma 3.1. □

In the proof of the main theorem, we will also use the following two classical maximum principles. For the convenience of the reader, we now show them in the version suitable for this article.

Lemma 3.2. *(Strong parabolic maximum principle for a not necessarily non-negative coefficient) For each $\lambda \in (0, 1)$, assume $z(x, t) \in C^{2,1}(\Sigma_\lambda \times (0, \infty)) \cap C\left(\overline{\Sigma_\lambda} \times [0, \infty)\right)$ and for $(x, t) \times \Sigma_\lambda \times (0, \infty)$, $z(x, t) \geqslant 0$ satisfies that*

$$\frac{\partial z(x, t)}{\partial t} - \Delta z(x, t) - c(x, t) z(x, t) \geqslant 0,$$

where $c(x, t)$ is bounded in $\Sigma_\lambda \times (0, \infty)$. If $z(x, t)$ attains its minimum 0 over $\overline{\Sigma_\lambda} \times [0, \infty)$ at a point $(x_0, t_0) \in \Sigma_\lambda \times (0, \infty)$, then $z(x, t) \equiv 0$ in $\Sigma_\lambda \times (0, t_0]$.

Proof. For each $\lambda \in (0, 1)$, we set $c_0 = \underset{\Sigma_\lambda \times (0, \infty)}{sup} c(x, t)$ and let

$$\bar{z}(x, t) = e^{-c_0 t} z(x, t),$$

and then for $(x, t) \times \Sigma_\lambda \times (0, \infty)$, $\bar{z}(x, t) = e^{-c_0 t} z(x, t) \geqslant 0$ and satisfies that

$$\begin{aligned}
&\frac{\partial \bar{z}(x, t)}{\partial t} - \Delta \bar{z}(x, t) + (c_0 - c(x, t)) \bar{z}(x, t) \\
&= -c_0 e^{-c_0 t} z(x, t) + e^{-c_0 t} \frac{\partial z(x, t)}{\partial t} - e^{-c_0 t} \Delta z(x, t) + c_0 e^{-c_0 t} z(x, t) - c(x, t) e^{-c_0 t} z(x, t) \\
&= e^{-c_0 t} \left(\frac{\partial z(x, t)}{\partial t} - \Delta z(x, t) - c(x, t) z(x, t) \right) \geqslant 0.
\end{aligned}$$

If $z(x,t)$ attains its minimum 0 at (x_0,t_0), then $\overline{z}(x,t)$ also gets its minimum 0 at the point (x_0,t_0) over $\overline{\Sigma_\lambda}\times[0,\infty)$. Considering $c_0-c(x,t)\geqslant 0$ in $\Sigma_\lambda\times(0,\infty)$, then from the strong parabolic maximum principle with $c(x,t)\geqslant 0$ (Theorem 12 in [46], Chapter 7), we can obtain that for $(x,t)\in\Sigma_\lambda\times(0,t_0]$, we have $\overline{z}(x,t)\equiv 0$ and therefore $z(x,t)=e^{c_0t}\,\overline{z}(x,t)\equiv 0$. □

Lemma 3.3 (Parabolic Hopf's lemma for a not necessarily non-negative coefficient)**.** *For each $\lambda\in(0,1)$, we let (x_0,t_0) be a point on the boundary of $\Sigma_\lambda\times(0,T)$ for $\forall T>0$ such that $z(x_0,t_0)=0$ is the minimum in $\overline{\Sigma_\lambda}\times[0,T]$. Assume that there exists a neighborhood $V:=|x-x_0|^2+|t-t_0|^2<R_0^2$ of (x_0,t_0) such that for $(x,t)\in V\cap(\Sigma_\lambda\times(0,T))$, $z(x,t)>0$ and satisfies*

$$\frac{\partial z(x,t)}{\partial t}-\Delta z(x,t)-c(x,t)z(x,t)\geqslant 0,$$

where $c(x,t)$ is bounded in $\Sigma_\lambda\times(0,T)$. If there exists a sphere $S:=|x-x'|^2+|t-t'|^2<R$ passing through (x_0,t_0) and contained in $\overline{\Sigma_\lambda}\times[0,T]$ and $(x_0,t_0)\neq(x',t')$, then under the assumptions made above, we have

$$\frac{\partial z}{\partial n}(x_0,t_0)<0,$$

where n is the unit outer normal of $\partial\Sigma_\lambda$ for the fixed t_0.

Proof. For each $\lambda\in(0,1)$, we set $c_0=\underset{\Sigma_\lambda\times(0,\infty)}{sup}\ c(x,t)$ and let

$$\overline{z}(x,t)=e^{-c_0t}z(x,t).$$

Then at the point (x_0,t_0), $z(x,t)$ also gets its minimum 0 and for $(x,t)\in V\cap(\Sigma_\lambda\times(0,T))$, $\overline{z}(x,t)>0$ and satisfies that

$$\frac{\partial\overline{z}(x,t)}{\partial t}-\Delta\overline{z}(x,t)+(c_0-c(x,t))\overline{z}(x,t)\geqslant 0.$$

Considering $(c_0-c(x,t))\geqslant 0$ in $\sigma_\lambda\times(0,T)$, then from the parabolic Hopf's lemma (Theorem 2 in [47]), we can derive that every outer non-tangential derivative $\partial\overline{z}\backslash\partial\nu$ at (x_0,t_0) is negative, where ν represents any outer non-tangential vector. Particularly, by the definition of $\overline{z}(x,t)$, for the fixed t_0 and the unit outer normal n of $\partial\Sigma_\lambda$, we have $\partial z\backslash\partial n(x_0,t_0)=\partial\overline{z}\backslash\partial n(x_0,t_0)<0$. □

4. Proof of the main theorem

We will carry out the proof in two steps. For simplicity, choose any direction within the region to be the x_1 direction. The first step is to show that for λ sufficiently close to the right end of the domain, the following holds for all $\varphi\in\omega(u)$:

$$\psi(x)\geqslant 0,\quad \forall x\in\Sigma_\lambda. \tag{4.1}$$

This provides the initial position to move the plane. We then move the plane T_λ to the left as long as the inequality (4.1) continues to hold, until it reaches its limiting position. Define

$$\lambda_0=\inf\{\lambda\geqslant 0\,|\,\psi_\mu(x)\geqslant 0,\ \text{for all}\ \varphi\in\omega(u),\ x\in\Sigma_\mu,\ \mu\geqslant\lambda\}. \tag{4.2}$$

We will show that $\lambda_0=0$. Since the x_1 direction can be chosen arbitrarily, this implies that for any $\varphi\in\omega(u)$, $\varphi(x)$ is radially symmetric and $(1+|x|^2)^{\frac{N-2}{2}}\varphi(x)$ is monotone decreasing about the origin. We will now detail these two steps.

Proof of Theorem 1.1. For all $\varphi \in \omega(u)$, we assume $\varphi \not\equiv 0$ in $B_1(0)$.

Step 1. We show that for $\lambda < 1$ and sufficiently close to 1, the following holds:

$$\psi_\lambda(x) \geqslant 0, \quad \forall x \in \Sigma_\lambda, \quad \forall \varphi \in \omega(u). \tag{4.3}$$

The Lipschitz continuity assumption (F1) on f implies that $c_\lambda(x,t)$ is bounded. Additionally, we have

$$z_\lambda(x,t) = \left(\frac{1+|x^\lambda|^2}{1+|x|^2}\right)^{\frac{N-2}{2}} u(x^\lambda, t) - u(x,t) > 0, \quad (x,t) \in \partial\Sigma_\lambda \times (0,\infty),$$

since $u(x,t) = 0$, for $(x,t) \in \partial B_1(0) \times (0,\infty)$, and u is positive in $B_1(0)$. Combining this with (2.14), we can apply Lemma 3.1 to conclude that (4.3) holds.

Step 2. We will demonstrate that the parameter λ_0, as defined in (4.2), is equal to zero, that is to say

$$\lambda_0 = 0. \tag{4.4}$$

Assume for contradiction that $\lambda_0 > 0$. We will demonstrate that T_{λ_0} can be shifted slightly to the left, thereby contradicting the definition of λ_0.

To begin with, we intend to determine the sign of ψ_{λ_0} for any $\varphi \in \omega(u)$ and all $x \in \Sigma_{\lambda_0}$ under the case $\lambda_0 > 0$. To achieve this, according to the definition (2.12), we now need to discuss the inequality that $z_{\lambda_0}(x,t)$ satisfies as $t \to \infty$. From the definition of $\omega(u)$, for each $\varphi \in \omega(u)$, there exists a sequence $\{t_k\}$ such that $u(x,t_k) \to \varphi(x)$ as $t_k \to \infty$. Define

$$u_k(x,t) = u(x, t + t_k - 1), \tag{4.5}$$

and

$$f_k(|x|, u, t) = f(|x|, u, t + t_k - 1). \tag{4.6}$$

Then we have $u_k(x,1) \to \varphi(x)$ in the sense of $C(B_1(0))$ as $k \to \infty$. Let $Q_1 := B_1(0) \times [1-\varepsilon_0, 1+\varepsilon_0]$. We now have

$$\begin{cases} \frac{\partial u_k}{\partial t}(x,t) - \Delta u_k(x,t) = f_k(|x|, u_k(x,t), t), & (x,t) \in Q_1, \\ u_k(x,t) = 0, & (x,t) \in \partial B_1(0) \times [1-\varepsilon_0, 1+\varepsilon_0]. \end{cases} \tag{4.7}$$

Similarly as in (2.8)–(2.11), we have the following definitions of functions:

$$v_k(x,t) = (1+|x|^2)^{\frac{N-2}{2}} u_k(x,t), \tag{4.8}$$

$$w_{\lambda_0,k}(x,t) = v_k(x^{\lambda_0}, t) - v_k(x,t), \tag{4.9}$$

$$z_{\lambda_0,k}(x,t) = z_{\lambda_0}(x, t+t_k-1) = \left(\frac{1+|x^{\lambda_0}|^2}{1+|x|^2}\right)^{\frac{N-2}{2}} u_k(x^{\lambda_0}, t) - u_k(x,t). \tag{4.10}$$

It follows from (4.7) and Lemma 2.1 that

$$\frac{\partial z_{\lambda_0,k}}{\partial t}(x,t) - \Delta z_{\lambda_0,k}(x,t) \geqslant c_{\lambda_0,k}(x,t) z_{\lambda_0,k}(x,t), \quad (x,t) \in \Sigma_{\lambda_0} \times [\bar{t}, \infty),$$

where

$$c_{\lambda_0,k}(x,t) = c_{\lambda_0}(x, t+t_k-1) = \frac{f_k(|x|, (1+|x|^2)^{-\frac{N-2}{2}} v_k(x^{\lambda_0}, t), t) - f_k(|x|, (1+|x|^2)^{-\frac{N-2}{2}} v_k(x,t), t)}{(1+|x|^2)^{-\frac{N-2}{2}} v_k(x^{\lambda_0}, t) - (1+|x|^2)^{-\frac{N-2}{2}} v_k(x,t)}.$$

Based on the relationship between the functions (4.8)–(4.10) and u_k defined as (4.5), using Lemma 5.3, we deduce the existence of subsequences v_k, $w_{\lambda_0,k}$, and $z_{\lambda_0,k}$ which converge uniformly to the respective functions v_∞, $w_{\lambda_0,\infty}$, and $z_{\lambda_0,\infty}$ all in the sense of $C^{2,1}\left(\Sigma_{\lambda_0} \times [1-\varepsilon_0, 1+\varepsilon_0]\right)$ as $k \to \infty$ and they satisfy

$$\begin{aligned} z_{\lambda_0,\infty}(x,t) &= (1+|x|^2)^{-\frac{N-2}{2}} w_{\lambda_0,\infty}(x,t) = (1+|x|^2)^{-\frac{N-2}{2}} \left(v_\infty(x^{\lambda_0}, t) - v_\infty(x,t)\right) \\ &= \left(\frac{1+|x^{\lambda_0}|^2}{1+|x|^2}\right)^{\frac{N-2}{2}} u_\infty(x^{\lambda_0}, t) - u_\infty(x,t). \end{aligned}$$

Furthermore, both in the sense of $C\left(\Sigma_{\lambda_0} \times [1-\varepsilon_0, 1+\varepsilon_0]\right)$ as $k \to \infty$ we have

$$\begin{aligned} \frac{\partial z_{\lambda_0,k}}{\partial t}(x,t) - \Delta z_{\lambda_0,k}(x,t) &\to \frac{\partial z_{\lambda_0,\infty}}{\partial t}(x,t) - \Delta z_{\lambda_0,\infty}(x,t), \\ c_{\lambda_0,k}(x,t) &\to c_{\lambda_0,\infty}(x,t), \end{aligned}$$

where $c_{\lambda_0,\infty}(x,t)$ is bounded in $\Sigma_{\lambda_0} \times [1-\varepsilon_0, 1+\varepsilon_0]$. This follows from the proof of Lemma 5.3, where it is established that the sequence f_k within the definition of $c_{\lambda_0,k}$ uniformly converges to f_∞ satisfying (F1) in the sense of $C\left(\Sigma_{\lambda_0} \times [1-\varepsilon_0, 1+\varepsilon_0]\right)$. For any $\varphi \in \omega(u)$, by the definition of the limit set $\omega(u)$, there exists t_k such that $z_{\lambda_0}(x,t_k) \to \psi_{\lambda_0}(x)$ in the sense of $C(\Sigma_{\lambda_0})$ as $t_k \to \infty$. Particularly, according to the convergence of $z_{\lambda_0,k}$, in the sense of $C^2(\Sigma_{\lambda_0})$ we have

$$z_{\lambda_0}(x,t_k) = z_{\lambda_0,k}(x,1) \to z_{\lambda_0,\infty}(x,1) = \psi_{\lambda_0}(x), \quad as\ k \to \infty. \tag{4.11}$$

Combining the continuity of $z_{\lambda_0,\infty}$ with respect to t and the definition of λ_0, we deduce that $z_{\lambda_0,\infty}(x,t)$ satisfies the following inequalities:

$$\begin{cases} \dfrac{\partial z_{\lambda_0,\infty}(x,t)}{\partial t} - \Delta z_{\lambda_0,\infty}(x,t) \geqslant c_{\lambda_0,\infty}(x,t) z_{\lambda_0,\infty}(x,t), & (x,t) \in \Sigma_{\lambda_0} \times [1-\varepsilon_0, 1+\varepsilon_0], \\ z_{\lambda_0,\infty}(x,t) \geqslant 0, & (x,t) \in \Sigma_{\lambda_0} \times [1-\varepsilon_0, 1+\varepsilon_0]. \end{cases} \tag{4.12}$$

We apply Lemma 3.2 to (4.12) in $\Sigma_{\lambda_0} \times [1-\varepsilon_0, 1+\varepsilon_0]$ to get either

$$z_{\lambda_0,\infty}(x,t) \equiv 0 \quad \text{for} \quad (x,t) \in \Sigma_{\lambda_0} \times [1-\varepsilon_0, 1+\varepsilon_0], \tag{4.13}$$

or

$$z_{\lambda_0,\infty}(x,t) > 0 \quad \text{for} \quad (x,t) \in \Sigma_{\lambda_0} \times (1-\varepsilon_0, 1+\varepsilon_0]. \tag{4.14}$$

In case (4.13), since $z_{\lambda_0,\infty}(x,t) \equiv 0$, for all $(x,t) \in \Sigma_{\lambda_0} \times [1-\varepsilon_0, 1+\varepsilon_0]$, we have

$$w_{\lambda_0,\infty}(x,t) \equiv 0, \quad i.e., \quad v_\infty(x^{\lambda_0}, t) \equiv v_\infty(x,t). \tag{4.15}$$

From (2.13), for $(x,t) \in \Sigma_{\lambda_0} \times (1-\varepsilon_0, 1+\varepsilon_0]$, we have

$$\Delta_{(g,x)} v_\infty(x,t) = \left(\frac{1+|x|^2}{2}\right)^2 \frac{\partial v_\infty}{\partial t}(x,t) - \frac{(1+|x|^2)^{\frac{N+2}{2}}}{4} f_\infty(|x|, (1+|x|^2)^{-\frac{N-2}{2}} v_\infty(x,t), t) + \frac{N(N-2)}{4} v_\infty(x,t), \tag{4.16}$$

and

$$\Delta_{(g,x)} v_\infty(x^{\lambda_0}, t) = \left(\frac{1+|x^{\lambda_0}|^2}{2}\right)^2 \frac{\partial v_\infty}{\partial t}(x^{\lambda_0}, t) - \frac{(1+|x^{\lambda_0}|^2)^{\frac{N+2}{2}}}{4} f_\infty(|x^{\lambda_0}|, (1+|x^{\lambda_0}|^2)^{-\frac{N-2}{2}} v_\infty(x^{\lambda_0}, t), t) + \frac{N(N-2)}{4} v_\infty(x^{\lambda_0}, t). \tag{4.17}$$

By (4.15) we can obtain that

$$\Delta_{(g,x)} w_{\lambda_0,\infty}(x,t) \equiv 0, \quad \left(\frac{1+|x^{\lambda}|^2}{2}\right)^2 \frac{\partial w_{\lambda_0,\infty}}{\partial t}(x^{\lambda}, t) \equiv 0, \quad \frac{N(N-2)}{4} w_{\lambda_0,\infty}(x,t) \equiv 0,$$

for $(x,t) \in \Sigma_{\lambda_0} \times [1-\varepsilon_0, 1+\varepsilon_0]$. Combining (4.16) and (4.17), we finally arrive at

$$(1+|x|^2)^{\frac{N+2}{2}} f_\infty(|x|, (1+|x|^2)^{-\frac{N-2}{2}} v_\infty(x,t), t) \equiv (1+|x^{\lambda_0}|^2)^{\frac{N+2}{2}} f_\infty(|x^{\lambda_0}|, (1+|x^{\lambda_0}|^2)^{-\frac{N-2}{2}} v_\infty(x,t), t), \tag{4.18}$$

for $(x,t) \in \Sigma_{\lambda_0} \times (1-\varepsilon_0, 1+\varepsilon_0]$. Since Lemma 5.1 implies that $|x| > |x^{\lambda_0}|$, (4.18) contradicts the assumption (F3) of f. It follows that case (4.13) is invalid.

Next from case (4.14), we can derive that for all $\varphi \in \omega(u)$,

$$z_{\lambda_0,\infty}(x, 1) = \psi_{\lambda_0}(x) > 0, \quad x \in \Sigma_{\lambda_0}. \tag{4.19}$$

Since $\lambda_0 > 0$, we now attempt to slightly shift λ_0 to the left. If λ_0 still meets definition (4.2), we can construct a contradiction. For any small $l > 0$, we set $\overline{V_{\lambda_0+l}} = x \in \left\{x \in \overline{B_1(0)} \,\middle|\, |x - e_{\lambda_0}| \geqslant \frac{1+\lambda_0^2}{2\lambda_0} + l\right\}$, and for ψ_{λ_0}, there exists a $C_\varphi > 0$, such that

$$\psi_{\lambda_0} \geqslant C_\varphi > 0, \quad x \in \overline{V_{\lambda_0+l}}. \tag{4.20}$$

We now show that, for all $\varphi \in \omega(u)$, there exists a universal constant C_0 such that

$$\psi_{\lambda_0} \geqslant C_0 > 0, \quad x \in \overline{V_{\lambda_0+l}}. \tag{4.21}$$

If not, there exists a sequence of functions $\{\psi_{\lambda_0}^k\}$ with respect to $\varphi^k \subset \omega(u)$ and a sequence of points $\{x^k\} \subset \overline{V_{\lambda_0+l}}$ such that for each k we have

$$\psi_{\lambda_0}^k(x^k) < \frac{1}{k}. \tag{4.22}$$

By the compactness of $\omega(u)$ in $C(\overline{B_1(0)})$, there exists $\psi_{\lambda_0}^0$ which corresponds to some $\varphi^0 \in \omega(u)$ and $x^0 \in \overline{V_{\lambda_0+l}}$ such that

$$\psi_{\lambda_0}^k(x^k) \to \psi_{\lambda_0}^0(x^0),$$

as $k \to \infty$ in the sense of $C(B_1(0))$. Now by (4.22) and the definition of λ_0, we obtain

$$\psi_{\lambda_0}^0(x^0) = 0,$$

which contradicts (4.19), since $\varphi^0 \in \omega(u)$. Thus (4.21) must be established.

From (4.21) and the continuity of ψ_λ with respect to λ, for each ψ_λ, under a fixed C_0, there exists $\varepsilon_\varphi > 0$ such that

$$\psi_\lambda(x) \geqslant \frac{C_0}{2} > 0, \quad x \in \overline{V_{\lambda_0+l}}, \quad \forall \lambda \in (\lambda_0 - \varepsilon_\varphi, \lambda_0).$$

Similarly due to the compactness of $\omega(u)$ in $C(\overline{B_1(0)})$, for all ψ_λ, there exists a universal $\varepsilon > 0$ such that

$$\psi_\lambda(x) \geqslant \frac{C_0}{2} > 0, \quad x \in \overline{V_{\lambda_0+l}}, \quad \forall \lambda \in (\lambda_0 - \varepsilon, \lambda_0). \tag{4.23}$$

Consequently, for t sufficiently large, we have

$$z_\lambda(x,t) \geqslant 0, \quad x \in \overline{V_{\lambda_0+l}}, \quad \forall \lambda \in (\lambda_0 - \varepsilon, \lambda_0).$$

Since $l > 0$ is small, by (4.22), we can choose $\varepsilon > 0$ small, such that $\Sigma_\lambda \backslash V_{\lambda_0+l}$ is a narrow region defined as (3.1) for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$, and then applying the *asymptotic narrow region principle* (Lemma 3.1), we arrive at

$$\psi_\lambda(x) \geqslant 0, \quad \forall x \in \Sigma_\lambda \backslash V_{\lambda_0+l}. \tag{4.24}$$

Combining (4.23) and (4.24), we derive that

$$\psi_\lambda(x) \geqslant 0, \quad \forall x \in \Sigma_\lambda, \quad \forall \lambda \in (\lambda_0 - \varepsilon, \lambda_0), \quad \forall \varphi \in \omega(u).$$

This contradicts the definition of λ_0. Therefore, $\lambda_0 = 0$ must be true.

As a result, (4.4) implies that for all $\varphi \in \omega(u)$,

$$\psi_0(x) \geqslant 0, \quad \forall x \in \Sigma_0,$$

that is to say, for all $\varphi \in \omega(u)$ and $x \in \Sigma_0$,

$$\begin{aligned} z_0(x,1) &= \left(\frac{1+|x^0|^2}{1+|x|^2}\right)^{\frac{N-2}{2}} u_\infty(x^0,1) - u_\infty(x,1) \\ &= \left(\frac{1+|x^0|^2}{1+|x|^2}\right)^{\frac{N-2}{2}} \varphi(x^0) - \varphi(x) \geqslant 0. \end{aligned}$$

Since $x^0 = (-x_1, x_2, \cdots, x_N)$ for $x \in \Sigma_0$ with $0 < x_1 < 1$, finally we can get that

$$\varphi(-x_1, x_2, \cdots, x_N) \geqslant \varphi(x_1, x_2, \cdots, x_N). \tag{4.25}$$

Since the x_1 direction can be chosen arbitrarily, (4.25) implies that all $\varphi(x)$ are radially symmetric about the origin. Combining with (4.12) and the proof of (4.19), we can derive that for $0 < \lambda < 1$, $z_{\lambda,\infty}$ satisfies

$$\begin{cases} \dfrac{\partial z_{\lambda,\infty}(x,t)}{\partial t} - \Delta z_{\lambda,\infty}(x,t) \geqslant c_{\lambda,\infty}(x,t) z_{\lambda,\infty}(x,t), & (x,t) \in \Sigma_\lambda \times [1-\varepsilon_0, 1+\varepsilon_0], \\ z_{\lambda,\infty}(x,t) = 0, & (x,t) \in T_\lambda \times [1-\varepsilon_0, 1+\varepsilon_0], \\ z_{\lambda,\infty}(x,t) > 0, & (x,t) \in \Sigma_\lambda \times [1-\varepsilon_0, 1+\varepsilon_0], \end{cases}$$

where $c_{\lambda,\infty}$ is bounded. Then we can apply Lemma 3.3 to obtain that

$$\frac{\partial z_{\lambda,\infty}(x,1)}{\partial n} = \frac{\partial \psi_\lambda(x)}{\partial n} < 0, \quad x \in T_\lambda, \quad \forall\, 0 < \lambda < 1,$$

where n is the outer normal vector of $\partial\overline{\Sigma_\lambda}$. From Lemma 2.2, we can conclude that

$$\frac{\partial[(1+|x|^2)^{\frac{N-2}{2}}\varphi(x)]}{\partial n} > 0. \tag{4.26}$$

Under the conclusion that all $\varphi(x)$ are radially symmetric about the origin, from (4.26) we can infer that for all $0 < r < 1$,

$$\partial_r\left((1+r^2)^{\frac{N-2}{2}}\varphi(r)\right) < 0, \tag{4.27}$$

which shows asymptotic monotonicity. Now we complete the proof of Theorem 1.1. □

5. Appendix

We note that $B_1(0) = \{x \in \mathbb{R}^N : |x| < 1\}$, $N \geqslant 3$, we set $S^+ = \{X \in \mathbb{R}^{N+1} : |X| = 1, X_{N+1} > 0\}$, and define a mapping $P : (S^+, |dX|^2) \to (B_1(0), g)$ by

$$P(X_1, \cdots, X_N, X_{N+1}) = \frac{1}{X_{N+1}+1}(X_1, \cdots, X_N).$$

The overall idea of the proof of (2.2)–(2.5) is similar to that in [32] in the case $a = 1$, so we omit the proof process. Strongly inspired by Lemma A.1 and Lemma A.2 in [31], here we give the proof of (2.6) and (2.16).

Lemma 5.1. *We have*

$$\frac{1+|x|^2}{1+|x^\lambda|^2} = \left(\frac{2\lambda}{1+\lambda^2}\right)^2 |x-e_\lambda|^2. \tag{5.1}$$

Therefore, we can derive $|x| > |x^\lambda|$ *for* $x \in \Sigma_\lambda$.

Proof. We define $a = -e_\lambda/|e_\lambda|^2 = (2\lambda/(1-\lambda^2), 0, \cdots, 0)$. From this definition, (2.3), and (2.5), by some elementary computations, we note that

$$\begin{aligned} a^\lambda &= e_\lambda + \left(\frac{1+\lambda^2}{2\lambda}\right)^2 \frac{a-e_\lambda}{|a-e_\lambda|^2} = 0, \\ |e_\lambda|^2 + 1 &= \left(\frac{1+\lambda^2}{2\lambda}\right)^2 = \frac{1-\lambda^2}{2\lambda}\cdot\frac{(1+\lambda^2)^2}{2\lambda(1-\lambda)^2} = |e_\lambda||a-e_\lambda|. \end{aligned}$$

Then we have

$$\begin{aligned} |x^\lambda|^2 &= |x^\lambda - a^\lambda|^2 \\ &= \left|\left(\frac{1+\lambda^2}{2\lambda}\right)^2\left(\frac{x-e_\lambda}{|x-e_\lambda|^2} - \frac{a-e_\lambda}{|a-e_\lambda|^2}\right)\right|^2 = (|e_\lambda|^2+1)^2\left|\frac{x-e_\lambda}{|x-e_\lambda|^2} - \frac{a-e_\lambda}{|a-e_\lambda|^2}\right|^2 \\ &= \frac{(|e_\lambda|^2+1)^2|x-a|^2}{|x-e_\lambda|^2|a-e_\lambda|^2} = \frac{|e_\lambda|^2|a-e_\lambda|^2|x-a|^2}{|x-e_\lambda|^2|a-e_\lambda|^2} = \frac{|e_\lambda|^2|x-a|^2}{|x-e_\lambda|^2}, \end{aligned}$$

where we use the property of distance operations that for $p, q \in \mathbb{R}^N\backslash\{0\}$,

$$\left|\frac{p}{|p|^2} - \frac{q}{|q|^2}\right| = \frac{|p-q|}{|p||q|}.$$

Thus we obtain

$$1+|x^\lambda|^2 = 1+\frac{|e_\lambda|^2|x-a|^2}{|x-e_\lambda|^2} = 1+\frac{|e_\lambda|^2|x+e_\lambda/|e_\lambda|^2|^2}{|x-e_\lambda|^2} = \frac{(|e_\lambda|^2+1)(1+|x|^2)}{|x-e_\lambda|^2} = \left(\frac{1+\lambda^2}{2\lambda}\right)^2\frac{(1+|x|^2)}{|x-e_\lambda|^2},$$

which implies (5.1). Due to the definition of Σ_λ, for $x\in\Sigma_\lambda$, we have

$$\frac{1+|x|^2}{1+|x^\lambda|^2} = \left(\frac{2\lambda}{1+\lambda^2}\right)^2|x-e_\lambda|^2 > \left(\frac{2\lambda}{1+\lambda^2}\right)^2\left(\frac{1+\lambda^2}{2\lambda}\right)^2 = 1,$$

and then we can find that $|x|>|x^\lambda|$. □

Lemma 5.2. *Assume that $v(x)\in C^2(B_1(0))$. Let $y=x^\lambda$ and $v(y)$ is a function with y as the independent variable. We define $v_\lambda(x)=v(x^\lambda)$ as a new form of the function with x as the independent variable. Then*

$$\begin{aligned}\Delta_{(g,x)}v_\lambda(x) &= \left(\frac{1+|x|^2}{2}\right)^2\left(\Delta_x v_\lambda(x) - \frac{2(N-2)}{1+|x|^2}x\cdot\nabla_x v_\lambda(x)\right)\\ &= \left(\frac{1+|y|^2}{2}\right)^2\left(\Delta_y v(y) - \frac{2(N-2)}{1+|y|^2}y\cdot\nabla_y v(y)\right)\Bigg|_{y=x^\lambda} = \Delta_{(g,y)}v(y)|_{y=x^\lambda},\end{aligned} \tag{5.2}$$

where $\Delta_x=\sum_{i=1}^N\partial^2/\partial x_i^2$, $\quad x\cdot\nabla_x=\sum_{i=1}^N x_i\partial/\partial x_i$, and $\Delta_{(g,x)}$ are defined as in (2.7).

Proof. To compare the values of $\Delta_{(g,x)}v_\lambda(x)$ and $\Delta_{(g,y)}v(y)$ through direct computation, we define $u(y)$ and $u_\lambda(x)$ as

$$\begin{aligned} u(y) &= (1+|y|^2)^{-(N-2)/2}v(y),\\ u_\lambda(x) &= (1+|x|^2)^{-(N-2)/2}v_\lambda(x).\end{aligned}$$

Then we can find that

$$\frac{1}{4}(1+|y|^2)^{(N+2)/2}\Delta_y u(y) = \left(\frac{1+|y|^2}{2}\right)^2\left(\Delta_y v - \frac{2(N-2)}{1+|y|^2}y\cdot\nabla_y v\right) - \frac{N(N-2)}{4}v, \tag{5.3}$$

$$\frac{1}{4}(1+|x|^2)^{(N+2)/2}\Delta_x u_\lambda(x) = \left(\frac{1+|x|^2}{2}\right)^2\left(\Delta_x v_\lambda - \frac{2(N-2)}{1+|x|^2}x\cdot\nabla_x v_\lambda\right) - \frac{N(N-2)}{4}v_\lambda, \tag{5.4}$$

$$u(y) = \left(\frac{1+|x|^2}{1+|y|^2}\right)^{(N-2)/2}u_\lambda(x). \tag{5.5}$$

For simplicity, we define

$$X=x-e_\lambda,\quad Y=y-e_\lambda,\quad U_\lambda(X)=u_\lambda(x),\ \text{and}\ U(Y)=u(y). \tag{5.6}$$

By (2.5), (5.1), and (5.5), it follows that

$$Y=\left(\frac{1+\lambda^2}{2\lambda}\right)^2\frac{X}{|X|^2},\quad X=\left(\frac{1+\lambda^2}{2\lambda}\right)^2\frac{Y}{|Y|^2}, \tag{5.7}$$

$$U(Y) = \left(\frac{2\lambda}{1+\lambda^2}\right)^{N-2} |X|^{N-2} U_\lambda(X). \tag{5.8}$$

Then we can calculate that

$$\begin{aligned}
\frac{\partial X_j}{\partial Y_i}\bigg|_{i=j} &= \left(\frac{1+\lambda^2}{2\lambda}\right)^2 \frac{|Y|^2 - 2Y_i^2}{|Y|^4}, \quad \frac{\partial X_j}{\partial Y_i}\bigg|_{i\neq j} = \left(\frac{1+\lambda^2}{2\lambda}\right)^2 \frac{-2Y_iY_j}{|Y|^4}, \\
\frac{\partial U(Y)}{\partial Y_i} &= \sum_{j=1}^N \frac{\partial U(Y)}{\partial X_j}\frac{\partial X_j}{\partial Y_i} \\
&= \left(\frac{2\lambda}{1+\lambda^2}\right)^{N-2} \sum_{j=1}^N \left(\frac{\partial |X|^{N-2}}{\partial X_j} U_\lambda(X) + \frac{\partial U_\lambda(X)}{\partial X_j}|X|^{N-2}\right)\frac{\partial X_j}{\partial Y_i}, \\
\frac{\partial^2 U(Y)}{\partial Y_i^2} &= \sum_{j=1}^N \left(\frac{\partial^2 U(Y)}{\partial X_j^2}\left(\frac{\partial X_j}{\partial Y_i}\right)^2 + \frac{\partial U(Y)}{\partial X_j}\frac{\partial^2 X_j}{\partial Y_i^2}\right) + \sum_{1\leqslant p\neq q}^N \frac{\partial^2 U(Y)}{\partial X_p \partial X_q}\frac{\partial X_p}{\partial Y_i}\frac{\partial X_q}{\partial Y_i}, \\
\frac{\partial^2 U(Y)}{\partial X_j^2} &= \left(\frac{2\lambda}{1+\lambda^2}\right)^{N-2}\left(\frac{\partial^2 |X|^{N-2}}{\partial X_j^2} U_\lambda(X) + 2\frac{\partial |X|^{N-2}}{\partial X_1}\frac{\partial U_\lambda(X)}{\partial X_j} + |X|^{N-2}\frac{\partial^2 U_\lambda(X)}{\partial X_j^2}\right), \\
\frac{\partial^2 U(Y)}{\partial X_p X_q} &= \left(\frac{2\lambda}{1+\lambda^2}\right)^{N-2}\left(\frac{\partial^2 |X|^{N-2}}{\partial X_p \partial X_q} U_\lambda(X) + |X|^{N-2}\frac{\partial U_\lambda(X)}{\partial X_p \partial X_q}\right) \\
&\quad + \left(\frac{2\lambda}{1+\lambda^2}\right)^{N-2}\left(\frac{\partial |X|^{N-2}}{\partial X_p}\frac{\partial U_\lambda(X)}{\partial X_q} + \frac{\partial |X|^{N-2}}{\partial X_q}\frac{\partial U_\lambda(X)}{\partial X_p}\right).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&\Delta_Y U(Y) = \sum_{i=1}^N \frac{\partial^2 U(Y)}{\partial Y_i^2} \\
= &\left(\frac{2\lambda}{1+\lambda^2}\right)^{N-2} |X|^{N-2}\left(\sum_{j=1}^N \frac{\partial^2 U_\lambda(X)}{\partial X_j^2}\sum_{i=1}^N\left(\frac{\partial X_j}{\partial Y_i}\right)^2 + \sum_{1\leqslant p\neq q}^N \frac{\partial^2 U_\lambda(X)}{\partial X_p \partial X_q}\sum_{i=1}^N \frac{\partial X_p}{\partial Y_i}\frac{\partial X_q}{\partial Y_i}\right) \\
&+ \left(\frac{2\lambda}{1+\lambda^2}\right)^{N-2}\sum_{j=1}^N \frac{\partial U_\lambda(X)}{\partial X_j}\left(2\frac{\partial |X|^{N-2}}{\partial X_j}\sum_{i=1}^N\left(\frac{\partial X_j}{\partial Y_i}\right)^2 + |X|^{N-2}\sum_{i=1}^N \frac{\partial^2 X_j}{\partial Y_i^2}\right) \\
&+ \left(\frac{2\lambda}{1+\lambda^2}\right)^{N-2}\sum_{j=1}^N \frac{\partial U_\lambda(X)}{\partial X_j}\left(\sum_{i\leqslant m\neq j}^N \frac{\partial |X|^{N-2}}{\partial X_m}\sum_{i=1}^N \frac{\partial X_j}{\partial Y_i}\frac{\partial X_m}{\partial Y_i}\right) \\
&+ \left(\frac{2\lambda}{1+\lambda^2}\right)^{N-2} U_\lambda(X)\left(\sum_{j=1}^N \frac{\partial^2 |X|^{N-2}}{\partial X_j^2}\sum_{i=1}^N\left(\frac{\partial X_j}{\partial Y_i}\right)^2 + \sum_{1\leqslant p\neq q}^N \frac{\partial^2 |X|^{N-2}}{\partial X_p \partial X_q}\sum_{i=1}^N \frac{\partial X_p}{\partial Y_i}\frac{\partial X_q}{\partial Y_i}\right) \\
&+ \left(\frac{2\lambda}{1+\lambda^2}\right)^{N-2} U_\lambda(X)\left(\sum_{j=1}^N \frac{\partial |X|^{N-2}}{\partial X_j}\sum_{i=1}^N \frac{\partial^2 X_j}{\partial Y_i^2}\right).
\end{aligned}$$

From (5.7) we also have

$$\sum_{i=1}^{N}\left(\frac{\partial X_j}{\partial Y_i}\right)^2=\left(\frac{1+\lambda^2}{2\lambda}\right)^4\frac{1}{|Y|^4}=\left(\frac{1+\lambda^2}{2\lambda}\right)^4\left(\frac{2\lambda}{1+\lambda^2}\right)^8|X|^4=\left(\frac{2\lambda}{1+\lambda^2}\right)^4|X|^4,\quad \sum_{i=1}^{N}\frac{\partial X_p}{\partial Y_i}\frac{\partial X_q}{\partial Y_i}=0,$$

$$2\frac{\partial|X|^{N-2}}{\partial X_j}\sum_{i=1}^{N}\left(\frac{\partial X_j}{\partial Y_i}\right)^2+|X|^{N-2}\sum_{i=1}^{N}\frac{\partial^2 X_j}{\partial Y_i^2}=0,\ \sum_{j=1}^{N}\frac{\partial^2|X|^{N-2}}{\partial X_j^2}\sum_{i=1}^{N}\left(\frac{\partial X_j}{\partial Y_i}\right)^2+\sum_{j=1}^{N}\frac{\partial|X|^{N-2}}{\partial X_j}\sum_{i=1}^{N}\frac{\partial^2 X_j}{\partial Y_i^2}=0.$$

Through the above calculation process, we finally get that

$$\Delta_Y U(Y)=\left(\frac{2\lambda}{1+\lambda^2}\right)^{N-2}|X|^{N-2}\left(\sum_{j=1}^{N}\frac{\partial^2 U_\lambda(X)}{\partial X_j^2}\sum_{i=1}^{N}\left(\frac{\partial X_j}{\partial Y_i}\right)^2\right)=\left(\frac{2\lambda}{1+\lambda^2}\right)^{N+2}|X|^{N+2}\Delta_X U_\lambda(X). \tag{5.9}$$

By (5.6), for each i, we have

$$\frac{\partial U_\lambda(X)}{\partial X_i}=\frac{\partial u_\lambda(x)}{\partial x_i}\frac{\partial x_i}{\partial X_i}=\frac{\partial u_\lambda(x)}{\partial x_i},\qquad \frac{\partial^2 U_\lambda(X)}{\partial X_i^2}=\frac{\partial\left(\frac{\partial U_\lambda(X)}{\partial X_i}\right)}{\partial X_i}=\frac{\partial\left(\frac{\partial u_\lambda(x)}{\partial x_i}\right)}{\partial x_i}\frac{\partial x_i}{\partial X_i}=\frac{\partial^2 u_\lambda(x)}{\partial x_i^2},$$

$$\frac{\partial U(Y)}{\partial Y_i}=\frac{\partial u(y)}{\partial y_i}\frac{\partial y_i}{\partial Y_i}=\frac{\partial u(y)}{\partial y_i},\qquad \frac{\partial^2 U(U)}{\partial Y_i^2}=\frac{\partial\left(\frac{\partial U(Y)}{\partial Y_i}\right)}{\partial Y_i}=\frac{\partial\left(\frac{\partial u(y)}{\partial y_i}\right)}{\partial y_i}\frac{\partial y_i}{\partial Y_i}=\frac{\partial^2 u(y)}{\partial y_i^2}.$$

Then (5.9) implies

$$\Delta_y u=\left(\frac{2\lambda}{1+\lambda^2}\right)^{N+2}|x-e_\lambda|^{N+2}\Delta_x u_\lambda.$$

By (5.1), we have

$$(1+|y|^2)^{\frac{N+2}{2}}\Delta_y u=(1+|x|^2)^{\frac{N+2}{2}}\Delta_x u_\lambda.$$

From (5.3) and (5.4), we can conclude that (5.2) holds. □

For each fixed $\tau>0$ and $0<\alpha<1$, we define the parabolic cylinder $Q_\tau:=B_1(0)\times[\tau-\varepsilon_0,\tau+\varepsilon_0]$, where ε_0 is a small constant. For any points $(x,t),(\tilde{x},\tilde{t})\in Q_\tau$, since $u(x,t)\in C^{2,1}(Q_\tau)$, there exist two constants $\xi_t\in(\min\{t,\tilde{t}\},\max\{t,\tilde{t}\})$, $\xi_x\in(0,1)$, such that

$$\begin{aligned}\frac{\left|u(x,t)-u(\tilde{x},\tilde{t})\right|}{|x-\tilde{x}|^\alpha+\left|t-\tilde{t}\right|^{\frac{\alpha}{2}}} &= \frac{\left|u(x,t)-u(x,\tilde{t})+u(x,\tilde{t})-u(\tilde{x},\tilde{t})\right|}{|x-\tilde{x}|^\alpha+|t-\tilde{t}|^{\frac{\alpha}{2}}}\\ &\leqslant \frac{\left|u(x,t)-u(x,\tilde{t})\right|+|u(x,\tilde{t})-u(\tilde{x},\tilde{t})|}{|x-\tilde{x}|^\alpha+|t-\tilde{t}|^{\frac{\alpha}{2}}}\\ &< |u_t(x,\xi_t)|\left|t-\tilde{t}\right|^{1-\frac{\alpha}{2}}+|\nabla u(x+\xi_x(\tilde{x}-x),t)|\,|x-\tilde{x}|^{1-\alpha}.\end{aligned}$$

Again by $u(x,t)\in C^{2,1}(Q_\tau)$ and the boundedness of Q_τ, we have $|u_t(x,\xi_t)|$, $|\nabla u(x+\xi_x(\tilde{x}-x),t)|$, $\left|t-\tilde{t}\right|^{1-\frac{\alpha}{2}}$, and $|x-\tilde{x}|^{1-\alpha}$ are all bounded. Then for some $0<\alpha<1$, there exists a constant $C_0>0$ such that

$$u(x,t)-u(\tilde{x},\tilde{t})\leqslant C_0\left(|x-\tilde{x}|^\alpha+\left|t-\tilde{t}\right|^{\frac{\alpha}{2}}\right). \tag{5.10}$$

Combining (5.10) with the boundedness of $u(x,t)$ in Q_τ, for any $\tau>0$, the orbit $\{u(\cdot,t),\ t\in[\tau-\varepsilon_0,\tau+\varepsilon_0]\}$ is relatively compact in $C(B_1(0))$. To directly address the properties of the functions in $\omega(u)$, we present the following lemma.

Lemma 5.3. *Let $M := sup\{\|u(\cdot,t)\|_{L^\infty} : t > 0\}$, and then, under definitions* (4.5) *and* (4.6), *for each k, u_k and f_k satisfy the problem* (4.7).

Then there exist some functions $u_\infty \in C^{2,1}(B_1(0)\times[1-\varepsilon_0, 1+\varepsilon_0])$ and $f_\infty \in C((0,1)\times[-M,M]\times[1-\varepsilon_0,1+\varepsilon_0])$ such that $u_k \to u_\infty$ in the sense of $C^{2,1}(B_1(0)\times[1-\varepsilon_0,1+\varepsilon_0])$ and $f_k \to f_\infty$ as $k\to\infty$ in the sense of $C((0,1)\times[-M,M]\times[1-\varepsilon_0,1+\varepsilon_0])$. With a constant ε_0, $u_\infty(x,t)$ satisfies

$$\begin{cases} \frac{\partial u_\infty}{\partial t} - \Delta u_\infty = f_\infty(|x|, u_\infty(x,t), t), & (x,t)\in B_1(0)\times[1-\varepsilon_0,1+\varepsilon_0], \\ u_\infty = 0, & (x,t)\in \partial B_1(0)\times[1-\varepsilon_0,1+\varepsilon_0]. \end{cases} \tag{5.11}$$

Proof. For each K and any $(x,t),(u,\tilde{u}),(\tilde{x},\tilde{t})$ in the bounded domain $Q_u := (0,1)\times[-M,M]\times[1-\varepsilon_0,1+\varepsilon_0]$, by (5.13) and definition (4.6), we set C' to be the maximum of C_u, C_x, and C_t appearing in (5.13), with any $\varepsilon>0$, if $\||x|-|\tilde{x}|\|^\alpha + |u-\tilde{u}| + \left|t-\tilde{t}\right|^{\frac{\alpha}{2}} < \delta(\varepsilon) = \varepsilon\backslash C'$, and then we have

$$|f_k(|x|,u(x,t),t) - f_k(|\tilde{x}|,u(\tilde{x},\tilde{t}),\tilde{t})| \leqslant C'(|x-\tilde{x}|^\alpha + |u-\tilde{u}| + |t-\tilde{t}|^{\frac{\alpha}{2}}) < C'\cdot\frac{\varepsilon}{C'} = \varepsilon,$$

which means the sequence $\{f_k\}_{k\in\mathbb{N}}$ is equicontinuous in Q_u. In addition, owing to the boundedness of Q_u and the continuity of f in Q_u, we deduce that $\{f_k\}_{k\in\mathbb{N}}$ is bounded in Q_u for each k. Then the Ascoli-Azela theorem implies that there exists a function $f_\infty \in C(Q_u)$ such that $f_k \to f_\infty$ in the sense of $C(Q_u)$ as $k\to\infty$, and f_∞ is Lipschitz continuous in u by (F1).

Next we discuss the convergence of $\{u_k\}_{k\in\mathbb{N}}$. Set $Q_1 := B_1(0)\times[1-\varepsilon_0,1+\varepsilon_0]$, and from (4.7), for each k, let $\tilde{f}_k(x,t) = f_k(|x|,u_k(x,t),t)$. Then we have

$$\frac{\partial u_k(x,t)}{\partial t} - \Delta u_k(x,t) = \tilde{f}_k(x,t), \quad (x,t)\in Q_1. \tag{5.12}$$

Additionally, for any $(x,t),(\tilde{x},\tilde{t})\in Q_1$, by conditions (F1) and (F2), we have three constants C_x, C_u, and C_t such that

$$\begin{aligned} &|\tilde{f}_k(x,t) - \tilde{f}_k(\tilde{x},\tilde{t}) = |f_k(|x|,u_k(x,t),t) - f_k(|\tilde{x}|,u_k(\tilde{x},\tilde{t}),\tilde{t})| \\ \leqslant\ & |f(|x|,u_k(x,t),t) - f(|\tilde{x}|,u_k(x,t),t)| + |f(|\tilde{x}|,u_k(x,t),t) - f(|\tilde{x}|,u_k(\tilde{x},\tilde{t}),t)| \\ &+ |f(|\tilde{x}|,u_k(\tilde{x},\tilde{t}),t) - f(|\tilde{x}|,u_k(\tilde{x},\tilde{t}),\tilde{t})| \\ \leqslant\ & C_x|x-\tilde{x}|^\alpha + C_u|u_k(x,t)-u_k(\tilde{x},\tilde{t})| + C_t|t-\tilde{t}|^{\frac{\alpha}{2}}. \end{aligned} \tag{5.13}$$

By (5.10), we set $C = \max\{C_x, C_uC_0, C_t\}$, and then for any $(x,t),(\tilde{x},\tilde{t})\in Q_1$, we have

$$|\tilde{f}_k(x,t) - \tilde{f}_k(\tilde{x},\tilde{t})| \leqslant C\left(|x-\tilde{x}|^\alpha + \left|t-\tilde{t}\right|^{\frac{\alpha}{2}}\right), \tag{5.14}$$

which means for any two points $P(x,t)$ and $\tilde{P}(\tilde{x},\tilde{t})$, we define $d(P,\tilde{P}) = (|x-\tilde{x}| + |t-\tilde{t}|^{\frac{1}{2}})$, and then we can get that

$$\sup_{\substack{P,\tilde{P}\in Q_1 \\ P\neq\tilde{P}}} \frac{|\tilde{f}_k(P) - \tilde{f}_k(\tilde{P})|}{d^\alpha(P,\tilde{P})} < +\infty. \tag{5.15}$$

Given that u_k is the solution of (4.7), then the existence and uniqueness theorem of classical solutions for heat equations (Theorem 3.3.7 in [48]) implies that

$$\begin{aligned}
&|u_k|_{2+\alpha,1+\frac{\alpha}{2};Q_1}\\
&:= \sup_{P\in Q_1}|u_k(P)| + \sup_{P\in Q_1}|Du_k(P)| + \sup_{P\in Q_1}|D^2u_k(P)| + \sup_{P\in Q_1}\left|\frac{\partial u_k}{\partial t}(P)\right| + \sup_{\substack{P,\tilde P\in Q_1\\ P\neq\tilde P}}\frac{|u_k(P)-u_k(\tilde P)|}{d^\alpha(P,\tilde P)}\\
&\quad+\sup_{\substack{P,\tilde P\in Q_1\\ P\neq\tilde P}}\frac{|Du_k(P)-Du_k(\tilde P)|}{d^\alpha(P,\tilde P)}+\sup_{\substack{P,\tilde P\in Q_1\\ P\neq\tilde P}}\frac{|D^2u_k(P)-D^2u_k(\tilde P)|}{d^\alpha(P,\tilde P)}+\sup_{\substack{P,\tilde P\in Q_1\\ P\neq\tilde P}}\frac{\left|\frac{\partial u_k}{\partial t}(P)-\frac{\partial u_k}{\partial t}(\tilde P)\right|}{d^\alpha(P,\tilde P)}\\
&\leqslant C<+\infty,
\end{aligned}\tag{5.16}$$

where C is a constant independent of k, $Du_k = (\partial u_k/\partial x_1, \partial u_k/\partial x_2, \cdots, \partial u_k/\partial x_N)$, and D^2u_k is the Hessian matrix of u_k. On the one hand (5.16) shows the equicontinuity of u_k, Du_k, D^2u_k, and $\partial u_k/\partial t$ in $C(Q_1)$, which means for a given $\varepsilon > 0$, we can choose $\delta > 0$ independent of k such that

$$|u_k(P)-u_k(\tilde P)| + \left|\frac{\partial u_k}{\partial t}(P)-\frac{\partial u_k}{\partial t}(\tilde P)\right| + |Du_k(P)-Du_k(\tilde P)| + |D^2u_k(P)-D^2u_k(\tilde P)| \leqslant C\delta^\alpha < \varepsilon,$$

for all $P, \tilde P \in Q_1$ and k. On the other hand (5.16) also ensures that for each k, u_k, Du_k, D^2u_k, and $\partial u_k/\partial t$ are all bounded in Q_1. By the Ascoli-Azela theorem, there exist functions u_∞, $u_{t\infty}$, $\{u_{i\infty}\}_{i=1}^N$, $\left\{u_{ij\infty}\right\}_{i,j=1}^N \in C(Q_1)$ such that as $k \to \infty$ for each i and j we have

$$u_k \to u_\infty, \quad \frac{\partial u_k}{\partial t} \to u_{t\infty}, \quad \frac{\partial u_k}{\partial x_i} \to u_{i\infty}, \quad \frac{\partial u_k}{\partial x_i \partial x_j} \to u_{ij\infty},$$

which are all in the sense of $C(Q_1)$. Employing the fundamental theorem of caculus and (5.16), for $\varepsilon > 0$, $(x,t) \in Q_1$, and fixed i, we can choose $\overline{\delta} = \min(\delta, 1) > 0$ independent of k such that $B_{\overline{\delta}}(x) \times [1-\varepsilon_0, 1+\varepsilon_0] \in Q_1$ and for all $|h| < \overline{\delta}$ and k we have

$$\begin{aligned}
&\left|\frac{\partial u_k}{\partial x_i}(x_1,\cdots,x_j+h,\cdots,x_N,t)-\frac{\partial u_k}{\partial x_i}(x_1,\cdots,x_j,\cdots,x_N,t)-\frac{\partial^2 u_k}{\partial x_i\partial x_j}(x,t)h\right|\\
&=\left|\int_0^1\frac{\partial^2 u_k}{\partial x_i\partial x_j}(x_1,\cdots,x_j+sh,\cdots,x_N,t)hds-\frac{\partial^2 u_k}{\partial x_i\partial x_j}(x,t)h\right|\\
&\leqslant\int_0^1\left|\frac{\partial^2 u_k}{\partial x_i\partial x_j}(x_1,\cdots,x_j+sh,\cdots,x_N,t)-\frac{\partial^2 u_k}{\partial x_i\partial x_j}(x,t)\right||h|ds\leqslant|h|^{1+\alpha}C<C\delta^\alpha|h|<\varepsilon|h|.
\end{aligned}$$

Similarly we can get that

$$\left|u_k(x_1,\cdots,x_i+h,\cdots,x_N,t)-u_k(x,t)-\frac{\partial u_k}{\partial x_i}(x,t)h\right|\leqslant\sup_{Q_1}\left|D^2u_k\right||h|^{1+\alpha}\leqslant C|h|^{1+\alpha}<\varepsilon|h|,$$

$$\left|u_k(x,t+h)-u_k(x,t)-\frac{\partial u_k(x,t)}{\partial t}h\right|\leqslant\int_0^1\left|\frac{\partial u_k}{\partial t}(x,t+sh)-\frac{\partial u_k}{\partial t}(x,t)\right||h|ds<\varepsilon|h|.$$

Letting $k \to \infty$, we can derive that

$$
\begin{aligned}
&|u_{i\infty}(x_1, \cdots, x_j + h_j, \cdots, x_N, t) - u_{i\infty}(x_1, \cdots, x_j, \cdots, x_N, t) - u_{ij\infty}(x, t)h| < \varepsilon|h|, \\
&|u_{\infty}(x_1, \cdots, x_i + h_i, \cdots, x_N, t) - u_{\infty}(x_1, \cdots, x_i, \cdots, x_N, t) - u_{i\infty}(x, t)h| < \varepsilon|h|, \\
&\qquad |u_\infty(x, t + h) - u_\infty(x, t) - u_{t\infty}h| < \varepsilon|h|.
\end{aligned} \tag{5.17}
$$

Then (5.17) means that for each i and j,

$$
u_{ij\infty} = \frac{\partial^2 u_\infty}{\partial x_i \partial x_j}, \quad u_{i\infty} = \frac{\partial u_\infty}{\partial x_i}, \quad u_{t\infty} = \frac{\partial u_\infty}{\partial t}, \tag{5.18}
$$

as $h \to 0$. The above discussion shows that $u_\infty \in C^{2,1}(Q_1)$ and as $k \to \infty$ we observe that $u_k \to u_\infty$ in the sense of $C^{2,1}(Q_1)$ and

$$
\frac{\partial u_k}{\partial t} \to \frac{\partial u_\infty}{\partial t}, \quad Du_k \to Du_\infty, \quad D^2 u_k \to D^2 u_\infty, \tag{5.19}
$$

all in the sense of $C(Q_1)$. By (5.19) and the existence of u_∞ and f_∞, considering the problem (4.7) as $k \to \infty$, we can deduce that u_∞ satisfies

$$
\begin{cases}
\dfrac{\partial u_\infty}{\partial t}(x, t) - \Delta u_\infty(x, t) = f_\infty(|x|, u_\infty(x, t), t), & (x, t) \in B_1(0) \times [1 - \varepsilon_0, 1 + \varepsilon_0], \\
u_\infty(x, t) = 0, & (x, t) \in \partial B_1(0) \times [1 - \varepsilon_0, 1 + \varepsilon_0].
\end{cases}
$$

Then we complete the proof of Lemma 5.3. □

Author contributions

Baiyu Liu: Methodology, writing–review and editing; Wenlong Yang: Writing–original draft preparation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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