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Research article

Positive and sign-changing solutions for Kirchhoff equations with indefinite potential

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Abstract: We deal with the nonlinear Kirchhoff problem

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=f(u),\quad x\in\mathbb{R}^3,\tag{$\mathcal{P}$}$$

where a is a positive constant, $b>0$ is a parameter, the potential function V is allowed to change its sign, and the nonlinearity $f\in C(\mathbb{R},\mathbb{R})$ exhibits subcritical growth. Under some suitable conditions on V, we first prove that the problem has a positive ground state solution for all $b>0$. Then, by using a more general global compactness lemma and a sign-changing Nehari manifold, combined with the method of constructing a sign-changing $(PS)_c$ sequence, we show the existence of a least energy sign-changing solution for $b>0$ that is sufficiently small. Moreover, the asymptotic behavior $b\searrow 0$ is established.

Keywords: Kirchhoff problem; positive solutions; sign-changing solutions; variational methods; indefinite potential
Mathematics Subject Classification: 35J20, 35J60

1. Introduction and main results

In the past decades, the following Kirchhoff problem

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+V(x)u=f(x,u),\quad x\in\mathbb{R}^3\tag{1.1}$$

has attracted considerable attention. As we know, the following Dirichlet problem is one of the important deformations of Equation (1.1), which can be degenerated from (1.1). That is, if $V(x)=0$

and Ω is a bounded subset of $\mathbb{R}^3$, then Equation (1.1) will become

$$\begin{cases} -\left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Problem (1.2) corresponds to

$$u_{tt} - \left(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\right)\Delta u = f(x,u),$$

which was advanced by Kirchhoff in [1]. As a generalization of the classical D'Alembert wave equation of the free vibration of elastic strings, the Kirchhoff model considers the changes of string length caused by lateral vibrations, which has important practical significance. For more mathematical and physical background about the Kirchhoff equations, we direct readers to [2, 3] and the references quoted within them.

When $b = 0$, Equation (1.1) degenerates to the Schrödinger problem

$$-a\Delta u + V(x)u = f(x,u), \quad x \in \mathbb{R}^N, \tag{1.3}$$

which has been studied in the past few decades; see, for instance, [4–9] and the references therein. One interesting characteristic is the potential V change sign on $\mathbb{R}^N$. In [6], Bahrouni, Ounaies, and Radulescu showed the existence of infinitely many solutions for Equation (1.3) with $f(x,u) = a(x)|u|^{q-1}u$ and $0 < q < 1$. Further, Furtado, Maia, and Medeiros [10] investigated the Schrödinger equation (1.3) with $a = 1$ and $f(x,u) = f(u)$. Concretely, the nonlinearity $f \in C^1(\mathbb{R},\mathbb{R})$ is superlinear with subcritical growth. In addition, it verifies the Ambrosetti-Rabinowitz condition: for some $\theta > 2$, there is

$(\widehat{f})$ $$0 < \theta F(t) \le tf(t) \text{ for all } t \neq 0, \text{ where } F(t) = \int_0^t f(s)\mathrm{d}s.$$

Besides, V satisfies the following assumptions:

(V_0) $V \in L^t_{loc}(\mathbb{R}^3)$ for some $t > \frac{3}{2}$;

(V_1) $0 < V_\infty := \lim_{|x|\to\infty} V(x) < +\infty$;

(V_2) $\int_{\mathbb{R}^3} |V^-(x)|^{\frac{3}{2}}\mathrm{d}x < S^{\frac{3}{2}}$, where $V^- := \max\{-V, 0\}$ and S is the best constant for the Sobolev embedding, given by

$$S := \inf_{u\in\mathcal{D}^{1,2}(\mathbb{R}^3)\setminus\{0\}} \frac{\int_{\mathbb{R}^3} |\nabla u|^2\mathrm{d}x}{\left(\int_{\mathbb{R}^3} |u|^6\mathrm{d}x\right)^{\frac{1}{3}}};$$

(V_3) $V(x) \le V_\infty$ for all $x \in \mathbb{R}^3$ and $V \not\equiv V_\infty$;

(V_4) there exist $\gamma > 0$ and $C_V > 0$ such that

$$V(x) \le V_\infty - C_V e^{-\gamma|x|}, \quad \text{for all } x \in \mathbb{R}^3.$$

By using variational methods and the concentration-compactness principle, they obtained a positive ground state solution, and also a nodal solution. Therefore, we know that Equation (1.3) has a least energy sign-changing solution $v_0 \in \mathcal{M}_0$ such that $\mathcal{I}_0(v_0) = c_{0,2} := \inf_{u \in \mathcal{M}_0} \mathcal{I}_0(u)$, where $\mathcal{M}_0 := \{u \in X : u^{\pm} \neq 0,\ \langle \mathcal{I}_0'(u), u^+\rangle = \langle \mathcal{I}_0'(u), u^-\rangle = 0\}$ with $\mathcal{I}_0(u) = \frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u|^2 + V(x)u^2)\mathrm{d}x - \int_{\mathbb{R}^3} F(u)\mathrm{d}x$. For more results about problem (1.3) with indefinite potential, see [10–13] and the references therein.

When $b \neq 0$, due to the presence of $\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \Delta u$, problem (1.1) becomes a nonlocal problem, which also brings some essential difficulties for our study. In the past, problems similar to (1.1) have attracted a lot of interest, and so there are many results. For instance, the existence of positive, ground state, and sign-changing solutions for problem (1.1) with various potential V and nonlinearity f has been extensively studied; see [14–21]. In a recent paper [22], Ni, Sun, and Chen also obtained the existence and multiplicity of normalized solutions for a Kirchhoff type problem by using minimization techniques and Lusternik-Schnirelmann theory.

In what follows, we are particularly interested in the case when the potential V involved in problem (1.1) is indefinite. When V can change its sign and satisfies conditions $(V_0)-(V_2)$ and (V_4), Batista and Furtado [23] studied problem (1.1) with $f(x,u) = a(x)|u|^{p-2}u$, that is,

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = a(x)|u|^{p-2}u, \quad x \in \mathbb{R}^3, \tag{1.4}$$

where $4 < p < 6$ and a satisfies the following assumptions:

(a_0) $a \in L^\infty(\mathbb{R}^3)$;

(a_1) there exist $C_a,\ \theta_0 > 0$ such that

$$a(x) \leqslant a_\infty - C_a e^{-\theta_0 |x|}, \quad \text{for a.e. } x \in \mathbb{R}^3,$$

where

$$a_\infty := \lim_{|x| \to +\infty} a(x) > 0.$$

Via the constraint variational methods and the quantitative deformation lemma, the authors not only obtained a non-negative ground state solution, but also a sign-changing solution of Equation (1.4). Here, we point out that the proof of the existence of sign-changing solutions depends on the radial symmetry of V in their paper. Indeed, once the potential V is radial, one can overcome the lack of compactness by considering in the radial subspace. However, if V is not radial, it makes no sense to restrict the problem to spaces of radial functions as the authors did in [23]. Besides, we know the embedding $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ is not compact for $2 < p < 6$. Therefore, one may ask if there will be a sign-changing solution for problem (1.4) or problem (1.1) when the indefinite potential V is not radially symmetric and f is a general nonlinearity.

Next, we will provide an answer to the questions raised above. Specifically, we are going to investigate the problem

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)\Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^3, \tag{1.5}$$

where $a, b > 0$ and V is a sign-changing potential.

Moreover, we shall impose that $f \in C(\mathbb{R}, \mathbb{R})$ is odd. In addition, f also satisfies the following assumptions:

(f_0) $\lim\limits_{t\to 0} \frac{f(t)}{t} = 0$;

(f_1) $\lim\limits_{|t|\to\infty} \frac{f(t)}{|t|^5} = 0$;

(f_2) $\lim\limits_{|t|\to\infty} \frac{F(t)}{|t|^4} = +\infty$, where $F(t) := \int_0^t f(s)\mathrm{d}s$;

(f_3) $\frac{f(t)}{|t|^3}$ is a non-decreasing function for $t \in \mathbb{R} \setminus \{0\}$;

(f_4) $F(t) \geq 0$, for all $t \in \mathbb{R}$.

Through (f_0) and (f_1), it can be known that for any $\varepsilon > 0$, there is a positive constant C_ε which makes

$$|f(t)| \leq \varepsilon(|t| + |t|^5) + C_\varepsilon |t|^{q-1}, \quad \text{for any } t \in \mathbb{R} \tag{1.6}$$

where $q \in (2, 6)$. Then, let $G(t) := \frac{1}{4} f(t)t - F(t)$. By using (f_3), we can easily obtain

$$0 \leq G(t_1) \leq G(t_2), \quad \text{for any } 0 \leq t_1 \leq t_2. \tag{1.7}$$

In fact, for any $t_2 \geq t_1 \geq 0$, from (f_3), one has $\frac{f(t_1)}{t_1^3} \leq \frac{f(t)}{t^3} \leq \frac{f(t_2)}{t_2^3}$ for all $t_1 \leq t \leq t_2$. Then,

$$\begin{aligned}
G(t_1) &= \frac{1}{4} f(t_1)t_1 + \int_{t_1}^{t_2} f(t)\mathrm{d}t - F(t_2) \\
&= \frac{1}{4} f(t_1)t_1 + \int_{t_1}^{t_2} \frac{f(t)}{t^3} t^3 \mathrm{d}t - F(t_2) \\
&\leq \frac{1}{4} f(t_1)t_1 + \int_{t_1}^{t_2} \frac{f(t_2)}{t_2^3} t^3 \mathrm{d}t - F(t_2) \\
&\leq \frac{1}{4} f(t_1)t_1 + \frac{1}{4} \frac{f(t_2)}{t_2^3}(t_2^4 - t_1^4) - F(t_2) \\
&\leq \frac{1}{4}\{f(t_1)t_1 + f(t_2)t_2 - f(t_1)t_1\} - F(t_2) \\
&= \frac{1}{4} f(t_2)t_2 - F(t_2) = G(t_2).
\end{aligned}$$

Before presenting our main results, we first discussed the basic framework in the space

$$X := \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 \mathrm{d}x < +\infty \right\}$$

and the corresponding norm is given by

$$\|u\|_X := \left(\int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2)\mathrm{d}x \right)^{\frac{1}{2}}, \quad \text{for any } u \in X.$$

Next, we present the main results.

Theorem 1.1. *Assume that* $(V_0)-(V_3)$ *and* $(f_0)-(f_3)$ *are satisfied. Then, problem* (1.5) *has a positive ground state solution.*

In the proof, we need to overcome the problems stemming from the lack of compactness of Sobolev embedding in the whole space $\mathbb{R}^3$. To solve this problem, we first analyze the relationship between the energy functional's minimax level and that of the limit problem. Subsequently, we obtain a positive ground state solution by applying a more general global compactness lemma (see Lemma 3.1).

In the second result, we mainly focused on researching the existence of least-energy sign-changing solutions. In order to obtain the result, we will use the method in [24].

Theorem 1.2. *Assume that* f *satisfies* $(f_0)-(f_4)$*. Assume also that* V *satisfies* $(V_0)-(V_2)$*, and there exist positive constants* M, C*, and* γ *such that, for* $|x| \geq M$*,*

$$(V_4') \qquad V(x) \leq V_\infty - \frac{C}{1+|x|^\gamma}.$$

Then, there exists $b^* > 0$ *small enough such that Equation* (1.5) *possesses a least energy sign-changing solution for every* $b \in (0, b^*)$*.*

Remark 1.3. *Here, we would like to provide an example of nonlinearity* f*, which is odd and satisfies the conditions* $(f_0)-(f_4)$*, as shown below:*

$$f(t) = \sum_{i=1}^{l} a_i |t|^{b_i} t, \quad \text{for any } t \in \mathbb{R},$$

where $a_i > 0$ *and* $2 < b_i < 4$ *for every* $i \in \{1, 2, \ldots, l\}$*.*

In addition, we can find that a more obvious example that satisfies conditions $(V_0)-(V_2), (V_4')$ *is to take* $V(x) = V_\infty - \frac{C}{1+|x|^\gamma}$*, where* C *and* γ *are positive given by* (V_4')*. Alternatively, we can give another example below that satisfies our hypothesis. That is, the potential function* V *is given by*

$$V(x) = \begin{cases} -\frac{\alpha}{|x|^\beta}, & \text{if } |x| < 1; \\ \frac{|x|^2}{1+|x|^2}, & \text{if } |x| \geq 1, \end{cases}$$

where $\beta \in (0, 2)$ *and* $\alpha > 0$ *is a sufficiently small.*

Remark 1.4. *Compared with [10], we apply weaker conditions* (f_2) *and* (f_4) *instead of the Ambrosetti-Rabinowitz condition to investigate the existence of sign-changing solutions. Moreover, condition* (f_4) *can be used to construct a sign-changing* $(PS)_{c_{b,2}}$*; please refer to Section 5 for the detailed process. We note that condition* (f_4) *is only used here. Besides, it is not necessary for* f *to be differentiable. By using the quantitative deformation lemma, we can prove that the minimizer on the Nehari manifold is a critical point of the energy functional* $\mathcal{I}_b$ *given in Section 2, please refer to Theorem 1.1 below.*

Remark 1.5. *In this result, we apply the weaker condition* (V_4') *instead of the condition* (V_4) *given in [23], which makes our results applicable to a wider range of potential functions. Furthermore, the potential function* V *does not need to be radial in our paper, which is different from [23]. Moreover, for the proof of Theorem 1.2, compared to [10], the essential problem we face is that the existence of the nonlocal term poses some difficulties to energy estimates. Here, the condition* (V_4') *and the restriction on the range of the parameter* b *are crucial for estimating energy (see Lemma 5.1). After that, we can restore the compactness of a bounded Palais–Smale sequence at a certain level with the help of a global compactness lemma, thereby obtaining the conclusion of Theorem 1.2.*

Theorem 1.6. *Assume that u_{b_n} are the least energy sign-changing solutions of Equation* (1.5) *obtained in Theorem 1.2. Then, for any sequence $\{b_n\}$ with $b_n \searrow 0$ as $n \to \infty$, there exists a subsequence, still denoted by $\{b_n\}$, such that $u_{b_n} \to u_0$ in X as $n \to \infty$. Moreover, u_0 is a least energy sign-changing solution of Equation* (1.3) *with $f(x,u) = f(u)$.*

Remark 1.7. *In this paper, we not only weaken the potential condition, but also increase asymptotic behavior, which is different from [23]. In particular, our result regarding asymptotic behavior is new. The asymptotic research enriches the results of our paper, and at the same time, the sign-changing solution is closely related to the results of the Schrödinger equation in [10]. In practical terms, we study their asymptotic behavior as $b \searrow 0$ under assumptions $(V_0) - (V_2)$ and (V_4), and we show that they converge to a least energy sign-changing solution of the Schrödinger equation* (1.3) *with $f(x,u) = f(u)$, $b \searrow 0$.*

Now, we introduce the organizational structure of this paper. We first provide the notations and some necessary lemmas in Section 2. Then, in Section 3, our main job is to establish a more general global compactness result, which will be well applied in the proof of our main theorems. In Section 4, we first give energy estimates, then Theorem 1.1 is verified. Finally, we prove Theorem 1.2. After that, Theorem 1.6 is verifed in Section 5.

2. Preliminaries

Next, we first give some notations and necessary lemmas, which are very helpful for us in proving the main theorems.

- " $\rightharpoonup$ " and " $\to$ " depict the weak and strong convergence, sequentially.
- $|u|_p = (\int_{\mathbb{R}^3} |u|^p \mathrm{d}x)^{\frac{1}{p}}$ denotes the norm in $L^p(\mathbb{R}^3)$ for $p \in [1, +\infty)$.
- Let $H^1(\mathbb{R}^3)$ be the Hilbert space with respect to the norm $\|u\|_{H^1}^2 := \int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)\mathrm{d}x$.
- We use $|\Omega|$ to represent the Lebesgue measure of the set Ω.
- $o(1)$ denotes a quantity which goes to zero as $n \to \infty$.
- C, C_i $(i = 1, 2, \ldots)$ represent different positive constants.
- We denote $V = V^+ - V^-$ with $V^\pm := \max\{\pm V, 0\}$.

Lemma 2.1. *Under the conditions of (V_1) and (V_2), the quadratic form*

$$u \mapsto \int_{\mathbb{R}^3} (a|\nabla u|^2 + V(x)u^2)\mathrm{d}x \tag{2.1}$$

defines a norm in $H^1(\mathbb{R}^3)$, which is equivalent to the usual one.

Proof. Due to (V_1), one has that there is $R > 0$, which satisfies

$$\frac{V_\infty}{2} \le V^+ \le \frac{3V_\infty}{2}, \quad \text{for any } x \in \mathbb{R}^3 \setminus B_R(0), \tag{2.2}$$

where $B_R(0) := \{x \in \mathbb{R}^3 : |x| < R\}$. In general, we take $a = 1$. Then, using the Hölder inequality, the definition of S, and (2.2), one can ascertain

$$\int_{\mathbb{R}^3} (|\nabla u|^2 + Vu^2)\mathrm{d}x = \int_{\mathbb{R}^3} (|\nabla u|^2 + V^+u^2)\mathrm{d}x - \int_{\mathbb{R}^3} V^-u^2\mathrm{d}x$$

$$
\begin{aligned}
&\le \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \int_{B_R(0)} V^+ u^2 \mathrm{d}x + \int_{|x|\ge R} V^+ u^2 \mathrm{d}x \\
&\le \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + |V^+|_{L^{3/2}(B_R(0))} |u|_6^2 + \frac{3V_\infty}{2} \int_{|x|\ge R} u^2 \mathrm{d}x \\
&\le (1 + S^{-1}|V^+|_{L^{3/2}(B_R(0))}) \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{3V_\infty}{2} \int_{\mathbb{R}^3} u^2 \mathrm{d}x \\
&\le \max\left\{1 + S^{-1}|V^+|_{L^{3/2}(B_R(0))}, \frac{3V_\infty}{2}\right\} \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) \mathrm{d}x.
\end{aligned}
$$

Furthermore, from [10, Lemma 2.1], one deduces that there is $C_1 > 0$, which holds the inequality $\int_{\mathbb{R}^3}(|\nabla u|^2 + V^+u^2)\mathrm{d}x \ge C_1 \int_{\mathbb{R}^3}(|\nabla u|^2 + u^2)\mathrm{d}x$. Moreover, from (V_2), one has

$$
\left|\int_{\mathbb{R}^3} V^- u^2 \mathrm{d}x\right| \le \int_{\mathbb{R}^3} |V^-| u^2 \mathrm{d}x \le |V^-|_{\frac{3}{2}} |u|_6^2 \le S^{-1} |V^-|_{\frac{3}{2}} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x.
$$

Hence, combining with $V = V^+ - V^-$, it can be obtained that

$$
\begin{aligned}
\int_{\mathbb{R}^3} (|\nabla u|^2 + V u^2)\mathrm{d}x &\ge \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \int_{\mathbb{R}^3} V^+ u^2 \mathrm{d}x - S^{-1}|V^-|_{\frac{3}{2}} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \\
&\ge \min\left\{1 - S^{-1}|V^-|_{\frac{3}{2}}, 1\right\} \int_{\mathbb{R}^3} (|\nabla u|^2 + V^+ u^2)\mathrm{d}x \\
&\ge \min\left\{1 - S^{-1}|V^-|_{\frac{3}{2}}, 1\right\} C_1 \int_{\mathbb{R}^3} (|\nabla u|^2 + u^2)\mathrm{d}x
\end{aligned}
$$

and the lemma is completed. □

Remark 2.2. *From Lemma 2.1, it is obvious to see that the norm given by* (2.1) *is equivalent to the usual norm of* $H^1(\mathbb{R}^3)$. *Moreover, we know that the embeddings* $H^1(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ *(see [9, Theorem 1.8]) and* $H^1(\mathbb{R}^3) \hookrightarrow L^q_{loc}(\mathbb{R}^3)$ *(see [9, Theorem 1.9]) are continuous and compact, respectively, where* $p \in [2, 6]$ *and* $q \in [1, 6)$. *Therefore, the embedding* $X \hookrightarrow L^p(\mathbb{R}^3)$ *is also continuous for all* $p \in [2, 6]$. *Besides, one can see that the continuity of the embedding mentioned above can be represented by the following inequalities:*

$$
|u|_p^p \le S_p^p \|u\|_X^p, \quad \text{for any } p \in [2, 6],
$$

in which $S_p^p > 0$ *is a constant.*

Define the energy functional $\mathcal{I}_b : X \to \mathbb{R}$ by

$$
\mathcal{I}_b(u) = \frac{1}{2}\|u\|_X^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} F(u)\mathrm{d}x.
$$

One can see that $u \mapsto \int F(u)\mathrm{d}x$ is well defined on X, so $\mathcal{I}_b(u)$ is also well defined. Through discussions, one can deduce that $\mathcal{I}_b \in C^1(X, \mathbb{R})$. Moreover, for any $v \in H^1(\mathbb{R}^3)$,

$$
\langle \mathcal{I}_b'(u), v\rangle = \int_{\mathbb{R}^3} (a\nabla u \cdot \nabla v + V(x)uv)\mathrm{d}x + b\int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x \int_{\mathbb{R}^3} \nabla u \cdot \nabla v \mathrm{d}x - \int_{\mathbb{R}^3} f(u)v \mathrm{d}x.
$$

The limit problem associated with (1.5) is the autonomous problem below:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)\Delta u+V_\infty u=f(u),\quad x\in\mathbb{R}^3.\qquad(\mathcal{P}_\infty)$$

The functional corresponding to Equation ($\mathcal{P}_\infty$) is

$$\mathcal{I}_{b,\infty}(u)=\frac{1}{2}\int_{\mathbb{R}^3}(a|\nabla u|^2+V_\infty u^2)\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x\right)^2-\int_{\mathbb{R}^3}F(u)\mathrm{d}x.$$

Let

$$c_{b,1}:=\inf_{u\in\mathcal{N}_b}\mathcal{I}_b(u)\quad\text{and}\quad c_{b,\infty}:=\inf_{u\in\mathcal{N}_{b,\infty}}\mathcal{I}_{b,\infty}(u),\qquad(2.3)$$

where

$$\mathcal{N}_b:=\{u\in X\setminus\{0\}:\langle\mathcal{I}_b'(u),u\rangle=0\}\quad\text{and}\quad\mathcal{N}_{b,\infty}:=\left\{u\in X\setminus\{0\}:\langle\mathcal{I}_{b,\infty}'(u),u\rangle=0\right\}.$$

In addition, we define

$$c_{b,2}:=\inf_{u\in\mathcal{M}_b}\mathcal{I}_b(u),$$

where

$$\mathcal{M}_b:=\{u\in X:u^\pm\neq0,\ \langle\mathcal{I}_b'(u),u^+\rangle=\langle\mathcal{I}_b'(u),u^-\rangle=0\}.$$

Next, we will provide some important lemmas to prove Theorem 1.1 and Theorem 1.2.

Lemma 2.3. *If $\{u_n\}\subset\mathcal{M}_b$ is a minimizing sequence for $\mathcal{I}_b$, then $C_1\le\|u_n^\pm\|_X\le C_2$ for some C_1, $C_2>0$.*

Proof. Let $\{u_n\}\subset\mathcal{M}_b$ and $\mathcal{I}_b(u_n)\to m$ as $n\to\infty$. On one hand, due to (f_0), (f_1), and the Sobolev inequality, one has

$$\begin{aligned}\|u_n^\pm\|_X^2&\le\|u_n^\pm\|_X^2+b\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\int_{\mathbb{R}^3}|\nabla u_n^\pm|^2\mathrm{d}x=\int_{\mathbb{R}^3}f(u_n)u_n^\pm\mathrm{d}x\\&\le\int_{\mathbb{R}^3}\left(\varepsilon|u_n|+C_\varepsilon|u_n|^5\right)u_n^\pm\mathrm{d}x\\&=\varepsilon\int_{\mathbb{R}^3}|u_n^\pm|^2\mathrm{d}x+C_\varepsilon\int_{\mathbb{R}^3}|u_n^\pm|^6\mathrm{d}x\\&\le\varepsilon S_2^2\|u_n^\pm\|_X^2+C_\varepsilon S_6^6\|u_n^\pm\|_X^6.\end{aligned}$$

When ε is sufficiently small to make $(1-\varepsilon S_2^2)>0$, we can obtain that $\|u_n^\pm\|_X\ge C_1$ for some $C_1>0$. On the other hand, in light of $\{u_n\}\subset\mathcal{M}_b\subset\mathcal{N}_b$, we have $\langle\mathcal{I}_b'(u_n),u_n\rangle=0$. In virtue of (1.7), one can arrive at

$$\begin{aligned}m+o(1)=\mathcal{I}_b(u_n)&=\mathcal{I}_b(u_n)-\frac{1}{4}\langle\mathcal{I}_b'(u_n),u_n\rangle\\&=\frac{1}{4}\|u_n\|_X^2+\int_{\mathbb{R}^3}\left(\frac{1}{4}f(u_n)u_n-F(u_n)\right)\mathrm{d}x\\&\ge\frac{1}{4}\|u_n\|_X^2,\end{aligned}$$

which means that $m>0$, so one can deduce that $\{u_n\}$ is bounded in X. Namely, $\|u_n^\pm\|_X\le C_2$ for some $C_2>0$. Therefore, $C_1\le\|u_n^\pm\|_X\le C_2$. □

Remark 2.4. *If $\{u_n\} \subset X$ be a $(PS)_c$ sequence, then by using* (1.7) *and similar arguments to Lemma 2.3, one can get $\|u_n\|_X \leq C_2$ for some $C_2 > 0$.*

Similar to the discussion in [25], one can derive Lemma 2.5 and Lemma 2.6. Here, we omit the proof process.

Lemma 2.5. *Assume that $(V_0) - (V_2)$ and $(f_0) - (f_3)$ hold. If $u \in X$ with $u^{\pm} \neq 0$, then there exists a unique pair $(s_u, t_u) \in (0, +\infty) \times (0, +\infty)$ such that $s_u u^+ + t_u u^- \in \mathcal{M}_b$. Moreover,*

$$\mathcal{I}_b(s_u u^+ + t_u u^-) = \max_{s,t \geq 0} \mathcal{I}_b(s u^+ + t u^-).$$

Lemma 2.6. *Assume that $(V_0) - (V_2)$ and $(f_0) - (f_3)$ hold. If $u \in X$ with $u \neq 0$, then there exists a unique $s_u > 0$ such that $s_u u \in \mathcal{N}_b$. Moreover,*

$$\mathcal{I}_b(s_u u) = \max_{s \geq 0} \mathcal{I}_b(su).$$

Lemma 2.7. *$c_{b,\infty}$ can be obtained by some positive and radially symmetric function $\bar{u} \in \mathcal{N}_{b,\infty}$, that corresponds to the ground state solution of $(\mathcal{P}_\infty)$ (see [26]). Furthermore, if f is odd, then for every $0 < \delta < \sqrt{V_\infty}$, there is $C = C(\delta) > 0$ that satisfies*

$$0 < \bar{u}(x) \leq C e^{-\frac{\delta}{\alpha}|x|}, \quad \text{for any } x \in \mathbb{R}^3, \tag{2.4}$$

where $\alpha = \left(a + b \int_{\mathbb{R}^3} |\nabla \bar{u}|^2 \mathrm{d}x\right)^{\frac{1}{2}}$.

Proof. The existence of the ground state solutions of $(\mathcal{P}_\infty)$ was proved in [26, Proposition 2.4]. For the properties of solutions of $(\mathcal{P}_\infty)$, see [27, Proposition 1.1]. □

3. A global compactness result

Now, we will give a more general global compactness lemma, which is very useful.

Lemma 3.1. *Let $\{u_n\} \subset X$ be a sequence such that*

$$\mathcal{I}_b(u_n) \to c \ \text{ and } \ \mathcal{I}_b'(u_n) \to 0, \quad \text{as } n \to \infty.$$

Then, there exist $u_0 \in H^1(\mathbb{R}^3)$ and $A \in \mathbb{R}^+$, such that $\mathcal{J}_A'(u_0) = 0$, where

$$\mathcal{J}_A(u) = \frac{a + bA}{2} \int_{\mathbb{R}^3} |\nabla u|^2 \mathrm{d}x + \frac{1}{2} \int_{\mathbb{R}^3} V(x) u^2 \mathrm{d}x - \int_{\mathbb{R}^3} F(u) \mathrm{d}x,$$

and either

(i) $u_n \to u_0$ in X, or

(ii) there exists a number $k \in \mathbb{N}^+$, k sequences of points $\{y_n^j\} \subset \mathbb{R}^3$ with $|y_n^j| \to +\infty$, $1 \leq j \leq k$, and k functions $\left\{u^1, u^2, \ldots, u^k\right\} \subset H^1(\mathbb{R}^3)$, which are nontrivial weak solutions to

$$-(a + bA)\Delta u + V_\infty u = f(u) \tag{3.1}$$

and

$$c+\frac{bA^2}{4}=\mathcal{J}_A(u_0)+\sum_{j=1}^{k}\mathcal{J}_A^{\infty}(u^j), \tag{3.2}$$

$$\left\|u_n-u_0-\sum_{j=1}^{k}u^j(\cdot\,-y_n^j)\right\|_X\to 0,$$

$$A=|\nabla u_0|_2^2+\sum_{j=1}^{k}|\nabla u^j|_2^2, \tag{3.3}$$

where

$$\mathcal{J}_A^{\infty}(u)=\frac{a+bA}{2}\int_{\mathbb{R}^3}|\nabla u|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}V_\infty u^2\mathrm{d}x-\int_{\mathbb{R}^3}F(u)\mathrm{d}x.$$

Proof. In view of Remark 2.4, we know $\{u_n\}$ is bounded in X. Then, there are $u_0\in X$ and $A\in\mathbb{R}^+$ that satisfy

$$u_n\rightharpoonup u_0\ \ in\ X\ \ and\ \ \int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\to A. \tag{3.4}$$

Due to (3.4) and $\mathcal{I}_b'(u_n)\to 0$ as $n\to\infty$, we arrive at

$$\int_{\mathbb{R}^3}(a\nabla u_0\cdot\nabla\varphi+V(x)u_0\varphi)\mathrm{d}x+bA\int_{\mathbb{R}^3}\nabla u_0\cdot\nabla\varphi\mathrm{d}x-\int_{\mathbb{R}^3}f(u_0)\varphi\mathrm{d}x=0,\quad\forall\varphi\in X.$$

That is, $\mathcal{J}_A'(u_0)=0$. On the other hand, it is clear to see that

$$\begin{aligned}\mathcal{J}_A(u_n)=&\frac{a+bA}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}V(x)u_n^2\mathrm{d}x-\int_{\mathbb{R}^3}F(u_n)\mathrm{d}x\\=&\frac{a}{2}\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^3}V(x)u_n^2\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\right)^2\\&-\int_{\mathbb{R}^3}F(u_n)\mathrm{d}x+\frac{bA^2}{4}+o(1)\\=&\mathcal{I}_b(u_n)+\frac{bA^2}{4}+o(1).\end{aligned} \tag{3.5}$$

Besides, for all $\varphi\in C_0^\infty(\mathbb{R}^3)$, one has

$$\begin{aligned}\langle\mathcal{J}_A'(u_n),\varphi\rangle=&(a+bA)\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla\varphi\mathrm{d}x+\int_{\mathbb{R}^3}V(x)u_n\varphi\mathrm{d}x-\int_{\mathbb{R}^3}f(u_n)\varphi\mathrm{d}x\\=&\int_{\mathbb{R}^3}(a\nabla u_n\cdot\nabla\varphi+V(x)u_n\varphi)\mathrm{d}x+b\int_{\mathbb{R}^3}|\nabla u_n|^2\mathrm{d}x\int_{\mathbb{R}^3}\nabla u_n\cdot\nabla\varphi\mathrm{d}x\\&-\int_{\mathbb{R}^3}f(u_n)\varphi\mathrm{d}x+o(1)\\=&\langle\mathcal{I}_b'(u_n),\varphi\rangle+o(1).\end{aligned} \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\mathcal{J}_A(u_n)\to c+\frac{bA^2}{4}\ \ \text{and}\ \ \mathcal{J}_A'(u_n)\to 0,\quad\text{as } n\to\infty.$$

Next, we will demonstrate in three steps and provide detailed proof process.

Step 1. Letting $u_n^1 := u_n - u_0$, we can obtain

(a_1) $\mathcal{J}_A^\infty(u_n^1) \to c + \frac{bA^2}{4} - \mathcal{J}_A(u_0)$,

(b_1) $\langle(\mathcal{J}_A^\infty)'(u_n^1), u_n^1\rangle = \langle\mathcal{J}_A'(u_n), u_n\rangle - \langle\mathcal{J}_A'(u_0), u_0\rangle + o(1) = o(1)$.

To prove (a_1), we can use the weak convergence of $\{u_n\}$ and [28, Lemma 3] to conclude that

$$|\nabla u_n^1|_2^2 = |\nabla u_n|_2^2 - |\nabla u_0|_2^2 + o(1), \tag{3.7}$$

$$|u_n^1|_2^2 = |u_n|_2^2 - |u_0|_2^2 + o(1), \tag{3.8}$$

$$\int_{\mathbb{R}^3} F(u_n^1)\mathrm{d}x = \int_{\mathbb{R}^3} F(u_n)\mathrm{d}x - \int_{\mathbb{R}^3} F(u_0)\mathrm{d}x + o(1), \tag{3.9}$$

$$\int_{\mathbb{R}^3} f(u_n^1)u_n^1\mathrm{d}x = \int_{\mathbb{R}^3} f(u_n)u_n\mathrm{d}x - \int_{\mathbb{R}^3} f(u_0)u_0\mathrm{d}x + o(1). \tag{3.10}$$

Moreover, by virtue of (V_1), we know $\forall\varepsilon > 0$, $\exists R > 0$ such that $|V_\infty - V(x)| < \varepsilon$ on $\mathbb{R}^3 \setminus B_R(0)$. Hence, it holds that

$$\begin{aligned}\left|\int_{\mathbb{R}^3}(V_\infty - V)(u_n^2 - u_0^2)\mathrm{d}x\right| &\le \int_{\mathbb{B}_R(0)} |V_\infty - V||u_n^2 - u_0^2|\mathrm{d}x + \varepsilon\int_{\mathbb{R}^3\setminus B_{B_R(0)}} |u_n^2 - u_0^2|\mathrm{d}x \\ &\le C_1|u_n - u_0|_{L^2(B_{B_R(0)})} + C_2\varepsilon = C_2\varepsilon + o(1).\end{aligned}$$

From the arbitrariness of ε, it can be concluded that

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}(V_\infty - V)(u_n^2 - u_0^2)\mathrm{d}x = 0. \tag{3.11}$$

Using (3.7)–(3.9) and (3.11), we can show that

$$\begin{aligned}\mathcal{J}_A^\infty(u_n^1) - \mathcal{J}_A(u_n) + \mathcal{J}_A(u_0) =&\frac{a + bA}{2}(|\nabla u_n^1|_2^2 - |\nabla u_n|_2^2 + |\nabla u_0|_2^2) \\ &+ \frac{1}{2}\int_{\mathbb{R}^3} V(x)(u_0^2 - u_n^2)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty(u_n^1)^2\mathrm{d}x \\ &+ \int_{\mathbb{R}^3}(F(u_n) - F(u_0) - F(u_n^1))\mathrm{d}x \\ =&o(1).\end{aligned}$$

That is, (a_1) is correct. As for (b_1), by using a similar argument as before and (3.10), it is sufficient to get (b_1). We omit the details here. Furthermore, by (a_1), we can obtain that $\mathcal{J}_A^\infty(u_n^1) \ge 0$.

Step 2. Define

$$\delta^{(1)} := \limsup_{n\to+\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)} |u_n^1|^2\mathrm{d}x.$$

Case 1 (*Vanishing*): $\delta^{(1)} = 0$. That is, as $n \to \infty$,

$$\sup_{y\in\mathbb{R}^3}\int_{B_1(y)} |u_n^1|^2\mathrm{d}x \to 0.$$

From the P.L. Lions lemma in [9], one has $u_n^1 \to 0$ in $L^t(\mathbb{R}^3)$ for any $t \in (2,6)$. Thus, we deduce that $\lim\limits_{n\to+\infty} \int_{\mathbb{R}^3} f(u_n^1)u_n^1 \mathrm{d}x = 0$ by (1.6). Besides, it is easy to get $\int_{\mathbb{R}^3}(V(x) - V_\infty)(u_n^1)^2 \mathrm{d}x = o(1)$. Hence, by (b_1), we can conclude $\|u_n^1\|_X \to 0$ as $n \to \infty$.

Case 2 (*Non-vanishing*): $\delta^{(1)} > 0$. Assume that there exists $\{y_n^1\} \subset \mathbb{R}^3$ such that

$$\int_{B_1(y_n^1)} |u_n^1|^2 \mathrm{d}x > \frac{\delta^{(1)}}{2} > 0.$$

We now define a new sequence $w_n^1 := u_n^1(\cdot + y_n^1)$. It is easy to get that $\{w_n^1\}$ is bounded in X. Moreover, we suppose that $w_n^1 \rightharpoonup u^1$ in X. Since

$$\int_{B_1(0)} |w_n^1|^2 \mathrm{d}x = \int_{B_1(y_n^1)} |u_n^1|^2 \mathrm{d}x > \frac{\delta^{(1)}}{2} > 0,$$

it follows from the Sobolev embedding that $u^1 \neq 0$. Moreover, $u_n^1 \rightharpoonup 0$ in X implies that $\{y_n^1\}$ is unbounded. That is, $|y_n^1| \to \infty$ as $n \to \infty$. Furthermore, we can show $(\mathcal{J}_A^\infty)'(u^1) = 0$.

Step 3. Setting $u_n^2 := u_n^1 - u^1(\cdot - y_n^1)$, we can check that

(a_2) $\mathcal{J}_A^\infty(u_n^2) \to c + \frac{bA^2}{4} - \mathcal{J}_A(u_0) - \mathcal{J}_A^\infty(u^1)$,

(b_2) $\langle (\mathcal{J}_A^\infty)'(u_n^2), u_n^2 \rangle = \langle \mathcal{J}_A'(u_n), u_n \rangle - \langle \mathcal{J}_A'(u_0), u_0 \rangle - \langle (\mathcal{J}_A^\infty)'(u^1), u^1 \rangle + o(1) = o(1)$.

Similar to Step 2, define

$$\delta^{(2)} := \limsup_{n\to+\infty} \sup_{y\in\mathbb{R}^3} \int_{B_1(y)} |u_n^2|^2 \mathrm{d}x.$$

If $\delta^{(2)} = 0$, we have $\|u_n^2\|_X \to 0$ as $n \to \infty$. That is, $\|u_n - u_0 - u^1(\cdot - y_n^1)\|_X \to 0$ as $n \to \infty$. By (3.7) and (a_2), we have $A = |\nabla u_0|_2^2 + |\nabla u^1|_2^2$ and $c + \frac{bA^2}{4} = \mathcal{J}_A(u_0) + \mathcal{J}_A^\infty(u^1)$. Furthermore, we know that $\mathcal{J}_A^\infty(u_n^2) = o(1)$. If $\delta^{(2)} > 0$, as the arguments as above, we know that there exists $\{y_n^2\} \subset \mathbb{R}^3$ unbounded, and a sequence $w_n^2 := u_n^2(\cdot + y_n^2)$ that satisfies $w_n^2 \rightharpoonup u^2$ in X and $u^2 \neq 0$. Besides, we can obtain $(\mathcal{J}_A^\infty)'(u^2) = 0$. Iterating the above process, we can show that

$$u_n^j = u_n^{j-1} - u^{j-1}(\cdot - y_n^{j-1})$$

with $|y_n^j| \to \infty$ and

$$w_n^{j-1} = u_n^{j-1}(\cdot + y_n^{j-1}) \rightharpoonup u^{j-1} \text{ in } X,$$

where u^j is the nontrivial weak solution of Equation (3.1). Moreover, we can conclude that

$$\mathcal{J}_A^\infty(u_n^j) = c + \frac{bA^2}{4} - \mathcal{J}_A(u_0) - \sum_{i=1}^{j-1} \mathcal{J}_A^\infty(u^i) + o(1).$$

Noticing that u^i is the nontrivial weak solution of Equation (3.1), in view of (1.7), we can obtain $\mathcal{J}_A^\infty(u^i) > 0$. Besides, similar to the above discussion, we know that when $\delta^{(j)} = 0$, $\mathcal{J}_A^\infty(u_n^j) = o(1)$. Hence, there is some finite constant $k \in \mathbb{N}$. Moreover, the above process will stop after k iterations. Namely, the proof is completed. □

Corollary 3.2. *The functional $\mathcal{I}_b$ satisfies $(PS)_c$ condition for $c \in (0, c_{b,\infty})$.*

Proof. Let $\{u_n\} \subset X$ be a $(PS)_c$ sequence for $c \in (0, c_{b,\infty})$. Then,

$$\mathcal{I}_b(u_n) \to c \in (0, c_{b,\infty}) \text{ and } \mathcal{I}_b'(u_n) \to 0, \quad \text{as } n \to \infty.$$

We only need to prove that $\{u_n\}$ has a convergent subsequence in X next. From Remak 2.4, we first conclude that $\{u_n\}$ is bounded in X. Hence, there is a subsequence of $\{u_n\}$, still denoted as $\{u_n\}$. Besides, there is also $u_0 \in H^1(\mathbb{R}^3)$ that satisfies $u_n \rightharpoonup u_0$ in X. If it is strongly convergent, then the proof is completed. If $u_n \nrightarrow u_0$ in X, from Lemma 3.1, there exist $A = |\nabla u_0|_2^2 + \sum_{j=1}^{k} |\nabla u^j|_2^2$, $k \in \mathbb{N}$ and $\{y_n^j\} \subset \mathbb{R}^3$ with $|y_n^j| \to +\infty$ for $j = 1, 2, \ldots, k$ and $u^j \in H^1(\mathbb{R}^3)$ such that

$$\mathcal{J}_A'(u_0) = 0, \quad \|u_n - u_0 - \sum_{j=1}^{k} u^j(\,\cdot\, - y_n^j)\|_X \to 0, \quad c + \frac{bA^2}{4} = \mathcal{J}_A(u_0) + \sum_{j=1}^{k} \mathcal{J}_A^\infty(u^j),$$

where u^j are nontrivial critical points of $\mathcal{J}_A^\infty$ for $j = 1, 2, \ldots, k$.

We first give the following two claims.

Claim 1: If $u_0 \neq 0$ and there exists $t_0 > 0$ such that $t_0 u_0 \in \mathcal{N}_b$, then we claim that $t_0 \leq 1$.

Since $t_0 u_0 \in \mathcal{N}_b$ and $\mathcal{J}_A'(u_0) = 0$, we can obtain that

$$\|t_0 u_0\|_X^2 + b\left(\int_{\mathbb{R}^3} |\nabla(t_0 u_0)|^2 \mathrm{d}x\right)^2 = \int_{\mathbb{R}^3} f(t_0 u_0)(t_0 u_0) \mathrm{d}x \tag{3.12}$$

and

$$(a + bA) \int_{\mathbb{R}^3} |\nabla u_0|^2 \mathrm{d}x + \int_{\mathbb{R}^3} V(x) u_0^2 \mathrm{d}x = \int_{\mathbb{R}^3} f(u_0) u_0 \mathrm{d}x. \tag{3.13}$$

It follows from (3.3), (3.12), and (3.13) that

$$\left(1 - \frac{1}{t_0^2}\right) \|u_0\|_X^2 \leq \int_{\mathbb{R}^3} \left(\frac{f(u_0)}{(u_0)^3} - \frac{f(t_0 u_0)}{(t_0 u_0)^3}\right) |u_0|^4 \mathrm{d}x. \tag{3.14}$$

From (f_3) and (3.14), it is easy to obtain $t_0 \leq 1$. Therefore, the claim is proved.

Claim 2: If $u_0 \neq 0$, we claim that

$$\mathcal{J}_A(u_0) \geq c_{b,1} + \frac{bA}{4} \int_{\mathbb{R}^3} |\nabla u_0|^2 \mathrm{d}x. \tag{3.15}$$

Combining $t_0 \leq 1$ and (1.7), we can arrive at

$$\int_{\mathbb{R}^3} \left(\frac{1}{4} f(t_0 u_0)(t_0 u_0) - F(t_0 u_0)\right) \mathrm{d}x \leq \int_{\mathbb{R}^3} \left(\frac{1}{4} f(u_0) u_0 - F(u_0)\right) \mathrm{d}x.$$

Hence, we can obtain

$$\begin{aligned} \mathcal{J}_A(u_0) &= \mathcal{J}_A(u_0) - \frac{1}{4}\langle \mathcal{J}_A'(u_0), u_0 \rangle \\ &= \frac{a + bA}{4} \int_{\mathbb{R}^3} |\nabla u_0|^2 \mathrm{d}x + \frac{1}{4} \int_{\mathbb{R}^3} V(x) u_0^2 \mathrm{d}x + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(u_0) u_0 - F(u_0)\right) \mathrm{d}x \end{aligned}$$

$$\begin{aligned}
&\geq \frac{t_0^2}{4}\int_{\mathbb{R}^3}(a|\nabla u_0|^2+V(x)u_0^2)\mathrm{d}x+\frac{bA}{4}\int_{\mathbb{R}^3}|\nabla u_0|^2\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^3}\left(\frac{1}{4}f(t_0u_0)(t_0u_0)-F(t_0u_0)\right)\mathrm{d}x\\
&=\mathcal{I}_b(t_0u_0)-\frac{1}{4}\langle\mathcal{I}_b'(t_0u_0),t_0u_0\rangle+\frac{bA}{4}\int_{\mathbb{R}^3}|\nabla u_0|^2\mathrm{d}x\\
&=\mathcal{I}_b(t_0u_0)+\frac{bA}{4}\int_{\mathbb{R}^3}|\nabla u_0|^2\mathrm{d}x\\
&\geq c_{b,1}+\frac{bA}{4}\int_{\mathbb{R}^3}|\nabla u_0|^2\mathrm{d}x.
\end{aligned}$$

Therefore, the claim is proved. Similarly, since $u^j\neq 0$ and $(\mathcal{J}_A^\infty)'(u^j)=0$, it is easy to see that

$$\mathcal{J}_A^\infty(u^j)\geq c_{b,\infty}+\frac{bA}{4}\int_{\mathbb{R}^3}|\nabla u^j|^2\mathrm{d}x. \tag{3.16}$$

Now, we return to our proof. If $u_0\equiv 0, k\geq 1$, then $A=\sum_{j=1}^k|\nabla u^j|_2^2$ and $c+\frac{bA^2}{4}=\sum_{j=1}^k\mathcal{J}_A^\infty(u^j)\geq kc_{b,\infty}+\frac{bA^2}{4}$. Noticing that $c<c_{b,\infty}$, this is absurd. If $u_0\not\equiv 0, k\geq 1$, then from (3.2), (3.15), and (3.16), we can obtain $c+\frac{bA^2}{4}\geq c_{b,1}+kc_{b,\infty}+\frac{bA^2}{4}$. By using $(f_0)-(f_3)$ and similar discussions to those in [9, Theorem 4.2], one ascertains $c_{b,1}>0$. Combining $c<c_{b,\infty}$, we know this case cannot occur. Therefore, $k=0$ and the proof is completed. □

4. Positive ground state solution

Now, we prove Theorem 1.1. First, due to conditions $(f_0)-(f_2)$, one can easily deduce that $\mathcal{I}_b$ satisfies the mountain pass geometry. Then, by conditions $(f_0)-(f_3)$ and similar discussions to those in [9, Theorem 4.2], we have

$$c_{b,1}=\inf_{\gamma\in\Gamma}\max_{t\in[0,1]}\mathcal{I}_b(\gamma(t))=\inf_{u\in X\setminus\{0\}}\max_{t\geq 0}\mathcal{I}_b(tu)>0,$$

where $\Gamma:=\{\gamma\in C([0,1],X):\gamma(0)=0,\ \mathcal{I}_b(\gamma(1))<0\}$.

Then, we show the relationship between $c_{b,1}$ and $c_{b,\infty}$. Note that $\bar{u}$ is a positive ground state solution of $(\mathcal{P}_\infty)$, so combined with Lemma 2.6, one can ascertain that there is $t_{\bar{u}}>0$, which makes $t_{\bar{u}}\bar{u}\in\mathcal{N}_b$.

Lemma 4.1. *Assume that* $(V_0)-(V_3)$ *and* $(f_0)-(f_3)$ *hold. Then,*

$$0<c_{b,1}<c_{b,\infty}.$$

Proof. We first claim that $t_{\bar{u}}<1$.

Since $\bar{u}\in\mathcal{N}_{b,\infty}$, one has that

$$\int_{\mathbb{R}^3}(a|\nabla\bar{u}|^2+V_\infty\bar{u}^2)\mathrm{d}x+b\left(\int_{\mathbb{R}^3}|\nabla\bar{u}|^2\mathrm{d}x\right)^2=\int_{\mathbb{R}^3}f(\bar{u})\bar{u}\mathrm{d}x. \tag{4.1}$$

Due to $t_{\bar{u}}\bar{u}\in\mathcal{N}_b$, we ascertain that

$$t_{\bar{u}}^2\int_{\mathbb{R}^3}(a|\nabla\bar{u}|^2+V(x)\bar{u}^2)\mathrm{d}x+t_{\bar{u}}^4b\left(\int_{\mathbb{R}^3}|\nabla\bar{u}|^2\mathrm{d}x\right)^2=\int_{\mathbb{R}^3}f(t_{\bar{u}}\bar{u})(t_{\bar{u}}\bar{u})\mathrm{d}x. \tag{4.2}$$

Combining (4.1), (4.2), and (V_3), we deduce that

$$\left(\frac{1}{t_{\bar{u}}^2}-1\right)\int_{\mathbb{R}^3}(a|\nabla\bar{u}|^2+V_\infty\bar{u}^2)\mathrm{d}x>\int_{\mathbb{R}^3}\left(\frac{f(t_{\bar{u}}\bar{u})}{(t_{\bar{u}}\bar{u})^3}-\frac{f(\bar{u})}{\bar{u}^3}\right)|\bar{u}|^4\mathrm{d}x.$$

If $t_{\bar{u}}\geq 1$, the above equation does not hold by (f_3). Thus, we can obtain that $t_{\bar{u}}<1$.

It follows from (1.7), (V_3), Lemma 2.6, and the above claim that

$$\begin{aligned}
c_{b,1}&\leq\max_{t\geq 0}\mathcal{I}_b(t\bar{u})=\mathcal{I}_b(t_{\bar{u}}\bar{u})\\
&<\frac{t_{\bar{u}}^2}{4}\int_{\mathbb{R}^3}(a|\nabla\bar{u}|^2+V_\infty\bar{u}^2)\mathrm{d}x+\int_{\mathbb{R}^3}\left(\frac{1}{4}f(t_{\bar{u}}\bar{u})(t_{\bar{u}}\bar{u})-F(t_{\bar{u}}\bar{u})\right)\mathrm{d}x\\
&<\frac{1}{4}\int_{\mathbb{R}^3}(a|\nabla\bar{u}|^2+V_\infty\bar{u}^2)\mathrm{d}x+\int_{\mathbb{R}^3}\left(\frac{1}{4}f(\bar{u})\bar{u}-F(\bar{u})\right)\mathrm{d}x\\
&=\mathcal{I}_{b,\infty}(\bar{u})=c_{b,\infty}.
\end{aligned}$$

The proof is completed. □

Proof of Theorem 1.1. Through using [9, Theorem 1.15], one knows that there exists a sequence $\{u_n\}\subset X$ such that

$$\mathcal{I}_b(u_n)\to c_{b,1}\quad and\quad \mathcal{I}_b'(u_n)\to 0,\quad as\ n\to\infty.$$

In view of Corollary 3.2 and Lemma 4.1, the sequence $\{u_n\}$ has a subsequence which strongly converges to $u\in X$. Besides, the function u satisfies $\mathcal{I}_b(u)=c_{b,1}>0$ and $\mathcal{I}_b'(u)=0$. We can easily get that $u\neq 0$. This indicates that u is a ground solution of Equation (1.5). Next, we prove that Equation (1.5) has a positive ground solution. First, based on f being an odd function, we can see $\mathcal{I}_b(|u|)=\mathcal{I}_b(u)=c_{b,1}$ and $|u|\in\mathcal{N}_b$. We can claim that $\mathcal{I}_b'(|u|)=0$ by using the deformation lemma, where f does not require differentiability. For convenience, let us note $w=|u|$. Then, we only need to prove $\mathcal{I}_b'(w)=0$.

By contradiction, we assume $\mathcal{I}_b'(w)\neq 0$. Then, there are $\varrho>0$ and $\delta>0$ that satisfy

$$\|\mathcal{I}_b'(u)\|_X\geq\varrho,\quad\forall u\in X\ \text{ with }\ \|u-w\|_X\leq 3\delta.$$

Let $D:=(1-\sigma,1+\sigma)$, where $\sigma\in(0,\min\{\frac{1}{2},\frac{\delta}{\sqrt{2}\|w\|_X}\})$. By using the fact that $w\in\mathcal{N}_b$ and the condition (f_3), we have

$$\langle\mathcal{I}_b'(tw),tw\rangle>0,\quad\text{if }\ t<1$$

and

$$\langle\mathcal{I}_b'(tw),tw\rangle<0,\quad\text{if }\ t>1.$$

Hence, we can take $t_1,t_2\in D\setminus\{1\}$, such that

$$\langle\mathcal{I}_b'(t_1w),t_1w\rangle>0,\quad\langle\mathcal{I}_b'(t_2w),t_2w\rangle<0.\tag{4.3}$$

It follows from Lemma 2.6 that

$$\bar{c}:=\max\{\mathcal{I}_b(t_1w),\mathcal{I}_b(t_2w)\}<\mathcal{I}_b(w)=c_{b,1}.\tag{4.4}$$

For $\varepsilon := \min\{(c_{b,1} - \bar{c})/3, \delta\varrho/8\}$ and $S_\delta := \{u \in X : \|u - w\|_X \leq \delta\}$, due to the deformation lemma in [9, Lemma 2.3], one deduces that there exists $\eta \in C([0,1] \times X, X)$ that satisfies

(a) $\eta(1,u) = u$ if $u \notin \mathcal{I}_b^{-1}([c_{b,1} - 2\varepsilon, c_{b,1} + 2\varepsilon]) \cap S_{2\delta}$;

(b) $\eta(1, \mathcal{I}_b^{c_{b,1}+\varepsilon} \cap S_\delta) \subset \mathcal{I}_b^{c_{b,1}-\varepsilon}$;

(c) $\mathcal{I}_b(\eta(1,u)) \leq \mathcal{I}_b(u)$, for any $u \in X$.

From this deformation, with the help of Lemma 2.6, we can claim that

$$\max_{t\in D} \mathcal{I}_b(\eta(1,tw)) < c_{b,1}. \tag{4.5}$$

Indeed, on the one hand, from Lemma 2.6, we know $\forall t \in D$,

$$\mathcal{I}_b(tw) \leq \mathcal{I}_b(w) = c_{b,1} \leq c_{b,1} + \varepsilon.$$

In addition, based on the definition of σ, it can be concluded that

$$\|tw - w\|_X^2 \leq \sigma^2 \|w\|_X^2 \leq \frac{1}{2}\delta^2 < \delta^2,$$

which means $tw \in S_\delta$. Hence, $tw \in \mathcal{I}_b^{c_{b,1}+\varepsilon} \cap S_\delta$. According to ($b$), it is easy to obtain (4.5).

In what follows, we can first claim that $\eta(1,tw) \cap \mathcal{N}_b \neq \emptyset$ for some $t \in D$. We define

$$\Phi(t) := \langle \mathcal{I}_b'(\eta(1,tw)), \eta(1,tw)\rangle, \quad \text{for } t > 0.$$

From (4.4), the definition of ε and (a), we have $\eta(1,t_1w) = t_1w$ and $\eta(1,t_2w) = t_2w$. From (4.3), one has

$$\Phi(t_1) = \langle \mathcal{I}_b'(t_1w), t_1w\rangle > 0, \quad \Phi(t_2) = \langle \mathcal{I}_b'(t_2w), t_2w\rangle < 0. \tag{4.6}$$

In view of (4.6) and the continuity of Φ, there exists $t_0 \in [t_1,t_2] \subset D$ such that $\Phi(t_0) = 0$. From the definition of Φ, one has $\eta(1,t_0w) \in \mathcal{N}_b$. Namely, $\eta(1,tw) \cap \mathcal{N}_b \neq \emptyset$ for some $t \in [t_1,t_2]$. Then, one gets $c_{b,1} \leq \mathcal{I}_b(\eta(1,t_0w))$, which is clearly contradictory to (4.5). Hence, the assumption is not valid and we ascertain that $\mathcal{I}_b'(w) = 0$. Therefore, w is a non-negative solution of Equation (1.5). Finally, according to the maximum principle [29, Theorem 3.5], one can obtain $w > 0$ in $\mathbb{R}^3$. Then, we say Equation (1.5) has a positive ground solution w. So far, the proof is completed. □

Remark 4.2. *Let $u_1 \in X$ be the solution obtained from Theorem 1.1. From the theory of classical Schrödinger equations (see [30, Theorem 3.1]), it is easy to conclude that u_1 decays exponentially as $|x| \to \infty$. Namely, for every $\delta > 0$, there exist $C = C(\delta) > 0$ and $R > 0$ such that*

$$u_1(x) \leq Ce^{-\delta|x|}, \quad \text{for any } |x| \geq R. \tag{4.7}$$

5. Existence and asymptotic behavior of least energy sign-changing solutions

In this section, we will prove the existence and asymptotic behavior of least-energy sign-changing solutions of Eq. (1.5). We first give the relationship among $c_{b,1}$, $c_{b,\infty}$ and $c_{b,2}$. Then, inspired by [24], we can construct a sign-changing $(PS)_{c_{b,2}}$ sequence for $\mathcal{I}_b$. After that, we can use Lemma 3.1 to prove Theorem 1.2. Finally, we prove Theorem 1.6.

Lemma 5.1. *Assume that* $(V_0)-(V_2)$, (V_4'), *and* $(f_0)-(f_3)$ *hold. Then, there exists* $b^*>0$ *small enough such that for* $0<b<b^*$ *we have*

$$0<c_{b,2}<c_{b,1}+c_{b,\infty}<2c_{b,\infty}.$$

Proof. Define $\bar{u}_n(x):=\bar{u}(x-ne_1)$ and $e_1:=(1,0,0)$, where $\bar{u}$ is given in Lemma 2.7. In what follows, u_1 represents the positive ground state solution of Equation (1.5), which is obtained from Theorem 1.1.

Claim 1: There exist $s_0,t_0>0$ such that $s_0u_1-t_0\bar{u}_{n_0}\in\mathcal{M}_b$ for some $n_0\in\mathbb{N}$ large enough.

In fact, denote $\chi(a)=\frac{1}{a}u_1-\bar{u}_n$ with $a>0$, and define a_1,a_2 by

$$a_1=\sup\{a\in\mathbb{R}^+:\chi^+(a)\neq 0\}\quad\text{and}\quad a_2=\inf\{a\in\mathbb{R}^+:\chi^-(a)\neq 0\}.$$

By Lemma 2.5, there exists $(s(\chi(a)),t(\chi(a)))$ such that $s(\chi(a))\chi^+(a)+t(\chi(a))\chi^-(a)\in\mathcal{M}_b$. Because u_1 is positive and $\bar{u}$ is radial, we can prove that $a_1=+\infty$. Indeed, we first notice that $\bar{u}$ is radially symmetric, as the same arguments presented in [31, Lemma 3.1.2] (see also [32, Radial Lemma 1]), one has that there exists a constant $C>0$ such that

$$|\bar{u}(x)|\le C\frac{\|\bar{u}\|_X}{|x|},\quad \text{for every } |x|\ge 1. \tag{5.1}$$

We can obtain that for every $x\in B_R(0)$, there is $|x-ne_1|\ge n-|x|\ge n-R$. By using (5.1), one can ascertain

$$\bar{u}_n(x)=\bar{u}(x-ne_1)\le C_1\frac{\|\bar{u}\|_X}{n-R}\le\frac{C_2}{n-R}.$$

Then, for fixed $x\in B_R(0)$, we will have

$$\chi(a)=\frac{1}{a}u_1-\bar{u}_n\ge\frac{1}{a}u_1-C_2\frac{1}{n-R}. \tag{5.2}$$

Therefore, we can take $\varepsilon=\frac{u_1}{2aC_2}>0$ and $n_0\in\mathbb{N}$, such that $\frac{1}{n-R}<\varepsilon$ for every $n\ge n_0$. Combining with (5.2), we know that for all $n\ge n_0$ and $a\in(0,+\infty)$, there is

$$\chi(a)=\frac{1}{a}u_1-\bar{u}_n>\frac{1}{2a}u_1>0.$$

Finally, we obtain that $a_1=+\infty$ by the definition of a_1.

If $a\to a_1=+\infty$, then $\frac{1}{a}u_1\to 0$ in X and $\chi^+(a)\to 0$. Similar to [33, Lemma 2.2 (*ii*)], one concludes that $s(\chi(a))\to+\infty$ and $\{t(\chi(a))\}$ is bounded in $\mathbb{R}^+$. Hence, as $a\to a_1$, one has

$$s(\chi(a))-t(\chi(a))\to+\infty. \tag{5.3}$$

Similarly, if $a\to a_2^+$, $\chi^-(a)\to 0$, one has that $t(\chi(a))\to+\infty$ and $\{s(\chi(a))\}$ is bounded in $\mathbb{R}^+$. So, as $a\to a_2^+$, we have

$$s(\chi(a))-t(\chi(a))\to-\infty. \tag{5.4}$$

From [33, Lemma 2.2 (i)], we can ascertain the continuity of s and t. Combining (5.3) and (5.4), we obtain that there exists $a_0 \in (a_2, a_1)$ such that $s(\chi(a_0)) = t(\chi(a_0))$ for n_0. Hence, let $s_0 = \frac{1}{a_0} s(\chi(a_0))$ and $t_0 = t(\chi(a_0))$, it is easy to show that

$$s(\chi(a_0))\chi(a_0) = s_0 u_1 - t_0 \bar{u}_{n_0} \in \mathcal{M}_b.$$

Claim 2: There exist $b^* > 0$ small enough and $n \in \mathbb{N}^+$ large enough such that for $0 < b < b^*$ we have

$$\max_{s,t>0} \mathcal{I}_b(s_0 u_1 - t_0 \bar{u}_n) < c_{b,1} + c_{b,\infty}. \tag{5.5}$$

Obviously, $\mathcal{I}_b(su_1 - t\bar{u}_n) < 0$ for s or t large enough. Next, we only need to consider this problem in a bounded interval. That is, we consider the case that $s, t \in (0, C)$, where $C > 0$. Moreover, one can check that there is $t_n > 0$ that satisfies $t_n \to 1$ as $n \to \infty$ and $t_n \bar{u}_n \in \mathcal{N}_b$. Then, by a direct calculation, we can obtain

$$\begin{aligned}
\mathcal{I}_b(su_1 - t\bar{u}_n) =& \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla(su_1 - t\bar{u}_n)|^2 + V(x)(su_1 - t\bar{u}_n)^2)\mathrm{d}x \\
&+ \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla(su_1 - t\bar{u}_n)|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} F(su_1 - t\bar{u}_n)\mathrm{d}x \\
=& \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla(su_1)|^2 + V(x)(su_1)^2)\mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla(su_1)|^2 \mathrm{d}x\right)^2 \\
&- \int_{\mathbb{R}^3} F(su_1)\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^3} (a|\nabla(t\bar{u}_n)|^2 + V(x)(t\bar{u}_n)^2)\mathrm{d}x \\
&+ \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla(t\bar{u}_n)|^2 \mathrm{d}x\right)^2 - \int_{\mathbb{R}^3} F(t\bar{u}_n)\mathrm{d}x - \int_{\mathbb{R}^3} a\nabla su_1 \cdot \nabla t\bar{u}_n \mathrm{d}x \\
&- \int_{\mathbb{R}^3} V(x)(su_1)(t\bar{u}_n)\mathrm{d}x \\
&+ \frac{b}{4}\Bigg\{4s^2t^2\left(\int_{\mathbb{R}^3} \nabla u_1 \cdot \nabla \bar{u}_n \mathrm{d}x\right)^2 - 4s^3 t\int_{\mathbb{R}^3} |\nabla u_1|^2 \mathrm{d}x \int_{\mathbb{R}^3} \nabla u_1 \cdot \nabla \bar{u}_n \mathrm{d}x \\
&+ 2s^2t^2 \int_{\mathbb{R}^3} |\nabla u_1|^2 \mathrm{d}x \int_{\mathbb{R}^3} |\nabla \bar{u}_n|^2 \mathrm{d}x - 4st^3 \int_{\mathbb{R}^3} |\nabla \bar{u}_n|^2 \mathrm{d}x \int_{\mathbb{R}^3} \nabla u_1 \cdot \nabla \bar{u}_n \mathrm{d}x\Bigg\} \\
&- \int_{\mathbb{R}^3} (F(su_1 - t\bar{u}_n) - F(su_1) - F(t\bar{u}_n))\mathrm{d}x \\
\le& \mathcal{I}_b(su_1) + \mathcal{I}_b(t_n\bar{u}_n) + B_n + C_n + D_n \\
=& \mathcal{I}_b(su_1) + \mathcal{I}_{b,\infty}(t_n\bar{u}_n) + A_n + B_n + C_n + D_n,
\end{aligned} \tag{5.6}$$

where

$$\begin{aligned}
A_n =& \frac{1}{2}\int_{\mathbb{R}^3} (V(x) - V_\infty)(t_n\bar{u}_n)^2 \mathrm{d}x, \\
B_n =& -st\int_{\mathbb{R}^3} (a\nabla u_1 \cdot \nabla \bar{u}_n + V(x)u_1\bar{u}_n)\,\mathrm{d}x,
\end{aligned}$$

$$C_n = \frac{b}{4}\Big\{4s^2t^2\left(\int_{\mathbb{R}^3}\nabla u_1\cdot\nabla\bar{u}_n\mathrm{d}x\right)^2 - 4s^3t\int_{\mathbb{R}^3}|\nabla u_1|^2\mathrm{d}x\int_{\mathbb{R}^3}\nabla u_1\cdot\nabla\bar{u}_n\mathrm{d}x$$
$$+ 2s^2t^2\int_{\mathbb{R}^3}|\nabla u_1|^2\mathrm{d}x\int_{\mathbb{R}^3}|\nabla\bar{u}_n|^2\mathrm{d}x - 4st^3\int_{\mathbb{R}^3}|\nabla\bar{u}_n|^2\mathrm{d}x\int_{\mathbb{R}^3}\nabla u_1\cdot\nabla\bar{u}_n\mathrm{d}x\Big\},$$
$$D_n = \int_{\mathbb{R}^3}(F(su_1) + F(t\bar{u}_n) - F(su_1 - t\bar{u}_n))\mathrm{d}x.$$

First, similar to the conclusion presented in [10], we can show that

$$\int_{\mathbb{R}^3} u_1|\bar{u}_n|^q\mathrm{d}x \le C_1e^{-\frac{\delta}{\alpha(q+1)}n} + C_2e^{-\frac{\delta}{q+1}n}, \tag{5.7}$$
$$\int_{\mathbb{R}^3} |u_1|^q\bar{u}_n\mathrm{d}x \le C_3e^{-\frac{\delta q}{\alpha(q+1)}n} + C_4e^{-\frac{\delta q}{q+1}n} \le C_5e^{-\frac{\delta}{\alpha(q+1)}n} + C_6e^{-\frac{\delta}{q+1}n}, \tag{5.8}$$

for any $q \in [1, 5]$.

For A_n, in view of (V_4'), we can arrive at

$$A_n \le -\frac{1}{4}\int_{B_1(0)}(V_\infty - V(x + ne_1))\bar{u}^2\mathrm{d}x \le -\frac{1}{4}C\int_{B_1(0)}\frac{\bar{u}^2}{1 + |x + ne_1|^\gamma}\mathrm{d}x \le -\frac{C_7}{n^\gamma}. \tag{5.9}$$

As for D_n, due to (1.6), one has

$$\begin{aligned} D_n &\le 2\int_{\mathbb{R}^3}(f(su_1)t\bar{u}_n + f(t\bar{u}_n)su_1)\,\mathrm{d}x \\ &\le 2\int_{\mathbb{R}^3}\Big(\varepsilon(|su_1|+|su_1|^5)t\bar{u}_n + C_\varepsilon|su_1|^{q-1}t\bar{u}_n + \varepsilon(|t\bar{u}_n|+|t\bar{u}_n|^5)su_1 + C_\varepsilon|t\bar{u}_n|^{q-1}su_1\Big)\,\mathrm{d}x \\ &\le C_8e^{-\frac{\delta}{6\alpha}n} + C_9e^{-\frac{\delta}{6}n}. \end{aligned} \tag{5.10}$$

In what follows, we estimate B_n. Since $\langle \mathcal{I}'_{b,\infty}(\bar{u}_n), u_1\rangle = 0$, we deduce that

$$\int_{\mathbb{R}^3}(a\nabla\bar{u}_n\cdot\nabla u_1 + V_\infty\bar{u}_nu_1)\,\mathrm{d}x + b\int_{\mathbb{R}^3}|\nabla\bar{u}_n|^2\mathrm{d}x\int_{\mathbb{R}^3}\nabla\bar{u}_n\cdot\nabla u_1\mathrm{d}x = \int_{\mathbb{R}^3}f(\bar{u}_n)u_1\mathrm{d}x. \tag{5.11}$$

From (1.6), (5.7), and (5.11), we derive that

$$\begin{aligned} \int_{\mathbb{R}^3}\nabla\bar{u}_n\cdot\nabla u_1\mathrm{d}x &= \frac{\int_{\mathbb{R}^3}f(\bar{u}_n)u_1\mathrm{d}x - \int_{\mathbb{R}^3}V_\infty\bar{u}_nu_1\mathrm{d}x}{a + b\int_{\mathbb{R}^3}|\nabla\bar{u}_n|^2\mathrm{d}x} \\ &\le C_{10}\left(\int_{\mathbb{R}^3}f(\bar{u}_n)u_1\mathrm{d}x + \int_{\mathbb{R}^3}V_\infty\bar{u}_nu_1\mathrm{d}x\right) \\ &\le C_{11}e^{-\frac{\delta}{6\alpha}n} + C_{12}e^{-\frac{\delta}{6}n}. \end{aligned} \tag{5.12}$$

Since $\langle \mathcal{I}'_b(u_1), \bar{u}_n\rangle = 0$, we can get that

$$\int_{\mathbb{R}^3}(a\nabla u_1\cdot\nabla\bar{u}_n + V(x)u_1\bar{u}_n)\,\mathrm{d}x + b\int_{\mathbb{R}^3}|\nabla u_1|^2\mathrm{d}x\int_{\mathbb{R}^3}\nabla u_1\cdot\nabla\bar{u}_n\mathrm{d}x = \int_{\mathbb{R}^3}f(u_1)\bar{u}_n\mathrm{d}x.$$

Let us go back to the term B_n now. Combining (1.6), (5.8), (5.12), and the above equation, we can obtain that

$$\begin{aligned}B_n &= st\left(b\int_{\mathbb{R}^3}|\nabla u_1|^2\mathrm{d}x\int_{\mathbb{R}^3}\nabla u_1\cdot\nabla\bar{u}_n\mathrm{d}x-\int_{\mathbb{R}^3}f(u_1)\bar{u}_n\mathrm{d}x\right)\\ &\le st\left(b\int_{\mathbb{R}^3}|\nabla u_1|^2\mathrm{d}x\int_{\mathbb{R}^3}\nabla u_1\cdot\nabla\bar{u}_n\mathrm{d}x+\int_{\mathbb{R}^3}|f(u_1)|\bar{u}_n\mathrm{d}x\right)\\ &\le C_{13}e^{-\frac{\delta}{6\alpha}n}+C_{14}e^{-\frac{\delta}{6}n}.\end{aligned}\tag{5.13}$$

By (5.12), it is easy to conclude that

$$\begin{aligned}C_n &\le C_{15}e^{-\frac{\delta}{6\alpha}n}+C_{16}e^{-\frac{\delta}{6}n}+\frac{C^4}{2}b\int_{\mathbb{R}^3}|\nabla u_1|^2\mathrm{d}x\int_{\mathbb{R}^3}|\nabla\bar{u}_n|^2\mathrm{d}x\\ &\le C_{15}e^{-\frac{\delta}{6\alpha}n}+C_{16}e^{-\frac{\delta}{6}n}+C_{17}b.\end{aligned}\tag{5.14}$$

Then, due to (5.9), (5.10), (5.13), and (5.14), we can ascertain that

$$A_n+B_n+C_n+D_n\le-\frac{C_7}{n^\gamma}+C_{18}e^{-\frac{\delta}{6\alpha}n}+C_{19}e^{-\frac{\delta}{6}n}+C_{17}b.\tag{5.15}$$

Therefore, choosing $n_0\in\mathbb{N}^+$ large enough, we can obtain $-\frac{C_7}{n_0^\gamma}+C_{18}e^{-\frac{\delta}{6\alpha}n_0}+C_{19}e^{-\frac{\delta}{6}n_0}<0$. Hence, we can take $b^*=\frac{C_7\frac{1}{n_0^\gamma}-C_{18}e^{-\frac{\delta}{6\alpha}n_0}-C_{19}e^{-\frac{\delta}{6}n_0}}{2C_{17}}>0$, and then, $\forall b\in(0,b^*)$, we have

$$A_n+B_n+C_n+D_n<0,$$

for all $n\ge n_0$. Noticing that (5.6), we deduce that

$$\begin{aligned}\max_{s,t\in(0,C)}\mathcal{I}_b(su_1-t\bar{u}_n) &< \max_{s\in(0,C)}\mathcal{I}_b(su_1)+\mathcal{I}_{b,\infty}(\bar{u}_n)\\ &\le \mathcal{I}_b(u_1)+\mathcal{I}_{b,\infty}(\bar{u}_n)\\ &= c_{b,1}+c_{b,\infty},\end{aligned}$$

for all $n\ge n_0$. Therefore, the claim is proved. Finally, combining Claim 1 and Claim 2, for any $b\in(0,b^*)$, we can show that

$$c_{b,2}\le\mathcal{I}_b(s_0u_1-t_0\bar{u}_{n_0})\le\max_{s,t>0}\mathcal{I}_b(su_1-t\bar{u}_{n_0})<c_{b,1}+c_{b,\infty}.$$

The proof is completed. □

In order to construct a sign-changing $(PS)_{c_{b,2}}$ sequence of the functional $\mathcal{I}_b$, we follow the method in [24]. Define

$$g(u,v):=\begin{cases}\frac{\int_{\mathbb{R}^3}f(u)u\,dx}{\|u\|_X^2+b(\int_{\mathbb{R}^3}|\nabla u|^2\,dx)^2+b\int_{\mathbb{R}^3}|\nabla u|^2\,dx\int_{\mathbb{R}^3}|\nabla v|^2\,dx}, & \text{if } u\ne0;\\ 0, & \text{if } u=0.\end{cases}\tag{5.16}$$

First, since f being an odd function, we can obtain $g(u, v) > 0$ if $u \neq 0$. Besides, we can see that $u \in \mathcal{M}_b$ if and only if $g(u^+, u^-) = g(u^-, u^+) = 1$. Moreover, we can construct the following set U, which is larger than the set $\mathcal{M}_b$. Namely, we define

$$U := \left\{ u \in X : |g(u^+, u^-) - 1| < \frac{1}{2}, |g(u^-, u^+) - 1| < \frac{1}{2} \right\}.$$

We know $\mathcal{M}_b \subset U$. Furthermore, from Lemma 2.5, it can be concluded that $\mathcal{M}_b \neq \emptyset$. That is, $\mathcal{M}_b \subset U \neq \emptyset$. Now, we use P to represent the cone of the non-negative functions in X, $D := [0, 1]$, $E := D \times D$, and Σ to represent the set of continuous maps σ, so that for all $s, t \in D$,

(*i*) $\sigma \in C(E, X)$;

(*ii*) $\sigma(s, 0) = 0$, $\sigma(0, t) \in P$ and $\sigma(1, t) \in -P$;

(*iii*) $(\mathcal{I}_b \circ \sigma)(s, 1) \leq 0$ and $\frac{\int_{\mathbb{R}^3} f(\sigma(s,1))\sigma(s,1)\,dx}{\|\sigma(s,1)\|_X^2 + b(\int_{\mathbb{R}^3} |\nabla(\sigma(s,1))|^2\,dx)^2} \geq 2$.

We can claim that $\Sigma \neq \emptyset$. In fact, for every $u \in X$ and $u^\pm \neq 0$, we can set $\sigma(s, t) = \mu(1-s)tu^+ + \mu stu^-$, where $\mu > 0$ and $s, t \in D$. Then, through simple calculations, one can conclude $\sigma(s, t) \in \Sigma$ for some $\mu > 0$. Moreover, we can also obtain the following lemmas.

Lemma 5.2. *Assume that* $(V_0) - (V_2)$ *and* $(f_0) - (f_4)$ *hold. Then,*

$$\inf_{\sigma \in \Sigma} \sup_{u \in \sigma(E)} \mathcal{I}_b(u) = \inf_{u \in \mathcal{M}_b} \mathcal{I}_b(u) = c_{b,2}.$$

Proof. On one hand, for all $u \in \mathcal{M}_b$ and $s, t \in D$, there is $\sigma(s, t) = \mu(1 - s)tu^+ + \mu stu^-$ such that when $\mu > 0$ is sufficiently large, $\sigma(s, t) \in \Sigma$. In light of Lemma 2.5, one concludes that $\mathcal{I}_b(u) = \max_{s,t \geq 0} \mathcal{I}_b(su^+ + tu^-)$. Hence, one obtains that

$$\mathcal{I}_b(u) \geq \sup_{u \in \sigma(E)} \mathcal{I}_b(u) \geq \inf_{\sigma \in \Sigma} \sup_{u \in \sigma(E)} \mathcal{I}_b(u),$$

which implies that

$$\inf_{u \in \mathcal{M}_b} \mathcal{I}_b(u) \geq \inf_{\sigma \in \Sigma} \sup_{u \in \sigma(E)} \mathcal{I}_b(u). \tag{5.17}$$

On the other hand, we can see that for every $\sigma \in \Sigma$, there is $u_\sigma \in \sigma(E) \cap \mathcal{M}_b$. Therefore,

$$\sup_{u \in \sigma(E)} \mathcal{I}_b(u) \geq \mathcal{I}_b(u_\sigma) \geq \inf_{u \in \mathcal{M}_b} \mathcal{I}_b(u).$$

Then, one can arrive at

$$\inf_{\sigma \in \Sigma} \sup_{u \in \sigma(E)} \mathcal{I}_b(u) \geq \inf_{u \in \mathcal{M}_b} \mathcal{I}_b(u). \tag{5.18}$$

Hence, combining (5.17) and (5.18), we can conclude that

$$\inf_{\sigma \in \Sigma} \sup_{u \in \sigma(E)} \mathcal{I}_b(u) = \inf_{u \in \mathcal{M}_b} \mathcal{I}_b(u) = c_{b,2}.$$

Now it remains to prove the claim. Indeed, due to the definition of Σ, we can know $\sigma(0,t) \in P$ and $\sigma(1,t) \in -P$, which holds for every $\sigma \in \Sigma$ and $t \in D$. Moreover, by using conditions (f_4) and (1.7), it is easy to ascertain that for every $t \in \mathbb{R}$, there is $\frac{1}{4}f(t)t \geq F(t) \geq 0$. Next, for convenience, one can define $l^{\pm}(s,t) := g(\sigma^+(s,t), \sigma^-(s,t)) \pm g(\sigma^-(s,t), \sigma^+(s,t))$. Then, one can deduce

$$l^-(0,t) \geq 0 \tag{5.19}$$

and

$$l^-(1,t) \leq 0. \tag{5.20}$$

Then, we can use property (*iii*) and the inequality $\frac{b}{a} + \frac{d}{c} \geq \frac{b+d}{a+c}$, where $a,b,c,d > 0$, to obtain that

$$l^+(s,1) \geq 2.$$

Thus,

$$l^+(s,1) - 2 \geq 0. \tag{5.21}$$

In addition, we can verify $l^+(s,0) = 0$. Hence,

$$l^+(s,0) - 2 = -2 < 0. \tag{5.22}$$

Therefore, using Miranda's theorem [34] and (5.19)–(5.22), there is $(s_\sigma, t_\sigma) \in E$ that satisfies

$$l^-(s_\sigma, t_\sigma) = l^+(s_\sigma, t_\sigma) - 2 = 0.$$

As a result, one has

$$g(\sigma^+(s_\sigma, t_\sigma), \sigma^-(s_\sigma, t_\sigma)) = g(\sigma^-(s_\sigma, t_\sigma), \sigma^+(s_\sigma, t_\sigma)) = 1.$$

Namely, there exists $u_\sigma = \sigma(s_\sigma, t_\sigma) \in \sigma(E) \cap \mathcal{M}_b$ for any $\sigma \in \Sigma$. The proof is completed. □

Lemma 5.3. *Assume that* $(V_0) - (V_2)$ *and* $(f_0) - (f_4)$ *hold, then there exists a sequence* $\{u_n\} \subset U$ *satisfying* $\mathcal{I}_b(u_n) \to c_{b,2}$ *and* $\mathcal{I}_b'(u_n) \to 0$ *as* $n \to \infty$.

Proof. Let $\{w_n\} \subset \mathcal{M}_b$ be a minimizing sequence and $\sigma(s,t) = \mu(1-s)tw_n^+ + \mu stw_n^- \in \Sigma$, so

$$\lim_{n\to\infty} \max_{w\in\sigma_n(E)} \mathcal{I}_b(w) = \lim_{n\to\infty} \mathcal{I}_b(w_n) = c_{b,2}. \tag{5.23}$$

We can claim that there is a sequence $\{u_n\} \subset X$, which satisfies, as $n \to \infty$,

$$\mathcal{I}_b(u_n) \to c_{b,2}, \quad \mathcal{I}_b'(u_n) \to 0, \quad \mathrm{dist}(u_n, \sigma_n(E)) \to 0. \tag{5.24}$$

Suppose there is a contradiction, then, there exists $\delta > 0$ such that $\sigma_n(E) \cap V_\delta = \emptyset$ for n, which is sufficiently large, where

$$V_\delta = \left\{u \in X : \exists v \in X,\ s.t.\ \|v-u\|_X \leq \delta,\ \|\mathcal{I}_b'(v)\|_X \leq \delta,\ |\mathcal{I}_b(v) - c_{b,2}| \leq \delta\right\}.$$

Through using [35], due to Hofer [36], there is $\eta \in C(D \times X, X)$, which satisfies the following properties for some $\varepsilon \in (0, \frac{c_{b,2}}{2})$ and every $t \in D$.

(*i*) $\eta(0,u) = u$, $\eta(t,-u) = -\eta(t,u)$;

(*ii*) $\eta(t,u) = u$ for any $u \in \mathcal{I}_b^{c_{b,2}-\varepsilon} \cup (X \setminus \mathcal{I}_b^{c_{b,2}+\varepsilon})$, where $\mathcal{I}_b^c = \{u \in X : \mathcal{I}_b(u) \le c\}$;

(*iii*) $\eta(1, \mathcal{I}_b^{c_{b,2}+\frac{\varepsilon}{2}} \setminus V_\delta) \subset \mathcal{I}_b^{c_{b,2}-\frac{\varepsilon}{2}}$;

(*iv*) $\eta(1, (\mathcal{I}_b^{c_{b,2}+\frac{\varepsilon}{2}} \cap P) \setminus V_\delta) \subset \mathcal{I}_b^{c_{b,2}-\frac{\varepsilon}{2}} \cap P$.

By (5.23), select n to be sufficiently large so that

$$\sigma_n(E) \subset \mathcal{I}_b^{c_{b,2}+\frac{\varepsilon}{2}}, \quad \sigma_n(E) \cap V_\delta = \emptyset. \tag{5.25}$$

Define $\tilde{\sigma}_n(s,t) := \eta(1, \sigma_n(s,t))$ for all $(s,t) \in E$. Then, it is clear to see that $\tilde{\sigma}_n \in \Sigma$. From (5.25) and property (*iii*), it can be inferred that $\tilde{\sigma}_n(E) \subset \mathcal{I}_b^{c_{b,2}-\frac{\varepsilon}{2}}$. Therefore,

$$c_{b,2} = \inf_{\sigma \in \Sigma} \sup_{w \in \sigma(E)} \mathcal{I}_b(w) \le \max_{w \in \tilde{\sigma}_n(E)} \mathcal{I}_b(w) \le c_{b,2} - \frac{\varepsilon}{2},$$

which is absurd, so the claim is valid.

Now, we begin to prove $\{u_n\} \subseteq U$ when n is large enough. Indeed, in light of $\mathcal{I}_b'(u_n) \to 0$ as $n \to \infty$, one can ascertain $\langle \mathcal{I}_b'(u_n), u_n^\pm \rangle = o(1)$. After that, it is sufficient to prove $u_n^\pm \neq 0$, which means $g(u_n^+, u_n^-) \to 1$ and $g(u_n^-, u_n^+) \to 1$. Therefore, one gets $\{u_n\} \subset U$ for n large enough. Using (5.24), there is a sequence $\{v_n\}$ that satisfies

$$v_n = s_n w_n^+ + t_n w_n^- \in \sigma_n(E), \quad \|v_n - u_n\|_X \to 0. \tag{5.26}$$

To get $u_n^\pm \neq 0$, we only need to show that $s_n w_n^+ \neq 0$ and $t_n w_n^- \neq 0$ for n large enough. It derives from Lemma 2.3 and the fact of $\{w_n\} \subset \mathcal{M}_b$ that $C_1 \le \|w_n^\pm\| \le C_2$ for some $C_1, C_2 > 0$. Next, we only need to prove $s_n \nrightarrow 0$ and $t_n \nrightarrow 0$ as $n \to \infty$. By contradiction, if $s_n \to 0$ as $n \to \infty$, in light of (5.26) and $\mathcal{I}_b$ being continuous, one can get

$$0 < c_{b,2} = \lim_{n\to\infty} \mathcal{I}_b(v_n) = \lim_{n\to\infty} \mathcal{I}_b(s_n w_n^+ + t_n w_n^-) = \lim_{n\to\infty} \mathcal{I}_b(t_n w_n^-).$$

Choosing $0 < \varepsilon < \frac{1}{2S_2^2}$ such that

$$\begin{aligned}
c_{b,2} &= \lim_{n\to\infty} \mathcal{I}_b(w_n) = \lim_{n\to\infty} \max_{s,t>0} \mathcal{I}_b(s w_n^+ + t w_n^-) \ge \lim_{n\to\infty} \max_{s>0} \mathcal{I}_b(s w_n^+ + t_n w_n^-) \\
&= \lim_{n\to\infty} \max_{s\ge 0} \left\{ \frac{1}{2}\|s w_n^+ + t_n w_n^-\|_X^2 + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla(s w_n^+ + t_n w_n^-)|^2\, dx \right)^2 \right. \\
&\quad \left. - \int_{\mathbb{R}^3} F(s w_n^+ + t_n w_n^-)\, dx \right\} \\
&\ge \lim_{n\to\infty} \mathcal{I}_b(t_n w_n^-) + \lim_{n\to\infty} \max_{s\ge 0} \left\{ \frac{s^2}{2}\|w_n^+\|_X^2 - \int_{\mathbb{R}^3} F(s w_n^+)\, dx \right\} \\
&\ge c_{b,2} + \lim_{n\to\infty} \max_{s\ge 0} \left\{ \frac{s^2}{2}\|w_n^+\|_X^2 - \varepsilon s^2 \int_{\mathbb{R}^3} |w_n^+|^2\, dx - C_\varepsilon s^6 \int_{\mathbb{R}^3} |w_n^+|^6\, dx \right\}
\end{aligned}$$

$$\geq c_{b,2} + \lim_{n\to\infty} \max_{s\geq 0} \left\{ \frac{s^2}{2} \left(1 - 2S_2^2\varepsilon\right) \|w_n^+\|_X^2 - C_\varepsilon S_6^6 \|w_n^+\|_X^6 s^6 \right\} > c_{b,2},$$

leads to a contradiction. Then, the above assumption is not valid. Therefore, $\{u_n\} \subseteq U$ for n large enough. □

Based on the previous lemma, we will now focus on the proof of Theorem 1.2.

Proof of Theorem 1.2. First, there is a $(PS)_{c_{b,2}}$ sequence $\{u_n\} \subseteq U$ by Lemma 2.3 and Lemma 5.3, which is bounded. Then, we assume that there exists a subsequence, which satisfies $u_n \rightharpoonup u_0$ in X. We can claim $u_n \to u_0$ in X, thus we deduce that

$$\mathcal{I}_b(u_0) = c_{b,2} \quad \text{and} \quad \mathcal{I}_b'(u_0) = 0.$$

From Lemma 2.3, by $u_n \to u_0$ in X, we get $\|u_0^\pm\|_X \geq C_1 > 0$, namely $u_0 \in \mathcal{M}_b$. Hence, $u_0 \in \mathcal{M}_b$ is a least energy sign-changing solution of Equation (1.5).

It remains to verify the above claim. In fact, due to Lemma 3.1, if case (i) occurs, the proof is completed. If case (ii) occurs, since $c_{b,1} < c_{b,\infty}$, it follows from (3.2) that $k \leq 1$. Hence, $k = 0$ or $k = 1$.

If $u_0 \equiv 0$, in light of $c_{b,2} > 0$, one can deduce for $k = 1$ and $A = |\nabla u^1|_2^2$,

$$u_n \to u^1(\cdot - y_n^1). \tag{5.27}$$

Since $|y_n^1| \to +\infty$ and $\{u_n\} \subset U$, (5.27) and Lemma 2.3 imply that $(u^1)^\pm \neq 0$. Besides, due to $(\mathcal{J}_A^\infty)'(u^1) = 0$, we get $\langle (\mathcal{J}_A^\infty)'(u^1)^\pm, (u^1)^\pm \rangle = \langle (\mathcal{J}_A^\infty)'(u^1), (u^1)^\pm \rangle = 0$. Hence, we have $\mathcal{J}_A^\infty((u^1)^\pm) \geq c_{b,\infty} + \frac{bA}{4} \int_{\mathbb{R}^3} |\nabla (u^1)^\pm|^2 \mathrm{d}x$ by (3.16). Therefore,

$$\begin{aligned} c_{b,1} + c_{b,\infty} + \frac{bA^2}{4} > c_{b,2} + \frac{bA^2}{4} &= \mathcal{J}_A^\infty(u^1) = \mathcal{J}_A^\infty((u^1)^+) + \mathcal{J}_A^\infty((u^1)^-) \\ &\geq c_{b,\infty} + \frac{bA}{4} \int_{\mathbb{R}^3} |\nabla (u^1)^+|^2 \mathrm{d}x + c_{b,\infty} + \frac{bA}{4} \int_{\mathbb{R}^3} |\nabla (u^1)^-|^2 \mathrm{d}x \\ &\geq 2c_{b,\infty} + \frac{bA^2}{4}, \end{aligned}$$

which contradicts with $c_{b,1} < c_{b,\infty}$. Thus, $u_0 \neq 0$, combining $c_{b,2} < c_{b,1} + c_{b,\infty}$, (3.2), (3.15), and (3.16), we can get that

$$c_{b,1} + c_{b,\infty} + \frac{bA^2}{4} > c_{b,2} + \frac{bA^2}{4} = \mathcal{J}_A(u_0) + \sum_{j=1}^{k} \mathcal{J}_A^\infty(u^j) \geq c_{b,1} + kc_{b,\infty} + \frac{bA^2}{4}.$$

Therefore, we can show that $k = 0$ and $u_n \to u_0$ in X. The proof is completed. □

From now on, we will study the asymptotic behavior of the above sign-changing solutions with respect to b. In order to facilitate research, we set $c_{0,1} := \inf_{u\in\mathcal{N}_0} \mathcal{I}_0(u)$, $c_{0,\infty} := \inf_{u\in\mathcal{N}_{0,\infty}} \mathcal{I}_{0,\infty}(u)$ and $c_{0,2} := \inf_{u\in\mathcal{M}_0} \mathcal{I}_0(u)$ with $\mathcal{N}_0 := \{u \in X \setminus \{0\} : \langle \mathcal{I}_0'(u), u\rangle = 0\}$, $\mathcal{N}_{0,\infty} := \{u \in X \setminus \{0\} : \langle \mathcal{I}_{0,\infty}'(u), u\rangle = 0\}$, and $\mathcal{M}_0 := \{u \in X : u^\pm \neq 0, \langle \mathcal{I}_0'(u), u^+\rangle = \langle \mathcal{I}_0'(u), u^-\rangle = 0\}$, where $\mathcal{I}_0(u)$ and $\mathcal{I}_{0,\infty}(u)$ respectively represent $\mathcal{I}_b(u)$ and $\mathcal{I}_{b,\infty}(u)$ with $b = 0$. Besides, from Theorem 1.2, we have obtained that Equation (1.5) has a least energy sign-changing solution for all $b \in (0, b^*)$ under hypothesis $(V_0) - (V_2)$ and (V_4). Next, we denote it as u_b and consider the problem with $b \in (0, \min\{b^*, 1\})$.

Proof of Theorem 1.6. First of all, in virtue of Theorem 1.2 and Lemma 5.1, we immediately know that $c_{b,2}$ is achievable by some u_b for all $b \in (0, b^*)$ and satisfies $\mathcal{I}_b(u_b) = c_{b,2} < c_{b,1} + c_{b,\infty}$. Hence, there is a sequence $\{u_{b_n}\}$ that satisfies $\mathcal{I}_{b_n}(u_{b_n}) = c_{b_n,2} < c_{b_n,1} + c_{b_n,\infty}$ and $\mathcal{I}'_{b_n}(u_{b_n}) = 0$. Similar to the proof in [37], we can get that $\mathcal{I}_b(u_b) = c_{b,2} \le M_0$, where M_0 is a positive constant. Due to (f_3), we have that

$$M_0 + 1 \ge \mathcal{I}_{b_n}(u_{b_n}) - \frac{1}{4}\langle \mathcal{I}'_{b_n}(u_{b_n}), u_{b_n}\rangle \ge \frac{1}{4}\|u_{b_n}\|_X^2,$$

and we can easily infer that $\{u_{b_n}\}$ is bounded. Thus, there is a subsequence of $\{u_{b_n}\}$ and $u_0 \in X$ that satisfies $u_{b_n} \rightharpoonup u_0$ in X. Besides that, we can easily check that the sequence $\{\int_{\mathbb{R}^3} |\nabla u_{b_n}|^2 dx \int_{\mathbb{R}^3} \nabla u_{b_n} \cdot \nabla \varphi dx\}$ is bounded for all $\varphi \in C_0^\infty(\mathbb{R}^3)$. Therefore, combining $b_n \to 0$ as $n \to +\infty$, we can arrive at

$$\lim_{n\to\infty} b_n \int_{\mathbb{R}^3} |\nabla u_{b_n}|^2 dx \int_{\mathbb{R}^3} \nabla u_{b_n} \cdot \nabla \varphi dx = 0.$$

Through simple calculations, one has

$$\begin{aligned}
\langle \mathcal{I}'_0(u_0), \varphi\rangle &= \int_{\mathbb{R}^3} (a\nabla u_0 \cdot \nabla\varphi + V(x)u_0\varphi) dx - \int_{\mathbb{R}^3} f(u_0)\varphi dx \\
&= \lim_{n\to\infty} \left\{ \int_{\mathbb{R}^3} (a\nabla u_{b_n} \cdot \nabla\varphi + V(x)u_{b_n}\varphi) dx - \int_{\mathbb{R}^3} f(u_n)\varphi dx \right\} \\
&= \lim_{n\to\infty} \langle \mathcal{I}'_{b_n}(u_{b_n}), \varphi\rangle = 0
\end{aligned}$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3)$, which implies that $\mathcal{I}'_0(u_0) = 0$.

In the following, we deduce that $\mathcal{I}_0(u_0) \le c_{0,2}$. Because of (f_2), we know that there is $A_0 > 0$ large enough such that $\mathcal{I}_{b_n}(sv_0^+ + tv_0^-) < 0$ for all $s + t \ge A_0$. Due to Lemma 2.5, we get that there is $(s_{b_n}, t_{b_n}) \in (0, +\infty) \times (0, +\infty)$ that satisfies $s_{b_n}v_0^+ + t_{b_n}v_0^- \in \mathcal{M}_{b_n}$, where $v_0 \in \mathcal{M}_0$ satisfies $\mathcal{I}_0(v_0) = c_{0,2}$. Hence, we have $\mathcal{I}_{b_n}(s_{b_n}v_0^+ + t_{b_n}v_0^-) \ge 0$ by (f_3), and then $0 < s_{b_n}, t_{b_n} < A_0$. Therefore, we can conclude that

$$\begin{aligned}
c_{0,2} &= \mathcal{I}_0(v_0) = \mathcal{I}_{b_n}(v_0) - \frac{b_n}{4}\left(\int_{\mathbb{R}^3} |\nabla v_0|^2 dx\right)^2 \\
&\ge \mathcal{I}_{b_n}(s_{b_n}v_0^+ + t_{b_n}v_0^-) + \frac{1 - s_{b_n}^4}{4}\langle \mathcal{I}'_{b_n}(v_0), v_0^+\rangle + \frac{1 - t_{b_n}^4}{4}\langle \mathcal{I}'_{b_n}(v_0), v_0^-\rangle - \frac{b_n}{4}\left(\int_{\mathbb{R}^3} |\nabla v_0|^2 dx\right)^2 \\
&\ge c_{b_n,2} - \frac{1 + A_0^4}{4}\left|\langle \mathcal{I}'_{b_n}(v_0), v_0^+\rangle\right| - \frac{1 + A_0^4}{4}\left|\langle \mathcal{I}'_{b_n}(v_0), v_0^-\rangle\right| - \frac{b_n}{4}\left(\int_{\mathbb{R}^3} |\nabla v_0|^2 dx\right)^2 \\
&= c_{b_n,2} - \frac{1 + A_0^4}{4} b_n \int_{\mathbb{R}^3} |\nabla v_0|^2 dx \int_{\mathbb{R}^3} |\nabla v_0^+|^2 dx - \frac{1 + A_0^4}{4} b_n \int_{\mathbb{R}^3} |\nabla v_0|^2 dx \int_{\mathbb{R}^3} |\nabla v_0^-|^2 dx \\
&\quad - \frac{b_n}{4}\left(\int_{\mathbb{R}^3} |\nabla v_0|^2 dx\right)^2.
\end{aligned}$$

Hence,

$$\limsup_{n\to+\infty} c_{b_n,2} \le c_{0,2}. \tag{5.28}$$

In light of (f_3) and Fatou's Lemma, one gets

$$\begin{aligned}\mathcal{I}_0(u_0) =&\mathcal{I}_0(u_0) - \frac{1}{4}\langle \mathcal{I}_0'(u_0), u_0\rangle\\ =&\frac{1}{4}\|u_0\|_X^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(u_0)u_0 - F(u_0)\right)\mathrm{d}x\\ \leq&\liminf_{n\to+\infty}\left\{\frac{1}{4}\|u_{b_n}\|_X^2 + \int_{\mathbb{R}^3}\left(\frac{1}{4}f(u_{b_n})u_{b_n} - F(u_{b_n})\right)\mathrm{d}x\right\}\\ =&\liminf_{n\to+\infty}\left\{\mathcal{I}_{b_n}(u_{b_n}) - \frac{1}{4}\langle \mathcal{I}_{b_n}'(u_{b_n}), u_{b_n}\rangle\right\}\\ =&\liminf_{n\to+\infty}\mathcal{I}_{b_n}(u_{b_n}) \leq \limsup_{n\to+\infty} c_{b_n,2} \leq c_{0,2}.\end{aligned} \tag{5.29}$$

Next, we show that $u_0 \in \mathcal{M}_0$ and $\mathcal{I}_0(u_0) \geq c_{0,2}$. First, combining the boundedness of $\{u_{b_n}\}$ and (5.28), one deduces that

$$\begin{aligned}o(1) \leq \lim_{n\to\infty}\mathcal{I}_0(u_{b_n}) &= \lim_{n\to\infty}\left\{\mathcal{I}_{b_n}(u_{b_n}) - \frac{b_n}{4}\left(\int_{\mathbb{R}^3}|\nabla u_{b_n}|^2\mathrm{d}x\right)^2\right\}\\ &= \lim_{n\to\infty}\mathcal{I}_{b_n}(u_{b_n}) \leq \limsup_{n\to+\infty} c_{b_n,2} \leq c_{0,2}.\end{aligned} \tag{5.30}$$

Besides, it is easy to check that

$$0 = \langle \mathcal{I}_{b_n}'(u_{b_n}), \varphi\rangle = \langle \mathcal{I}_0'(u_{b_n}), \varphi\rangle + o(1)$$

for all $\varphi \in C_0^\infty(\mathbb{R}^3)$, which means that

$$\lim_{n\to\infty}\mathcal{I}_0'(u_{b_n}) = 0. \tag{5.31}$$

Hence, owing to (5.30) and (5.31), we know that there is a subsequence of $\{u_{b_n}\}$, still denoted by $\{u_{b_n}\}$, which satisfies

$$\lim_{n\to\infty}\mathcal{I}_0(u_{b_n}) := c \leq c_{0,2} \quad \text{and} \quad \lim_{n\to\infty}\mathcal{I}_0'(u_{b_n}) = 0. \tag{5.32}$$

In virtue of [10, Proposition 4.1], we get that $c_{0,2} < c_{0,1} + c_{0,\infty}$. According to [10], we know that $\mathcal{I}_0$ satisfies the $(PS)_{c^*}$ condition, where $c^* < c_{0,1} + c_{0,\infty}$. Hence, by (5.32), we get that the $(PS)_c$ sequence $\{u_{b_n}\}$ has a convergent subsequence, still denoted by $\{u_{b_n}\}$. Then, one has $u_{b_n} \to u_0$ in X. Moreover, in light of $\{u_{b_n}\} \subset \mathcal{M}_{b_n}$ and Lemma 2.3, we can arrive at $\|u_{b_n}^{\pm}\|_X \geq C_1 > 0$, and then $\|u_0^{\pm}\|_X \geq C_1 > 0$. Noting that $\mathcal{I}_0'(u_0) = 0$, we can show that $u_0 \in \mathcal{M}_0$. Therefore, we can obtain that $\mathcal{I}_0(u_0) \geq c_{0,2}$. Combining with (5.29), we conclude that $u_0 \in \mathcal{M}_0$ and $\mathcal{I}_0(u_0) = c_{0,2}$. Namely, u_0 is a least energy sign-changing solution of Equation (1.3). The proof is completed. □

Author contributions

Yan-Fei Yang: Writing-original draft, Writing-review & editing; Chun-Lei Tang: Formal analysis, Methodology, Supervision, Writing-review & editing.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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