Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 17(1): 145–158.
DOI: 10.3934/cam.2025007
Received: 02 July 2024
Revised: 20 January 2025
Accepted: 10 February 2025
Published: 18 February 2025

Research article

Serrin-type blowup Criterion for the degenerate compressible Navier-Stokes equations

Zhigang Wang*

School of Mathematics and Statistics, Fuyang Normal University, Anhui 236037, P.R.China

* **Correspondence:** Email: Zhigangwang@fynu.edu.cn; Tel: +86-558-259-1133; Fax: +86-558-259-1133.

Abstract: In this paper, we consider the Cauchy problem of the three-dimensional isentropic compressible Navier-Stokes equations with degenerate viscosities. When the shear and bulk viscosity coefficients are both given as a constant multiple of the mass density's power (ρ^δ with $\delta > 1$), we show that the L^∞ norms of ∇u, $\nabla\rho^{\frac{\gamma-1}{2}}$ and $\nabla\rho^{\frac{\delta-1}{2}}$ control the possible breakdown of classical solutions with far-field vacuum; this criterion is analogous to Serrin's blowup criterion for the compressible Navier–Stokes equations.

Keywords: Compressible Navier–Stokes equations; degenerate viscosity; far-field vacuum; classical solutions; Serrin-type blowup Criterion
Mathematics Subject Classification: Primary:35A01, 35B40; Secondary:35B65, 35A09

1. Introduction

The motion of a general viscous isentropic compressible fluid occupying a spatial domain $\Omega \subset \mathbb{R}^3$ can be described by the following isentropic compressible Navier–Stokes equations:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla P = \mathrm{div}\mathbb{T}, \end{cases} \tag{1.1}$$

where $\rho \geq 0$ is the mass density; $u = (u_1, u_2, u_3)$ is the velocity of the fluid; $x = (x_1, x_2, x_3) \in \Omega$, $t \geq 0$ are the space and time variables, respectively. For the polytropic gases, the constitutive relation, which is also called the equations of state, is given by

$$P = A\rho^\gamma, \quad \gamma > 1. \tag{1.2}$$

Here P is the pressure, $A > 0$ is an entropy constant and γ is the adiabatic exponent. $\mathbb{T}$ denotes the viscous stress tensor with the form

$$\mathbb{T} = \mu(\rho)(\nabla u + (\nabla u)^\top) + \lambda(\rho)\mathrm{div}u\,\mathbb{I}_3, \tag{1.3}$$

where $\mathbb{I}_3$ is the 3×3 identity matrix,

$$\mu(\rho) = \alpha\rho^\delta, \quad \lambda(\rho) = \beta\rho^\delta, \tag{1.4}$$

for some constant $\delta \geq 0$, $\mu(\rho)$ is the shear viscosity coefficient, $\lambda(\rho) + \frac{2}{3}\mu(\rho)$ is the bulk viscosity coefficient, α and β are both constants satisfying

$$\alpha > 0 \quad \text{and} \quad 2\alpha + 3\beta \geq 0. \tag{1.5}$$

This system can be derived from the Boltzmann equations through the Chapman–Enskog expansion, cf. Chapman–Cowling [1] and Li–Qin [2].

In this paper, we consider the Cauchy problem for (1.1) with the following initial data and far-field behavior:

$$(\rho, u)|_{t=0} = (\rho_0(x) \geq 0,\ u_0(x)) \quad \text{for} \quad x \in \mathbb{R}^3, \tag{1.6}$$

$$(\rho, u)(t, x) \to (0, 0) \quad \text{as} \quad |x| \to \infty \ \text{ for } \ t \geq 0. \tag{1.7}$$

In addition, we will use the following simplified notations; most of them are introduced in the standard homogeneous and inhomogeneous Sobolev spaces [3]:

$$\begin{aligned}
&\|f\|_s = \|f\|_{H^s(\mathbb{R}^3)}, \ \ |f|_p = \|f\|_{L^p(\mathbb{R}^3)}, \ \ \|f\|_{m,p} = \|f\|_{W^{m,p}(\mathbb{R}^3)},\\
&D^{k,r} = \{f \in L^1_{loc}(\mathbb{R}^3) : |f|_{D^{k,r}} = |\nabla^k f|_r < +\infty\},\\
&D^1_* = \{f \in L^6(\mathbb{R}^3) : |f|_{D^1} = |\nabla f|_2 < \infty\},\\
&X([0,T]; Y(\mathbb{R}^3)) = X([0,T]; Y), \quad \int_{\mathbb{R}^3} f\mathrm{d}x = \int f\mathrm{d}x.
\end{aligned}$$

For the initial data away from vacuum, the local existence of classical solutions for (1.1)-(1.6)-(1.7) is proved due to the standard symmetric hyperbolic-parabolic structure which satisfies the well-known Kawashima's condition [4, 5]. However, for the initial data including vacuum, such an approach does not work because of the degeneration in the momentum equation. Generally, a vacuum often occurs in some physical requirements, such as the finite total initial mass and the finite total initial energy. It is well known that the main difficulty is to understand the behavior of the velocity field near the vacuum.

When the coefficients of viscosity μ and λ are constant ($\delta = 0$), the degeneration occurs only in the time evolution of the momentum equation, Cho–Choe–Kim [6] introduce a remedy by this initial compatibility condition:

$$-\mathrm{div}\mathbb{T}_0 + \nabla P(\rho_0) = \sqrt{\rho_0}g, \quad \text{for some} \quad g \in L^2(\mathbb{R}^3),$$

which implies that $(\sqrt{\rho}u_t, \nabla u_t)$ in $L^\infty([0, T^*]; L^2)$ for a short time T^*. Then they obtain successfully the local well-posedness of smooth solutions with vacuum in some three-dimensional Sobolev spaces, and also show the necessity of the initial compatibility condition in their solution class. And Huang, Li, and Xin [7] establish the global well-posedness of these classical solutions with small energy and vacuum. For some related results, please refer to [8–15].

When the coefficients of viscosity μ and λ are dependent on the density ($\delta > 0$), the system (1.1) has received a lot of attention, see [16–22]. However, both the time evolution and viscosities are degenerate near the vacuum in the momentum equation, which prevents us from utilizing a similar remedy proposed in [6]. Recently, Zhu and his collaborators obtained some important advances on the well-posedness of classical solutions in this case; see [23–29]. Based on this observation for the momentum equation in (1.1), this one can also be rewritten in the following form:

$$u_t + \mathrm{div}(u \otimes u) + \frac{A\gamma}{\gamma - 1}\nabla\rho^{\gamma-1} + \rho^{\delta-1}Lu = \psi \cdot Q(u), \tag{1.8}$$

where

$$\begin{aligned} &\psi \triangleq \nabla \log \rho \text{ when } \delta = 1; \quad \psi \triangleq \frac{\delta}{\delta - 1}\nabla\rho^{\delta-1} \text{ when } \delta \neq 1; \\ &Lu \triangleq -\alpha\Delta u - (\alpha + \beta)\nabla\mathrm{div}u; \\ &Q(u) \triangleq \alpha(\nabla u + (\nabla u)^\top) + \beta\mathrm{div}u\mathbb{I}_3. \end{aligned} \tag{1.9}$$

As $\delta = 1$, they [24] introduce uninformative a priori estimates of $\nabla \log \rho$ in $L^6 \cap D^1 \cap D^2$ to establish the existence of a 2-D local classical solution with far-field vacuum, which is also extended to the three-dimensional spaces in [29]. As $\delta \in (1, +\infty)$, by using some hyperbolic approaches when $\rho > 0$ in (1.8) and the hyperbolic one

$$u_t + u \cdot \nabla u = 0 \tag{1.10}$$

when $\rho = 0$, they [25] establish the existence of a 3-D local classical solution with a vacuum. The corresponding global well-posedness under some initial smallness assumptions is also established in [26]. As $\delta \in (0, 1)$, they [27] introduce an elaborate elliptic approach on the operators $L(\rho^{\delta-1}u)$ and some initial compatibility conditions, to obtain the well-posedness of a local regular solution with far-field vacuum in some inhomogeneous Sobolev spaces.

In the current paper, we are concerned with the main mechanism for possible breakdown of classical solutions for the Cauchy problem (1.1)-(1.6)-(1.7) with $\delta > 1$ obtained in [25]. Our result shows that, the L^∞ norms of ∇u, $\nabla\rho^{\frac{\delta-1}{2}}$ and $\nabla\rho^{\frac{\gamma-1}{2}}$ control the possible breakdown of this solution, which means that if a solution of (1.1)-(1.6)-(1.7) is initially regular and loses its regularity at some later time, then the formation of singularity must be caused by losing the bound of ∇u, $\nabla\rho^{\frac{\delta-1}{2}}$ or $\nabla\rho^{\frac{\gamma-1}{2}}$ as the critical time approaches; equivalently, if they all remain bounded, a regular solution persists. This conclusion can be stated precisely as follows.

Theorem 1.1. *If (ρ, u) is the unique classical solution obtained in Theorem 2.1, and $\overline{T} < +\infty$ is the maximal existence time of (ρ, u), then*

$$\lim_{T \mapsto \overline{T}} \int_0^T \left(|\nabla u|_\infty^2 + |\nabla\rho^{\frac{\delta-1}{2}}|_\infty^2 + |\nabla\rho^{\frac{\gamma-1}{2}}|_\infty\right) \mathrm{d}t = \infty. \tag{1.11}$$

Furthermore, if $\gamma \leq \delta$, then

$$\lim_{T \mapsto \overline{T}} \int_0^T \left(|\nabla u|_\infty^2 + |\nabla\rho^{\frac{\gamma-1}{2}}|_\infty^2\right) \mathrm{d}t = \infty. \tag{1.12}$$

This paper is organized as follows. In §2, we introduce some known well-posedenss theories of the Cauchy problem (1.1)-(1.6)-(1.7) and some fundamental lemmas; in §3, we provide the detailed proof of Theorem 1.1.

2. Preliminaries

In this section, we introduce some known well-posedness theories of the Cauchy problem (1.1)-(1.6)-(1.7) and some fundamental lemmas that are frequently used in our proof.

In order to state our results clearly, we introduce the following regular solutions of the Cauchy problem (1.1)-(1.6)-(1.7) from [23].

Definition 2.1 (**Regular solutions**). *[23] Let $T > 0$ be a positive time. The function pair $(\rho(t,x), u(t,x))$ is called a regular solution to the Cauchy problem (1.1)-(1.6)-(1.7) in $[0,T] \times \mathbb{R}^3$ if $(\rho(t,x), u(t,x))$ satisfies this problem in the sense of distributions and:*

$$
\begin{aligned}
&(A)\ \rho \geq 0,\ \rho^{\frac{\delta-1}{2}} \in C([0,T];H^3),\ \ \rho^{\frac{\gamma-1}{2}} \in C([0,T];H^3);\\
&(B)\ u \in C([0,T];H^{s'}) \cap L^\infty([0,T];H^3),\ \ \rho^{\frac{\delta-1}{2}}\nabla^4 u \in L^2([0,T];L^2),\\
&(C)\ u_t + u\cdot\nabla u = 0\ \textit{as}\ \rho(t,x) = 0,
\end{aligned}
$$

where $s' \in [2,3)$ is an arbitrary constant.

The well-posedness of these regular solutions has been established in [23] as follow:

Theorem 2.1. *[23] Assume $\delta > 1$ in (1.4). If initial data (ρ_0, u_0) satisfy*

$$
\rho_0 \geq 0, \quad (\rho_0^{\frac{\gamma-1}{2}}, \rho_0^{\frac{\delta-1}{2}}, u_0) \in H^3, \tag{2.1}
$$

then there exists a time $T_* > 0$, and a unique regular solution (ρ, u) in $[0,T_*]\times\mathbb{R}^3$ to the Cauchy problem (1.1)-(1.6)-(1.7) satisfying

$$
\begin{aligned}
&\rho \geq 0,\ \ (\rho^{\frac{\gamma-1}{2}}, \rho^{\frac{\delta-1}{2}}) \in C([0,T_*];H^3),\\
&u \in C([0,T_*];H^{s'}) \cap L^\infty([0,T_*];H^3),\ \ \rho^{\frac{\delta-1}{2}}\nabla^4 u \in L^2([0,T_*];L^2),
\end{aligned} \tag{2.2}
$$

where $s' \in [2,3)$ is an arbitrary constant. Moreover, if $1 < \min\{\gamma,\delta\} \leq 3$, (ρ, u) is indeed a classical solution of the Cauchy problem (1.1)-(1.6)-(1.7) in $(0,T_*]\times\mathbb{R}^3$.

Next, we introduce the well-known Gagliardo–Nirenberg inequality and Moser-type calculus inequality.

Lemma 2.1. *[30] Let function $u \in L^q \cap D^{1,r}(\mathbb{R}^n)$ for $1 \leq q, r \leq \infty$. Suppose also that a real number θ and natural numbers m and j satisfy*

$$
\frac{1}{p} = \frac{j}{n} + \left(\frac{1}{r} - \frac{m}{n}\right)\theta + \frac{1-\theta}{q} \quad \textit{and} \quad \frac{j}{m} \leq \theta \leq 1.
$$

Then $u \in D^{j,p}(\mathbb{R}^n)$, and there exists a constant C depending only on m, n, j, q, r and θ such that

$$
|\mathrm{D}^j u|_p \leq C|\mathrm{D}^m u|_r^\theta |u|_q^{1-\theta}. \tag{2.3}
$$

Moreover, if $j = 0$, $mr < n$ and $q = \infty$, then it is necessary to make the additional assumption that either u tends to zero at infinity or that u lies in $L^s(\mathbb{R}^n)$ for some finite $s > 0$; if $1 < r < \infty$ and $m - j - n/r$ is a non-negative integer, then it is necessary to assume also that $\theta \neq 1$.

Lemma 2.2. *[31] Let r, a and b be constants such that*

$$\frac{1}{r} = \frac{1}{a} + \frac{1}{b}, \quad \text{and} \quad 1 \leq a,\ b,\ r \leq \infty.$$

$\forall s \geq 1$, if $f, g \in W^{s,a} \cap W^{s,b}(\mathbb{R}^n)$, then it holds that

$$|\nabla^s(fg) - f\nabla^s g|_r \leq C_s\big(|\nabla f|_a |\nabla^{s-1} g|_b + |\nabla^s f|_b |g|_a\big), \tag{2.4}$$

$$|\nabla^s(fg) - f\nabla^s g|_r \leq C_s\big(|\nabla f|_a |\nabla^{s-1} g|_b + |\nabla^s f|_a |g|_b\big), \tag{2.5}$$

where $C_s > 0$ is a constant depending only on s, and $\nabla^s f$ ($s \geq 1$) is the set of all $\partial_x^\zeta f$ with $|\zeta| = s$. Here $\zeta = (\zeta_1, ..., \zeta_n) \in \mathbb{R}^n$ is a multi-index.

3. Serrin-type blow-up criterion

The purpose of this section is to prove Theorem 1.1. Let (ρ, u) be the classical solution to (1.1)-(1.6)-(1.7) obtained in Theorem 2.1 in $[0, T] \times \mathbb{R}^3$. Suppose that $\overline{T}$ is the maximal existence time, and the opposite of (1.11) holds, i.e.,

$$\lim_{T \mapsto \overline{T}} \int_0^T \big(|\nabla u|_\infty^2 + |\nabla \rho^{\frac{\delta-1}{2}}|_\infty^2 + |\nabla \rho^{\frac{\gamma-1}{2}}|_\infty\big)\, \mathrm{d}t = \bar{c} < \infty, \tag{3.1}$$

where $\bar{c} > 0$ is some finite constant.

3.1. Reformulation

By the two new quantities:

$$\varphi = \rho^{\frac{\delta-1}{2}} \quad \text{and} \quad \phi = \rho^{\frac{\gamma-1}{2}},$$

we can rewrite system (1.1) into a new system, which consists of a "quasi-symmetric hyperbolic"-"degenerate elliptic" coupled system with some special lower-order source terms for (ϕ, u), and a transport equation for φ:

$$\begin{cases} A_0 W_t + \sum_{j=1}^{3} A_j(W)\partial_j W + \varphi^2 \mathbb{L}(W) = \mathbb{H}(\varphi) \cdot \mathbb{Q}(W), \\ \varphi_t + u \cdot \nabla \varphi + \dfrac{\delta - 1}{2} \varphi \mathrm{div} u = 0, \\ (W, \varphi)|_{t=0} = (W_0, \varphi_0), \quad x \in \mathbb{R}^3, \\ (W, \varphi) \to (0, 0), \quad \text{as} \quad |x| \to \infty, \quad t \geq 0, \end{cases} \tag{3.2}$$

where $W = (\phi, u)^\top$ and

$$\mathbb{L}(W) = \begin{pmatrix} 0 \\ a_1 L u \end{pmatrix}, \quad \mathbb{H}(\varphi) = \begin{pmatrix} 0 \\ \nabla \varphi^2 \end{pmatrix}, \quad \mathbb{Q}(W) = \begin{pmatrix} 0 & 0 \\ 0 & a_1 Q(u) \end{pmatrix}, \tag{3.3}$$

with $a_1 = \frac{(\gamma-1)^2}{4A\gamma} > 0$ and $Q(u) = \frac{\delta}{\delta-1}\mathbb{S}(u)$. Meanwhile, $\partial_j W = \partial x_j W$, and

$$A_0 = \begin{pmatrix} 1 & 0 \\ 0 & a_1\mathbb{I}_3 \end{pmatrix}, \quad A_j = \begin{pmatrix} u^{(j)} & \frac{\gamma-1}{2}\phi e_j \\ \frac{\gamma-1}{2}\phi e_j^\top & a_1 u^{(j)}\mathbb{I}_3 \end{pmatrix}, \quad j = 1, 2, 3. \tag{3.4}$$

Here $e_j = (\delta_{1j}, \delta_{2j}, \delta_{3j})$ $(j = 1, 2, 3)$ is the Kronecker symbol satisfying $\delta_{ij} = 1$, when $i = j$ and $\delta_{ij} = 0$, otherwise. For any $\xi \in \mathbb{R}^4$, we have

$$\xi^\top A_0 \xi \geq a_2 |\xi|^2 \quad \text{with} \quad a_2 = \min\{1, a_1\} > 0. \tag{3.5}$$

Moreover,

$$(W_0, \varphi_0) = (\phi, u, \varphi)|_{t=0} = (\rho_0^{\frac{\gamma-1}{2}}(x), u_0(x), \rho_0^{\frac{\delta-1}{2}}(x)), \quad x \in \mathbb{R}^3. \tag{3.6}$$

3.2. Uniform estimates for arbitrarily large time

Based on (3.1), we will make some uniform estimates for arbitrarily large times.

Lemma 3.1. *If* (3.1) *holds, then*

$$\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3)} \leq C, \tag{3.7}$$

for any $0 < T \leq \overline{T}$*, where the constant* $C > 0$ *is only dependent on* (ρ_0, u_0)*,* $\bar{c}$*,* α*,* β*,* A*,* γ *and* δ*.*

Proof. First, it is obvious that ρ can be represented by

$$\rho(t, x) = \rho_0(\Phi(0, t, x)) \exp\Big(- \int_0^t \mathrm{div} u(s, \Phi(s, t, x)) \mathrm{d}s \Big), \tag{3.8}$$

where $\Phi \in C^1([0, T] \times [0, T] \times \mathbb{R}^3)$ is the solution to the initial value problem

$$\begin{cases} \dfrac{d}{ds}\Phi(s, t, x) = u(s, \Phi(s, t, x)), & 0 \leq s \leq T, \\ \Phi(t, t, x) = x, & 0 \leq s \leq T, \quad x \in \mathbb{R}^3. \end{cases} \tag{3.9}$$

Then it is easy to derive that

$$\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3)} \leq |\rho_0|_\infty \exp(\bar{c}) \quad \text{for} \quad 0 \leq T < \overline{T}. \tag{3.10}$$

□

Next, we are going to establish the H^3 estimates of (ϕ, u, φ). Let

$$\zeta = \zeta^1 + \zeta^2 + \zeta^3$$

for three multi-indexes $\zeta^i \in \mathbb{R}^3$ $(i = 1, 2, 3)$ satisfying $|\zeta^i| = 0$ or 1. On the one hand, we apply the operator ∂_x^ζ to $(3.2)_1$, multiply the resulting equations by $\partial_x^\zeta W$ on both sides and integrate over $\mathbb{R}^3$ to have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int((\partial_x^\zeta W)^\top A_0\partial_x^\zeta W)\mathrm{d}x+a_1\alpha|\varphi\nabla\partial_x^\zeta u|_2^2+a_1(\alpha+\beta)|\varphi\mathrm{div}\partial_x^\zeta u|_2^2\\
=&\frac{1}{2}\int(\partial_x^\zeta W)^\top \mathrm{div}A(W)\partial_x^\zeta W\mathrm{d}x\\
&-\sum_{j=1}^{3}\int(\partial_x^\zeta(A_j(W)\partial_j W)-A_j(W)\partial_j\partial_x^\zeta W)\cdot\partial_x^\zeta W\mathrm{d}x\\
&-a_1\int(\partial_x^\zeta(\varphi^2 Lu)-\varphi^2 L\partial_x^\zeta u)\cdot\partial_x^\zeta u\mathrm{d}x\\
&-a_1\int\Big(\frac{\delta-1}{\delta}\nabla\varphi^2\cdot Q(\partial_x^\zeta u)-\partial_x^\zeta(\nabla\varphi^2\cdot Q(u))\Big)\cdot\partial_x^\zeta u\mathrm{d}x\triangleq\sum_{i=1}^{5}J_i.
\end{aligned}\tag{3.11}$$

On the other hand, we apply the operator ∂_x^ζ to $(3.2)_2$, multiply the resulting equations by $\partial_x^\zeta\varphi$ on both sides and integrate over $\mathbb{R}^3$ to have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|\partial_x^\zeta\varphi|_2^2=&\frac{1}{2}\int\mathrm{div}u|\partial_x^\zeta\varphi|^2\mathrm{d}x-\int(\partial_x^\zeta(u\cdot\nabla\varphi)-u\cdot\nabla\partial_x^\zeta\varphi)\partial_x^\zeta\varphi\mathrm{d}x\\
&-\frac{\delta-1}{2}\int\partial_x^\zeta(\varphi\mathrm{div}u)\partial_x^\zeta\varphi\mathrm{d}x\triangleq\sum_{i=6}^{8}J_i.
\end{aligned}\tag{3.12}$$

Some terms in (3.11) and (3.12) are easy to be estimated in the following lemma.

Lemma 3.2. *For $|\zeta|=k$, $k=0,1,2,3$, one has*

$$\begin{aligned}
J_1+J_2+J_4+J_6+J_7\le& C|\nabla W|_\infty(|\nabla^k W|_2^2+|\nabla^k\varphi|_2^2)\\
&+C|\nabla\varphi|_\infty(|\varphi\nabla^{k+1}u|_2|\nabla^k u|_2+|\nabla^k\varphi|_2|\nabla^k u|_2),
\end{aligned}\tag{3.13}$$

where the constant $C>0$ is only dependent on (ρ_0,u_0), $\bar{c}$, α, β, A, γ and δ.

Proof. For $|\zeta|=k=0,1,2,3$, from the Hölder and Sobolev inequalities and Lemma 2.2 we can derive that

$$\begin{aligned}
J_1=&\frac{1}{2}\int(\partial_x^\zeta W)^\top\mathrm{div}A(W)\partial_x^\zeta W\mathrm{d}x\le C|\nabla W|_\infty|\partial_x^\zeta W|_2^2,\\
J_2=&-\sum_{j=1}^{3}\int\Big(\partial_x^\zeta(A_j(W)\partial_j W)-A_j(W)\partial_j\partial_x^\zeta W\Big)\partial_x^\zeta W\mathrm{d}x\\
\le& C|\nabla W|_\infty|\nabla^k W|_2^2,\\
J_4=&-\frac{\delta-1}{\delta}a_1\int\nabla\varphi^2\cdot Q(\partial_x^\zeta u)\cdot\partial_x^\zeta u\mathrm{d}x\le C|\nabla\varphi|_\infty|\varphi\nabla\partial_x^\zeta u|_2|\partial_x^\zeta u|_2,\\
J_6=&\frac{1}{2}\int\mathrm{div}u|\partial_x^\zeta\varphi|^2\mathrm{d}x\le C|\nabla u|_\infty|\partial_x^\zeta\varphi|_2^2,\\
J_7=&-\int(\partial_x^\zeta(u\cdot\nabla\varphi)-u\cdot\nabla\partial_x^\zeta\varphi)\partial_x^\zeta\varphi\mathrm{d}x\\
\le& C(|\nabla u|_\infty|\nabla^k\varphi|_2^2+|\nabla\varphi|_\infty|\nabla^k\varphi|_2|\nabla^k u|_2),
\end{aligned}\tag{3.14}$$

which implies the desired conclusion.

□

Based on the relations (3.11)-(3.12) and Lemma 3.2, we can obtain the following lower order estimates.

Lemma 3.3. *If* (ρ, u) *satisfies (3.1), then it holds that for any* $0 < T < \overline{T}$

$$\sup_{0\le t\le T}(\|\varphi\|_1 + \|\phi\|_1 + \|u\|_1)(t) + \int_0^T (|\varphi\nabla u|_2^2 + |\varphi\nabla^2 u|_2^2)\mathrm{d}t \le C, \tag{3.15}$$

where the constant $C > 0$ *is only dependent on* (ρ_0, u_0), $\bar{c}$, α, β, A, γ *and* δ.

Proof. **Step 1**. First, for the L^2-estimate, from (3.11)-(3.12), we can see that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int W^\top A_0 W\mathrm{d}x + a_1\alpha|\varphi\nabla u|_2^2 + a_1(\alpha+\beta)|\varphi\mathrm{div}u|_2^2\\
=&\frac{1}{2}\int W^\top \mathrm{div}A(W)W\mathrm{d}x + \frac{1}{\delta}a_1\int\nabla\varphi^2\cdot Q(u)\cdot u\mathrm{d}x\\
\le& C(|\nabla W|_\infty|W|_2^2 + |\nabla\varphi|_\infty|\varphi\nabla u|_2|u|_2),\\
&\frac{1}{2}\frac{d}{dt}|\varphi|_2^2 = \frac{2-\delta}{2}\int \mathrm{div}u|\varphi|^2\mathrm{d}x \le C|\nabla u|_\infty|\varphi|_2^2.
\end{aligned}\tag{3.16}$$

By (3.16), (3.1), Young's inequality, and Grönwall's inequality, we can conclude that

$$\sup_{0\le t\le T}(|W|_2^2 + |\varphi|_2^2)(t) + \int_0^T|\varphi\nabla u|_2^2\mathrm{d}t \le C.$$

Step 2. Second, for the D^1-estimate, i.e., $|\zeta| = 1$, we have

$$\begin{aligned}
J_3 =& -a_1\int(\partial_x^\zeta(\varphi^2 Lu) - \varphi^2 L\partial_x^\zeta u)\cdot\partial_x^\zeta u\mathrm{d}x \le C|\nabla\varphi|_\infty|\varphi Lu|_2|\nabla u|_2,\\
J_8 =& -\frac{\delta-1}{2}\int\partial_x^\zeta(\varphi\mathrm{div}u)\partial_x^\zeta\varphi\mathrm{d}x \le C(|\nabla u|_\infty|\nabla\varphi|_2^2 + |\varphi\nabla^2 u|_2|\nabla\varphi|_2),\\
J_5 =& a_1\int\partial_x^\zeta(\nabla\varphi^2\cdot Q(u))\cdot\partial_x^\zeta u\mathrm{d}x\\
=& -a_1\int\nabla\varphi^2\cdot Q(u)\cdot\partial_x^\zeta\partial_x^\zeta u\mathrm{d}x \le C|\nabla\varphi|_\infty|\varphi\nabla^2 u|_2|\nabla u|_2.
\end{aligned}\tag{3.17}$$

Based on (3.17), (3.1), Lemma 3.2, Young's inequality, and Grönwall's inequality, we can conclude that

$$\sup_{0\le t\le T}(|\nabla W|_2^2 + |\nabla\varphi|_2^2)(t) + \int_0^T|\varphi\nabla^2 u|_2^2\mathrm{d}t \le C.$$

□

The higher-order estimates for W are also obtained as follow:

Lemma 3.4. *If* (ρ, u) *satisfies (3.1), then it holds that for any* $0 < T < \overline{T}$

$$\begin{aligned}\sup_{0\le t\le T}(\|\nabla^2\varphi\|_1 + \|\nabla^2\phi\|_1 + \|\nabla^2 u\|_1)(t)&\\ +\int_0^T(|\varphi\nabla^3u|_2^2 + |\varphi\nabla^4 u|_2^2)\mathrm{d}t \le C,&\end{aligned}\tag{3.18}$$

where the constant $C > 0$ *is only dependent on* (ρ_0, u_0), $\bar{c}$, α, β, A, γ *and* δ.

Proof. **Step 1**. First, for the D^2-estimate, i.e., $|\zeta| = 2$, we have

$$\begin{aligned}J_3 =& - a_1\int(\partial_x^\zeta(\varphi^2 Lu) - \varphi^2 L\partial_x^\zeta u)\cdot\partial_x^\zeta u\mathrm{d}x\\ \le& C(|\nabla\varphi|_\infty|\nabla^2 u|_2 + |\varphi\nabla^3 u|_2)|\nabla\varphi|_\infty|\nabla^2 u|_2 + C|J_3^*|,\\ J_5 =& a_1\int\partial_x^\zeta(\nabla\varphi^2\cdot Q(u))\cdot\partial_x^\zeta u\mathrm{d}x\\ =& C(|\varphi|_\infty|\nabla^3\varphi|_2|\nabla u|_\infty + |\nabla\varphi|_\infty^2|\nabla^2 u|_2)|\nabla^2 u|_2\\ &+ C(|\varphi\nabla^3 u|_2 + |\nabla^2\varphi|_2|\nabla u|_\infty)|\nabla\varphi|_\infty|\nabla^2 u|_2 + C|J_5^*|,\\ J_8 =& -\frac{\delta-1}{2}\int\partial_x^\zeta(\varphi\mathrm{div}u)\partial_x^\zeta\varphi\mathrm{d}x\\ \le& C(|\nabla u|_\infty|\nabla^2\varphi|_2 + |\nabla\varphi|_\infty|\nabla^2 u|_2 + |\varphi\nabla^3 u|_2)|\nabla^2\varphi|_2,\end{aligned}\tag{3.19}$$

where the terms J_3^* and J_5^*, via integration by parts, can be estimated as

$$\begin{aligned}J_3^* + J_5^* =& \int(\varphi\partial_x^\zeta\varphi Lu + \varphi\nabla\partial_x^{\zeta_*}\varphi\cdot\partial_x^{\zeta-\zeta_*}Q(u)))\cdot\partial_x^\zeta u\mathrm{d}x\\ \le& C(|\nabla\varphi|_\infty|\nabla^2 u|_2 + |\varphi\nabla^3 u|_2)|\nabla\varphi|_\infty|\nabla^2 u|_2,\end{aligned}\tag{3.20}$$

for some multi-indexe $\zeta_* \in \mathbb{R}^3$ satisfying $|\zeta_*| = 1$.

Thus, by (3.11)-(3.12), Lemma 3.2, (3.19)-(3.20), and Young's inequality, we can conclude that

$$\begin{aligned}&\frac{d}{dt}(|\nabla^2 W|_2^2 + |\nabla^2\varphi|_2^2)(t) + |\varphi\nabla^3 u|_2^2\\ \le& C(|\nabla W|_\infty + |\nabla\varphi|_\infty^2 + 1)(|\nabla^2 W|_2^2 + |\nabla^2\varphi|_2^2)\\ &+ C|\nabla u|_\infty|\nabla\varphi|_\infty|\nabla^2\varphi|_2|\nabla^2 u|_2 + C|\varphi|_\infty|\nabla u|_\infty|\nabla^3\varphi|_2|\nabla^2 u|_2.\end{aligned}\tag{3.21}$$

Step 2. Second, for the D^3-estimate, i.e., $|\zeta| = 3$, it follows from the Hölder and Sobolev inequalities that

$$\begin{aligned}J_3 =& - a_1\int(\partial_x^\zeta(\varphi^2 Lu) - \varphi^2 L\partial_x^\zeta u)\cdot\partial_x^\zeta u\mathrm{d}x\\ \le& C(|\nabla\varphi|_\infty|\nabla^3 u|_2 + |\varphi\nabla^4 u|_2)|\nabla\varphi|_\infty|\nabla^3 u|_2\\ &+ C|\nabla^2\varphi|_4|\nabla\varphi|_\infty|\nabla^2 u|_4|\nabla^3 u|_2 + C|J_3^{**}|\\ \le& C(|\nabla\varphi|_\infty|\nabla^3 u|_2 + |\varphi\nabla^4 u|_2)|\nabla\varphi|_\infty|\nabla^3 u|_2\\ &+ C|\nabla\varphi|_\infty^{\frac{3}{2}}|\nabla^3 u|_2^{\frac{3}{2}}|\nabla u|_\infty^{\frac{1}{2}}|\nabla^3\varphi|_2^{\frac{1}{2}} + C|J_3^{**}|,\end{aligned}\tag{3.22}$$

where we have used the Gagliardo–Nirenberg inequality in Lemma 2.1:

$$|\nabla^2\varphi|_4 \le C|\nabla\varphi|_\infty^{\frac{1}{2}}|\nabla^3\varphi|_2^{\frac{1}{2}} \quad \text{and} \quad |\nabla^2 u|_4 \le C|\nabla u|_\infty^{\frac{1}{2}}|\nabla^3 u|_2^{\frac{1}{2}},$$

and the term J_3^{**}, via integration by parts, can be estimated as

$$\begin{aligned}
J_3^{**} =& - a_1 \int (\varphi\partial_x^{\zeta_{**}}\varphi\partial_x^{\zeta-\zeta_{**}} Lu + \varphi\partial_x^{\zeta}\varphi Lu)\cdot\partial_x^{\zeta} u\mathrm{d}x\\
\le& C(|\nabla\varphi|_\infty|\nabla^3 u|_2 + |\varphi\nabla^4 u|_2)|\nabla\varphi|_\infty|\nabla^3 u|_2\\
&+ C(|\nabla\varphi|_\infty|\nabla^3 u|_2 + |\varphi\nabla^4 u|_2)|\nabla^2\varphi|_4|\nabla^2 u|_4\\
\le& C(|\nabla\varphi|_\infty|\nabla^3 u|_2 + |\varphi\nabla^4 u|_2)|\nabla\varphi|_\infty|\nabla^3 u|_2\\
&+ C(|\nabla\varphi|_\infty|\nabla^3 u|_2 + |\varphi\nabla^4 u|_2)|\nabla\varphi|_\infty^{\frac{1}{2}}|\nabla u|_\infty^{\frac{1}{2}}|\nabla^3\varphi|_2^{\frac{1}{2}}|\nabla^3 u|_2^{\frac{1}{2}},
\end{aligned} \tag{3.23}$$

for some multi-indexes $\zeta_{**} \in \mathbb{R}^3$ satisfying $|\zeta_{**}| = 2$.

For the term J_5, we have

$$\begin{aligned}
J_5 =& a_1 \int \partial_x^{\zeta}(\nabla\varphi^2 \cdot Q(u))\cdot\partial_x^{\zeta} u\mathrm{d}x\\
\le& C(|\nabla\varphi|_\infty|\nabla^3 u|_2 + |\varphi\nabla^4 u|_2)|\nabla\varphi|_\infty|\nabla^3 u|_2\\
&+ C|\nabla^2\varphi|_4|\nabla\varphi|_\infty|\nabla^2 u|_4|\nabla^3 u|_2 + C\sum_{i=1}^{3}|J_{5i}^{**}|\\
\le& C(|\nabla\varphi|_\infty|\nabla^3 u|_2 + |\varphi\nabla^4 u|_2)|\nabla\varphi|_\infty|\nabla^3 u|_2\\
&+ C|\nabla^3\varphi|_2^{\frac{1}{2}}|\nabla\varphi|_\infty^{\frac{3}{2}}|\nabla u|_\infty^{\frac{1}{2}}|\nabla^3 u|_2^{\frac{3}{2}} + C\sum_{i=1}^{3}|J_{5i}^{**}|,
\end{aligned} \tag{3.24}$$

where the term J_{5i}^{**} $(i = 1, 2, 3)$, via integration by parts, can be estimated as

$$\begin{aligned}
J_{51}^{**}+J_{52}^{**} =& \int (\varphi\nabla\partial_x^{\zeta_*}\varphi\cdot\partial_x^{\zeta-\zeta_*}Q(u) + \varphi\nabla\partial_x^{\zeta_{**}}\varphi\cdot\partial_x^{\zeta-\zeta_{**}}Q(u))\cdot\partial_x^{\zeta} u\mathrm{d}x\\
\le& C(|\nabla\varphi|_\infty|\nabla^3 u|_2 + |\varphi\nabla^4 u|_2)|\nabla\varphi|_\infty|\nabla^3 u|_2\\
&+ C|\nabla\varphi|_\infty|\nabla^3 u|_2|\nabla^2\varphi|_4|\nabla^2 u|_4 + C|\nabla^2\varphi|_4|\nabla^2 u|_4|\varphi\nabla^4 u|_2\\
\le& C(|\nabla\varphi|_\infty|\nabla^3 u|_2 + |\varphi\nabla^4 u|_2)|\nabla\varphi|_\infty|\nabla^3 u|_2\\
&+ C(|\nabla\varphi|_\infty^{\frac{3}{2}}|\nabla^3 u|_2^{\frac{3}{2}} + |\nabla\varphi|_\infty^{\frac{1}{2}}|\nabla^3 u|_2^{\frac{1}{2}}|\varphi\nabla^4 u|_2)|\nabla u|_\infty^{\frac{1}{2}}|\nabla^3\varphi|_2^{\frac{1}{2}},\\
J_{53}^{**} =& \int \partial_x^{\zeta}\nabla\varphi^2\cdot Q(u)\cdot\partial_x^{\zeta} u\mathrm{d}x\\
=& - \int \partial_x^{\zeta-\zeta_*}\nabla\varphi^2\cdot\partial_x^{\zeta_*}Q(u)\cdot\partial_x^{\zeta} u\mathrm{d}x - \int \partial_x^{\zeta-\zeta_*}\nabla\varphi^2\cdot Q(u)\cdot\partial_x^{\zeta+\zeta_*} u\mathrm{d}x\\
\le& C(|\nabla\varphi|_\infty|\nabla^3 u|_2 + |\varphi\nabla^4 u|_2)|\nabla\varphi|_\infty|\nabla^3 u|_2\\
&+ C|\nabla\varphi|_\infty|\nabla^3 u|_2|\nabla^2\varphi|_4|\nabla^2 u|_4 + C|\nabla^2\varphi|_4|\nabla^2 u|_4|\varphi\nabla^4 u|_2\\
&- \int \partial_x^{\zeta-\zeta_*}\nabla\varphi^2\cdot Q(u)\cdot\partial_x^{\zeta+\zeta_*} u\mathrm{d}x (= J_*)\\
\le& C(|\nabla\varphi|_\infty|\nabla^3 u|_2 + |\varphi\nabla^4 u|_2)|\nabla\varphi|_\infty|\nabla^3 u|_2\\
&+ C(|\nabla\varphi|_\infty^{\frac{3}{2}}|\nabla^3 u|_2^{\frac{3}{2}} + |\nabla\varphi|_\infty^{\frac{1}{2}}|\nabla^3 u|_2^{\frac{1}{2}}|\varphi\nabla^4 u|_2)|\nabla u|_\infty^{\frac{1}{2}}|\nabla^3\varphi|_2^{\frac{1}{2}} + J_*,
\end{aligned} \tag{3.25}$$

for some multi-indexes $\zeta_*, \zeta_{**} \in \mathbb{R}^3$ satisfying $|\zeta_*| = 1$ and $|\zeta_{**}| = 2$. For the last term in J_{53}^{**}, we still need the integration by parts to estimate it,

$$\begin{aligned}
J_* =& -\int \partial_x^{\zeta-\zeta_*}\nabla\varphi^2 \cdot Q(u) \cdot \partial_x^{\zeta+\zeta_*} u \mathrm{d}x \\
\leq& C|\nabla u|_\infty |\nabla^3\varphi|_2 |\varphi\nabla^4 u|_2 + C\left|\int \partial_x^{\zeta-\zeta_*}\varphi\nabla\varphi \cdot Q(u) \cdot \partial_x^{\zeta+\zeta_*} u \mathrm{d}x\right| \\
\leq& C|\nabla u|_\infty |\nabla^3\varphi|_2 |\varphi\nabla^4 u|_2 + C|\nabla\varphi|_\infty |\nabla^3 u|_2 |\nabla^2\varphi|_4 |\nabla^2 u|_4 \\
&+ C|\nabla u|_\infty |\nabla^3 u|_2 |\nabla^2\varphi|_4^2 + C|\nabla\varphi|_\infty |\nabla u|_\infty |\nabla^3 u|_2 |\nabla^3\varphi|_2 \\
\leq& C|\nabla u|_\infty |\nabla^3\varphi|_2 |\varphi\nabla^4 u|_2 + C|\nabla\varphi|_\infty^{\frac{3}{2}} |\nabla^3 u|_2^{\frac{3}{2}} |\nabla^3\varphi|_2^{\frac{1}{2}} |\nabla u|_\infty^{\frac{1}{2}} \\
&+ C|\nabla\varphi|_\infty |\nabla u|_\infty |\nabla^3 u|_2 |\nabla^3\varphi|_2.
\end{aligned} \tag{3.26}$$

For the term J_8, we have

$$\begin{aligned}
J_8 =& \frac{\delta-1}{2}\int \partial_x^\zeta(\varphi \mathrm{div} u)\partial_x^\zeta\varphi \mathrm{d}x \\
\leq& C|\nabla u|_\infty |\nabla^3\varphi|_2^2 + C|\nabla^2\varphi|_4 |\nabla^2 u|_4 |\nabla^3\varphi|_2 \\
&+ C|\nabla\varphi|_\infty |\nabla^3\varphi|_2 |\nabla^3 u|_2 + C|\varphi\nabla^4 u|_2 |\nabla^3\varphi|_2 \\
\leq& C|\nabla u|_\infty |\nabla^3\varphi|_2^2 + C|\nabla\varphi|_\infty^{\frac{1}{2}} |\nabla u|_\infty^{\frac{1}{2}} |\nabla^3 u|_2^{\frac{1}{2}} |\nabla^3\varphi|_2^{\frac{3}{2}} \\
&+ C|\nabla\varphi|_\infty |\nabla^3\varphi|_2 |\nabla^3 u|_2 + C|\varphi\nabla^4 u|_2 |\nabla^3\varphi|_2.
\end{aligned} \tag{3.27}$$

Then, according to (3.11)-(3.12), Lemma 3.1-3.2, (3.22)-(3.27), and Young's inequality, we can conclude that

$$\begin{aligned}
&\frac{d}{dt}(|\nabla^3 W|_2^2 + |\nabla^3\varphi|_2^2)(t) + |\varphi\nabla^4 u|_2^2 \\
\leq& C(|\nabla W|_\infty + |\nabla u|_\infty^2 + |\nabla\varphi|_\infty^2 + 1)(|\nabla^3 W|_2^2 + |\nabla^2\varphi|_2^2),
\end{aligned} \tag{3.28}$$

which, together with Grönwall's inequality, (3.1) and (3.20), gives (3.18). Hence the proof is finished. □

3.3. Proof of Theorem 1.1

Now, we know that if the classical solution $(\rho, u)(t, x)$ exists up to the time $\overline{T}$, where $\overline{T} < +\infty$ is the maximal existence time such that the assumption (3.1) holds, then we have Lemmas 3.1-3.4. Thus, by the standard weak compactness theory, we can see that for any sequence $\{t_k\}_{k=1}^\infty$ with $t_k \in (0, \overline{T})$ and $t_k \to \overline{T} (k \to \infty)$, there exists one subsequence $\{t_{n_k}\}_{k=1}^\infty$ and functions $(\phi, u, \varphi)(\overline{T}, x)$ such that

$$\begin{aligned}
&\phi(t_{n_k}, x) \rightharpoonup \phi(\overline{T}, x) \text{ in } H^3 \text{ as } k \to \infty, \\
&u(t_{n_k}, x) \rightharpoonup u(\overline{T}, x) \text{ in } H^3 \text{ as } k \to \infty, \\
&\varphi(t_{n_k}, x) \rightharpoonup \varphi(\overline{T}, x) \text{ in } H^3 \text{ as } k \to \infty,
\end{aligned} \tag{3.29}$$

which implies that $(\phi, u, \varphi)(\overline{T}, x)$ satisfies the initial data (2.1) in Theorem 2.1. Thus, by Theorem 2.1, the classical solution (ρ, u), can be extended beyond $[0, \overline{T}]$. This contradicts to the fact that $\overline{T}$ is the maximal existence time. Thus, we obtain (1.11).

Furthermore, when $\gamma \le \delta$, by Lemma 3.1 we have

$$\|\rho\|_{L^\infty([0,T]\times\mathbb{R}^3)} \le C.$$

Due to

$$\nabla\rho^{\frac{\delta-1}{2}} = \frac{\delta-1}{\gamma-1}\rho^{\frac{\delta-\gamma}{2}}\nabla\rho^{\frac{\gamma-1}{2}},$$

it is obvious that

$$\int_0^T |\nabla\rho^{\frac{\delta-1}{2}}|_\infty^2 \,\mathrm{d}t \le C\|\rho\|^{\delta-\gamma}_{L^\infty([0,T]\times\mathbb{R}^3)} \int_0^T |\nabla\rho^{\frac{\gamma-1}{2}}|_\infty^2 \,\mathrm{d}t,$$

which implies that we can obtain (1.12) similar to (1.11). Therefore, we complete the proof of Theorem 1.1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research is supported by the Outstanding Youth Research Project of Anhui Universities (2023AH030075); the Research and Innovation Team Project of Fuyang Normal University (kytd202003).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. S. Chapman, T. Cowling, *The Mathematical Theory of Non-Uniform Gases: An Account of the Kinetic Theory of Viscosity, Thermal Conduction and Diffusion in Gases*, Cambridge University Press, 1990. https://doi.org/10.2307/3609795
2. T. Li, T. Qin, *Physics and Partial Differential Equations*, Vol. II., Beijing: Higher Education Press, 2014.
3. G. Galdi, *An Introduction to the Mathmatical Theory of the Navier–Stokes Equations*, New York: Springer, 1994. https://doi.org/10.1007/978-0-387-09620-9
4. S. Kawashima, *Systems of A Hyperbolic-Parabolic Composite Type, with Applications to the Equations of Magnetohydrodynamics*, Ph.D thesis, Kyoto University in Kyoto, 1983. https://doi.org/10.14989/doctor.k3193
5. J. Nash, Le probleme de Cauchy pour les équations différentielles dún fluide général, *Bull. Soc. Math. France*, **90** (1962), 487–491. https://doi.org/10.24033/bsmf.1586
6. Y. Cho, H. Choe, H. Kim, Unique solvability of the initial boundary value problems for compressible viscous fluids, *J. Math. Pures Appl.*, **83** (2004), 243–275. https://doi.org/10.1016/j.matpur.2003.11.004

7. X. Huang, J. Li, Z. Xin, Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations, *Commun. Pure Appl. Math.*, **65** (2012), 549–585. https://doi.org/10.1002/cpa.21382
8. Y. Cho, B. Jin, Blow-up of viscous heat-conducting compressible flows, *J. Math. Anal. Appl.*, **320** (2006), 819–826. https://doi.org/10.1016/j.jmaa.2005.08.005
9. Y. Cho, H. Kim, On classical solutions of the compressible Navier–Stokes equations with nonnegative initial densities, *Manu. Math.*, **120** (2006), 91–129. https://doi.org/10.1007/s00229-006-0637-y
10. X. Huang, J. Li, Z. Xin, Blow-up criterion for viscous baratropic flows with vacuum states, *Commun. Math. Phys.*, **301** (2010) 23–35. https://doi.org/10.1007/s00220-010-1148-y
11. H. Li, Y. Wang, Z. Xin, Non-existence of classical solutions with finite energy to the Cauchy problem of the compressible Navier–Stokes equations, *Arch. Rational. Mech. Anal.*, **232** (2019), 557–590. https://doi.org/10.1007/s00205-018-1328-z
12. O. Rozanova, Blow-up of smooth highly decreasing at infinity solutions to the compressible Navier–Stokes equations, *J. Differential Equations*, **245** (2010), 1762–1774. https://doi.org/10.1016/j.jde.2008.07.007
13. Y. Sun, C. Wang, Z. Zhang, A Beale–Kato–Majda blow-up criterion to the compressible Navier–Stokes equation, *J. Math. Pures Appl.*, **95** (2011), 36–47. https://doi.org/10.1016/j.matpur.2010.08.001
14. H. Wen, C. Zhu, Blow-up criterions of strong solutions to 3D compressible Navier–Stokes equations with vacuum, *Adv. Math.*, **248** (2013), 534–572. https://doi.org/10.1016/j.aim.2013.07.018
15. Z. Xin, W. Yan, On blow-up of classical solutions to the compressible Navier–Stokes equations, *Commun. Math. Phys.*, **321** (2013), 529–541. https://doi.org/10.1007/s00220-012-1610-0
16. R. Danchin, Well-posedness in critical spaces for barotropic viscous fluids with truly not constant density, *Commun. Partial Differ. Equ.*, **32** (2007), 1373–1397. https://doi.org/10.1080/03605300600910399
17. Q. Jiu, Y. Wang, Z. Xin, Remarks on blow-up of smooth solutions to the compressible fluid with constant and degenerate viscosities, *J. Differential Equations*, **259** (2015), 2981–3003. https://doi.org/10.1016/j.jde.2015.04.007
18. T. Liu, Z. Xin, T. Yang, Vacuum states for compressible flow, *Discrete Contin. Dyn. Syst.*, **4** (1998), 1–32. https://doi.org/10.3934/dcds.1998.4.1
19. L. Sundbye, Global existence for Dirichlet problem for the viscous shallow water equation, *J. Math. Anal. Appl.*, **202** (1996), 236–258. https://doi.org/10.1006/jmaa.1996.0315
20. L. Sundbye, Global existence for the Cauchy problem for the viscous shallow water equations, *Rocky Mt. J. Math.*, **28** (1998), 1135–1152. https://doi.org/10.1216/rmjm/1181071760
21. T. Yang, C. Zhu, Compressible Navier–Stokes equations with degnerate viscosity coefficient and vacuum, *Commun. Math. Phys.*, **230** (2002), 329–363. https://doi.org/10.1007/s00220-002-0703-6
22. T. Zhang, D. Fang, Compressible flows with a density-dependent viscosity coefficient, *SIAM. J. Math. Anal.*, **41** (2010), 2453–2488. https://doi.org/10.1137/090758878

23. Y. Geng, Y. Li, S. Zhu, Vanishing viscosity limit of the Navier–Stokes equations to the Euler equations for compressible fluid flow with vacuum, *Arch Rational Mech. Anal.*, **234** (2019), 727–775. https://doi.org/10.1007/s00205-019-01401-9
24. Y. Li, R. Pan, S. Zhu, On classical solutions to 2D Shallow water equations with degenerate viscosities, *J. Math. Fluid Mech.*, **19** (2017), 151–190. https://doi.org/10.1007/s00021-016-0276-3
25. Y. Li, R. Pan, S. Zhu, On classical solutions for viscous polytropic fluids with degenerate viscosities and vacuum, *Arch. Rational. Mech. Anal.*, **234** (2019), 1281–1334. https://doi.org/10.1007/s00205-019-01412-6
26. Z. Xin, S. Zhu, Global well-posedness of regular solutions to the three-dimensional isentropic compressible Navier-Stokes equations with degenerate viscosities and vacuum, *Adv. Math.*, **393** (2021), 108072. https://doi.org/10.1016/j.aim.2021.108072
27. Z. Xin, S. Zhu, Well-posedness of three-dimensional isentropic compressible Navier–Stokes equations with degenerate viscosities and far field vacuum, *J. Math. Pures Appl.*, **152** (2021), 94–144. https://doi.org/10.48550/arXiv.1811.04744
28. S. Zhu, On the breakdown of regular solutions with finite energy for 3D degenerate compressible Navier–Stokes equations, *J. Math. Fluid Mech.*, **23** (2021), 52. https://doi.org/10.48550/arXiv.1911.07965
29. S. Zhu, Existence results for viscous polytropic fluids with degenerate viscosity coefficients and vacuum, *J. Differential Equations*, **259** (2015), 84–119. https://doi.org/10.1016/j.jde.2015.01.048
30. O. Ladyzenskaja, N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, American Mathematical Society, Providence, RI, 1968. https://doi.org/10.1090/mmono/023
31. A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, New York: Spinger-Verlag, 1986. https://doi.org/10.1007/978-1-4612-1116-7