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Research article

On periodic solutions of second-order partial difference equations involving p-Laplacian

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Abstract: By combining variational techniques with the saddle point theorem, we investigate the existence and nonexistence of periodic solutions to second-order partial difference equations involving p-Laplacians. Our obtained results generalize and complement some known ones. Finally, we display some examples and numerical simulations to show the validity of our main results.

Keywords: existence; nonexistence; periodic solution; partial difference equation involving p-Laplacian; variational method
Mathematics Subject Classification: 39A14, 34C37

1. Introduction

Let $\mathbb{Z}$, $\mathbb{N}$, and $\mathbb{R}$ stand for the sets of integers, natural numbers and real numbers, respectively. Consider the existence and nonexistence of periodic solutions of a partial difference equation in the following form:

$$-\triangle_1\left[\phi_p\left(\triangle_1 x(n-1,m)\right)\right]-\triangle_2\left[\phi_p\left(\triangle_2 x(n,m-1)\right)\right]=f\left((n,m),x(n,m)\right),\quad n,m\in\mathbb{Z}. \tag{1.1}$$

Here, $\triangle_i$ $(i=1,2)$ represents the forward difference operator, and $\triangle_1 x(n-1,m)=x(n,m)-x(n-1,m)$, $\triangle_2 x(n,m-1)=x(n,m)-x(n,m-1)$. The p-Laplacian operator is defined as $\phi_p(x)=|x|^{p-2}x$ for $1<p<+\infty$ and $x\in\mathbb{R}$. Given integers $T_1,T_2>0$, $x=\{x(n,m)\}$ is (T_1,T_2)-periodic, which means that $x(n+T_1,m)=x(n,m)=x(n,m+T_2)$ for all $(n,m)\in\mathbb{Z}^2$. The nonlinearity $f\in C(\mathbb{Z}^2\times\mathbb{R},\mathbb{R})$ is T_1-periodic in n and T_2-periodic in m. Denote $F((n,m),x)=\int_0^x f((n,m),s)\,ds$ for all $(n,m)\in\mathbb{Z}^2$.

Owing to both in our real life and scientific research, many phenomena and data are recorded with discrete data; difference equations have a wide range of applications and a long research history in various fields to describe discrete phenomena [1,2]. With the popularization of computers and the rapid development of computer technology, the study of difference equation theory has made great progress in various aspects since Guo and Yu [3] first applied the variational method to difference equations. For example, the authors obtained periodic solutions [4], homoclinic solutions [5] of second-order difference equations, and standing waves solutions [6] for the discrete Schrödinger equations. As to difference equations involving p-Laplacians, here is a list of a few:

$$\triangle(\phi_p(\triangle x_{n-1})) + f(n, x_n) = 0, \qquad n \in \mathbb{Z}, \tag{1.2}$$

where $\triangle x_n = x_{n+1} - x_n$, is a special case of Equation (1.1). Results on periodic solutions and positive solutions of (1.2) were given in [7] and [8], respectively. The authors [9] studied periodic solutions of

$$\triangle(\phi_p(\triangle x_{n-1}) + f(n, x_{n+1}, x_n, x_{n-1}) = 0, \qquad n \in \mathbb{Z}. \tag{1.3}$$

As (1.3) is in higher-order, homoclinic solutions and periodic solutions were displayed in [10] and [11].

Nowadays, more and more phenomena need to be described by two or more multi-variables. Subsequently, both partial differential equations and partial difference equations, containing two or more than two variables, have caught the keen attention of many scholars, and rich results have emerged. Here mention a few; in [12–14], authors obtained a series of results for partial differential equations. Long studied discrete Kirchhoff-type problems and obtained a series of results on multiple solutions [15, 16], least energy solutions [17] and infinitely many large energy solutions [18] (see also [19–21] and reference therein). In [22], the authors gave results on periodic solutions for a second- order difference equation. When partial difference equations contain p-Laplacian, multiple existence results were given in [23]. As to homoclinic solutions, Mei and Zhou [24] gave results for partial difference equations with mixed nonlinearities, and Long [25] considered nonlinear (p, q)-Laplacian partial difference equations with a parameter $\lambda > 0$.

Motivated by the above mentioned results, we deal with periodic solutions of (1.1) by variational techniques together with the saddle point theorem. To demonstrate the validity of our main results, we also present some examples and numerical simulations. Our results generalize and complement some known ones, as detailed in Remark 1.2.

Now we state our main results as follows:

Theorem 1.1. *Assume the following suppositions are fulfilled.*
(A_1) *There exists a constant* $M_0 > 0$ *such that*

$$| f((n,m), x) | \leq M_0, \qquad \forall\ ((n,m), x) \in \mathbb{Z}^2 \times \mathbb{R}.$$

(A_2)

$$\lim_{|x| \to +\infty} F((n,m), x) = +\infty, \qquad \forall (n,m) \in \mathbb{Z}^2.$$

Then Equation (1.1) possesses at least a (T_1, T_2)*-periodic solution.*

Theorem 1.2. *Let* f *satisfy*
(A_3) *there exist positive constants* R_1 *and* α $(\frac{2}{p} < \alpha < 2)$ *such that*

$$0 < xf((n,m), x) \leq \frac{\alpha p}{2} F((n,m), x), \qquad \forall\ (n,m) \in \mathbb{Z}^2 \text{ and } | x | \geq R_1;$$

(A_4) *there exist positive constants* b_1, b_2, *and* β $(\frac{2}{p} < \beta \leq \alpha)$ *such that*

$$F((n,m),x) \geq b_1 \mid x \mid^{\frac{\beta p}{2}} - b_2, \qquad \forall\ ((n,m),x) \in \mathbb{Z}^2 \times \mathbb{R}.$$

Then Equation (1.1) admits at least a (T_1, T_2)*-periodic solution.*

Remark 1.1. *Substitute* (A_3) *by*
(A_3') *there exist constants* $a_1, a_2 > 0$ *such that*

$$F((n,m),x) \leq a_1 \mid x \mid^{\frac{\alpha p}{2}} + a_2, \qquad \forall\ ((n,m),x) \in \mathbb{Z}^2 \times \mathbb{R}.$$

The conclusion of Theorem 1.2 is still valid.

Further, to obtain nontrivial periodic solutions, we have

Theorem 1.3. *Assume the following conditions hold*
(A_5) $F((n,m),0) = 0, \qquad \forall\ (n,m) \in \mathbb{Z}^2$;
(A_6) *there exists a constant* $\frac{2}{p} < \alpha < 2$ *such that*

$$0 < xf((n,m),x) \leq \frac{\alpha p}{2} F((n,m),x), \qquad \forall\ (n,m) \in \mathbb{Z}^2 \text{ and } x \neq 0;$$

(A_7) *there exist constants* $b_3 > 0$ *and* $\frac{2}{p} < \beta \leq \alpha$ *such that*

$$F((n,m),x) \geq b_3 \mid x \mid^{\frac{\beta p}{2}}, \qquad \forall\ ((n,m),x) \in \mathbb{Z}^2 \times \mathbb{R}.$$

Then Equation (1.1) has at least one nontrivial (T_1, T_2)*-periodic solution.*

Theorem 1.4. *Suppose* (A_1), (A_2), *and* (A_5) *hold. Moreover,*
(A_8) *there exist constants* $b_4 > 0$ *and* $0 < \gamma < 2$ *such that*

$$F((n,m),x) \geq b_4 \mid x \mid^{\frac{\gamma p}{2}}, \qquad \forall\ ((n,m),x) \in \mathbb{Z}^2 \times \mathbb{R}.$$

Then Equation (1.1) possesses at least one nontrivial (T_1, T_2)*-periodic solution.*

Theorem 1.5. *If for all* $(n,m) \in \mathbb{Z}^2$ *and* $x \neq 0$, *there holds*

$$xf((n,m),x) < 0.$$

Then Equation (1.1) has no nontrivial (T_1, T_2)*-periodic solution.*

Remark 1.2. *Our Theorems 1.1, 1.2, 1.3 and 1.4 are generalizations of Theorems 1.1, 1.2, 1.3, and 1.4 in [9], respectively. Moreover, Theorem 1.5 supplements the nonexistence of periodic solutions of Equations (1.1) and (1.2).*

The rest of this paper is organized as follows. In Section 2, we establish the variational framework corresponding to Equation (1.1) and give some basic lemmas that play a vital role in proving our main results. Section 3 presents detailed proofs of our main results. Finally, three examples and numerical simulations are provided in Section 4.

2. Preliminaries

For convenience, we give some notations. Denote $\mathbb{Z}(t,s) := \{t, t+1, \cdots, s\}$ with integers $t \leq s$ and $\Omega := \mathbb{Z}(1,T_1) \times \mathbb{Z}(1,T_2)$. Let

$$x = \{x(n,m)\}_{n,m\in\mathbb{Z}} = (\cdots;\cdots, x(1,0), x(2,0), \cdots;\cdots, x(1,1), x(2,1), \cdots;\cdots).$$

Define a T_1T_2-dimensional subspace E of vector space $S = \{x = \{x(n,m)\} | x(n,m) \in \mathbb{R}, n, m \in \mathbb{Z}\}$ by

$$E = \{x = \{x(n,m)\} \in S \,|\, x(n+T_1,m) = x(n,m) = x(n,m+T_2), \quad n,m \in \mathbb{Z}\},$$

which is endowed with the inner product

$$\langle x, y\rangle = \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} x(n,m)y(n,m), \qquad \forall x, y \in E.$$

Thus, the induced norm $\|\cdot\|$ is

$$\|x\| = \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |x(n,m)|^2\right)^{\frac{1}{2}}, \qquad \forall x \in E,$$

and E is isomorphic to $\mathbb{R}^{T_1T_2}$.

Write

$$\|x\|_p = \left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |x(n,m)|^p\right)^{\frac{1}{p}}, \qquad \forall x \in E.$$

It follows that $\|x\|_2 = \|x\|$ and there exist positive constants ζ_p and ξ_p with $\frac{\zeta_p}{\xi_p} = (T_1T_2)^{\frac{-|2-p|}{2p}}$ such that

$$\zeta_p\|x\| \leq \|x\|_p \leq \xi_p\|x\|, \qquad \forall x \in E. \tag{2.1}$$

Further, we have, for any $x \in E$, there exist positive constants C_1, C_2, C_3 such that

$$C_1\|x\|_{\frac{\alpha p}{2}} \leq \|x\| \leq C_2\|x\|_{\frac{\beta p}{2}}, \tag{2.2}$$

$$C_1\|x\|_{\frac{\alpha p}{2}} \leq \|x\| \leq C_3\|x\|_{\frac{\gamma p}{2}}. \tag{2.3}$$

Consider the associated functional $I : E \to \mathbb{R}$ in the form as

$$I(x) = -\frac{1}{p}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[|\triangle_1 x(n-1,m)|^p + |\triangle_2 x(n,m-1)|^p\right] + \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F\left((n,m), x(n,m)\right). \tag{2.4}$$

Then I is C^1. Using periodic conditions, simple calculation yields that

$$\frac{\partial I}{\partial x(n,m)} = \triangle_1\left[\phi_p\left(\triangle_1 x(n-1,m)\right)\right] + \triangle_2\left[\phi_p\left(\triangle_2 x(n,m-1)\right)\right] + f\left((n,m), x(n,m)\right),$$

which means that Equation (1.1) is the corresponding Euler-Lagrange equation for I. Consequently, we transform the problem to find (T_1,T_2)-periodic solutions of Equation (1.1) to the problem to seek critical points of I in E.

Identify $x = \{x(n,m)\}_{n,m\in\mathbb{Z}} \in E$ with

$$x = (x(1,1), \cdots, x(T_1,1); x(1,2), \cdots, x(T_1,2); \cdots; x(1,T_2), \cdots, x(T_1,T_2))^T,$$

and write

$$x' = Dx = (x(1,1), \cdots, x(1,T_2); x(2,1), \cdots, x(2,T_2); \cdots; x(T_1,1), \cdots, x(T_1,T_2))^T\ ,$$

where

$$D = \begin{array}{c} \begin{array}{cccccccccccccc} & & & T_1 & & & & 2T_1 & & (T_2-1)T_1+1 & & & & \end{array} \\ \left(\begin{array}{ccccccccccccc} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1 \end{array}\right) \begin{array}{l} \\ \\ \\ T_2 \\ \\ \\ \\ 2T_2 \\ \\ (T_1-1)T_2+1. \\ \\ \\ \\ \end{array} \end{array}$$

Then $\|x\|_s = \|x'\|_s$ for all $s > 1$.

Let

$$A_{kl} = \begin{pmatrix} B_k & & & 0 \\ & B_k & & \\ & & \ddots & \\ 0 & & & B_k \end{pmatrix}_{kl\times kl}$$

with

$$B_k = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{k\times k}.$$

By [9], the eigenvalues of matrix $A_{T_1T_2}$ are $\lambda_i = 2(1 - \cos\frac{2i\pi}{T_1})$, $i = 0, 1, 2, \cdots, T_1 - 1$. Thus $\lambda_0 = 0$ and $\lambda_i > 0$ for $1 \le i \le T_1 - 1$. Further, each λ_i is T_2-multiple and

$$\begin{cases} \underline{\lambda} = \min\{\lambda_1, \lambda_2, \cdots, \lambda_{T_1-1}\} = 4\sin^2\frac{\pi}{T_1}, \\ \overline{\lambda} = \max\{\lambda_1, \lambda_2, \cdots, \lambda_{T_1-1}\} = 4\cos^2\frac{1-(-1)^{T_1}}{4T_1}\pi. \end{cases} \tag{2.5}$$

Similarly, $A_{T_2T_1}$ has eigenvalues μ_j $(0 \le j \le T_2 - 1)$ and

$$\begin{cases}\underline{\mu} = \min\{\mu_1, \mu_2, \cdots, \mu_{T_2-1}\} = 4\sin^2\frac{\pi}{T_2},\\ \overline{\mu} = \max\{\mu_1, \mu_2, \cdots, \mu_{T_2-1}\} = 4\cos^2\frac{1-(-1)^{T_2}}{4T_2}\pi.\end{cases} \tag{2.6}$$

We split E as $E = V \oplus Y$ with $Y = \{y \in E | y = \{c, c, \cdots, c\}, c \in \mathbb{R}\}$. It follows that

$$\begin{aligned}&\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |\triangle_1 x(n-1, m)|^p \le \xi_p^p \overline{\lambda}^{\frac{p}{2}} \|x\|^p, \quad \forall x \in E,\\ &\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |\triangle_2 x(n, m-1)|^p \le \xi_p^p \overline{\mu}^{\frac{p}{2}} \|x'\|^p = \xi_p^p \overline{\mu}^{\frac{p}{2}} \|x\|^p, \quad \forall x \in E,\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}&\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |\triangle_1 x(n-1, m)|^p \ge \zeta_p^p \underline{\lambda}^{\frac{p}{2}} \|x\|^p, \quad \forall x \in V,\\ &\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |\triangle_2 x(n, m-1)|^p \ge \zeta_p^p \underline{\mu}^{\frac{p}{2}} \|x'\|^p = \zeta_p^p \underline{\mu}^{\frac{p}{2}} \|x\|^p, \quad \forall x \in V.\end{aligned} \tag{2.8}$$

Now, we state some basic definitions. Let X be a real Banach space. $I \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale ($P.S.$ for short) condition, which states that any sequence $\{x_n\} \subset X$ such that $\{I(x_n)\}$ is bounded and $\lim\limits_{n\to\infty} I'(x_n) \to 0$ possesses a convergent subsequence.

We denote by B_ρ, the open ball with center 0 and radius ρ in X, and ∂B_ρ its boundary. Recall the Saddle Point Theorem, introduced in [26], which plays a crucial role in proofs of our main results.

Lemma 2.1. *(Saddle Point Theorem [26]) Let $X = X_1 \oplus X_2$ be a real Banach space with finite-dimensional subspace $X_1 \neq \{0\}$. Suppose $I \in C^1(X, \mathbb{R})$ fulfills the $P.S.$ condition and*
(J_1) $I|_{\partial B_\rho \cap X_1} \le \sigma$ for constants σ and $\rho > 0$;
(J_2) $I|_{e+X_2} \ge \omega$ for constants $e \in B_\rho \cap X_1$ and $\omega > \sigma$.
Then I admits a critical value $c \ge \omega$ with

$$c = \inf_{h\in\Gamma}\max_{x\in B_\rho\cap X_1} I(h(x)) \quad \textit{and} \quad \Gamma = \left\{h \in C(\bar{B}_\rho \cap X_1, X) \mid h|_{\partial B_\rho \cap X_1} = id\right\}.$$

3. Proofs of main results

In this section, we present detailed proofs of our main results.

Proof of Theorem 1.1 We complete the proof by Lemma 2.1 in three steps.

Step 1 I satisfies the $P.S.$ condition on E.

Assume that $\{x_k\} \subset E$ is a $P.S.$ sequence, that is, $\lim\limits_{k\to\infty} I'(x_k) = 0$ and there exists a constant $M_1 > 0$ such that $|I(x_k)| \le M_1$. Then for k large enough and any $x \in E$, we have

$$\langle I'(x_k), x\rangle \ge -\|x\|. \tag{3.1}$$

Take $x_k = v_k + y_k \in V \oplus Y$, it follows that

$$
\begin{aligned}
&\langle I'(x_k), v_k\rangle \\
=&\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left\{\triangle_1\left[\phi_p\left(\triangle_1 x_k(n-1,m)\right)\right] + \triangle_2\left[\phi_p\left(\triangle_2 x_k(n,m-1)\right)\right] + f\left((n,m), x_k(n,m)\right)\right\}\cdot v_k(n,m) \\
=&\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[\phi_p\left(\triangle_1 v_k(n,m)\right) - \phi_p\left(\triangle_1 v_k(n-1,m)\right)\right]\cdot v_k(n,m) \\
&+\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[\phi_p\left(\triangle_2 v_k(n,m)\right) - \phi_p\left(\triangle_2 v_k(n,m-1)\right)\right]\cdot v_k(n,m) + \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f\left((n,m), x_k(n,m)\right)\cdot v_k(n,m) \\
=&\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[\phi_p\left(\triangle_1 v_k(n-1,m)\right)\cdot v_k(n-1,m) - \phi_p\left(\triangle_1 v_k(n-1,m)\right)\cdot v_k(n,m)\right] \\
&+\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[\phi_p\left(\triangle_2 v_k(n,m-1)\right)\cdot v_k(n,m-1) - \phi_p\left(\triangle_2 v_k(n,m-1)\right)\cdot v_k(n,m)\right] \\
&+\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f\left((n,m), x_k(n,m)\right)\cdot v_k(n,m) \\
=&-\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[\phi_p\left(\triangle_1 v_k(n-1,m)\right)\cdot \triangle_1 v_k(n-1,m) + \phi_p\left(\triangle_2 v_k(n,m-1)\right)\cdot \triangle_2 v_k(n,m-1)\right] \\
&+\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f\left((n,m), x_k(n,m)\right)\cdot v_k(n,m) \\
=&-\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[|\triangle_1 v_k(n-1,m)|^p + |\triangle_2 v_k(n,m-1)|^p\right] + \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f\left((n,m), x_k(n,m)\right)\cdot v_k(n,m).
\end{aligned}
$$

Together with (A_1) and (3.1), we deduce that

$$
\begin{aligned}
&\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[|\triangle_1 v_k(n-1,m)|^p + |\triangle_2 v_k(n,m-1)|^p\right] \\
\leq&\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[f\left((n,m), x_k(n,m)\right)\cdot v_k(n,m)\right] + \|v_k\| \\
\leq& M_0\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} \mid v_k(n,m)\mid + \|v_k\| \\
\leq&(M_0\sqrt{T_1T_2}+1)\|v_k\|.
\end{aligned}
\tag{3.2}
$$

By (2.8), we have

$$
\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[|\triangle_1 v_k(n-1,m)|^p + |\triangle_2 v_k(n,m-1)|^p\right] \geq \zeta_p^p\left(\underline{\lambda}^{\frac{p}{2}} + \underline{\mu}^{\frac{p}{2}}\right)\|v_k\|^p. \tag{3.3}
$$

Thus, combining (3.2) with (3.3), we obtain

$$\zeta_p^p\left(\underline{\lambda}^{\frac{p}{2}}+\underline{\mu}^{\frac{p}{2}}\right)\|v_k\|^p \leq (M_0\sqrt{T_1T_2}+1)\|v_k\|. \tag{3.4}$$

Since $p>1$, (3.4) ensures that $\|v_k\|$ has a maximum value. Thus, $\{v_k\}$ is a bounded sequence.

Next, we show that $\{y_k\}$ is also a bounded sequence. Owing to (A_1), (2.7) and

$$\begin{aligned}M_1 \geq & I(x_k)=-\frac{1}{p}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[|\triangle_1 x_k(n-1,m)|^p+|\triangle_2 x_k(n,m-1)|^p]+\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}F((n,m),x_k(n,m))\\ =&-\frac{1}{p}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[|\triangle_1 v_k(n-1,m)|^p+|\triangle_2 v_k(n,m-1)|^p]+\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}F((n,m),y_k(n,m))\\ &+\sum_{n=1}^{T}\sum_{m=1}^{T_1}\left[F((n,m),x_k(n,m))-F((n,m),y_k(n,m))\right],\end{aligned}$$

we attain that, for $\theta\in(0,1)$, there holds

$$\begin{aligned}&\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}F((n,m),y_k(n,m))\\ \leq & M_1+\frac{1}{p}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[|\triangle_1 v_k(n-1,m)|^p+|\triangle_2 v_k(n,m-1)|^p]\\ &+\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\mid F((n,m),x_k(n,m))-F((n,m),y_k(n,m))\mid\\ \leq & M_1+\frac{\xi_p^p}{p}\left(\overline{\lambda}^{\frac{p}{2}}+\overline{\mu}^{\frac{p}{2}}\right)\|v_k\|^p+\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\mid f((n,m),(y_k+\theta v_k)(n,m))\mid\cdot\mid v_k(n,m)\mid\\ \leq & M_1+\frac{\xi_p^p}{p}\left(\overline{\lambda}^{\frac{p}{2}}+\overline{\mu}^{\frac{p}{2}}\right)\|v_k\|^p+M_0\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\mid v_k(n,m)\mid\\ \leq & M_1+\frac{\xi_p^p}{p}\left(\overline{\lambda}^{\frac{p}{2}}+\overline{\mu}^{\frac{p}{2}}\right)\|v_k\|^p+M_0\sqrt{T_1T_2}\|v_k\|.\end{aligned}$$

Notice that $\{v_k\}$ is bounded, then $\left\{\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}F((n,m),y_k(n,m))\right\}$ is bounded. We claim that $\{y_k\}$ is bounded. Otherwise, we assume that $\lim_{k\to\infty}\|y_k\|=\infty$. Let $y_k=(c_k,c_k,\cdots,c_k)^T\in Y$ where $c_k\in\mathbb{R}$, $k\in\mathbb{N}$, then

$$\|y_k\|=\left(\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\mid c_k\mid^2\right)^{\frac{1}{2}}=\sqrt{T_1T_2}\mid c_k\mid\to+\infty \quad \text{as} \quad k\to+\infty.$$

In view of (A_2),

$$F((n,m),y_k(n,m))=F((n,m),c_k)\to+\infty \text{ as } k\to\infty.$$

Thus, $\left\{\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}F((n,m),y_k(n,m))\right\}\to+\infty$, which is a contradiction. Therefore, $\{y_k\}$ is bounded. Consequently, $\{x_k\}\subset E$ is a bounded sequence on the finite-dimensional space E, and the $P.S.$ condition is verified.

Step 2 (J_1) of Lemma 2.1 is fulfilled.

From (A_1), there exists a constant $M_0' > 0$ such that

$$| F((n,m),z) |\leq M_0 | z | + M_0', \qquad \forall\ ((n,m),z) \in \mathbb{Z}^2 \times \mathbb{R}.$$

Utilizing (2.8), for any $v \in V$, it follows that

$$\begin{aligned} I(v) =& -\frac{1}{p}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[|\triangle_1 v(n-1,m)|^p + |\triangle_2 v(n,m-1)|^p] + \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m),v(n,m)) \\ \leq& -\frac{\zeta_p^p}{p}\left(\underline{\lambda}^{\frac{p}{2}} + \underline{\mu}^{\frac{p}{2}}\right)\|v\|^p + M_0\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} | v(n,m) | + M_0' T_1 T_2 \\ \leq& -\frac{\zeta_p^p}{p}\left(\underline{\lambda}^{\frac{p}{2}} + \underline{\mu}^{\frac{p}{2}}\right)\|v\|^p + M_0\sqrt{T_1 T_2}\|v\| + M_0' T_1 T_2 \\ \to& -\infty, \qquad\qquad \text{as} \quad \|v\| \to +\infty. \end{aligned}$$

Therefore, (J_1) holds.

Step 3 (J_2) of Lemma 2.1 is satisfied.

For any $y \in Y$, there is $c \in \mathbb{R}$ such that

$$y = (c, c, \cdots, c)^T.$$

Taking account of (A_2), one gets that there exists a constant $R_0 > 0$ such that $F((n,m),x) > 0$ for any $(n,m) \in \mathbb{Z}^2$ and $| x |> R_0$. Let

$$M_2 = \min_{|x|\leq R_0} F((n,m),x), \qquad\qquad M_2' = \min\{0, M_2\}.$$

Then

$$F((n,m),x) \geq M_2', \qquad \forall\ ((n,m),x) \in \mathbb{Z}^2 \times \mathbb{R}.$$

Hence, we have

$$I(y) = \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m),c) \geq M_2' T_1 T_2, \qquad \forall y \in Y,$$

which indicates that I satisfies (J_2) of Lemma 2.1 with $e = 0$. Thus, the desired result follows.

Proof of Theorem 1.2 To prove Theorem 1.2 by Lemma 2.1, it is necessary to verify that I satisfies the $P.S.$ condition on E and the geometry conditions (J_1) and (J_2) of Lemma 2.1.

First, we testify that the $P.S.$ condition is satisfied. Suppose $\{x_k\} \subset E$ and there is a constant $M_3 > 0$ such that

$$\lim_{k\to\infty} I'(x_k) = 0 \qquad\qquad \text{and} \qquad | I(x_k) |\leq M_3, \quad \forall k \in \mathbb{N}. \tag{3.5}$$

Then for k is large enough, there holds

$$| \langle I'(x_k), x_k\rangle |\leq \|x_k\|. \tag{3.6}$$

Recall

$$\langle I'(x_k), x_k\rangle = -\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[|\triangle_1 x_k(n-1,m)|^p + |\triangle_2 x_k(n,m-1)|^p] + \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f((n,m), x_k(n,m))\cdot x_k(n,m).$$

In combination with (3.5) and (3.6), we have

$$\begin{aligned} M_3 + \frac{1}{p}\|x_k\| &\geq I(x_k) - \frac{1}{p}\langle I'(x_k), x_k\rangle \\ &= \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[F((n,m), x_k(n,m)) - \frac{1}{p} f((n,m), x_k(n,m))\cdot x_k(n,m)\right]. \end{aligned} \tag{3.7}$$

Write $\Omega = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = \{(n,m)\in\Omega \mid |x_k(n,m)| \geq R_1\}, \qquad \Omega_2 = \{(n,m)\in\Omega \mid |x_k(n,m)| < R_1\}. \tag{3.8}$$

Then (3.7) and (A_3) imply that

$$\begin{aligned} M_3 + \frac{1}{p}\|x_k\| \geq & \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m), x_k(n,m)) - \frac{1}{p}\sum_{(n,m)\in\Omega_1} f((n,m), x_k(n,m))\cdot x_k(n,m) \\ & - \frac{1}{p}\sum_{(n,m)\in\Omega_2} f((n,m), x_k(n,m))\cdot x_k(n,m) \\ \geq & \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m), x_k(n,m)) - \frac{\alpha}{2}\sum_{(n,m)\in\Omega_1} F((n,m), x_k(n,m)) \\ & - \frac{1}{p}\sum_{(n,m)\in\Omega_2} f((n,m), x_k(n,m))\cdot x_k(n,m) \\ = & (1-\frac{\alpha}{2})\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m), x_k(n,m)) \\ & + \frac{1}{p}\sum_{(n,m)\in\Omega_2}\left[\frac{\alpha}{2} pF((n,m), x_k(n,m)) - f((n,m), x_k(n,m))\cdot x_k(n,m)\right]. \end{aligned}$$

Moreover, $\frac{\alpha}{2} pF((n,m), z) - f((n,m), z)\cdot z$ is continuous with respect to z, which means that there is a constant $M_4 > 0$ such that

$$\frac{\alpha}{2} pF((n,m), z) - f((n,m), z)\cdot z \geq -M_4, \qquad \forall (n,m)\in\mathbb{Z}^2 \quad \text{and} \quad |z| \leq R_1.$$

Thus,

$$M_3 + \frac{1}{p}\|x_k\| \geq (1-\frac{\alpha}{2})\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m), x_k(n,m)) - \frac{1}{p} T_1 T_2 M_4. \tag{3.9}$$

By (A_4) and (3.9), we achieve

$$M_3 + \frac{1}{p}\|x_k\| \geq (1-\frac{\alpha}{2})b_1 \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} \mid x_k(n,m) \mid^{\frac{\beta p}{2}} -(1-\frac{\alpha}{2})b_2T_1T_2 - \frac{1}{p}T_1T_2M_4$$
$$= (1-\frac{\alpha}{2})b_1 \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} \mid x_k(n,m) \mid^{\frac{\beta p}{2}} -M_5,$$

where $M_5 = (1-\frac{\alpha}{2})b_2T_1T_2 + \frac{1}{p}T_1T_2M_4$. Joint with (2.3), we obtain

$$M_3 + \frac{1}{p}\|x_k\| \geq (1-\frac{\alpha}{2})b_1 C_2^{\frac{\beta p}{2}}\|x_k\|^{\frac{\beta p}{2}} - M_5,$$

that is,

$$(1-\frac{\alpha}{2})b_1 C_2^{\frac{\beta p}{2}}\|x_k\|^{\frac{\beta p}{2}} - \frac{1}{p}\|x_k\| \leq M_3 + M_5. \tag{3.10}$$

Remind $\frac{2}{p} < \beta \leq \alpha < 2$, (3.10) guarantees $\{x_k\}$ is bounded. Since E is finite dimensional, the $P.S.$ condition is satisfied.

Second, we complete the proof by verifying that I satisfies the geometry conditions (J_1) and (J_2) of Lemma 2.1. For any $y = (c, c, \cdots, c) \in Y$ with $c \in \mathbb{R}$, (A_4) means that

$$I(y) = \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m), y(n,m)) \geq b_1 \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} \mid y(n,m) \mid^{\frac{\beta p}{2}} -b_2T_1T_2$$
$$= b_1T_1T_2 \mid c \mid^{\frac{\beta p}{2}} -b_2T_1T_2 =: \omega_0.$$

For any $v \in V$, (A_3'), (2.2) and (2.8) yield

$$\begin{aligned} I(v) &= -\frac{1}{p}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[|\triangle_1 v(n-1,m)|^p + |\triangle_2 v(n,m-1)|^p] + \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m), v(n,m)) \\ &\leq -\frac{\zeta_p^p}{p}\left(\underline{\lambda}^{\frac{p}{2}} + \underline{\mu}^{\frac{p}{2}}\right)\|v\|^p + a_1\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} \mid v(n,m) \mid^{\frac{\alpha p}{2}} +a_2T_1T_2 \\ &\leq -\frac{\zeta_p^p}{p}\left(\underline{\lambda}^{\frac{p}{2}} + \underline{\mu}^{\frac{p}{2}}\right)\|v\|^p + a_1C_1^{\frac{\alpha p}{2}}\|v\|^{\frac{\alpha p}{2}} + a_2T_1T_2. \end{aligned} \tag{3.11}$$

Notice that $\frac{2}{p} < \alpha < 2$, (3.11) indicates that there is a constant $\rho_0 > 0$ large enough such that

$$I(v) \leq \omega_0 - 1 < \omega_0, \qquad \forall v \in V, \quad \|v\| = \rho_0.$$

Thus, both (J_1) and (J_2) are satisfied. Therefore, Lemma 2.1 ensures that Equation (1.1) possesses at least a (T_1, T_2)-periodic solution. The proof is completed.

Proof of Theorem 1.3 To obtain nontrivial solutions, we divide the proof of Theorem 1.3 in four steps.

Step 1 I satisfies the $P.S.$ condition on E.

Let sequence $\{x_k\} \subset E$ such that

$$\lim_{k\to\infty} I'(x_k) = 0, \qquad\qquad | I(x_k) | \le M_6, \quad \forall k \in \mathbb{N},$$

where $M_6 > 0$ is a constant. For k large enough, one obtains

$$| \langle I'(x_k), x_k\rangle | \le \|x_k\|.$$

Moreover,

$$\begin{aligned}\langle I'(x_k), x_k\rangle = &- \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} [|\triangle_1 x_k(n-1,m)|^p + |\triangle_2 x_k(n,m-1)|^p]\\ &+ \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} f((n,m), x_k(n,m)) \cdot x_k(n,m).\end{aligned}$$

Together with (2.3), (A_6) and (A_7), it follows that

$$\begin{aligned}M_6 + \frac{1}{p}\|x_k\| &\ge I(x_k) - \frac{1}{p}\langle I'(x_k), x_k\rangle\\ &= \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[F((n,m), x_k(n,m)) - \frac{1}{p} f((n,m), x_k(n,m)) \cdot x_k(n,m)\right]\\ &\ge \sum_{n=1}^{T_1}\sum_{m=1}^{T_2}\left[F((n,m), x_k(n,m)) - \frac{\alpha}{2} F((n,m), x_k(n,m))\right]\\ &= (1-\frac{\alpha}{2})\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m), x_k(n,m))\\ &\ge (1-\frac{\alpha}{2}) b_3 \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} | x_k(n,m) |^{\frac{\beta p}{2}}\\ &\ge (1-\frac{\alpha}{2}) b_3 C_2^{\frac{\beta p}{2}} \|x_k\|^{\frac{\beta p}{2}}.\end{aligned}$$

Namely,

$$(1-\frac{\alpha}{2}) b_3 C_2^{\frac{\beta p}{2}} \|x_k\|^{\frac{\beta p}{2}} - \frac{1}{p}\|x_k\| \le M_6. \tag{3.12}$$

Recall $\frac{2}{p} < \beta \le \alpha < 2$, (3.12) implies that $\{x_k\}$ is a bounded sequence. Due to the fact that E is a finite-dimensional space, then I satisfies the $P.S.$ condition.

Step 2 I meets (J_1) of Lemma 2.1.

For any $v \in V$, (3.11) gives

$$I(v) \le -\frac{\zeta_p^p}{p}\left(\underline{\lambda}^{\frac{p}{2}} + \underline{\mu}^{\frac{p}{2}}\right)\|v\|^p + a_1 C_1^{\frac{\alpha p}{2}}\|v\|^{\frac{\alpha p}{2}} + a_2 T_1 T_2 \ \to -\infty \text{ as } \|v\| \to +\infty.$$

Therefore, (J_1) of Lemma 2.1 is satisfied.

Step 3 I fulfills (J_2) of Lemma 2.1.

Given $x = v_0 + y$ with $v_0 \in V$ and $y \in Y$, from (A_7), (2.3), and (2.7), we have that

$$\begin{aligned} I(x) &= -\frac{1}{p}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[|\triangle_1 x(n-1,m)|^p + |\triangle_2 x(n,m-1)|^p] + \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m),x(n,m)) \\ &\geq -\frac{\xi_p^p}{p}\left(\overline{\lambda}^{\frac{p}{2}} + \overline{\mu}^{\frac{p}{2}}\right)\|v_0\|^p + b_3\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |\ (v_0+y)(n,m)\ |^{\frac{\beta p}{2}} \\ &\geq -\frac{\xi_p^p}{p}\left(\overline{\lambda}^{\frac{p}{2}} + \overline{\mu}^{\frac{p}{2}}\right)\|v_0\|^p + b_3 C_2^{\frac{\beta p}{2}}\|v_0\|^{\frac{\beta p}{2}} + b_3 C_2^{\frac{\beta p}{2}}\|y\|^{\frac{\beta p}{2}}, \end{aligned}$$

which means that there is a sufficiently small positive constant δ_1 satisfying

$$I(v_0+y) \geq \delta_1^{\frac{\beta p}{2}}\left(b_3 C_2^{\frac{\beta p}{2}} - \frac{\xi_p^p}{p}\left(\overline{\lambda}^{\frac{p}{2}} + \overline{\mu}^{\frac{p}{2}}\right)\delta_1^{p-\frac{\beta p}{2}}\right) := \omega_1 > 0,$$

for $v_0 \in \partial B_{\delta_1} \cap V$ and $y \in Y$. Then (J_2) is valid.

Step 4 I has a nontrivial critical point.

Applying the saddle point theorem, we find a critical value $c \geq \omega_1 > 0$ of I. Let $\bar{x} \in E$ be the corresponding critical point, that is,

$$I(\bar{x}) = c \geq \omega_1 > 0. \tag{3.13}$$

Further, $\bar{x}$ is a nontrivial critical point, that is, $\bar{x} \neq 0$. Or else, if $\bar{x} = 0$, by (A_5), we have

$$I(\bar{x}) = \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m),0) = 0.$$

This contradicts (3.13). Hence, $\bar{x} \neq 0$ and the proof is completed.

Proof of Theorem 1.4 From the proof of Theorem 1.1, we know that (A_1) and (A_2) ensure that the $P.S.$ condition and (J_1) of Lemma 2.1 are valid. Then we only need to prove that (J_2) of Lemma 2.1 also holds.

Taking $x = v_0 + y$, where $v_0 \in V$ and $y \in Y$, by (A_8), (2.3) and (2.8), we obtain

$$\begin{aligned} I(x) &= -\frac{1}{p}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[|\triangle_1 x(n-1,m)|^p + |\triangle_2 x(n,m-1)|^p] + \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m),x(n,m)) \\ &= -\frac{1}{p}\sum_{n=1}^{T_1}\sum_{m=1}^{T_2}[|\triangle_1 v_0(n-1,m)|^p + |\triangle_2 v_0(n,m-1)|^p] + \sum_{n=1}^{T_1}\sum_{m=1}^{T_2} F((n,m),(v_0+y)(n,m)) \\ &\geq -\frac{\xi_p^p}{p}\left(\overline{\lambda}^{\frac{p}{2}} + \overline{\mu}^{\frac{p}{2}}\right)\|v_0\|^p + b_4\sum_{n=1}^{T_1}\sum_{m=1}^{T_2} |\ (v_0+y)(n,m)\ |^{\frac{\gamma p}{2}} \\ &\geq -\frac{\xi_p^p}{p}\left(\overline{\lambda}^{\frac{p}{2}} + \overline{\mu}^{\frac{p}{2}}\right)\|v_0\|^p + b_4 C_3^{\frac{\gamma p}{2}}\|v_0\|^{\frac{\gamma p}{2}} + b_4 C_3^{\frac{\gamma p}{2}}\|y\|^{\frac{\gamma p}{2}}, \end{aligned}$$

which means that, for $v_0 \in \partial B_{\delta_2} \cap V$ and any $y \in Y$, there exists a sufficiently small constant $\delta_2 > 0$ such that

$$I(v_0+y) \geq \delta_2^{\frac{\gamma p}{2}}\left(b_4 C_3^{\frac{\gamma p}{2}} - \frac{\xi_p^p}{p}\left(\overline{\lambda}^{\frac{p}{2}} + \overline{\mu}^{\frac{p}{2}}\right)\delta_2^{p-\frac{\gamma p}{2}}\right) := \omega_2 > 0.$$

Thus (J_2) is verified. Therefore, Lemma 2.1 means that I admits a critical value $c \geq \omega_2 > 0$. Denote the corresponding critical point by $\bar{x}$, that is, $I(\bar{x}) = c > 0$. With (A_5), we get

$$I(0) = \sum_{n=1}^{T_1} \sum_{m=1}^{T_2} F((n,m),0) = 0,$$

which implies that $\bar{x} \neq 0$. Thus our proof is done.

Proof of Theorem 1.5 For the sake of contradiction, we assume that x^* is a nontrivial (T_1, T_2)-periodic solution of Equation (1.1), which is equivalent to the fact that x^* is a nontrivial critical point of I on E. Hence, $I'(x^*) = 0$ with $x^* \neq 0$. Direct computation gives that

$$\begin{aligned}\langle I'(x), x\rangle = &- \sum_{n=1}^{T_1} \sum_{m=1}^{T_2} [|\triangle_1 x(n-1,m)|^p + |\triangle_2 x(n,m-1)|^p] \\ &+ \sum_{n=1}^{T_1} \sum_{m=1}^{T_2} f\left((n,m), x(n,m)\right) \cdot x(n,m).\end{aligned}$$

Therefore,

$$\sum_{n=1}^{T_1} \sum_{m=1}^{T_2} f\left((n,m), x^*(n,m)\right) \cdot x^*(n,m) = \sum_{n=1}^{T_1} \sum_{m=1}^{T_2} [|\triangle_1 x^*(n-1,m)|^p + |\triangle_2 x^*(n,m-1)|^p] \geq 0. \tag{3.14}$$

On the other hand, (A_9) gives

$$\sum_{n=1}^{T_1} \sum_{m=1}^{T_2} f\left((n,m), x^*(n,m)\right) \cdot x^*(n,m) < 0.$$

This is in conflict with (3.14). Consequently, the proof is finished.

4. Examples

We give three examples to illustrate applications of our main results. Write

$$\bar{E} = \{x = \{x(n,m)\} \in S \,|\, x(n+2,m) = x(n,m) = x(n,m+2), \quad n,m \in \mathbb{Z}\}.$$

To facilitate the presentation of numerical simulations, we abbreviate $x \in \bar{E}$ as

$$x = (x(1,1), x(2,1), x(1,2), x(2,2)).$$

Example 4.1. *Take* $T_1 = T_2 = 2$ *and* $p = 3$. *Consider*

$$\triangle_1 \left[\phi_3 \left(\triangle_1 x(n-1,m)\right)\right] + \triangle_2 \left[\phi_3 \left(\triangle_2 x(n,m-1)\right)\right] + \frac{16x(n,m)}{1+x^2(n,m)} = 0, \quad n,m \in \mathbb{Z}. \tag{4.1}$$

Here

$$f((n,m),x) = \frac{16x}{1+x^2}, \qquad x \in \mathbb{R}.$$

By integration, it yields

$$F((n,m),x) = 8\ln(1+x^2), \qquad x \in \mathbb{R}.$$

Direct calculations give

$$| f((n,m),x) |=| \frac{16x}{1+x^2} |\leq 8, \qquad \lim_{|x|\to+\infty} F((n,m),x) = +\infty.$$

Thus, Equation (4.1) fulfills all the assumptions of Theorem 1.1, which guarantees that Equation (4.1) has at least a $(2,2)$-periodic solution. Use Matlab; a solution $x \in \bar{E}$ of Equation (4.1) is presented as

$$x = (1, -1, 1, -1).$$

Example 4.2. *Take* $p = 3$ *and* $T_1 = T_2 = 2$. *Consider*

$$\triangle_1 [\phi_3 (\triangle_1 x(n-1,m))] + \triangle_2 [\phi_3 (\triangle_2 x(n,m-1))] + 4x(n,m) = 0, \quad n,m \in \mathbb{Z}. \tag{4.2}$$

Here

$$f((n,m),x) = 4x, \qquad x \in \mathbb{R},$$

then

$$F((n,m),x) = 2x^2, \qquad x \in \mathbb{R}.$$

Take $\alpha = \frac{4}{3}$ and $\beta = \frac{4}{3}$, then (A_3) and (A_4) of Theorem 1.2 hold. Thus, Equation (4.2) has at least a $(2,2)$-periodic solution.

Further, we know that $F((n,m),0) = 0$ for any $n,m \in \mathbb{Z}$. Thus, (A_5), (A_6), and (A_7) of Theorem 1.3 are also true. Thereby, it can be further confirmed that Equation (4.2) holds at least one nontrivial $(2,2)$-periodic solution. We list a solution $x \in \bar{E}$ of Equation (4.2) as follows:

$$x = (\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}).$$

Example 4.3. *Consider Equation (1.1) with* $f((n,m),x) = -x$. *Then Equation (1.1) turns into*

$$\triangle_1 [\phi_3 (\triangle_1 x(n-1,m))] + \triangle_2 [\phi_3 (\triangle_2 x(n,m-1))] - x(n,m) = 0. \tag{4.3}$$

It is clear that the condition (A_9) of Theorem 1.5 is valid. Therefore, Equation (4.3) has no nontrivial (T_1, T_2)-periodic solution.

Author contributions

Dan Li: Writing-original draft, formal analysis; Yuhua Long: Methodology, writing-review editing.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. Y. C. Zhou, H. Cao, Y. N. Xiao, *Difference Equations and Their Applications*, Beijing, Science Press, 2014.
2. Y. H. Long, X. F. Pang, Q. Q. Zhang, Codimension-one and codimension-two bifurcations of a discrete Leslie-Gower type predator-prey model, *Discrete Contin. Dyn. Syst. Ser. B*, **30** (2025), 1357–1389. https://doi.org/10.3934/dcdsb.2024132
3. Z. M. Guo, J. S. Yu, The existence of periodic and subharmonic solutions of subquadratic second order difference equations, *J. London Math. Soc.*, **68** (2003), 419–430. https://doi.org/10.1112/S0024610703004563
4. Z. M. Guo, J. S. Yu, Existence of periodic and subharmonic solutions for second-order superlinear difference equations, *Sci. China Ser. A-Math.*, **46** (2003), 506–515. https://doi.org/10.1007/BF02884022
5. M. J. Ma, Z. M. Guo, Homoclinic orbits and subharmonics for nonlinear second order difference equations, *Nonlinear Anal.*, **67** (2007), 1737–1745. https://doi.org/10.1016/j.na.2006.08.014
6. Z. G. Wang, Q. Y. Li, Standing waves solutions for the discrete Schrödinger equations with resonance, *Bull. Malays. Math. Sci. Soc.*, **46** (2023), 171. https://doi.org/10.1007/s40840-023-01530-1
7. T. S. He, W. G. Chen, Periodic solutions of second order discrete convex systems involving the p-Laplacian, *Appl. Math. Comput.*, **206** (2008), 124–132. https://doi.org/10.1016/j.amc.2008.08.037
8. Z. M. He, On the existence of positive solutions of p-Laplacian difference equations, *J. Comput. Appl. Math.*, **161** (2003), 193–201. https://doi.org/10.1016/j.cam.2003.08.004
9. X. Liu, H. P. Shi, Y. B. Zhang, Existence of periodic solutions of second order nonlinear p-Laplacian difference equations, *Acta Math. Hungar.*, **133** (2011), 148–165. https://doi.org/10.1007/s10474-011-0137-8
10. J. H. Kuang, Existence of homoclinic solutions for higher-order periodic difference equations with p-Laplacian, *J. Math. Anal. Appl.*, **417** (2014), 904–917. https://doi.org/10.1016/j.jmaa.2014.03.077
11. P. Mei, Z. Zhou, Periodic and subharmonic solutions for a 2nth-order p-Laplacian difference equation containing both advances and retardations, *Open Math.*, **16** (2018), 1435–1444. https://doi.org/10.1515/math-2018-0123
12. Q. Li, V. D. Radulescu, W. Zhang, Normalized ground states for the Sobolev critical Schrödinger equation with at least mass critical growth, *Nonlinearity*, **37** (2024), 025018. https://doi.org/10.1088/1361-6544/ad1b8b
13. N. S. Papageorgiou, J. Zhang, W. Zhang, Solutions with sign information for noncoercive double phase equations, *J. Geom. Anal.*, **34** (2024), 14. https://doi.org/10.1007/s12220-023-01463-y

14. D. Qin, X. Tang, J. Zhang, Ground states for planar Hamiltonian elliptic systems with critical exponential growth, *J. Differential Equations*, **308** (2022), 130–159. https://doi.org/10.1016/j.jde.2021.10.063
15. Y. H. Long, Nontrivial solutions of discrete Kirchhoff type problems via Morse theory, *Adv. Nonlinear Anal.*, **11** (2022), 1352–1364. https://doi.org/10.1515/anona-2022-0251
16. Y. H. Long, Multiple results on nontrivial solutions of discrete Kirchhoff type problems, *J. Appl. Math. Comput.*, **69** (2023), 1–17. https://doi.org/10.1007/s12190-022-01731-0
17. Y. H. Long, Least energy sign-changing solutions for discrete Kirchhoff-type problems, *Appl. Math. Lett.*, **150** (2024), 108968. https://doi.org/10.1016/j.aml.2023.108968
18. Y. H. Long, Q. Q. Zhang, Infinitely many large energy solutions to a class of nonlocal discrete elliptic boundary value problems, *Comm. Pure Appl. Math.*, **22** (2023), 1545–1564. https://doi.org/10.3934/cpaa.2023037
19. Y. Bo, D. Tian, X. Liu, Y. F. Jin, Discrete maximum principle and energy stability of the compact difference scheme for two-dimensional Allen-Cahn equation, *J. Funct. Spaces*, **2022** (2022), 8522231. https://doi.org/10.1155/2022/8522231
20. M. A. Ragusa, A. Tachikawa, On some regularity results of minimizers of energy functionals, *AIP Conference Proceedings*, **637** (2014), 854–863. https://doi.org/10.1063/1.4904658
21. F. Wu, Global energy conservation for distributional solutions to incompressible Hall-MHD equations without resistivity, *Filomat*, **37** (2023) 9741–9751. https://doi.org/10.2298/FIL2328741W
22. S. H. Wang, Z. Zhou, Periodic solutions for a second-order partial difference equation, *J. Appl. Math. Comput.*, **69** (2023), 731–752. https://doi.org/10.1007/s12190-022-01769-0
23. H. T. He, M. Ousbika, Z. Allali, J. B. Zuo, Non-trivial solutions for a partial discrete Dirichlet nonlinear problem with p-Laplacian, *Comm. Anal. Mech.*, **15** (2023), 598–610. https://doi.org/10.3934/cam.2023030
24. P. Mei, Z. Zhou, Homoclinic solutions for partial difference equations with mixed nonlinearities, *J. Geom. Anal.*, **33** (2023), 117. https://doi.org/10.1007/s12220-022-01166-w
25. Y. H. Long, On homoclinic solutions of nonlinear Laplacian partial difference equations with a parameter, *Discrete Contin. Dyn. Syst. Ser. S*, **17** (2024), 2489–2510. https://doi.org/10.3934/dcdss.2024005
26. P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, American Mathematical Society, 1986. https://doi.org/10.1090/cbms/065