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Research article

The stability and decay of 2D incompressible Boussinesq equation with partial vertical dissipation

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Abstract: This paper studies a special 2D anisotropic incompressible Boussinesq equation in $\mathbb{T}^2$ with $\mathbb{T} = [-\frac{1}{2}, \frac{1}{2}]$ being a 1D periodic box. The system concerned here possesses vertical dissipation only in the vertical component of the velocity and vertical heat diffusion. When the buoyancy forcing is not present, the 2D Boussinesq equation is a 2D Navier-Stokes equation with vertical dissipation only in the vertical component. The stability and large-time behavior problem on the solutions to the 2D Navier-Stokes equation with only vertical or horizontal dissipation remains unknown. When coupled with the temperature, the global regularity to the system with vertical dissipation and vertical diffusion in $\mathbb{R}^2$ has been solved by Cao and Wu (*Arch. Ration. Mech. Anal., 208(2013), 985-1004*). The stability with horizontal dissipation and horizontal diffusion in the periodic domain $\mathbb{T} \times \mathbb{R}$ has also been established by Dong, Wu, Xu, and Zhu (*Calc. Var. Partial Differential Equations, 60(2021)*) recently. Now whether the solution of the 2D system remains stable has yet to be solved when the velocity has vertical dissipation only in the u_2 equation. This paper aims to solve the problem and investigates the stability and large-time behavior of the solution to the special 2D Boussinesq equations on perturbations near the hydrostatic equilibrium. The basic idea here is to decompose the physical quantity f into its horizontal average, vertical average, and their corresponding oscillations. By establishing the strong Poincaré-type inequalities and several anisotropic inequalities related to the oscillations, we are able to obtain H^2-stability of the solution under the assumptions that the initial data is sufficiently small and obeys some symmetries. Furthermore, the exponential decay rates for the oscillation parts in H^1 are also established.

Keywords: 2D anisotropic Boussinesq equations; partial dissipation; stability; vertical thermal diffusion
Mathematics Subject Classification: 35A01, 35Q35, 76D03

1. Introduction

The Boussinesq equations model buoyancy-driven flows such as geophysical fluids and various Rayleigh-Bénard convection (see, e.g., [1–4]). The Boussinesq equations are mathematically significant [3]. This paper concerns a special anisotropic 2D incompressible Boussinesq equation with only vertical dissipations.

$$\begin{cases} \partial_t U + U\cdot\nabla U + \nabla P = \mu \begin{pmatrix} 0 \\ \partial_{22}U_2 \end{pmatrix} + \Theta e_2, \quad x\in\Omega,\ t>0, \\ \partial_t\Theta + u\cdot\nabla\Theta = \eta\partial_{22}\Theta, \\ \nabla\cdot U = 0, \\ U(x,0) = U_0(x), \quad \Theta(x,0) = \Theta_0(x), \end{cases} \tag{1.1}$$

where U represents the velocity field of the fluid, P the pressure, Θ the temperature, and $e_2 = (0,1)$ is the unit vector in the vertical direction. Here $\mu > 0$ is the kinematic viscosity, $\eta > 0$ is the thermal diffusivity and the spatial domain Ω is given by

$$\Omega = \mathbb{T}^2$$

with $\mathbb{T} = [-\frac{1}{2}, \frac{1}{2}]$ being a 1D periodic box.

This paper attempts to achieve two main goals. The first is to understand the stability and large-time behavior of perturbations near hydrostatic fluid equilibrium given by

$$U_{he} = 0,\ \Theta_{he} = x_2,\ P_{he} = \frac{1}{2}x_2^2.$$

It is easy to verify that hydrostatic fluid equilibrium$(U_{he}, \Theta_{he}, P_{he})$ is a steady state solution of (1.1). We consider the perturbation (u,θ) with

$$u = U - U_{he},\ \theta = \Theta - \Theta_{he}.$$

Then (u,θ) satisfies

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \mu \begin{pmatrix} 0 \\ \partial_{22}u_2 \end{pmatrix} + \theta e_2, \quad x\in\Omega,\ t>0, \\ \partial_t\theta + u\cdot\nabla\theta + u_2 = \eta\partial_{22}\theta, \\ \nabla\cdot u = 0, \\ u(x,0) = u_0(x), \theta(x,0) = \theta_0(x). \end{cases} \tag{1.2}$$

The second is to help better reveal the smoothing and stabilization effect of the temperature by considering the system (1.2) with the vertical dissipation in only the second component of the velocity.

The standard incompressible Boussinesq equations with full dissipation read as

$$\begin{cases} \partial_t U + U\cdot\nabla U + \nabla P = \mu\Delta U + \Theta e_2, \\ \partial_t\Theta + u\cdot\nabla\Theta = \eta\Delta\Theta, \\ \nabla\cdot U = 0, \\ U(x,0) = U_0(x), \quad \Theta(x,0) = \Theta_0(x). \end{cases} \tag{1.3}$$

The physical background and mathematical features of (1.3) make the model a rich area for mathematical investigations. Over the past decades, the Boussinesq equations have attracted considerable interest from mathematical scholars. Major concerns are oriented around the global well-posedness and finite-time blow up of large-data classical solutions and global regularity for the Boussinesq equations with full partial dissipation, i.e., $\mu = 0$ or $\eta = 0$, or the mixed partial dissipation case (see, e.g., [5–13])

In recent years, the problems of stability and large-time behavior of its solutions has garnered a lot attention, and significant progress have been made. For the 2D case, Doering, Wu, Zhao, and Zheng [14] rigorously proved the global asymptotic stability near a special type of hydrostatic equilibrium without buoyancy diffusion on a bounded domain subject to stress-free boundary conditions. Later, Tao and Wu [15] resolved some of the problems left open in [14]. They studied the stability problem for perturbations near hydrostatic equilibrium of the 2D Boussinesq equations without thermal diffusion in the periodic domain $\mathbb{T}^2$. Ben Said, Pandey, and Wu [16] solved the stability problem for a 2D Boussinesq system with only vertical dissipation and horizontal thermal diffusion in $\mathbb{R}^2$. Furthermore, when the dissipation is the opposite of that in [16], i.e. the horizontal dissipation and the vertical thermal diffusion, [17] established the stability in the Sobolev space H^2 and obtained algebraic decay rates for the oscillation parts in the H^1-norm when the spatial domain Ω is $\mathbb{T} \times \mathbb{R}$. More results with partial dissipation in two dimensions can be found in [15, 18–25]. For the 3D case, there are also some developments on the stability of solutions (see, e.g., [26–32]). Here we recall a recent result obtained by Wu and Zhang in [32]. They considered a 3D anisotropic Boussinesq system in the periodic domain $\Omega = \mathbb{R}^2 \times \mathbb{T}$. The stability and large-time behavior problem on perturbations near the hydrostatic balance were established.

Our paper here focuses on the 2D Boussinesq equations with only vertical dissipations. In order to better understand relevant progress and our difficulties, let's review some related results, which means the system with partial dissipation only in one direction. Cao and Wu [33] established the global-in-time existence of classical solutions to the 2D anisotropic Boussinesq equations with vertical dissipation in $\mathbb{R}^2$ and solved the global regularity problem. The stability of the 2D Boussinesq equations with only horizontal or vertical dissipation remains an open problem. Some recent works are devoted to this system in the periodic domain. Dong, Wu, Xu, and Zhu [34] investigated the stability and exponential decay of the 2D Boussinesq equations with horizontal dissipation in the domain $\mathbb{T} \times \mathbb{R}$. Also, [35] proved the nonlinear stability of Couette flow in a uniform magnetic field with only vertical dissipation in the same domain as [34]. Now whether the solution of the 2D system remains stable in a periodic domain if the velocity has horizontal or vertical dissipation only in one component equation, say, u_1 or u_2 equation.

Motivated by the above works related to only one-direction dissipation, we examine the 2D Boussinesq (1.2) in $\mathbb{T}^2$ and establish the stability result and the exponential decay rates of the solution. Before stating our results, we first assume that u_0 and θ_0 satisfy the symmetry as follows:

$$u_{01} \textit{ is odd in } x_1, \ u_{02} \textit{ and } \theta_0 \textit{ are even in } x_1. \tag{1.4}$$

Theorem 1.1. *Consider the 2D Boussinesq equation (1.2) with the initial data* $(u_0, \theta_0) \in H^2(\Omega)$ *satisfying* $\nabla \cdot u_0 = 0$ *and the symmetry condition (1.4). Then there exists* $\delta > 0$ *such that, if*

$$\|(u_0, \theta_0)\|_{H^2} \leq \delta,$$

then (1.1) possesses a unique global solution satisfying, for any $t > 0$,

$$\|u(\tau)\|_{H^2}^2 + \|\theta(\tau)\|_{H^2}^2 + 2\mu \int_0^t \|\partial_2 u_2\|_{H^2}^2 d\tau + 2\eta \int_0^t \|\partial_2 \theta\|_{H^2}^2 d\tau \le C\delta^2 \tag{1.5}$$

for some universal constant $C > 0$.

Remark 1.2. *The symmetry property (1.4) for the solution (u, θ) at $t = 0$ can persist for any time $t > 0$, namely,*

$$u_1 \text{ is odd in } x_1,\ u_2 \text{ and } \theta \text{ are even in } x_1. \tag{1.6}$$

A similar proof can be found in [32] and [36].

Remark 1.3. *If we consider the 2D Boussinesq equation with horizontal dissipation*

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \mu \begin{pmatrix} \partial_{11} u_1 \\ 0 \end{pmatrix} + \theta e_2, \quad x \in \Omega,\ t > 0, \\ \partial_t \theta + u \cdot \nabla \theta + u_2 = \eta \partial_{11} \theta, \\ \nabla \cdot u = 0, \\ u(x,0) = u_0(x), \theta(x,0) = \theta_0(x). \end{cases} \tag{1.7}$$

the stability result in Theorem 1.1 still holds provided that the symmetry condition (1.4) is replaced by

$$u_{01} \text{ is even in } x_1,\ u_{02} \text{ and } \theta \text{ are odd in } x_1. \tag{1.8}$$

Theorem 1.1 assesses the global-in-time existence and stability of small solutions to (1.2). Due to the lack of the horizontal dissipation, the proof of Theorem 1.1 is nontrivial. Especially, the velocity equation involves only vertical dissipation of u_2; it is extremely challenging to control the growth of the Navier-Stokes nonlinear term, i.e., $u \cdot \nabla u$. In fact, when Navier-Stokes possesses the dissipation in one direction, namely

$$\partial_t u + u \cdot \nabla u + \nabla p = \begin{pmatrix} 0 \\ \partial_{22} u_2 \end{pmatrix},$$

the global existence in time of solutions in the whole space $\mathbb{R}^2$ remains an open problem. Here we consider the periodic domain $\mathbb{T}^2$, which will greatly help solve this problem. More precisely, our proof will take advantage of the domain and explore many significant properties. Based on these properties, several key anisotropic inequalities will then be introduced. There are two important observations. The first is that by separating a physical quantity into its average, including both horizontal and vertical directions and the corresponding oscillations, we are able to establish the strong Poincaré-type inequalities, which are very powerful tools and also play a crucial role in the proof. The second observation is that if (u_0, θ_0) satisfies the symmetry given in (1.4), then (u, θ) maintains the same symmetries, namely,

$$u_1 \text{ is odd in } x_1, u_2 \text{ and } \theta \text{ are even in } x_1.$$

This can be achieved via the uniqueness of the solution. Specifically, define

$$U_1(x_1, x_2, t) = -u_1(-x_1, x_2, t), \quad U_2(x_1, x_2, t) = u_2(-x_1, x_2, t),$$

$$P(x_1, x_2, t) = p(-x_1, x_2, t), \quad \Theta(x_1, x_2, t) = \theta(-x_1, x_2, t).$$

It easily verifies that $U = (U_1, U_2)$, P, and Θ are still the solution of (1.2). Then the uniqueness implies the symmetries (1.8). Based on the symmetric property, another strong version of the Poincaré inequality can be obtained. With these properties and inequalities at our disposal, we can resolve all the difficult items.

Let us briefly outline the sketch of the proof. The framework in the proof of Theorem 1.1 is the bootstrapping argument. We first introduce some notations. For a sufficiently smooth function $f = f(x_1, x_2)$, we define its horizontal average $\overline{f}^{(1)}$ and vertical average $\overline{f}^{(2)}$ by

$$\overline{f}^{(1)} = \int_{\mathbb{T}} f(x_1, x_2)\, dx_1, \ \overline{f}^{(2)} = \int_{\mathbb{T}} f(x_1, x_2)\, dx_2, \tag{1.9}$$

and the corresponding oscillation part

$$\widetilde{f}^{(1)} = f - \overline{f}^{(1)}, \ \widetilde{f}^{(2)} = f - \overline{f}^{(2)}. \tag{1.10}$$

This decomposition is extremely useful due to some of the related properties (see Lemma 2.1). We remark that the most important property is $\overline{\widetilde{f}^{(i)}}^{(i)} = 0$ for $i = 1, 2$, which allows us to establish a strong Poincaré-type inequality,

$$\|\widetilde{f}^{(i)}\|_{L^2} \le C\|\partial_i \widetilde{f}^{(i)}\|_{L^2}. \tag{1.11}$$

Meanwhile, from the symmetries (1.1) we can also obtain another strong Poincaré-type inequality

$$\|\widetilde{f}^{(2)}\|_{L^2} \le C\|\partial_1 \widetilde{f}^{(2)}\|_{L^2}. \tag{1.12}$$

Furthermore, to deal with the triple products that stem from the nonlinear terms, the anisotropic inequality involving triple products associated with $\widetilde{f}^{(2)}$ is provided,

$$\int_{\Omega} |f\widetilde{g}^{(2)}h|\, dx \le C\|f\|_{L^2}^{\frac{1}{2}}(\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{2}}\|\partial_2 \widetilde{g}^{(2)}\|_{L^2}\|h\|_{L^2}. \tag{1.13}$$

To obtain the global existence of the solutions in the Sobolev setting H^2, we now introduce the H^2-energy $E(t)$ defined by

$$E(t) = \sup_{0\le\tau\le t}(\|u\|_{H^2}^2 + \|\theta\|_{H^2}^2) + 2\mu \int_0^t \|\partial_2 u_2\|_{H^2}^2 d\tau + 2\eta \int_0^t \|\partial_2 \theta\|_{H^2}^2 d\tau.$$

As aforementioned, the most difficult term is the integral involving the nonlinear term in the velocity, i.e.

$$\int \partial_1^2(u \cdot \nabla u) \cdot \nabla \partial_1^2 u\, dx + \int \partial_2^2(u \cdot \nabla u) \cdot \nabla \partial_2^2 u\, dx.$$

However, with the help of the strong Poincaré-type inequality and anisotropic inequalities, we are able to settle the difficulty. Take one term for instance, by integrations by parts, (1.11) and (1.13), the following nonlinear integral can be bounded as

$$\int \partial_1^2 u_1 \partial_1 u_2 \partial_1^2 u_2\, dx = \int \partial_1 \widetilde{u}_2^{(2)} \partial_1 \partial_2 u_2 \partial_1^2 u_2\, dx + \int \partial_1 \widetilde{u}_2^{(2)} \partial_1 u_2 \partial_1^2 \partial_2 u_2\, dx$$

$$
\begin{aligned}
&\le C\|\partial_1\widetilde{u_2}^{(2)}\|_{L^4}\|\partial_1\partial_2 u_2\|_{L^4}\|\partial_1^2 u_2\|_{L^2}\\
&\quad + C\|\partial_1^2\partial_2 u_2\|_{L^2}\|\partial_1 u_2\|_{L^2}^{\frac12}(\|\partial_1 u_2\|_{L^2}+\|\partial_1^2 u_2\|_{L^2})^{\frac12}\|\partial_1\partial_2\widetilde{u_2}^{(2)}\|_{L^2}\\
&\le C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2.
\end{aligned}
\tag{1.14}
$$

Therefore, through a series of subtle bounds, we can control the growth of all nonlinear terms and establish the closed priori estimate:

$$
E(t) \le C_0 E(0) + C_1 E^{\frac32}(t). \tag{1.15}
$$

Then applying a bootstrapping argument to (1.15) implies the uniform upper bound (1.5) for the initial data is small enough.

Next, we show the second theorem assessing the large-time behavior of the solutions of (1.2). More precisely, the exponential decay rates for the oscillation part of the solution are established.

Theorem 1.4. *Assume the initial data $(u_0,\theta_0)\in H^2(\Omega)$ with $\nabla\cdot u_0 = 0$ satisfying the symmetry condition (1.4) and*

$$
\|(u_0,\theta_0)\|_{H^2(\Omega)} \le \delta
$$

for some $\delta > 0$ small enough. Let (u,θ) be the corresponding solution of (1.2). Then the oscillation part $(\widetilde{u}^{(2)},\widetilde{\theta}^{(2)})$ decays exponentially in time,

$$
\|(\widetilde{u}^{(2)},\widetilde{\theta}^{(2)})\|_{H^1(\Omega)} \le C\delta e^{-Ct}, \tag{1.16}
$$

$$
\|(\partial_2\nabla\widetilde{u}_2^{(2)},\partial_2\nabla\widetilde{\theta}^{(2)})\|_{L^2(\Omega)} \le C\delta e^{-Ct} \tag{1.17}
$$

for all $t \ge 0$ and some constant $C > 0$.

Remark 1.5. *Following the decay results from Theorem 1.4, the solution (u,θ) of (1.2) is asymptotically close to the vertical average (u,θ) in $H^1(\Omega)$ satisfying $(\overline{u}^{(2)},\overline{\theta}^{(2)})$ satisfies*

$$
\begin{cases}
\partial_t\overline{u}_1^{(2)} + \partial_1(\overline{u_1^2}^{(2)}) + \partial_2(\overline{u_1u_2}^{(2)}) + \partial_1\overline{p}^{(2)} = 0,\\
\partial_t\overline{u}_2^{(2)} + \partial_1(\overline{u_1u_2}^{(2)}) + \partial_2(\overline{u_2^2}^{(2)}) = \overline{\theta}^{(2)},\\
\partial_t\overline{\theta}^{(2)} + \partial_1(\overline{u_1\theta}^{(2)}) + \overline{u_2}^{(2)} = 0.
\end{cases}
$$

We explain the main idea in the proof of Theorem 1.4. Due to the degeneracy in the viscous dissipation and the heat diffusion, especially, the very weak dissipation for the velocity, it is impossible to establish the large-time behavior for (u,θ). We remark that classical approaches such as Schonbek's Fourier splitting method [37, 38] that solve the fully dissipated system in whole space no longer apply. Therefore, we have to develop some new techniques. Based on one key observation, i.e., the strong Poincaré-type inequality

$$
\|\widetilde{f}^{(2)}\|_{L^2} \le C\|\partial_2\widetilde{f}^{(2)}\|_{L^2},
$$

we are content to investigate the decay of $(\widetilde{u}^{(2)},\widetilde{\theta}^{(2)})$ of the Boussinesq system (1.2) with $\widetilde{u}^{(2)}$ and $\widetilde{\theta}^{(2)}$ obeying

$$
\begin{cases}
\partial_t\widetilde{u}_1^{(2)} + \partial_1(u_1^2 - \overline{u_1^2}^{(2)}) + \partial_2(u_1u_2 - \overline{u_1u_2}^{(2)}) + \partial_1\widetilde{p}^{(2)} = 0,\\
\partial_t\widetilde{u}_2^{(2)} + \partial_1(u_1u_2 - \overline{u_1u_2}^{(2)}) + \partial_2(u_2^2 - \overline{u_2^2}^{(2)}) + \partial_2\widetilde{p}^{(2)} = \mu\partial_2^2\widetilde{u}_2^{(2)} + \widetilde{\theta}^{(2)},\\
\partial_t\widetilde{\theta}^{(2)} + \partial_1(u_1\theta - \overline{u_1\theta}^{(2)}) + \partial_2\widetilde{(u_2\theta)}^{(2)} + \widetilde{u}_2^{(2)} = \partial_2^2\widetilde{\theta}^{(2)}.
\end{cases}
$$

Our goal is to derive a differential inequality of the form

$$\frac{d}{dt}X(t) + cX(t) \leq 0,$$

which implies the exponential decay rates $X(t) \leq Ce^{-ct}$. The proof of Theorem 1.4 is divided into two stages. The first stage proves the exponential decay rate for $\|(\widetilde{u}^{(2)}, \widetilde{\theta}^{(2)})\|_{H^1}$ while the second is to estimate $\|(\partial_2\nabla\widetilde{u}_2^{(2)}, \partial_2\nabla\widetilde{\theta}^{(2)})\|_{L^2}$. The estimates are more complicated than that of the stability. Besides the inequalities (1.11), (1.12), and (1.13), we need to introduce two additional anisotropic inequalities associated with the L^4-norm and L^∞-norm (see Lemma 2.4 for details), which will be frequently used in the proof of the decay rates. After a long and delicate estimate, it is obtained that

$$\frac{d}{dt}(\|\widetilde{u}^{(2)}\|_{H^1}^2 + \|\widetilde{\theta}^{(2)}\|_{H^1}^2) + \min\{\mu, \eta\}(\|\partial_2\widetilde{u}_2^{(2)}\|_{H^1}^2 + \|\partial_2\widetilde{\theta}^{(2)}\|_{H^1}^2) \leq 0,$$

and

$$\frac{d}{dt}(\|\partial_2\nabla\widetilde{u}^{(2)}\|_{L^2}^2 + \|\partial_2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2) + \min\{\mu, \eta\}(\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2 + \|\partial_2^2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2) \leq 0.$$

Using (1.11) again yields the desired exponential decay (1.16) and (1.17) in Theorem 1.4. More technical details can be found in the proof of Theorem 1.4 in Section 4.

The rest of this paper is organized as follows. Section 2 presents four tool lemmas to be used in the proof of Theorems 1.1 and 1.4. Section 3 is devoted to the proof of Theorem 1.1. Section 4 proves the exponential decay estimate of Theorem 1.4. At the end, we claim that C may be different for each line in this article.

2. Decomposition and anisotropic inequalities

To prepare for the proofs in the subsequent sections, we provide some preliminary lemmas. The first presents some properties on $\widetilde{f}^{(i)}, \overline{f}^{(i)}$ for $i = 1, 2$, and their derivatives. The second contains three Poincaré-type inequalities, which provide the powerful tools for proving our theorems. The third proposes an anisotropic upper bound for triple products, whereas the last states anisotropic inequalities related to the L^4-norm and L^∞-norm that serve the proof of the large-time behavior.

We start with the properties of the composition for f, which can be derived via the definitions (1.9) and (1.10).

Lemma 2.1. *Let $\widetilde{f}^{(i)}$ and $\overline{f}^{(i)}$ for $i = 1, 2$ be defined as in (1.9) and (1.10). Then we have*

(1) *The average operator and the oscillation operator can commute with the derivatives, i.e.*

$$\overline{\partial_1 f}^{(1)} = 0,\ \overline{\partial_2 f}^{(1)} = \partial_2\overline{f}^{(1)},\ \widetilde{\partial_1 f}^{(1)} = \partial_1 f,\ \widetilde{\partial_2 f}^{(1)} = \partial_2\widetilde{f}^{(1)}.$$

$$\overline{\partial_1 f}^{(2)} = \partial_1\overline{f}^{(2)},\ \overline{\partial_2 f}^{(2)} = 0,\ \widetilde{\partial_1 f}^{(2)} = \partial_1\widetilde{f}^{(2)},\ \widetilde{\partial_2 f}^{(2)} = \partial_2 f.$$

In particular, if $\nabla \cdot f = 0$, then

$$\nabla \cdot \overline{f}^{(i)} = 0,\ \nabla \cdot \widetilde{f}^{(i)} = 0.$$

(2) *The corresponding average of the oscillation* $\widetilde{f}^{(i)}$ *is zero, for* $i = 1, 2$

$$\overline{\widetilde{f}^{(i)}}^{(i)} = 0.$$

(3) *For any* $k \geq 0$, $\widetilde{f}^{(i)}$ *and* $\overline{f}^{(i)}$ *are orthogonal in Sobolev space* H^k.

$$(\overline{f}^{(i)}, \widetilde{f}^{(i)})_{H^k} = 0, \quad \|f\|_{H^k}^2 = \|\overline{f}^{(i)}\|_{H^k}^2 + \|\widetilde{f}^{(i)}\|_{H^k}^2.$$

The second lemma provides the strong Poincaré-type inequalities associated with the oscillation $\widetilde{f}^{(i)}$ for $i = 1, 2$.

Lemma 2.2. *Assume* $f \in H^1(\Omega)$. *Then for* $i = 1, 2$ *it holds,*

$$\|\widetilde{f}^{(i)}\|_{L^2} \leq C\|\partial_i \widetilde{f}^{(i)}\|_{L^2}. \tag{2.1}$$

where $C > 0$ *is a pure constant. In addition, if we further assume* $\overline{f}^{(1)} = 0$, *then*

$$\|\widetilde{f}^{(2)}\|_{L^2} \leq C\|\partial_1 \widetilde{f}^{(2)}\|_{L^2}. \tag{2.2}$$

Proof. Without loss of generality, we prove (2.1) for the case $i = 2$. Thanks to the fact that the vertical average of $\widetilde{f}^{(2)}$ is zero, the proof for (2.1) is easy. In fact, by the integral mean value theorem, for any $x_2 \in \mathbb{T}$, there exists $y \in \mathbb{T}$ such that

$$\int_{\mathbb{T}} \widetilde{f}^{(2)}(x_1, x_2)\, dx_2 = \widetilde{f}^{(2)}(x_1, y) = 0.$$

Using Leibniz's formula yields

$$(\widetilde{f}^{(2)})^2 = \int_y^{x_2} \partial_2 (\widetilde{f}^{(2)})^2 (x_1, s) ds = 2\int_y^{x_2} \widetilde{f}^{(2)} \partial_2 \widetilde{f}^{(2)} ds.$$

By Hölder's inequality,

$$(\widetilde{f}^{(2)})^2 \leq C\|\widetilde{f}^{(2)}\|_{L^2_{x_2}} \|\partial_2 \widetilde{f}^{(2)}\|_{L^2_{x_2}}$$

Then integrating in space Ω, we obtain

$$\|\widetilde{f}^{(2)}\|_{L^2} \leq C\|\partial_2 \widetilde{f}^{(2)}\|_{L^2}.$$

(2.2) follows from $\overline{\widetilde{f}^{(2)}}^{(1)} = 0$. By the definition of $\overline{\widetilde{f}^{(2)}}^{(1)}$, $f = \overline{f}^{(2)} + \widetilde{f}^{(2)}$ and $\overline{f}^{(1)} = 0$, we obtain

$$\overline{\widetilde{f}^{(2)}}^{(1)} = \int_{\mathbb{T}} \widetilde{f}^{(2)}\, dx_1 = \int_{\mathbb{T}} (f - \overline{f}^{(2)})\, dx_1 = \int_{\mathbb{T}} f\, dx_1 - \int_{\mathbb{T}^2} f\, dx_1\, dx_2 = 0.$$

Then a similar argument to (2.1) yields the desired strong Poincaré-type inequality (2.2) in x_1-direction. This concludes the proof of Lemma 2.2. □

The third lemma assesses an anisotropic upper bound for triple products, which will be used frequently in both Theorem 1.1 and 1.4. Similar anisotropic inequalities in $\mathbb{R}^2$ are also available (see e.g. [39]). We are able to use a similar proof to that in [39] together with the Poincaré-type inequality (2.1) to obtain the anisotropic inequality in periodic domain Ω.

Lemma 2.3. *For any functions $f, g, h, \partial_2 g, \partial_1 f \in L^2(\Omega)$, then*

$$\int_\Omega |f\widetilde{g}^{(2)}h|\,dx \le C\|f\|_{L^2}^{\frac{1}{2}}(\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{2}}\|\partial_2\widetilde{g}^{(2)}\|_{L^2}\|h\|_{L^2}. \tag{2.3}$$

Proof. By Hölder's inequality, Minkowski's inequality, (2.1), (2.7), and (2.8), we have

$$\begin{aligned}
\int_\Omega |f\widetilde{g}^{(2)}h|\,dx &\le \|\|f\|_{L^\infty_{x_1}}\|_{L^2_{x_2}}\|\|\widetilde{g}^{(2)}\|_{L^2_{x_1}}\|_{L^\infty_{x_2}}\|h\|_{L^2}\\
&\le C\|\|f\|_{L^\infty_{x_1}}\|_{L^2_{x_2}}\|\|\widetilde{g}^{(2)}\|_{L^\infty_{x_2}}\|_{L^2_{x_1}}\|h\|_{L^2}\\
&\le C\Big\|\|f\|_{L^2_{x_1}}^{\frac{1}{2}}(\|f\|_{L^2_{x_1}} + \|\partial_1 f\|_{L^2_{x_1}})^{\frac{1}{2}}\Big\|_{L^2_{x_2}}\\
&\quad\times\Big\|\|\widetilde{g}^{(2)}\|_{L^2_{x_2}}^{\frac{1}{2}}\|\partial_2\widetilde{g}^{(2)}\|_{L^2_{x_2}}^{\frac{1}{2}}\Big\|_{L^2_{x_1}}\|h\|_{L^2}\\
&\le C\|f\|_{L^2}^{\frac{1}{2}}(\|f\|_{L^2} + \|\partial_1 f\|_{L^2})^{\frac{1}{2}}\|\partial_2\widetilde{g}^{(2)}\|_{L^2}\|h\|_{L^2}.
\end{aligned}$$

□

We now state the last lemma, which provides two anisotropic upper bounds on L^4-norm and L^∞-norm of $\widetilde{f}^{(2)}$. It can be achieved via 1D inequalities of L^∞-norm and the strong Poincaré-type inequality in Lemma 2.2.

Lemma 2.4. *Assume $\partial_1 f \in L^2(\Omega)$ and $\partial_2 f \in H^1(\Omega)$. Then the following inequalities holds,*

$$\begin{aligned}
\|\widetilde{f}^{(2)}\|_{L^\infty} &\le C\|\partial_2\widetilde{f}^{(2)}\|_{L^2}^{\frac{1}{2}}(\|\widetilde{f}^{(2)}\|_{L^2} + \|\partial_1\widetilde{f}^{(2)}\|_{L^2})^{\frac{1}{4}}\\
&\quad\times(\|\partial_2\widetilde{f}^{(2)}\|_{L^2} + \|\partial_1\partial_2\widetilde{f}^{(2)}\|_{L^2})^{\frac{1}{4}} && (2.4)\\
&\le C\|\partial_2\nabla\widetilde{f}^{(2)}\|_{L^2}. && (2.5)\\
\|\partial_2\widetilde{f}^{(2)}\|_{L^4} &\le C\|\partial_2\nabla\widetilde{f}^{(2)}\|_{L^2} && (2.6)
\end{aligned}$$

where $C > 0$ are some pure constants.

Proof. To prove inequality (2.4), we need the 1D inequalities of L^∞-norm,

$$\|f\|_{L^\infty(\mathbb{T})} \le C\|f\|_{L^2(\mathbb{T})}^{\frac{1}{2}}(\|f\|_{L^2(\mathbb{T})} + \|Df\|_{L^2(\mathbb{T})})^{\frac{1}{2}}, \tag{2.7}$$

$$\|\widetilde{f}^{(i)}\|_{L^\infty(\mathbb{T})} \le C\|\widetilde{f}^{(i)}\|_{L^2(\mathbb{T})}^{\frac{1}{2}}\|\partial_i\widetilde{f}^{(i)}\|_{L^2(\mathbb{T})}^{\frac{1}{2}}, \tag{2.8}$$

which can be obtained through a slight modification of the proof for (2.1).

Applying Hölder's inequality in one component and Minkowski's inequality, combining (2.1) and (2.8), we have

$$\|\widetilde{f}^{(2)}\|_{L^\infty} \le C\Big\|\|\widetilde{f}^{(2)}\|_{L^2_{x_2}}^{\frac{1}{2}}\|\partial_2\widetilde{f}^{(2)}\|_{L^2_{x_2}}^{\frac{1}{2}}\Big\|_{L^\infty_{x_1}}$$

$$
\begin{aligned}
&\leq C\Big\|\|\widetilde{f}^{(2)}\|_{L^\infty_{x_1}}\Big\|^{\frac{1}{2}}_{L^2_{x_2}}\Big\|\|\partial_2\widetilde{f}^{(2)}\|_{L^\infty_{x_1}}\Big\|^{\frac{1}{2}}_{L^2_{x_2}}\\
&\leq C\Big\|\|\widetilde{f}^{(2)}\|^{\frac{1}{2}}_{L^2_{x_1}}(\|\widetilde{f}^{(2)}\|_{L^2_{x_1}}+\|\partial_1\widetilde{f}^{(2)}\|_{L^2_{x_1}})^{\frac{1}{2}}\Big\|^{\frac{1}{2}}_{L^2_{x_2}}\\
&\quad\times\Big\|\|\partial_2\widetilde{f}^{(2)}\|^{\frac{1}{2}}_{L^2_{x_1}}(\|\partial_2\widetilde{f}^{(2)}\|_{L^2_{x_1}}+\|\partial_1\partial_2\widetilde{f}^{(2)}\|_{L^2_{x_1}})^{\frac{1}{2}}\Big\|^{\frac{1}{2}}_{L^2_{x_2}}\\
&\leq C\|\partial_2\widetilde{f}^{(2)}\|^{\frac{1}{2}}_{L^2}(\|\widetilde{f}^{(2)}\|_{L^2}+\|\partial_1\widetilde{f}^{(2)}\|_{L^2})^{\frac{1}{4}}\\
&\quad\times(\|\partial_2\widetilde{f}^{(2)}\|_{L^2}+\|\partial_1\partial_2\widetilde{f}^{(2)}\|_{L^2})^{\frac{1}{4}},
\end{aligned}
$$

which, combining with $\|\widetilde{f}^{(2)}\| \leq \|\partial_2\widetilde{f}^{(2)}\|$, derives (2.4).

(2.6) is the direct consequence of Hölder's inequality and Poincaré inequality (2.1).

$$
\begin{aligned}
\|\partial_2\widetilde{f}^{(2)}\|_{L^4} &\leq \|\partial_2\widetilde{f}^{(2)}\|_{L^2}+\|\partial_2\nabla\widetilde{f}^{(2)}\|_{L^2}\\
&\leq C\|\partial_2\nabla\widetilde{f}^{(2)}\|_{L^2}.
\end{aligned}
$$

This completes the proof of Lemma 2.4. □

3. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1, which claims the global existence and stability of solutions of (1.2). To obtain this result, we need to establish a global priori estimate of the energy $E(t)$, as shown in Proposition 3.1. With the energy inequality at our disposal, we are then able to prove Theorem 1 by using the bootstrapping argument (see [40, p.21]).

Proposition 3.1. *Assume the initial data* (u_0, θ_0) *satisfies the conditions in (1.4). Let* $E(t)$ *be an energy functional defined by*

$$
E(t) = \sup_{0\leq\tau\leq t}(\|u\|^2_{H^2}+\|\theta\|^2_{H^2}) + 2\mu\int_0^t\|\partial_2u_2\|^2_{H^2}d\tau + 2\eta\int_0^t\|\partial_2\theta\|^2_{H^2}d\tau.
$$

Then there exist two constants C_0 *and* C_1*, depending on* μ *and* η *such that, for* $0 < t < T$,

$$
E(t) \leq C_0E(0) + C_1E^{\frac{3}{2}}(t). \tag{3.1}
$$

Proof of proposition 3.1. First, we have the L^2-bound

$$
(\|u\|^2_{L^2}+\|\theta\|^2_{L^2}) + 2\mu\int_o^t\|\partial_2u_2\|^2_{L^2}d\tau + 2\eta\int_o^t\|\partial_2\theta\|^2_{L^2}d\tau = \|u_0\|^2_{L^2}+\|\theta_0\|^2_{L^2}. \tag{3.2}
$$

Note that the norm $\|(u(t),\theta(t))\|_{H^2}$ is equivalent to $\|(u(t),\theta(t))\|_{L^2} + \|(u(t),\theta(t))\|_{\dot{H}^2}$. Thus it suffices to bound $\|(u(t),\theta(t))\|_{\dot{H}^2}$. Applying $\partial_i^2(i = 1,2)$ to (1.2), taking the L^2-inner product of the resulted equations with $(\partial_i^2u, \partial_i^2\theta)$, and using divergence-free condition for u, we obtain

$$
\frac{1}{2}\frac{d}{dt}\sum_{i=1}^2(\|\partial_i^2u\|^2_{L^2}+\|\partial_i^2\theta\|^2_{L^2}) + \mu\sum_{i=1}^2\|\partial_i^2\partial_2u_2\|^2_{L^2} + \eta\sum_{i=1}^2\|\partial_i^2\partial_2\theta\|^2_{L^2}
$$

$$
\begin{aligned}
&= -\sum_{i=1}^{2}\int \partial_i^2(u\cdot\nabla u)\cdot\partial_i^2 u\,dx - \sum_{i=1}^{2}\int \partial_i^2(u\cdot\nabla\theta)\cdot\partial_i^2\theta\,dx\\
&:= I_1 + I_2,
\end{aligned}
\tag{3.3}
$$

where we have used

$$\int \partial_i^2\nabla p\cdot\partial_i^2 u\ dx = 0.$$

To make full use of the anisotropic dissipation, by integrations by parts and $\nabla\cdot u = 0$, we first split I_1 into four parts.

$$
\begin{aligned}
I_1 \quad &= \quad -\int \partial_1^2 u\cdot\nabla u\cdot\partial_1^2 u\,dx - 2\int \partial_1 u\cdot\partial_1\nabla u\cdot\partial_1^2 u\,dx\\
&\quad\quad -\int \partial_2^2 u\cdot\nabla u\cdot\partial_2^2 u\,dx - 2\int \partial_2 u\cdot\partial_2\nabla u\cdot\partial_2^2 u\,dx\\
&:= \quad I_{11} + I_{12} + I_{13} + I_{14}.
\end{aligned}
$$

We now bound I_{11} through I_{14} one by one. For I_{11}, we further decompose it as follows:

$$
\begin{aligned}
I_{11} \quad &= \quad -\int \partial_1^2 u_1\partial_1 u_1\partial_1^2 u_1\ dx - \int \partial_1^2 u_1\partial_1 u_2\partial_1^2 u_2\ dx\\
&\quad\quad -\int \partial_1^2 u_2\partial_2 u_1\partial_1^2 u_1\ dx - \int \partial_1^2 u_2\partial_2 u_2\partial_1^2 u_2\ dx\\
&:= \quad I_{11,1} + I_{11,2} + I_{11,3} + I_{11,4}.
\end{aligned}
$$

Thanks to the dissipation of u_2 in the x_2-direction, direct applications of Hölder's inequality, Sobolev's inequality, and $\nabla\cdot u = 0$ can show that

$$I_{11,1} \le C\|\partial_1 u_1\|_{L^4}\|\partial_1^2 u_1\|_{L^4}\|\partial_1^2 u_1\|_{L^2} \le C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2. \tag{3.4}$$

According to Lemma 2.1, we obtain $\partial_1\partial_2 u_2 = \partial_1\partial_2\widetilde{u}_2^{(2)}$. Then by integration by parts, Hölder's inequality, Sobolev's inequality, (2.1), and Lemma 2.3, $I_{11,2}$ can be bounded as

$$
\begin{aligned}
I_{11,2} &= -\int \partial_1\widetilde{u}_2^{(2)}\partial_1\partial_2 u_2\partial_1^2 u_2\,dx - \int \partial_1\widetilde{u}_2^{(2)}\partial_1 u_2\partial_1^2\partial_2 u_2\,dx\\
&\le C\|\partial_1\widetilde{u}_2^{(2)}\|_{L^4}\|\partial_1\partial_2 u_2\|_{L^4}\|\partial_1^2 u_2\|_{L^2}\\
&+ C\|\partial_1^2\partial_2 u_2\|_{L^2}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}(\|\partial_1 u_2\|_{L^2} + \|\partial_1^2 u_2\|_{L^2})^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u_2}^{(2)}\|_{L^2}\\
&\le C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2.
\end{aligned}
\tag{3.5}
$$

Similarly,

$$
\begin{aligned}
I_{11,4} &= 2\int \widetilde{u}_2^{(2)}\partial_1^2\partial_2 u_2\partial_1^2 u_2\,dx\\
&\le C\|\widetilde{u}_2^{(2)}\|_{L^\infty}\|\partial_1^2\partial_2 u_2\|_{L^2}\|\partial_1^2 u_2\|_{L^2}\\
&\le C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2.
\end{aligned}
\tag{3.6}
$$

For $I_{11,3}$, with the help of the symmetry $\overline{u}_1^{(1)} = 0$ together with (2.2), it is easy to obtain

$$\begin{aligned} I_{11,3} &\le C\|\partial_1^2 u_2\|_{L^2}\|\partial_2 u_1\|_{L^4}\|\partial_1^2 u_1\|_{L^4} \\ &\le C\|\partial_1^2 u_2\|_{L^2}\|\partial_2\partial_1 u_1\|_{H^1}\|\partial_1^2 u_1\|_{H^1} \\ &\le C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2. \end{aligned}$$

Combining all estimates above for $I_{11,1}$ through $I_{11,4}$ yields

$$I_{11} \le C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2.$$

I_{12} can be handled similarly to I_{11}. We first rewrite it as follows:

$$\begin{aligned} I_{12} = &-2\int \partial_1 u_1 \partial_1^2 u_1 \partial_1^2 u_1\, dx - 2\int \partial_1 u_1 \partial_1^2 u_2 \partial_1^2 u_2\, dx \\ &-2\int \partial_1 u_2 \partial_1\partial_2 u_1 \partial_1^2 u_1\, dx - 2\int \partial_1 u_2 \partial_1\partial_2 u_2 \partial_1^2 u_2\, dx. \end{aligned}$$

Then invoking (3.4), (3.5), and (3.6) and applying Hölder's inequality, Sobolev's inequality to the third term yields

$$I_{12} \le C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2.$$

We proceed to bound I_{13}. As I_{11}, I_{13} is first divided into four parts.

$$\begin{aligned} I_{13} = &-\int \partial_2^2 u_1 \partial_1 u_1 \partial_2^2 u_1\, dx - \int \partial_2^2 u_1 \partial_1 u_2 \partial_2^2 u_2\, dx \\ &-\int \partial_2^2 u_2 \partial_2 u_1 \partial_2^2 u_1\, dx - \int \partial_2^2 u_2 \partial_2 u_2 \partial_2^2 u_2\, dx \\ := &\; I_{13,1} + I_{13,2} + I_{13,3} + I_{13,4}. \end{aligned}$$

Lemma 2.1, integration by parts, Hölder's inequality together with (2.1) lead to

$$\begin{aligned} I_{13,1} &= 2\int \widetilde{u}_1^{(1)} \partial_1 \partial_2^2 u_1 \partial_2^2 u_1\, dx \le C\|\widetilde{u}_1^{(1)}\|_{L^\infty}\|\partial_1\partial_2^2 u_1\|_{L^2}\|\partial_2^2 u_1\|_{L^2} \\ &\le \|\partial_2 u_2\|_{H^2}^2\|\partial_2^2 u_1\|_{L^2}. \end{aligned}$$

Similarly, $I_{13,3}$ can be estimated as

$$\begin{aligned} I_{13,3} &= -\int \partial_2\widetilde{u}_1^{(1)}\partial_1\partial_2 u_1 \partial_2^2 u_1\, dx - \int \partial_2\widetilde{u}_1^{(1)}\partial_2 u_1 \partial_1\partial_2^2 u_1\, dx \\ &\le \|\partial_2\widetilde{u}_1^{(1)}\|_{L^4}\|\partial_1\partial_2 u_1\|_{L^4}\|\partial_2^2 u_1\|_{L^2} + \|\partial_2\widetilde{u}_1^{(1)}\|_{L^4}\|\partial_2 u_1\|_{L^4}\|\partial_1\partial_2^2 u_1\|_{L^2} \\ &\le C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2. \end{aligned}$$

By (2.3) and $\|u_1\|_{L^2} \le C\|\partial_1 u_1\|_{L^2}$, we obtain

$$\begin{aligned} I_{13,2} &\le \|\partial_2^3 u_2\|_{L^2}\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}}(\|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} + \|\partial_1^2 u_2\|_{L^2}^{\frac{1}{2}})\|\partial_2^2 u_1\|_{L^2} \\ &\le C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2. \end{aligned}$$

Also, we have

$$I_{13,4} \leq \|\partial_2 u_2\|_{L^4}\|\partial_2^2 u_2\|_{L^4}\|\partial_2^2 u_2\|_{L^2} \leq C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2.$$

Thus, we obtain

$$I_{13} \leq C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2.$$

With a nearly same argument with I_{13}, we derive

$$I_{14} = -2I_{13,3} - 2I_{13,1} + 2I_{13,4} - 2\int \partial_2 u_1 \partial_2 \partial_1 u_2 \partial_2^2 u_2\, dx \leq C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2.$$

In summary, we obtain the upper bound for I_1

$$I_1 \leq C\|u\|_{H^2}\|\partial_2 u_2\|_{H^2}^2. \tag{3.7}$$

Next, we turn to deal with I_2. It remains to be divided into four parts:

$$\begin{aligned} I_2 &= -\int \partial_1^2 u \cdot \nabla\theta \cdot \partial_1^2\theta\, dx - 2\int \partial_1 u \cdot \partial_1 \nabla\theta \cdot \partial_1^2\theta\, dx \\ &\quad -\int \partial_2^2 u \cdot \nabla\theta \cdot \partial_2^2\theta\, dx - 2\int \partial_2 u \cdot \partial_2 \nabla\theta \cdot \partial_2^2\theta\, dx \\ &:= I_{21} + I_{22} + I_{23} + I_{24}. \end{aligned}$$

We first split I_{21} into two terms, then use integration by parts and combine with Hölder's inequality, Sobolev's inequality, and (2.1) to obtain

$$\begin{aligned} I_{21} &= -\int \partial_1 \widetilde{u}_2^{(2)} \partial_1\partial_2\theta \partial_1^2\theta\, dx - \int \partial_1 \widetilde{u}_2^{(2)} \partial_1\theta \partial_1^2\partial_2\theta\, dx \\ &\quad + \int \partial_1^2\partial_2 u_2\, \widetilde{\theta}^{(2)} \partial_1^2\theta\, dx + \int \partial_1^2 u_2\, \widetilde{\theta}^{(2)} \partial_1^2\partial_2\theta\, dx \\ &\leq C\|\partial_1 \widetilde{u}_2^{(2)}\|_{L^4}\|\partial_1\partial_2\theta\|_{L^4}\|\partial_1^2\theta\|_{L^2} + C\|\partial_1 \widetilde{u_2}^{(2)}\|_{L^4}\|\partial_1\theta\|_{L^4}\|\partial_1^2\partial_2\theta\|_{L^2} \\ &\quad + C\|\partial_1^2\partial_2 u_2\|_{L^2}\|\widetilde{\theta}^{(2)}\|_{L^\infty}\|\partial_1^2\theta\|_{L^2} + C\|\partial_1^2 u_2\|_{L^2}\|\widetilde{\theta}^{(2)}\|_{L^\infty}\|\partial_1^2\partial_2\theta\|_{L^2} \\ &\leq C(\|u\|_{H^2} + \|\theta\|_{H^2})(\|\partial_2 u_2\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2). \end{aligned}$$

where we have used $\partial_1\partial_2 u_2 = \partial_1\partial_2\widetilde{u_2}^{(2)}$ and $\partial_2\theta = \partial_2\widetilde{\theta}^{(2)}$ by Lemma 2.1. Similarly, I_{22} can be estimated as follows:

$$\begin{aligned} I_{22} &= -4\int \widetilde{u}_2^{(2)} \partial_1^2\partial_2\theta \partial_1^2\theta\, dx + 2\int \partial_1\partial_2 u_2 \partial_1 \widetilde{\theta}^{(2)} \partial_1^2\theta\, dx + 2\int \partial_1 u_2 \partial_1 \widetilde{\theta}^{(2)} \partial_1^2\partial_2\theta\, dx \\ &\leq C\|\widetilde{u}_2^{(2)}\|_{L^\infty}\|\partial_1^2\partial_2\theta\|_{L^2}\|\partial_1^2\theta\|_{L^2} + C\|\partial_1\partial_2 u_2\|_{L^4}\|\partial_1\widetilde{\theta}^{(2)}\|_{L^4}\|\partial_1^2\theta\|_{L^2} \\ &\quad + C\|\partial_1 u_2\|_{L^4}\|\partial_1\widetilde{\theta}^{(2)}\|_{L^4}\|\partial_1^2\partial_2\theta\|_{L^2} \\ &\leq C(\|u\|_{H^2} + \|\theta\|_{H^2})(\|\partial_2 u_2\|_{H^2}^2 + \|\partial_2\theta\|_{H^2}^2). \end{aligned}$$

To bound I_{23}, we need to resort to the fact that $\overline{u}_1^{(1)} = 0$, i.e., $u_1 = \widetilde{u}_1^{(1)}$. Then by (2.3) and (2.1), we find

$$I_{23} = -\int \partial_2^2 \widetilde{u}_1^{(1)} \partial_1\theta \partial_2^2\theta\, dx - \int \partial_2^2 u_2 \partial_2\theta \partial_2^2\theta\, dx$$

$$
\begin{aligned}
&\leq C\|\partial_2^3\theta\|_{L^2}\|\partial_1\theta\|_{L^2}^{\frac{1}{2}}(\|\partial_1\theta\|_{L^2}+\|\partial_1^2\theta\|_{L^2})^{\frac{1}{2}}\|\partial_2^2\widetilde{u}_1^{(1)}\|_{L^2}\\
&\quad +C\|\partial_2^2u_2\|_{L^2}\|\partial_2\theta\|_{L^4}\|\partial_2^2\theta\|_{L^4}\\
&\leq C\|\theta\|_{H^2}(\|\partial_2u_2\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2).
\end{aligned}
$$

For the last term I_{24}, it is clear that

$$
I_{24}\leq C\|\partial_2u\|_{L^4}\|\partial_2\nabla\theta\|_{L^4}\|\partial_2^2\theta\|_{L^2}\leq C\|u\|_{H^2}\|\partial_2\theta\|_{H^2}^2.
$$

As a result of the above estimates, we obtain

$$
I_2\leq C(\|u\|_{H^2}+\|\theta\|_{H^2})(\|\partial_2u_2\|_{H^2}^2+\|\partial_2\theta\|_{H^2}^2). \tag{3.8}
$$

Inserting (3.7), (3.8) into (3.3) and integrating it in time, we conclude

$$
\begin{aligned}
&\sum_{i=1}^{2}(\|\partial_i^2u\|_{L^2}^2+\|\partial_i^2\theta\|_{L^2}^2)+2\int_0^t(\mu\|\partial_i^2\partial_2u_2\|_{L^2}^2+\eta\|\partial_i^2\partial_2u\theta\|_{L^2}^2)\,d\tau\\
&\leq\sum_{i=1}^{2}\|(\partial_i^2u_0,\partial_i^2\theta_0)\|_{L^2}^2+C\sup_{0\leq t\leq T}\|(u,\theta)\|_{H^2}^2\int_0^t(\|\partial_i^2\partial_2u_2\|_{L^2}^2+\|\partial_i^2\partial_2\theta\|_{L^2}^2)\,d\tau.
\end{aligned}
$$

which together with (3.2) implies the desired estimates (3.1). The proof of Proposition 3.1 is thus completed. □

We now prove Theorem 1.1.

Proof of Theorem1.1. We now have established a priori estimate on the H^2-norm of (u,θ), namely,

$$
E(t)\leq C_0E(0)+C_1E^{\frac{3}{2}}(t). \tag{3.9}
$$

The bootstrapping argument then allows us to prove the stability of the solution, provided that the initial data is sufficiently small, i.e.

$$
E(0)=\|(u_0,\theta_0)\|_{H^2}^2\leq\delta^2\leq\frac{1}{16C_0C_1^2}. \tag{3.10}
$$

To apply the bootstrapping argument, we start with the ansatz that

$$
E(t)\leq\frac{1}{4C_1^2}.
$$

Then (3.9) together with the small assumption (3.10) implies

$$
E(t)\leq C_0E(0)+C_1E^{\frac{1}{2}}(t)E(t)\leq C_0E(0)+\frac{1}{2}E(t),
$$

or

$$
E(t)\leq 2C_0E(0)\leq\frac{1}{8C_1^2}.
$$

Thus, the bootstrapping argument asserts for any $t\geq 0$,

$$
E(t)\leq C\delta^2.
$$

which means the perturbed solution of (1.2) exists globally for all time. We complete the proof of Theorem 1.1. □

4. Proof of Theorem 1.4

This section is committed to proving Theorem 1.4. As aforementioned in the introduction, to obtain the exponential decay, we will derive the following differential inequality:

$$\frac{d}{dt}X(t) + CX(t) \le 0. \tag{4.1}$$

In the proof, we will make extensive use of the anisotropic inequalities and Poincaré inequality presented in Section 2, which play a crucial role in establishing this type of inequality (4.1).

Proof of Theorem1.4. We first construct the equations of $(\widetilde{u}^{(2)}, \widetilde{\theta}^{(2)})$. By taking the vertical average of (1.2), it is easy to verify that $(\overline{u}^{(2)}, \overline{\theta}^{(2)})$ satisfies

$$\begin{cases} \partial_t \overline{u}_1^{(2)} + \partial_1(\overline{u_1^2}^{(2)}) + \partial_2(\overline{u_1 u_2}^{(2)}) + \partial_1 \overline{p}^{(2)} = 0, \\ \partial_t \overline{u}_2^{(2)} + \partial_1(\overline{u_1 u_2}^{(2)}) + \partial_2(\overline{u_2^2}^{(2)}) = \overline{\theta}^{(2)}, \\ \partial_t \overline{\theta}^{(2)} + \partial_1(\overline{u_1\theta}^{(2)}) + \overline{u_2}^{(2)} = 0. \end{cases} \tag{4.2}$$

Taking the difference between (1.2) and (4.2), we obtain

$$\begin{cases} \partial_t \widetilde{u}_1^{(2)} + \partial_1(u_1^2 - \overline{u_1^2}^{(2)}) + \partial_2(u_1 u_2 - \overline{u_1 u_2}^{(2)}) + \partial_1 \widetilde{p}^{(2)} = 0, \\ \partial_t \widetilde{u}_2^{(2)} + \partial_1(u_1 u_2 - \overline{u_1 u_2}^{(2)}) + \partial_2(u_2^2 - \overline{u_2^2}^{(2)}) + \partial_2 \widetilde{p}^{(2)} = \mu \partial_2^2 \widetilde{u}_2^{(2)} + \widetilde{\theta}^{(2)}, \\ \partial_t \widetilde{\theta}^{(2)} + \partial_1(u_1\theta - \overline{u_1\theta}^{(2)}) + \partial_2 \widetilde{(u_2\theta)}^{(2)} + \widetilde{u}_2^{(2)} = \eta \partial_2^2 \widetilde{\theta}^{(2)}. \end{cases} \tag{4.3}$$

Step 1. Decay for $\|(\widetilde{u}^{(2)}, \widetilde{\theta}^{(2)})\|_{H^1}$

Dotting the system (4.3) by $(\widetilde{u}^{(2)}, \widetilde{\theta}^{(2)})$ yields, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|\widetilde{u}^{(2)}\|_{L^2}^2 + \|\widetilde{\theta}^{(2)}\|_{L^2}^2) + \mu\|\partial_2 \widetilde{u}_2^{(2)}\|_{L^2}^2 + \eta\|\partial_2 \widetilde{\theta}^{(2)}\|_{L^2}^2 \\ &:= J_1 + J_2 + J_3 + J_4 + J_5 + J_6, \end{aligned} \tag{4.4}$$

where

$$\begin{aligned} J_1 &= -\int \partial_1(u_1^2 - \overline{u_1^2}^{(2)})\widetilde{u}_1^{(2)}\,dx,\ J_2 = -\int \partial_2(u_1u_2 - \overline{u_1u_2}^{(2)})\widetilde{u}_1^{(2)}\,dx, \\ J_3 &= -\int \partial_1(u_1u_2 - \overline{u_1u_2}^{(2)})\widetilde{u}_2^{(2)}\,dx,\ J_4 = -\int \partial_2(u_2^2 - \overline{u_2^2}^{(2)})\widetilde{u}_2^{(2)}\,dx, \\ J_5 &= -\int \partial_1(u_1\theta - \overline{u_1\theta}^{(2)})\widetilde{\theta}^{(2)}\,dx,\ J_6 = -\int \partial_2 \widetilde{(u_2\theta)}^{(2)}\widetilde{\theta}^{(2)}\,dx. \end{aligned}$$

Before bounding J_1 through J_6, we first make the following decompositions by $u = \overline{u}^{(2)} + \widetilde{u}^{(2)}$.

$$u_1^2 - \overline{u_1^2}^{(2)} = 2\overline{u}_1^{(2)}\widetilde{u}_1^{(2)} + \widetilde{(\widetilde{u}_1^{(2)})^2}^{(2)}. \tag{4.5}$$

$$u_2^2 - \overline{u_2^2}^{(2)} = 2\overline{u}_2^{(2)}\widetilde{u}_2^{(2)} + \widetilde{(\widetilde{u}_2^{(2)})^2}^{(2)}. \tag{4.6}$$

$$u_1u_2 - \overline{u_1u_2}^{(2)} = \overline{u}_1^{(2)}\widetilde{u}_2^{(2)} + \widetilde{u}_1^{(2)}\overline{u}_2^{(2)} + \widetilde{\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)}}^{(2)}. \tag{4.7}$$

$$u_1\theta - \overline{u_1\theta}^{(2)} = \overline{u}_1^{(2)}\widetilde{\theta}^{(2)} + \widetilde{u}_1^{(2)}\overline{\theta}^{(2)} + \widetilde{\widetilde{u}_1^{(2)}\widetilde{\theta}^{(2)}}^{(2)}. \tag{4.8}$$

Subsisting (4.5) in J_1 and using (2.2) and $\|\widetilde{f}^{(2)}\|_{L^2} \leq \|f\|_{L^2}$ yields

$$\begin{aligned} J_1 &= -2\int \partial_1\widetilde{u}_1^{(2)}\overline{u}_1^{(2)}\widetilde{u}_1^{(2)}\,dx - 2\int \partial_1\widetilde{(\widetilde{u}_1^{(2)})^2}^{(2)}\widetilde{u}_1^{(2)}\,dx \\ &\leq C\|\partial_1\widetilde{u}_1^{(2)}\|_{L^2}\|\overline{u}_1^{(2)}\|_{L^\infty}\|\widetilde{u}_1^{(2)}\|_{L^2} + C\|\partial_1(\widetilde{u}_1^{(2)})^2\|_{L^2}\|\widetilde{u}_1^{(2)}\|_{L^2} \\ &\leq C\|\partial_1\widetilde{u}_1^{(2)}\|_{L^2}\|\overline{u}_1^{(2)}\|_{L^\infty}\|\widetilde{u}_1^{(2)}\|_{L^2} + C\|\partial_1\widetilde{u}_1^{(2)}\|_{L^2}\|\widetilde{u}_1^{(2)}\|_{L^\infty}\|\widetilde{u}_1^{(2)}\|_{L^2} \\ &\leq C\|u\|_{H^2}\|\partial_2\widetilde{u}_2^{(2)}\|_{L^2}^2. \end{aligned}$$

where we have used, due to $\partial_1\overline{u}_1^{(2)} = -\partial_2\overline{u}_2^{(2)} = 0$,

$$\int \partial_1\overline{u}_1^{(2)}\widetilde{u}_1^{(2)}\widetilde{u}_1^{(2)}\,dx = 0.$$

Using a similar argument and replacing to apply the Poincaré inequality (2.2) by (2.1), J_4 can be estimated as

$$\begin{aligned} J_4 &= -2\int \partial_2\widetilde{u}_2^{(2)}\overline{u}_2^{(2)}\widetilde{u}_2^{(2)}\,dx - 2\int \partial_1\widetilde{(\widetilde{u}_2^{(2)})^2}^{(2)}\widetilde{u}_2^{(2)}\,dx \\ &\leq C\|u\|_{H^2}\|\partial_2\widetilde{u}_2^{(2)}\|_{L^2}^2. \end{aligned}$$

Invoking $\partial_2\widetilde{f}^{(2)} = \partial_2 f$ and $\partial_2\overline{f}^{(2)} = 0$, and noticing the following facts

$$\int \partial_2\widetilde{u}_1^{(2)}\overline{u}_2^{(2)}\widetilde{u}_1^{(2)}\,dx = 0,$$

we have

$$J_2 = -\int \partial_2\widetilde{u}_2^{(2)}\overline{u}_1^{(2)}\widetilde{u}_1^{(2)}\,dx - \int \partial_2\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)}\widetilde{u}_1^{(2)}\,dx - \int \partial_2\widetilde{u}_2^{(2)}\widetilde{u}_1^{(2)}\widetilde{u}_1^{(2)}\,dx.$$

Then Hölder's inequality, (2.2) and (2.4) lead to

$$\begin{aligned} J_2 &= -\int \partial_2\widetilde{u}_2^{(2)}\overline{u}_1^{(2)}\widetilde{u}_1^{(2)}\,dx - \int \partial_2\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)}\widetilde{u}_1^{(2)}\,dx - \int \partial_2\widetilde{u}_2^{(2)}\widetilde{u}_1^{(2)}\widetilde{u}_1^{(2)}\,dx \\ &\leq C\|\partial_2\widetilde{u}_2^{(2)}\|_{L^2}\|\overline{u}_1^{(2)}\|_{L^\infty}\|\widetilde{u}_1^{(2)}\|_{L^2} + C\|\partial_2\widetilde{u}_2^{(2)}\|_{L^2}\|\widetilde{u}_1^{(2)}\|_{L^\infty}\|\widetilde{u}_1^{(2)}\|_{L^2} \\ &\quad + C\|\widetilde{u}_1^{(2)}\|_{L^2}^{\frac{1}{2}}(\|\widetilde{u}_1^{(2)}\|_{L^2} + \|\partial_1\widetilde{u}_1^{(2)}\|_{L^2})^{\frac{1}{2}}\|\partial_2\widetilde{u}_2^{(2)}\|_{L^2}\|\partial_2\widetilde{u}_1^{(2)}\|_{L^2} \\ &\leq C\|u\|_{H^2}\|\partial_2\widetilde{u}_2^{(2)}\|_{L^2}^2. \end{aligned}$$

Similarly, J_3 is first rewritten as three parts and employing $\|\widetilde{u}_2^{(2)}\| \leq C\|\partial_2\widetilde{u}_2^{(2)}\|$ and $\|\widetilde{u}_1^{(2)}\| \leq C\|\partial_1\widetilde{u}_1^{(2)}\|$ yields

$$J_3 = -\int \widetilde{u}_1^{(2)}\partial_1\overline{u}_2^{(2)}\widetilde{u}_2^{(2)}\,dx - \int \partial_1(\widetilde{\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)}}^{(2)})\widetilde{u}_2^{(2)}\,dx$$

$$\begin{aligned}
&\leq \|\widetilde{u}_1^{(2)}\|_{L^2}\|\partial_1\widetilde{u}_2^{(2)}\|_{L^4}\|\widetilde{u}_2^{(2)}\|_{L^4}+\|\partial_1\widetilde{u}_1^{(2)}\|_{L^2}\|\widetilde{u}_2^{(2)}\|_{L^\infty}\|\widetilde{u}_2^{(2)}\|_{L^2}\\
&\quad+\|\partial_1\widetilde{u}_2^{(2)}\|_{L^2}\|\widetilde{u}_1^{(2)}\|_{L^\infty}\|\widetilde{u}_2^{(2)}\|_{L^2}\\
&\leq C\|u\|_{H^2}\|\partial_2\widetilde{u}_2^{(2)}\|_{H^1}^2.
\end{aligned}$$

Next, we estimate J_5. By means of Hölder's inequality, Lemma 2.2, Lemma 2.3, and $\|\widetilde{f}^{(2)}\|_{L^2}\leq\|f\|_{L^2}$, J_5 is bounded by

$$\begin{aligned}
J_5 &= -\int\partial_1\widetilde{\theta}^{(2)}\overline{u}_1^{(2)}\widetilde{\theta}^{(2)}\,dx-\int\partial_1\widetilde{u}_1^{(2)}\overline{\theta}^{(2)}\widetilde{\theta}^{(2)}\,dx\\
&\quad-\int\partial_1\overline{\theta}^{(2)}\widetilde{u}_1^{(2)}\widetilde{\theta}^{(2)}\,dx-\int\partial_1\left(\widetilde{\widetilde{u}_1^{(2)}\widetilde{\theta}^{(2)}}\right)^{(2)}\widetilde{\theta}^{(2)}\,dx\\
&\leq \|\partial_1\widetilde{\theta}^{(2)}\|_{L^2}\|\overline{u}_1^{(2)}\|_{L^\infty}\|\widetilde{\theta}^{(2)}\|_{L^2}+\|\partial_1\widetilde{u}_1^{(2)}\|_{L^2}\|\overline{\theta}^{(2)}\|_{L^\infty}\|\widetilde{\theta}^{(2)}\|_{L^2}\\
&\quad+C\|\widetilde{u}_1^{(2)}\|_{L^2}^{\frac12}(\|\widetilde{u}_1^{(2)}\|_{L^2}+\|\partial_1\widetilde{u}_1^{(2)}\|_{L^2})^{\frac12}\|\partial_2\widetilde{\theta}^{(2)}\|_{L^2}\|\partial_1\overline{\theta}^{(2)}\|_{L^2}\\
&\quad+\|\partial_1\widetilde{u}_1^{(2)}\|_{L^2}\|\widetilde{\theta}^{(2)}\|_{L^\infty}\|\widetilde{\theta}^{(2)}\|_{L^2}+\|\widetilde{u}_1^{(2)}\|_{L^\infty}\|\partial_1\widetilde{\theta}^{(2)}\|_{L^2}\|\widetilde{\theta}^{(2)}\|_{L^2}\\
&\leq C(\|u\|_{H^2}+\|\theta\|_{H^2})\|\partial_2\widetilde{u}_2^{(2)}\|_{L^2}\|\partial_2\widetilde{\theta}^{(2)}\|_{H^1},
\end{aligned}$$

where we have used

$$\int\partial_1\overline{u}_1^{(2)}\widetilde{\theta}^{(2)}\widetilde{\theta}^{(2)}\,dx=0.$$

Now to start estimating the last term J_6, we use the above Lemma 2.2 and Hölder's inequality, we get

$$\begin{aligned}
J_6 &= -\int\partial_2\widetilde{u}_2^{(2)}\theta\widetilde{\theta}^{(2)}\,dx-\int\partial_2\widetilde{\theta}^{(2)}u_2\widetilde{\theta}^{(2)}\,dx\\
&\leq \|\partial_2\widetilde{u}_2^{(2)}\|_{L^2}\|\theta\|_{L^\infty}\|\widetilde{\theta}^{(2)}\|_{L^2}+\|\partial_2\widetilde{\theta}^{(2)}\|_{L^2}\|u_2\|_{L^\infty}\|\widetilde{\theta}^{(2)}\|_{L^2}\\
&\leq C(\|u\|_{H^2}+\|\theta\|_{H^2})(\|\partial_2\widetilde{u}_2^{(2)}\|_{L^2}^2+\|\partial_2\widetilde{\theta}^{(2)}\|_{L^2}^2).
\end{aligned}$$

Collecting all estimates above yields

$$\begin{aligned}
&\frac{d}{dt}(\|\widetilde{u}^{(2)}\|_{L^2}^2+\|\widetilde{\theta}^{(2)}\|_{L^2}^2)+2\mu\|\partial_2\widetilde{u}_2^{(2)}\|_{L^2}^2+2\eta\|\partial_2\widetilde{\theta}^{(2)}\|_{L^2}^2\\
&\leq C\|(u,\theta)\|_{H^2}(\|\partial_2\widetilde{u}_2^{(2)}\|_{H^1}^2+\|\partial_2\widetilde{\theta}^{(2)}\|_{H^1}^2).
\end{aligned}\tag{4.9}$$

In what follows, we show the differential inequality of $\|(\nabla\widetilde{u}^{(2)},\nabla\widetilde{\theta}^{(2)})\|_{L^2}$. Taking the gradient of (4.3) and multiplying the resulting equations by $(\nabla\widetilde{u}^{(2)},\nabla\widetilde{\theta}^{(2)})$, we have

$$\begin{aligned}
&\frac12\frac{d}{dt}(\|\nabla\widetilde{u}^{(2)}\|_{L^2}^2+\|\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2)+\mu\|\partial_2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2+\eta\|\partial_2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2\\
&=-\int\partial_1\nabla(u_1^2-\overline{u_1^2}^{(2)})\cdot\nabla\widetilde{u}_1^{(2)}\,dx-\int\partial_2\nabla(u_1u_2-\overline{u_1u_2}^{(2)})\cdot\nabla\widetilde{u}_1^{(2)}\,dx\\
&\quad-\int\partial_1\nabla(u_1u_2-\overline{u_1u_2}^{(2)})\cdot\nabla\widetilde{u}_2^{(2)}\,dx-\int\partial_2\nabla(u_2^2-\overline{u_2^2}^{(2)})\cdot\nabla\widetilde{u}_2^{(2)}\,dx\\
&\quad-\int\partial_1\nabla(u_1\theta-\overline{u_1\theta}^{(2)})\cdot\nabla\widetilde{\theta}^{(2)}\,dx-\int\partial_2\nabla\widetilde{(u_2\theta)}^{(2)}\cdot\nabla\widetilde{\theta}^{(2)}\,dx.
\end{aligned}$$

$$:= K_1 + K_2 + K_3 + K_4 + K_5 + K_6.$$

By integration by parts and (4.5), K_1 is divided into two parts.

$$\begin{aligned} K_1 &= 2\int \nabla(\overline{u}_1^{(2)}\widetilde{u}_1^{(2)}) \cdot \partial_1 \nabla \widetilde{u}_1^{(2)}\, dx + 2\int \nabla \widetilde{\widetilde{u}_1^{(2)}\widetilde{u}_1^{(2)}}^{(2)} \cdot \partial_1 \nabla \widetilde{u}_1^{(2)}\, dx \\ &:= K_{11} + K_{12}. \end{aligned}$$

Due to Hölder's inequality and Lemma 2.2, we obtain

$$\begin{aligned} K_{11} &= 2\int \nabla \overline{u}_1^{(2)} \cdot \widetilde{u}_1^{(2)} \cdot \partial_1 \nabla \widetilde{u}_1^{(2)}\, dx + 2\int \overline{u}_1^{(2)} \cdot \nabla \widetilde{u}_1^{(2)} \cdot \partial_1 \nabla \widetilde{u}_1^{(2)}\, dx \\ &\leq C\|\nabla \overline{u}_1^{(2)}\|_{L^4}\|\widetilde{u}_1^{(2)}\|_{L^4}\|\partial_1 \nabla \widetilde{u}_1^{(2)}\|_{L^2} + C\|\nabla \widetilde{u}_1^{(2)}\|_{L^2}\|\overline{u}_1^{(2)}\|_{L^\infty}\|\partial_1 \nabla \widetilde{u}_1^{(2)}\|_{L^2} \\ &\leq C\|u\|_{H^2}\|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}^2. \end{aligned}$$

Also,

$$\begin{aligned} K_{12} &\leq C\|\nabla \widetilde{u}_1^{(2)} \cdot \widetilde{u}_1^{(2)}\|_{L^2}\|\partial_1 \nabla \widetilde{u}_1^{(2)}\|_{L^2} \\ &\leq C\|\nabla \widetilde{u}_1^{(2)}\|_{L^2}\|\widetilde{u}_1^{(2)}\|_{L^\infty}\|\partial_1 \nabla \widetilde{u}_1^{(2)}\|_{L^2} \\ &\leq C\|u\|_{H^2}\|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}^2. \end{aligned}$$

which together with the estimates for K_{11} gives

$$K_1 \leq C\|u\|_{H^2}\|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}^2. \tag{4.10}$$

Going through a similar process as in the derivation of (4.10), we have

$$K_4 \leq C\|u\|_{H^2}\|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}^2 \tag{4.11}$$

The estimates of K_2 are similar to those of J_2. We divide K_2 into three parts.

$$\begin{aligned} K_2 &= -\int \partial_2 \nabla(\overline{u}_1^{(2)}\widetilde{u}_2^{(2)}) \cdot \nabla \widetilde{u}_1^{(2)} - \int \partial_2 \nabla(\widetilde{u}_1^{(2)}\overline{u}_2) \cdot \nabla \widetilde{u}_1^{(2)} - \int \partial_2 \nabla(\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)}) \cdot \nabla \widetilde{u}_1^{(2)}\, dx \\ &:= K_{21} + K_{22} + K_{23}. \end{aligned}$$

We first bound K_{21}, K_{22}. At first glance, it seems there are eight terms that need to be estimated. However, due to the fact that $\partial_2 \overline{u}^{(2)} = 0$, the decomposition for K_{21} and K_{22} is reduced to three items. Then applying Hölder's inequality, Lemma 2.2, and Lemma 2.3, we obtain

$$\begin{aligned} K_{21} + K_{22} &= -\int \partial_2 \nabla \widetilde{u}_2^{(2)} \cdot \overline{u}_1^{(2)} \cdot \nabla \widetilde{u}_1^{(2)}\, dx - \int \partial_2 \widetilde{u}_2^{(2)} \cdot \nabla \overline{u}_1^{(2)} \cdot \nabla \widetilde{u}_1^{(2)}\, dx \\ &\quad - \int \partial_2 \widetilde{u}_1^{(2)} \cdot \partial_1 \overline{u}_2^{(2)} \cdot \partial_1 \widetilde{u}_1^{(2)}\, dx \\ &\leq C(\|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}\|\overline{u}_1^{(2)}\|_{L^\infty} + \|\partial_2 \widetilde{u}_2^{(2)}\|_{L^4}\|\nabla \overline{u}_1^{(2)}\|_{L^4})\|\nabla \widetilde{u}_1^{(2)}\|_{L^2} \\ &\quad + C\|\partial_2 \widetilde{u}_1^{(2)}\|_{L^2}^{\frac{1}{2}}(\|\partial_2 \widetilde{u}_1^{(2)}\|_{L^2} + \|\partial_1 \partial_2 \widetilde{u}_1^{(2)}\|_{L^2})^{\frac{1}{2}}\|\partial_2 \partial_1 \widetilde{u}_1^{(2)}\|_{L^2}\|\partial_1 \overline{u}_2^{(2)}\|_{L^2} \\ &\leq C\|u\|_{H^2}\|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}^2. \end{aligned}$$

Here we have used

$$-\int \partial_2 \nabla \widetilde{u}_1^{(2)} \cdot \overline{u}_2^{(2)} \cdot \nabla \widetilde{u}_1^{(2)}\, dx = 0,$$

which can be proved by integration by parts and $\partial_2 \overline{u_2}^{(2)} = 0$. For K_{23}, invoking Poincaré inequality (2.1), (2.2), and the anisotropic inequalities (2.3) and (2.4) yields

$$\begin{aligned}
K_{23} &= -\int \partial_2 \nabla \widetilde{u}_1^{(2)} \cdot \widetilde{u}_2^{(2)} \cdot \nabla \widetilde{u}_1^{(2)}\, dx - \int \partial_2 \nabla \widetilde{u}_2^{(2)} \cdot \widetilde{u}_1^{(2)} \cdot \nabla \widetilde{u}_1^{(2)}\, dx \\
&\quad - \int \partial_2 \widetilde{u}_1^{(2)} \cdot \nabla \widetilde{u}_2^{(2)} \cdot \nabla \widetilde{u}_1^{(2)}\, dx - \int \partial_2 \widetilde{u}_2^{(2)} \cdot \nabla \widetilde{u}_1^{(2)} \cdot \nabla \widetilde{u}_1^{(2)}\, dx \\
&\leq C(\|\partial_2 \nabla \widetilde{u}_1^{(2)}\|_{L^2} \|\widetilde{u}_2^{(2)}\|_{L^\infty} + \|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2} \|\widetilde{u}_1^{(2)}\|_{L^\infty}) \|\nabla \widetilde{u}_1^{(2)}\|_{L^2} \\
&\quad + C\|\nabla \widetilde{u}_1^{(2)}\|_{L^2}^{\frac{1}{2}} (\|\nabla \widetilde{u}_1^{(2)}\|_{L^2} + \|\partial_1 \nabla \widetilde{u}_1^{(2)}\|_{L^2})^{\frac{1}{2}} \|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2} \|\partial_2 \widetilde{u}_1^{(2)}\|_{L^2} \\
&\quad + C\|\partial_2 \widetilde{u}_2^{(2)}\|_{L^4} \|\nabla \widetilde{u}_1^{(2)}\|_{L^4} \|\nabla \widetilde{u}_1^{(2)}\|_{L^2} \\
&\leq C\|u\|_{H^2} \|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}^2.
\end{aligned}$$

We now deal with K_3. By (4.7), it naturally divides K_3 into three parts.

$$\begin{aligned}
K_3 &= -\int \partial_1 \nabla (\overline{u}_1^{(2)} \widetilde{u}_2^{(2)}) \cdot \nabla \widetilde{u}_2^{(2)}\, dx - \int \partial_1 \nabla (\widetilde{u}_1^{(2)} \overline{u}_2^{(2)}) \cdot \nabla \widetilde{u}_2^{(2)}\, dx - \int \partial_1 \nabla \widetilde{(\widetilde{u}_1^{(2)} \widetilde{u}_2^{(2)})}^{(2)} \cdot \nabla \widetilde{u}_2^{(2)}\, dx \\
&:= K_{31} + K_{32} + K_{33}.
\end{aligned}$$

Again by the good property $\partial_2 \overline{f}^{(2)} = 0$, K_{31} and K_{32} can be reformulated as

$$\begin{aligned}
K_{31} + K_{32} &= -\int \overline{u}_2^{(2)} \partial_1 \partial_2 \widetilde{u}_2^{(2)} \partial_1 \widetilde{u}_2^{(2)}\, dx - \int \partial_2 \overline{u}_1^{(2)} \partial_1 \widetilde{u}_2^{(2)} \partial_2 \widetilde{u}_2^{(2)}\, dx \\
&\quad - \int \partial_1^2 \overline{u}_2^{(2)} \widetilde{u}_1^{(2)} \partial_1 \widetilde{u}_2^{(2)}\, dx - \int \nabla \widetilde{u}_1^{(2)} \partial_1 \overline{u}_2^{(2)} \nabla \widetilde{u}_2^{(2)}\, dx \\
&\quad - \int \nabla \overline{u}_2^{(2)} \partial_1 \widetilde{u}_1^{(2)} \nabla \widetilde{u}_2^{(2)}\, dx,
\end{aligned}$$

where we use

$$\int \overline{u}_1^{(2)} \partial_1 \nabla \widetilde{u}_2^{(2)} \cdot \nabla \widetilde{u}_2^{(2)}\, dx = 0 \text{ and } \int \partial_1^2 \widetilde{u}_1^{(2)} \overline{u}_1^{(2)}\, \partial_1 \widetilde{u}_2^{(2)}\, dx = 0.$$

Then making full use of (2.1), (2.2), and (2.3), we infer

$$\begin{aligned}
K_{31} + K_{32} &\leq C\|\overline{u}_2^{(2)}\|_{L^\infty} \|\partial_1 \partial_2 \widetilde{u}_2^{(2)}\|_{L^2} \|\partial_1 \widetilde{u}_2^{(2)}\|_{L^2} + C\|\nabla \overline{u}^{(2)}\|_{L^4} \|\partial_2 \widetilde{u}_2^{(2)}\|_{L^4} \|\nabla \widetilde{u}_2^{(2)}\|_{L^2} \\
&\quad + C\|\partial_1^2 \overline{u}_2^{(2)}\|_{L^2} \|\widetilde{u}_1^{(2)}\|_{L^2}^{\frac{1}{2}} (\|\widetilde{u}_1^{(2)}\|_{L^2} + \|\partial_1 \widetilde{u}_1^{(2)}\|_{L^2})^{\frac{1}{2}} \|\partial_1 \partial_2 \widetilde{u}_2^{(2)}\|_{L^2} \\
&\quad + C\|\nabla \widetilde{u}_1^{(2)}\|_{L^2}^{\frac{1}{2}} (\|\nabla \widetilde{u}_1^{(2)}\|_{L^2} + \|\partial_1 \nabla \widetilde{u}_1^{(2)}\|_{L^2})^{\frac{1}{2}} \|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2} \|\partial_1 \overline{u}_2^{(2)}\|_{L^2} \\
&\leq C\|u\|_{H^2} \|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}^2.
\end{aligned}$$

For K_{33}, noticing that by (2.4) and Lemma 2.2

$$\|\widetilde{u}_1^{(2)}\|_{L^\infty} \leq C\|\partial_2 \widetilde{u}_1^{(2)}\|_{L^2}^{\frac{1}{2}} (\|\widetilde{u}_1^{(2)}\|_{L^2} + \|\partial_1 \widetilde{u}_1^{(2)}\|_{L^2})^{\frac{1}{4}}$$

$$
\begin{aligned}
&\times (\|\partial_2 \widetilde{u}_1^{(2)}\|_{L^2} + \|\partial_1\partial_2 \widetilde{u}_1^{(2)}\|_{L^2})^{\frac{1}{4}} \\
&\leq C\|\partial_1\partial_2 \widetilde{u}_1^{(2)}\|_{L^2}.
\end{aligned} \tag{4.12}
$$

Then it can be estimated as follows:

$$
\begin{aligned}
K_{33} &= -\int \partial_1^2 \left(\widetilde{\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)}}\right)^{(2)} \partial_1 \widetilde{u}_2 \, dx + \int \partial_1 \left(\widetilde{\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)}}\right)^{(2)} \partial_2^2 \widetilde{u}_2 \, dx \\
&\leq C\|\partial_1^2(\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)})\|_{L^2}\|\partial_1 \widetilde{u}_2^{(2)}\|_{L^2} + C\|\partial_1(\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)})\|_{L^2}\|\partial_2^2 \widetilde{u}_2^{(2)}\|_{L^2} \\
&\leq C(\|\partial_1^2 \widetilde{u}_1^{(2)}\|_{L^2}\|\widetilde{u}_2^{(2)}\|_{L^\infty} + \|\partial_1^2 \widetilde{u}_2^{(2)}\|_{L^2}\|\widetilde{u}_1^{(2)}\|_{L^\infty} + \|\partial_1 \widetilde{u}_1^{(2)}\|_{L^4}\|\partial_1 \widetilde{u}_2^{(2)}\|_{L^4})\|\partial_1 \widetilde{u}_2^{(2)}\|_{L^2} \\
&\quad + C(\|\partial_1 \widetilde{u}_1^{(2)}\|_{L^2}\|\widetilde{u}_2^{(2)}\|_{L^\infty} + \|\partial_1 \widetilde{u}_2^{(2)}\|_{L^2}\|\widetilde{u}_1^{(2)}\|_{L^\infty})\|\partial_2^2 \widetilde{u}_2^{(2)}\|_{L^2} \\
&\leq C\|u\|_{H^2}\|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}^2.
\end{aligned}
$$

Now we focus on estimating the term K_5, we are able to establish the upper bound in a similar way as in J_5. Since K_5 has more terms, according to (4.8), this can be divided into three terms,

$$
\begin{aligned}
K_5 &= -\int \partial_1 \nabla \left(\overline{u}_1^{(2)}\widetilde{\theta}^{(2)} + \widetilde{u}_1^{(2)}\overline{\theta}^{(2)} + \left(\widetilde{\widetilde{u}_1^{(2)}\widetilde{\theta}^{(2)}}\right)^{(2)}\right) \cdot \nabla \widetilde{\theta}^{(2)} \, dx \\
&:= K_{51} + K_{52} + K_{53}.
\end{aligned}
$$

We proceed to estimate each of these three items separately, owing to $\nabla \cdot \overline{u}^{(2)} = 0$ and $\partial_2 \overline{f}^{(2)} = 0$, K_{51} passes through the decomposition with only one term,

$$
\begin{aligned}
K_{51} &= -\int \partial_1 \widetilde{\theta}^{(2)} \nabla \overline{u}_1^{(2)} \cdot \nabla \widetilde{\theta}^{(2)} \, dx \\
&\leq C\|\nabla \overline{u}_1^{(2)}\|_{L^2}^{\frac{1}{2}}(\|\nabla \overline{u}_1^{(2)}\|_{L^2} + \|\partial_1 \nabla \overline{u}_1^{(2)}\|_{L^2})^{\frac{1}{2}}\|\partial_2\partial_1 \widetilde{\theta}^{(2)}\|_{L^2}\|\nabla \widetilde{\theta}^{(2)}\|_{L^2} \\
&\leq C\|u\|_{H^2}\|\partial_2 \nabla \widetilde{\theta}^{(2)}\|_{L^2}^2.
\end{aligned}
$$

Applying (2.1) and (2.3) yields

$$
\begin{aligned}
K_{52} &= -\int (\partial_1 \nabla \widetilde{u}_1^{(2)}\overline{\theta}^{(2)} + \partial_1 \nabla \overline{\theta}^{(2)}\widetilde{u}_1^{(2)} + \partial_1 \widetilde{u}_1^{(2)} \nabla \overline{\theta}^{(2)} + \partial_1 \overline{\theta}^{(2)} \nabla \widetilde{u}_1^{(2)}) \cdot \nabla \widetilde{\theta}^{(2)} \, dx \\
&\leq \|\partial_1 \nabla \widetilde{u}_1^{(2)}\|_{L^2}\|\overline{\theta}^{(2)}\|_{L^\infty}\|\nabla \widetilde{\theta}^{(2)}\|_{L^2} + \|\partial_1 \widetilde{u}_1^{(2)}\|_{L^4}\|\nabla \overline{\theta}^{(2)}\|_{L^4}\|\nabla \widetilde{\theta}^{(2)}\|_{L^2} \\
&\quad + C\|\widetilde{u}_1^{(2)}\|_{L^2}^{\frac{1}{2}}(\|\widetilde{u}_1^{(2)}\|_{L^2} + \|\partial_1 \widetilde{u}_1^{(2)}\|_{L^2})^{\frac{1}{2}}\|\partial_2 \nabla \widetilde{\theta}^{(2)}\|_{L^2}\|\partial_1 \nabla \overline{\theta}^{(2)}\|_{L^2} \\
&\quad + C\|\nabla \widetilde{u}_1^{(2)}\|_{L^2}^{\frac{1}{2}}(\|\nabla \widetilde{u}_1^{(2)}\|_{L^2} + \|\partial_1 \nabla \widetilde{u}_1^{(2)}\|_{L^2})^{\frac{1}{2}}\|\partial_2 \nabla \widetilde{\theta}^{(2)}\|_{L^2}\|\partial_1 \overline{\theta}^{(2)}\|_{L^2} \\
&\leq C(\|u\|_{H^2} + \|\theta\|_{H^2})(\|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}^2 + \|\partial_2 \nabla \widetilde{\theta}^{(2)}\|_{L^2}^2).
\end{aligned}
$$

By Hölder's inequality and (4.12), we obtain

$$
\begin{aligned}
K_{53} &= -\int \partial_1^2 \left(\widetilde{\widetilde{u}_1^{(2)}\widetilde{\theta}^{(2)}}\right)^{(2)} \partial_1 \widetilde{\theta}^{(2)} \, dx + \int \partial_1 \left(\widetilde{\widetilde{u}_1^{(2)}\widetilde{\theta}^{(2)}}\right)^{(2)} \partial_2^2 \widetilde{\theta}^{(2)} \, dx. \\
&\leq \|\partial_1^2(\widetilde{u}_1^{(2)}\widetilde{\theta}^{(2)})\|_{L^2}\|\partial_1 \widetilde{\theta}^{(2)}\|_{L^2} + \|\partial_1(\widetilde{u}_1^{(2)}\widetilde{\theta}^{(2)})\|_{L^2}\|\partial_2^2 \widetilde{\theta}^{(2)}\|_{L^2} \\
&\leq (\|\partial_1^2 \widetilde{u}_1^{(2)}\|_{L^2}\|\widetilde{\theta}^{(2)}\|_{L^\infty} + \|\partial_1^2 \widetilde{\theta}^{(2)}\|_{L^2}\|\widetilde{u}_1^{(2)}\|_{L^\infty} + \|\partial_1 \widetilde{u}_1^{(2)}\|_{L^4}\|\partial_1 \widetilde{\theta}^{(2)}\|_{L^4})\|\partial_1 \widetilde{\theta}^{(2)}\|_{L^2} \\
&\quad + (\|\partial_1 \widetilde{u}_1^{(2)}\|_{L^2}\|\widetilde{\theta}^{(2)}\|_{L^\infty} + \|\partial_1 \widetilde{\theta}^{(2)}\|_{L^2}\|\widetilde{u}_1^{(2)}\|_{L^\infty})\|\partial_2^2 \widetilde{\theta}^{(2)}\|_{L^2}
\end{aligned}
$$

$$\leq \quad C(\|u\|_{H^2}+\|\theta\|_{H^2})(\|\partial_2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2+\|\partial_2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2).$$

Finally, we estimate the last term, this can be decomposed into four terms by a similar method as in J_6 to obtain

$$K_6 \leq C(\|u\|_{H^2}+\|\theta\|_{H^2})(\|\partial_2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2+\|\partial_2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2).$$

Combining the estimates for K_1 through K_6, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla\widetilde{u}^{(2)}\|_{L^2}^2+\|\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2)+\mu\|\partial_2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2+\eta\|\partial_2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2\\ &\leq C(\|u\|_{H^2}+\|\theta\|_{H^2})(\|\partial_2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2+\|\partial_2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2),\end{aligned}$$

which together with (4.9) derives, for a pure constant C_1

$$\begin{aligned}&\frac{d}{dt}(\|\widetilde{u}^{(2)}\|_{H^1}^2+\|\widetilde{\theta}^{(2)}\|_{H^1}^2)+(2\mu-C_1(\|u\|_{H^2}+\|\theta\|_{H^2}))\|\partial_2\widetilde{u}_2^{(2)}\|_{H^1}^2\\ &+(2\eta-C_1(\|u\|_{H^2}+\|\theta\|_{H^2}))\|\partial_2\widetilde{\theta}^{(2)}\|_{H^1}^2\leq 0.\end{aligned}$$

Recalling the stability result in Theorem 1.1, we can select $\delta>0$ in (1.5) to be sufficiently small such that

$$2\mu-C_1(\|u\|_{H^2}+\|\theta\|_{H^2})>\mu,\ \ 2\eta-C_1(\|u\|_{H^2}+\|\theta\|_{H^2})>\eta.$$

Moreover, using Poincaré-type inequalities in Lemma 2.2 yields

$$\begin{aligned}\|\widetilde{u}^{(2)}\|_{H^1}^2 &= \|\widetilde{u}_1^{(2)}\|_{H^1}^2+\|\widetilde{u}_2^{(2)}\|_{H^1}^2\leq C\|\partial_1\widetilde{u}_1^{(2)}\|_{H^1}^2+C\|\partial_2\widetilde{u}_2^{(2)}\|_{H^1}^2\leq C\|\partial_2\widetilde{u}_2^{(2)}\|_{H^1}^2,\\ \|\widetilde{\theta}^{(2)}\|_{H^1}^2 &= \|\widetilde{\theta}^{(2)}\|_{L^2}^2+\|\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2\leq C\|\partial_2\widetilde{\theta}_2\|_{H^1}^2.\end{aligned}$$

Then we have

$$\frac{d}{dt}(\|\widetilde{u}^{(2)}\|_{H^1}^2+\|\widetilde{\theta}^{(2)}\|_{H^1}^2)+\min\{\mu,\eta\}(\|\widetilde{u}^{(2)}\|_{H^1}^2+\|\widetilde{\theta}^{(2)}\|_{H^1}^2)\leq 0.$$

which implies the exponential decay (1.16) in Theorem1.4.

Step 2. Decay for $\|(\partial_2\nabla\widetilde{u}^{(2)},\partial_2\nabla\widetilde{\theta}^{(2)})\|_{L^2}$

The routine and the procedure of the proof are similar to $\|(\widetilde{u}^{(2)},\widetilde{\theta}^{(2)})\|_{H^1}^2$. Applying $\partial_2\nabla$ operator to (4.3), multiplying the resulting equations by $(\partial_2\nabla\widetilde{u}^{(2)},\partial_2\nabla\widetilde{\theta}^{(2)})$ and then integrating over Ω yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\partial_2\nabla\widetilde{u}^{(2)}\|_{L^2}^2+\|\partial_2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2)+\mu\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2+\eta\|\partial_2^2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2\\ =\ &-\int\partial_1\partial_2\nabla(u_1^2-\overline{u}_1^{(2)})\cdot\partial_2\nabla\widetilde{u}_1^{(2)}\,dx-\int\partial_2^2\nabla(u_1u_2-\overline{u_1u_2}^{(2)})\cdot\partial_2\nabla\widetilde{u}_1^{(2)}\,dx\\ &-\int\partial_1\partial_2\nabla(u_1u_2-\overline{u_1u_2}^{(2)})\cdot\partial_2\nabla\widetilde{u}_2^{(2)}\,dx-\int\partial_2^2\nabla(u_2^2-\overline{u}_2^{(2)})\cdot\partial_2\nabla\widetilde{u}_2^{(2)}\,dx\\ &-\int\partial_1\partial_2\nabla(u_1\theta-\overline{u_1\theta}^{(2)})\cdot\partial_2\nabla\widetilde{\theta}^{(2)}\,dx-\int\partial_2^2\nabla\widetilde{(u_2\theta)}^{(2)}\cdot\partial_2\nabla\widetilde{\theta}^{(2)}\,dx\\ :=\ &L_1+L_2+L_3+L_4+L_5+L_6.\end{aligned}$$

The estimates for L_1 and L_4 are simple. Invoking (4.5) along with $\partial_2 \overline{f}^{(2)} = 0$, applying integration by parts, $u_1 = \widetilde{u}_1^{(2)} + \overline{u}_1^{(2)}$ and $\nabla u_1 = \nabla \widetilde{u}_1^{(2)} + \nabla \overline{u}_1^{(2)}$, we obtain

$$\begin{aligned} L_1 &= \int \partial_2 \nabla (2\overline{u}_1^{(2)} \widetilde{u}_1^{(2)} + \widetilde{(\widetilde{u}_1)^2}^{(2)}) \partial_1 \partial_2 \nabla \widetilde{u}_1^{(2)} \, dx \\ &= 2 \int (\partial_2 \nabla \widetilde{u}_1^{(2)} u_1 + \partial_2 \widetilde{u}_1^{(2)} \nabla u_1) \partial_1 \partial_2 \nabla \widetilde{u}_1^{(2)} \, dx \\ &\le C \|\partial_2 \nabla \widetilde{u}_1^{(2)}\|_{L^2} \|u_1\|_{L^\infty} \|\partial_1 \partial_2 \nabla \widetilde{u}_1^{(2)}\|_{L^2} \\ &\quad + C \|\nabla u_1\|_{L^2}^{\frac{1}{2}} (\|\nabla u_1\|_{L^2} + \|\partial_1 \nabla u_1\|_{L^2})^{\frac{1}{2}} \|\partial_2^2 \widetilde{u}_1^{(2)}\|_{L^2} \|\partial_1 \partial_2 \nabla \widetilde{u}_1^{(2)}\|_{L^2} \\ &\le C \|u\|_{H^2} \|\partial_2^2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}^2. \end{aligned} \tag{4.13}$$

Similarly,

$$\begin{aligned} L_4 &= \int \partial_2 \nabla (2\overline{u}_2^{(2)} \widetilde{u}_2^{(2)} + \widetilde{(\widetilde{u}_2^{(2)})^2}^{(2)}) \partial_2^2 \nabla \widetilde{u}_2^{(2)} \, dx \\ &= 2 \int (\partial_2 \nabla \widetilde{u}_2^{(2)} \overline{u}_2^{(2)} + \partial_2 \widetilde{u}_2^{(2)} \nabla \overline{u}_2^{(2)} + \partial_2 \nabla \widetilde{u}_2^{(2)} \widetilde{u}_2^{(2)} + \partial_2 \widetilde{u}_2^{(2)} \nabla \widetilde{u}_2^{(2)}) \partial_2^2 \nabla \widetilde{u}_2^{(2)} \, dx \\ &\le C \|\partial_2 \nabla \widetilde{u}_2^{(2)}\|_{L^2} \|u_2\|_{L^\infty} \|\partial_2^2 \nabla \widetilde{u}_2^{(2)}\|_{L^2} \\ &\quad + C \|\nabla u_2\|_{L^2}^{\frac{1}{2}} (\|\nabla u_2\|_{L^2} + \|\partial_1 \nabla u_2\|_{L^2})^{\frac{1}{2}} \|\partial_2^2 \widetilde{u}_2^{(2)}\|_{L^2} \|\partial_2^2 \nabla \widetilde{u}_2^{(2)}\|_{L^2} \\ &\le C \|u\|_{H^2} \|\partial_2^2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}^2. \end{aligned} \tag{4.14}$$

To estimate L_2, we first divide it into two terms according to $i = 1$ and $i = 2$.

$$\begin{aligned} L_2 &= \int \partial_1 \partial_2 (u_1 u_2 - \overline{u_1 u_2}^{(2)}) \cdot \partial_1 \partial_2^2 \widetilde{u}_1^{(2)} - \int \partial_2^3 (u_1 u_2 - \overline{u_1 u_2}^{(2)}) \cdot \partial_2^2 \widetilde{u}_1^{(2)} \, dx \\ &:= L_{21} + L_{22}. \end{aligned}$$

Again, based on the fact $\partial_2 \overline{f}^{(2)} = 0$, L_{21} is further decomposed as follows:

$$\begin{aligned} L_{21} &= \int \partial_1 \partial_2 (\overline{u}_1^{(2)} \widetilde{u}_2^{(2)} + \widetilde{u}_1^{(2)} \overline{u}_2^{(2)} + \widetilde{\widetilde{u}_1^{(2)} \widetilde{u}_2^{(2)}}^{(2)}) \partial_1 \partial_2^2 \widetilde{u}_1^{(2)} \, dx \\ &= \int \partial_1 \partial_2 \widetilde{u}_2^{(2)} \overline{u}_1^{(2)} \partial_1 \partial_2^2 \widetilde{u}_1^{(2)} \, dx + \int (\partial_1 \partial_2 \widetilde{u}_1^{(2)} \overline{u}_2^{(2)} + \partial_1 \overline{u}_2^{(2)} \partial_2 \widetilde{u}_1^{(2)}) \partial_1 \partial_2^2 \widetilde{u}_1^{(2)} \, dx \\ &\quad + \int (\partial_1 \partial_2 \widetilde{u}_1^{(2)} \widetilde{u}_2^{(2)} + \partial_1 \partial_2 \widetilde{u}_2^{(2)} \widetilde{u}_1^{(2)} + \partial_1 \widetilde{u}_1^{(2)} \partial_2 \widetilde{u}_2^{(2)} + \partial_1 \widetilde{u}_2^{(2)} \partial_2 \widetilde{u}_1^{(2)}) \partial_1 \partial_2^2 \widetilde{u}_1^{(2)} \, dx. \end{aligned}$$

Then Poincaré inequality (2.2), (2.6) along with $\|\widetilde{f}^{(2)}\|_{L^2}, \|\overline{f}^{(2)}\|_{L^2} \le \|f\|_{L^2}$ leads to

$$\begin{aligned} L_{21} &\le C (\|\partial_1 \partial_2 \widetilde{u}_2^{(2)}\|_{L^2} \|u_1\|_{L^\infty} + \|\partial_1 \partial_2 \widetilde{u}_1^{(2)}\|_{L^2} \|u_2\|_{L^\infty} + \|\partial_1 \widetilde{u}_1^{(2)}\|_{L^4} \|\partial_2 \widetilde{u}_2^{(2)}\|_{L^4}) \|\partial_1 \partial_2^2 \widetilde{u}_1^{(2)}\|_{L^2} \\ &\quad + C \|\partial_1 u_2\|_{L^2}^{\frac{1}{2}} (\|\partial_1 u_2\|_{L^2} + \|\partial_1^2 u_2\|_{L^2})^{\frac{1}{2}} \|\partial_2^2 \widetilde{u}_1^{(2)}\|_{L^2} \|\partial_1 \partial_2^2 \widetilde{u}_1^{(2)}\|_{L^2} \\ &\le C \|u\|_{H^2} \|\partial_2^2 \nabla \widetilde{u}_2^{(2)}\|_{L^2}^2. \end{aligned}$$

Similarly, by Lemma 2.2 and Lemma 2.3 we obtain

$$L_{22} = \int \partial_2^3 (\overline{u}_1^{(2)} \widetilde{u}_2^{(2)} + \widetilde{u}_1^{(2)} \overline{u}_2^{(2)} + \widetilde{\widetilde{u}_1^{(2)} \widetilde{u}_2^{(2)}}^{(2)}) \partial_2^2 \widetilde{u}_1^{(2)} \, dx$$

$$\begin{aligned}
&= \int (\overline{u}_1^{(2)}\partial_2^3\widetilde{u}_2^{(2)} + \widetilde{u}_1^{(2)}\partial_2^3\widetilde{u}_2^{(2)} + 3\partial_2\overline{u}_1^{(2)}\partial_2^2\widetilde{u}_2^{(2)} + \frac{5}{2}\partial_2^2\widetilde{u}_1^{(2)}\partial_2\widetilde{u}_2^{(2)})\partial_2^2\widetilde{u}_1^{(2)}\,dx \\
&\le C\|u\|_{L^\infty}\|\partial_2^3\widetilde{u}_2^{(2)}\|_{L^2}\|\partial_2^2\widetilde{u_1}^{(2)}\|_{L^2} + C\|\partial_2\widetilde{u}_2^{(2)}\|_{L^\infty}\|\partial_2^2\widetilde{u}_1^{(2)}\|_{L^2}^2 \\
&\quad +C\|\partial_2\widetilde{u}_1^{(2)}\|_{L^2}^{\frac{1}{2}}(\|\partial_2\widetilde{u}_1^{(2)}\|_{L^2} + \|\partial_1\partial_2\widetilde{u}_1^{(2)}\|_{L^2})^{\frac{1}{2}}\|\partial_2^3\widetilde{u}_2^{(2)}\|_{L^2}\|\partial_2^2\widetilde{u}_1^{(2)}\|_{L^2} \\
&\le C\|u\|_{H^2}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2,
\end{aligned}$$

where we have used, by integration by parts,

$$\int \partial_2^3\widetilde{u}_1^{(2)}\,\overline{u}_2^{(2)}\,\partial_1^2\widetilde{u}_1^{(2)} = 0,\ \int \widetilde{u}_2^{(2)}\partial_1^3\widetilde{u}_1^{(2)}\,\partial_1^2\widetilde{u}_1^{(2)}\,dx = -\frac{1}{2}\int \partial_2^2\widetilde{u}_1^{(2)}\partial_2\widetilde{u}_2^{(2)}\partial_2^2\widetilde{u}_1^{(2)}\,dx.$$

Thus,

$$L_2 \le C\|u\|_{H^2}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2. \tag{4.15}$$

The bound for L_3 is subtle. We first rewrite it as

$$\begin{aligned}
L_3 &= \int \partial_1\nabla(\overline{u}_1^{(2)}\widetilde{u}_2^{(2)} + \widetilde{u}_1^{(2)}\overline{u}_2^{(2)} + \widetilde{\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)}}^{(2)})\partial_2^2\nabla\widetilde{u}_2^{(2)}\,dx \\
&:= L_{31} + L_{32} + L_{33}.
\end{aligned}$$

Observe that $\nabla\overline{u}_1^{(2)} = 0$ and

$$\int \partial_1^2\widetilde{u}_2^{(2)}\overline{u}_1^{(2)}\partial_2^2\partial_1\widetilde{u}_2^{(2)}\,dx = 0,$$

which can be verified by integration by parts. It is easy to see

$$\begin{aligned}
L_{31} &= \int \partial_1\partial_2\widetilde{u}_2^{(2)}\overline{u}_1^{(2)}\partial_2^3\widetilde{u}_2^{(2)}\,dx \\
&\le C\|\partial_1\partial_2\widetilde{u}_2^{(2)}\|_{L^2}\|\overline{u}_1^{(2)}\|_{L^\infty}\|\partial_2^3\widetilde{u}_2^{(2)}\|_{L^2} \\
&\le C\|u\|_{H^2}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2.
\end{aligned}$$

Also, by Lemma 2.2 and (2.4), we infer

$$\begin{aligned}
L_{32} &= \int (\partial_1\nabla\widetilde{u}_1^{(2)}\overline{u}_2^{(2)} + \partial_1\nabla\widetilde{u}_2^{(2)}\overline{u}_1^{(2)} + \partial_1\widetilde{u}_1^{(2)}\nabla\overline{u}_2^{(2)} + \partial_1\overline{u}_2^{(2)}\nabla\widetilde{u}_1^{(2)})\cdot\partial_2^2\nabla\widetilde{u}_2^{(2)}\,dx \\
&\le C\|\partial_1\nabla\widetilde{u}_1^{(2)}\|_{L^2}\|\overline{u}_2^{(2)}\|_{L^\infty}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2} + C\|\partial_1\nabla\widetilde{u}_2^{(2)}\|_{L^2}\|\overline{u}_1^{(2)}\|_{L^\infty}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2} \\
&\quad +C\|\nabla\overline{u}_2^{(2)}\|_{L^2}^{\frac{1}{2}}(\|\nabla\overline{u}_2^{(2)}\|_{L^2} + \|\partial_1\nabla\overline{u}_2^{(2)}\|_{L^2})^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u}_1^{(2)}\|_{L^2}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2} \\
&\quad +C\|\partial_1\overline{u}_2^{(2)}\|_{L^2}^{\frac{1}{2}}(\|\partial_1\overline{u}_2^{(2)}\|_{L^2} + \|\partial_1^2\overline{u}_2^{(2)}\|_{L^2})^{\frac{1}{2}}\|\partial_2\nabla\widetilde{u}_1^{(2)}\|_{L^2}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2} \\
&\le C\|u\|_{H^2}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2.
\end{aligned}$$

where we have used

$$\|\widetilde{u}_1^{(2)}\|_{L^\infty} \le C\|\partial_2\nabla\widetilde{u}_1^{(2)}\|_{L^2} \le C\|\partial_2\partial_1\nabla\widetilde{u}_1^{(2)}\|_{L^2}.$$

For L_{33}, according to Lemma 2.1, $\partial_1\nabla\widetilde{f}^{(2)} = \widetilde{\partial_1\nabla f}^{(2)}$, and $\partial_1\nabla(\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)})$ can be decomposed into these four terms:

$$\partial_1\nabla(\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)}) = \partial_1\nabla\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)} + \partial_1\nabla\widetilde{u}_2^{(2)}\widetilde{u}_1^{(2)} + \partial_1\widetilde{u}_1^{(2)}\nabla\widetilde{u}_2^{(2)} + \partial_1\widetilde{u}_2^{(2)}\nabla\widetilde{u}_1^{(2)}.$$

Then, invoking $\|\widetilde{f}^{(2)}\|_{L^2} \le \|f\|_{L^2}$, (2.1), (2.5), and (4.12), L_{33} can be bounded,

$$\begin{aligned}
L_{33} \le\ & C(\|\partial_1\nabla\widetilde{u}_1^{(2)}\widetilde{u}_2^{(2)}\|_{L^2} + \|\partial_1\nabla\widetilde{u}_2^{(2)}\widetilde{u}_1^{(2)}\|_{L^2})\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2} \\
&+C(\|\partial_1\widetilde{u}_1^{(2)}\nabla\widetilde{u}_2^{(2)}\|_{L^2} + \|\partial_1\widetilde{u}_2^{(2)}\nabla\widetilde{u}_1^{(2)}\|_{L^2})\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2} \\
\le\ & C(\|\partial_1\nabla\widetilde{u}_1^{(2)}\|_{L^2}\|\widetilde{u}_2^{(2)}\|_{L^\infty} + \|\partial_1\nabla\widetilde{u}_2^{(2)}\|_{L^2}\|\widetilde{u}_1^{(2)}\|_{L^\infty})\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2} \\
&+C\|\partial_1\widetilde{u}_1^{(2)}\|_{L^\infty}\|\nabla\widetilde{u}_2^{(2)}\|_{L^2}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2} \\
&+C\|\nabla\widetilde{u}_1^{(2)}\|_{L^2}^{\frac{1}{2}}(\|\nabla\widetilde{u}_1^{(2)}\|_{L^2} + \|\partial_1\nabla\widetilde{u}_1^{(2)}\|_{L^2})^{\frac{1}{2}}\|\partial_1\partial_2\widetilde{u}_2^{(2)}\|_{L^2}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2} \\
\le\ & C\|u\|_{H^2}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2.
\end{aligned}$$

Therefore,

$$L_3 \le C\|u\|_{H^2}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2. \tag{4.16}$$

With reference to K_5, L_5 can be first shown as follows:

$$\begin{aligned}
L_5 &= \int \partial_1\nabla(\overline{u}_1^{(2)}\widetilde{\theta}^{(2)} + \widetilde{u}_1^{(2)}\overline{\theta}^{(2)} + \widetilde{(\widetilde{u}_1^{(2)}\widetilde{\theta}^{(2)})}^{(2)})\cdot\partial_2^2\nabla\widetilde{\theta}^{(2)}\,dx \\
&:= L_{51} + L_{52} + L_{53}.
\end{aligned}$$

Applying the equality $\partial_2\overline{f}^{(2)} = 0$ and $\partial_1\overline{u}_1^{(2)} = 0$, using integration by parts, we then obtain

$$\begin{aligned}
L_{51} &= -\int \overline{u}_1^{(2)}\partial_1\partial_2\nabla\widetilde{\theta}\cdot\partial_2\nabla\widetilde{\theta}^{(2)}\,dx \\
&= \frac{1}{2}\int \partial_1\overline{u}_1^{(2)}\partial_2\nabla\widetilde{\theta}^{(2)}\cdot\partial_2\nabla\widetilde{\theta}^{(2)}\,dx = 0.
\end{aligned}$$

Using (2.3) again together with (2.6) and (2.2), L_{52} can be bounded as

$$\begin{aligned}
L_{52} =\ & \int(\partial_1\nabla\widetilde{u}_1^{(2)}\overline{\theta}^{(2)} + \partial_1\nabla\overline{\theta}^{(2)}\widetilde{u}_1^{(2)} + \partial_1\widetilde{u}_1^{(2)}\nabla\overline{\theta}^{(2)} + \partial_1\overline{\theta}^{(2)}\nabla\widetilde{u}_1^{(2)})\cdot\partial_2^2\nabla\widetilde{\theta}^{(2)}\,dx \\
\le\ & (\|\partial_1\nabla\widetilde{u}_1^{(2)}\|_{L^2}\|\overline{\theta}^{(2)}\|_{L^\infty} + \|\partial_1\nabla\overline{\theta}^{(2)}\|_{L^2}\|\widetilde{u}_1^{(2)}\|_{L^\infty} + \|\partial_1\widetilde{u}_1^{(2)}\|_{L^4}\|\nabla\overline{\theta}^{(2)}\|_{L^4})\|\partial_2^2\nabla\widetilde{\theta}^{(2)}\|_{L^2} \\
&+C\|\nabla\widetilde{u}_1^{(2)}\|_{L^2}^{\frac{1}{2}}(\|\nabla\widetilde{u}_1^{(2)}\|_{L^2} + \|\nabla\partial_1\widetilde{u}_1^{(2)}\|_{L^2})^{\frac{1}{2}}\|\partial_1\partial_2\overline{\theta}_2^{(2)}\|_{L^2}\|\partial_2^2\nabla\widetilde{\theta}^{(2)}\|_{L^2} \\
\le\ & C\|\theta\|_{H^2}\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}\|\partial_2^2\nabla\widetilde{\theta}^{(2)}\|_{L^2}.
\end{aligned}$$

Similarly to K_{53}, L_{53} can be obtained as follows:

$$L_{53} \le C\|(u,\theta)\|_{H^2}(\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2 + \|\partial_2^2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2).$$

Combining all estimates above for L_{51}, L_{52}, and L_{53}, we obtain

$$L_5 \le C\|(u,\theta)\|_{H^2}(\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2 + \|\partial_2^2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2). \tag{4.17}$$

After integration by parts, L_6 is split into four terms

$$L_6 = \int(\partial_2\nabla\widetilde{u}_2^{(2)}\theta + \partial_2\nabla\widetilde{\theta}^{(2)}u_2 + \partial_2\widetilde{u}_2^{(2)}\nabla\theta + \partial_2\widetilde{\theta}^{(2)}\nabla u_2)\cdot\partial_2^2\nabla\widetilde{\theta}^{(2)}\,dx,$$

Similarly,

$$L_6 \leq C\|(u,\theta)\|_{H^2}(\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2 + \|\partial_2^2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2). \tag{4.18}$$

As a consequence of (4.13), (4.14), (4.15), (4.16), (4.17), and (4.18), we conclude that there exist two constants C_3 and C_4 such that

$$\begin{aligned}&\frac{d}{dt}(\|\partial_2\nabla\widetilde{u}^{(2)}\|_{L^2}^2 + \|\partial_2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2) + (2\mu - C_3(\|u\|_{H^2} + \|\theta\|_{H^2}))\|\partial_2^2\nabla\widetilde{u}_2^{(2)}\|_{L^2}^2\\&+(2\eta - C_4(\|u\|_{H^2} + \|\theta\|_{H^2}))\|\partial_2^2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2 \leq 0.\end{aligned} \tag{4.19}$$

Then (4.19) along with the stability result of Theorem 1.1 implies

$$\|\partial_2\nabla\widetilde{u}^{(2)}\|_{L^2}^2 + \|\partial_2\nabla\widetilde{\theta}^{(2)}\|_{L^2}^2 \leq Ce^{-c_1 t},$$

for some positive constants C_3, C_4, provided that the initial data is suitable to satisfy

$$2\mu - C_3(\|u\|_{H^2} + \|\theta\|_{H^2}) > \mu,$$

$$2\eta - C_4(\|u\|_{H^2} + \|\theta\|_{H^2}) > \eta.$$

We thus complete the proof of Theorem 1.4.

□

Author contributions

All authors have accepted responsibility for the entire content of this manuscript and consented to its submission to the journal, reviewed all the results, and approved the final version of the manuscript. Hongxia Lin, Sabana, and Qing Sun made for mathematical analysis and the derivation of the proof. Qing Sun and Ruiqi You prepared the original manuscript with contributions from all co-authors. Sabana Checked all English editing and grammar. Xiaochuan Guo performed the review and revision.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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