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Research article

Boundary Riesz potential estimates for parabolic equations with measurable nonlinearities

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Abstract: We obtain a boundary pointwise gradient estimate on a parabolic half cube $Q_{2R}\cap\{(x^1,x',t)\in\mathbb{R}^{n+1} : x^1>0\}$ for nonlinear parabolic equations with measurable nonlinearities, which are only assumed to be measurable in x^1-variable. The estimates are obtained in terms of Riesz potential of the right-hand side measure and the oscillation of the boundary data, where the boundary data is given on $Q_{2R}\cap\{(x^1,x',t)\in\mathbb{R}^{n+1} : x^1=0\}$.

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1. Introduction

In this paper, we consider parabolic equations with measurable nonlinearities and measure data

$$\begin{cases} u_t - \operatorname{div} a(Du,x^1,x',t) &= \mu \quad \text{in} \quad Q_{2R}^+, \\ u &= \psi \quad \text{on} \quad T_{2R}, \end{cases} \tag{1.1}$$

where μ is a radon measure with $|\mu|(Q_{2R}^+)<\infty$. Here, the parabolic half cube with the size ρ is denoted as $Q_\rho^+=(0,\rho)\times(-\rho,\rho)^{n-1}\times(-\rho^2,0)$ and the parabolic hyperplane with the size ρ is denoted as $T_\rho=\{0\}\times(-\rho,\rho)^{n-1}\times(-\rho^2,0)$.

We will obtain the pointwise gradient estimate of u in terms of Riesz potential of the right-hand side μ. Here, Riesz potential of μ is defined as

$$I_\alpha^{|\mu|}(x,t,r)=\int_0^r \frac{|\mu|(Q_\rho(x,t))}{\rho^{n+2-\alpha}}\frac{d\rho}{\rho} \qquad \Big((x,t)\in\mathbb{R}^{n+1},\ r>0,\ 0<\alpha<n\Big).$$

For the boundary data ψ, we measure the pointwise oscillation of the gradient in x'-variable and L^2-norm of the time derivative:

$$\int_0^r \left(\operatorname*{osc}_{T_\rho} D_{x'}\psi + \rho^2 \|\partial_t \psi\|_{L^2(T_\rho)} \right) \frac{d\rho}{\rho}.$$

For the ellipticity constants $0 < \lambda \leq \Lambda$, suppose that the nonlinearities $a(\xi, x, t)$ satisfy that

$$\begin{cases} a(\xi, x, t) \text{ is measurable in } (x, t) \text{ for every } \xi \in \mathbb{R}^n, \\ a(\xi, x, t) \text{ is } C^1\text{-regular in } \xi \text{ for almost every } (x, t) \in \mathbb{R}^{n+1}, \end{cases}$$

and

$$\begin{cases} |a(\xi, x, t)| & \leq & \Lambda|\xi|, \\ |D_\xi a(\xi, x, t)| & \leq & \Lambda, \\ \langle D_\xi a(\xi, x, t)\zeta, \zeta\rangle & \geq & \lambda|\zeta|^2, \end{cases} \tag{1.2}$$

for any $(x, t) \in \mathbb{R}^{n+1}$, $\xi \in \mathbb{R}^n$ and $\zeta \in \mathbb{R}^n$. Also suppose that $a(\xi, x, t) = a(\xi, x^1, x', t)$ is Dini-continuous in (x', t)-variables, i.e., that there exists a function $\omega : [0, \infty) \to [0, 1]$ which is non-decreasing concave with $\lim_{\rho \searrow 0} \omega(\rho) = \omega(0) = 0$ and

$$\int_0^1 \frac{\omega(\rho)\, d\rho}{\rho} < \infty,$$

satisfying that

$$|a(\xi, x^1, x', t) - a(\xi, x^1, y', s)| \leq \omega\left(|(x', t) - (y', s)|\right) |\xi| \tag{1.3}$$

for every $(x^1, x', t) \in \mathbb{R}^{n+1}$, $\xi \in \mathbb{R}^n$ and $y' \in \mathbb{R}^{n-1}$. Note that the nonlinearities are only merely measurable on x^1-variable.

For nonlinear parabolic equations, many authors obtained the pointwise gradient estimates by using potentials. Duzaar and Mingione considered linear growth condition in [1]. Kuusi and Mingione considered p-growth conditions and obtained Wolff potential type estimates in [2, 3] and Riesz potential type estimates in [4]. Also for elliptic equations with measurable nonlinearities, the boundary pointwise gradient estimates by using Riesz potentials were obtained in [5], where they used the excess decay estimates of the gradient in [6]. In this paper, we will extend the result [5] to nonlinear parabolic equations with measurable nonlinearities and obtain the boundary pointwise gradient estimates by using Riesz potentials.

For the reader's further interest, we refer to [7] for Morrey space estimates to linear parabolic systems with measurable coefficients and measure data. We refer to [8] for weighted Lebesgue estimates to linear parabolic systems with measurable coefficients. Potential can be used not only for the right-hand side data of the equation but also be considered as a multiplier of the solution, see for instance [9], which considers existence and blow-up solution with singular potentials multiplied to the solution.

We use the following notations in this paper. Let z be a typical point in $\mathbb{R}^n$, s be a typical time variable and $r > 0$ be a size.

1. $z = (z^1, \cdots, z^n) = (z^1, z')$.
2. $\mathbb{R}^{n+1}_+ = \{(x^1, x', t) \in \mathbb{R}^{n+1} : x^1 > 0\}$, $\mathbb{R}^{n+1}_0 = \{(x^1, x', t) \in \mathbb{R}^{n+1} : x^1 = 0\}$.
3. $Q_r(z, s) = (z^1 - r, z^1 + r) \times \cdots \times (z^n - r, z^n + r) \times (s - r^2, s)$, $Q_r = Q_r(\mathbf{0}, 0)$.
4. $Q^+_r(z, s) = Q_r(z, s) \cap \mathbb{R}^{n+1}_+$, $Q^+_r = Q_r \cap \mathbb{R}^{n+1}_+$.
5. $T_r(z', s) = \{0\} \times (z^2 - r, z^2 + r) \times \cdots \times (z^n - r, z^n + r) \times (s - r^2, s) = Q_r(0, z', s) \cap \mathbb{R}^{n+1}_0$.

6. $K_r(z) = (z^1 - r, z^1 + r) \times \cdots \times (z^n - r, z^n + r)$.
7. $K_r^+(z) = (z^1 - r, z^1 + r) \times \cdots \times (z^n - r, z^n + r) \cap \mathbb{R}^n_+$.
8. For $g \in L^1(U)$, $(g)_U = \fint_U g\, dx = \frac{1}{|U|}\int_U g\, dx$ when $|U| \neq 0$.

In view of the available approximation theory, we assume that $\mu \in L^1(Q^+_{2R})$. Without loss of generality, we shall assume that

$$\mu \in L^1(\mathbb{R}^{n+1}), \tag{1.4}$$

by letting $\mu\big|_{\mathbb{R}^{n+1}\setminus Q^+_{2R}} = 0$. By using the concept of SOLA (Solutions Obtained by Limit of Approximations), we will remove this assumption in Corollary 1.3. Also for the boundary data, let $\psi : T_{2R} \to \mathbb{R}$ be a function such that

$$D_{x'}\psi \in L^\infty(T_{2R}) \qquad \text{and} \qquad \partial_t\psi \in L^2(T_{2R}). \tag{1.5}$$

We obtain the following boundary pointwise gradient estimate in this paper.

Theorem 1.1. *There exists a constant $c_1 = c_1(n, \lambda, \Lambda) \geq 1$ such that the following holds. For some $r \in (0, R]$, assume that*

$$c_1 \int_0^{2r} \frac{\omega(\rho)d\rho}{\rho} \leq 1. \tag{1.6}$$

If $u \in C^0\left(-4R^2, 0\, ;\, L^2(K^+_{2R})\right) \cap L^2\left(-4R^2, 0\, ;\, W^{1,1}(K^+_{2R})\right)$ is a weak solution of

$$\begin{cases} u_t - \operatorname{div} a(Du, x^1, x', t) &= \mu \quad \text{in} \quad Q^+_{2R}, \\ u &= \psi \quad \text{on} \quad T_{2R} \end{cases} \tag{1.7}$$

with the assumptions (1.2)–(1.5)*, then we have that*

$$\begin{aligned} |Du(x_0, t_0)| &\leq c\left[\fint_{Q_r^+(x_0,t_0)} |Du|\, dxdt + |D_{x'}\psi(x_0', t_0)|\right] \\ &\quad + c\left[\int_0^{2r}\left(\frac{|\mu|(Q^+_\rho(x_0, t_0))}{\rho^{n+1}} + \operatorname*{osc}_{T_\rho(x_0',t_0)} D_{x'}\psi + \rho^2 \fint_{Q^+_\rho(x_0,t_0)} |\partial_t\psi|^2\, dxdt\right)\frac{d\rho}{\rho}\right], \end{aligned} \tag{1.8}$$

for any Lebesgue point $(x_0, t_0) = (x_0^1, x_0', t_0) \in \overline{Q^+_R}$ of Du with $c = c(n, \lambda, \Lambda)$.

To deal with SOLA, we now remove the assumption (1.4).

Definition 1.2. A SOLA of (1.7) is a distributional solution $u \in L^2\left(-4R^2, 0\, ;\, W^{1,1}(K^+_{2R})\right)$ to (1.7) such that u is the limit of solutions $u_h \in C^0\left(-4R^2, 0\, ;\, L^2(K^+_{2R})\right) \cap L^2\left(-4R^2, 0\, ;\, W^{1,1}(K^+_{2R})\right)$ to

$$\begin{cases} (u_h)_t - \operatorname{div} a(Du_h, x^1, x', t) &= \mu_h \quad \text{in } Q^+_{2R}, \\ u_h &= \psi \quad \text{on } T_{2R}, \end{cases}$$

in the sense that $u_h \to u$ in $L^2\left(-4R^2, 0\, ;\, W^{1,1}(K^+_{2R})\right)$ and $L^\infty \ni \mu_h \rightharpoonup \mu$ in the sense of measures satisfying

$$\limsup_{h\to\infty} |\mu_h|(Q^+_\rho(x_0, t_0)) \leq |\mu|(\lfloor Q^+_\rho(x_0, t_0)\rfloor) \quad \text{for any} \quad Q^+_\rho(x_0, t_0) \subset Q^+_{2R},$$

where $\lfloor Q \rfloor$ denotes the parabolic closure of Q.

We finally state our main result for SOLA.

Corollary 1.3. *Without* (1.4)*, Theorem 1.1 continues to hold for SOLA of* (1.7)*, with the estimate* (1.8) *for any Lebesgue point* $(x_0, t_0) = (x_0^1, x_0', t_0) \in \overline{Q_R^+}$ *of Du.*

Remark 1.4. For the sake of convenience and simplicity, we employ the letter $c > 0$ and $\alpha \in (0, 1]$ throughout this paper to denote any constants which can be explicitly computed in terms of known quantities such as n, λ, Λ. Thus the exact values denoted by c and α may change from line to line in each given computation.

2. Excess decay estimates

In this section, the nonlinearities are assumed to be depending only on ξ and x^1-variables with the following assumptions:

$$\begin{cases} a(\xi, x^1) \text{ is measurable in } x^1 \in \mathbb{R} \text{ for every } \xi \in \mathbb{R}^n, \\ a(\xi, x^1) \text{ is } C^1\text{-regular in } \xi \in \mathbb{R}^n \text{ for every } x^1 \in \mathbb{R}. \end{cases} \tag{2.1}$$

Also we assume that $a(\xi, x^1) : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfies

$$\begin{cases} |a(\xi, x^1)| \leq \Lambda|\xi| + \Gamma, \\ |D_\xi a(\xi, x^1)| \leq \Lambda, \\ \langle D_\xi a(\xi, x^1)\zeta, \zeta\rangle \geq \lambda|\zeta|^2, \end{cases} \tag{2.2}$$

for every $x^1 \in \mathbb{R}$, $\xi \in \mathbb{R}^n$, $\zeta \in \mathbb{R}^n$ and for some constants $0 < \lambda \leq \Lambda$, $\Gamma \geq 0$.

We obtain the boundary excess-decay estimates for parabolic equations in this section. Under the assumptions (2.1), (2.2) and

$$0 \leq x_0^1 \leq r, \tag{2.3}$$

let g be a weak solution of

$$\begin{cases} g_t - \operatorname{div} a(Dg, x^1) = 0 & \text{in } Q_{3r}^+(x_0, t_0), \\ g = \gamma' \cdot x' & \text{on } T_{3r}(x_0', t_0), \end{cases} \tag{2.4}$$

where $\gamma' = (\gamma^2, \cdots, \gamma^n) \in \mathbb{R}^{n-1}$ and $(x_0, t_0) = (x_0^1, x_0^2, \cdots, x_0^n) = (x_0^1, x_0', t_0)$.

Remark 2.1. If the nonlinearity $a(\xi, x^1)$ is only defined on $0 < x^1 < 3r$, then one can easily extend $a(\xi, x^1)$ to satisfy (2.1) and (2.2), which does not effect the results in this paper.

The nonlinearity $a(\xi, x^1)$ only depends on x^1-variable and $g = \gamma' \cdot x'$ on $T_{3r}(x_0', t_0)$. So one can use the difference quotient method to find that $D_k g - \gamma^k$ ($k \in \{2, 3, \cdots, n\}$) is weakly differentiable in $Q_{2r}^+(x_0, t_0)$. Moreover, one can show that $D_k g - \gamma^k \in W^{1,2}(Q_{2r}^+(x_0, t_0))$. By differentiating (2.4) with respect to x^k-variable, one can get that

$$\begin{cases} \partial_t \left(D_k g - \gamma^k\right) - D_i \left[a_{ij}(x, t) D_j \left(D_k g - \gamma^k\right)\right] = 0 & \text{in} \quad Q_{2r}^+(x_0, t_0), \\ D_k g - \gamma^k = 0 & \text{on} \quad T_{2r}(x_0', t_0), \end{cases} \tag{2.5}$$

where $a_{ij}(x,t) = \dfrac{\partial a^i}{\partial \xi_j}(Dg, x^1)$ satisfies that

$$\begin{cases} a_{ij}(x,t)\zeta_i\zeta_j & \geq & \lambda|\zeta|^2, \\ |a_{ij}(x,t)| & \leq & \Lambda, \end{cases} \tag{2.6}$$

for any $(x,t) \in Q^+_{2r}(x_0,t_0)$ and $\zeta \in \mathbb{R}^n$ with $1 \leq i, j \leq n$.

To obtain boundary estimates, we next extend the equation (2.5) from $Q^+_{2r}(x_0,t_0)$ to $Q_{2r}(x_0,t_0)$. For $k \in \{2, 3, \cdots, n\}$, we let

$$g_k \text{ be the odd extension of } D_k g - \gamma^k \text{ from } Q^+_{2r}(x_0,t_0) \text{ to } Q_{2r}(x_0,t_0). \tag{2.7}$$

Then

$$\begin{cases} \partial_t g_k - D_i[a_{ij}(x,t)D_j g_k] & = & 0 & \text{in} & Q^+_{2r}(x_0,t_0), \\ g_k & = & 0 & \text{on} & T_{2r}(x_0',t_0). \end{cases} \tag{2.8}$$

Let $\hat{a}_{ij}(x,t)$ be an extension of $a_{ij}(x,t)$ from $Q^+_{2r}(x_0,t_0)$ to $Q_{2r}(x_0,t_0)$ defined as

$$\begin{cases} \hat{a}_{11}(-x^1,x',t) = a_{11}(x^1,x',t), & \\ \hat{a}_{ij}(-x^1,x',t) = a_{ij}(x^1,x',t) & \text{when } 1 < i \leq n, 1 < j \leq n, \\ \hat{a}_{1j}(-x^1,x',t) = -a_{1j}(x^1,x',t) & \text{when } 1 < j \leq n, \\ \hat{a}_{i1}(-x^1,x',t) = -a_{i1}(x^1,x',t) & \text{when } 1 < i \leq n, \end{cases} \tag{2.9}$$

for $(x^1,x',t) \in Q_{2r}(x_0,t_0) \setminus Q^+_{2r}(x_0,t_0)$. Then one can check from (2.6) that

$$\begin{cases} \hat{a}_{ij}(x,t)\zeta_i\zeta_j & \geq & \lambda|\zeta|^2, \\ |\hat{a}_{ij}(x,t)| & \leq & \Lambda, \end{cases} \tag{2.10}$$

for any $(x,t) \in Q_{2r}(x_0,t_0)$ and $\zeta \in \mathbb{R}^n$. Then we obtain from (2.8) and (2.9) that g_k is a weak solution of the parabolic equation

$$\partial_t g_k - D_i\left[\hat{a}_{ij}(x,t)D_j g_k\right] \;=\; 0 \quad \text{in} \quad Q_{2r}(x_0,t_0). \tag{2.11}$$

From [10, Chapter 6], we have an excess decay estimate for linear parabolic equations, which can be applied to (2.11).

Lemma 2.2. *Under the assumptions*

$$\begin{cases} a_{ij}(x,t)\zeta_i\zeta_j \geq \lambda|\zeta|^2 & ((x,t) \in Q_r,\ \zeta \in \mathbb{R}^n), \\ \|a_{ij}\|_{L^\infty(Q_r)} \leq \Lambda, & \end{cases}$$

let w be a weak solution of

$$\partial_t w - D_i[a_{ij}(x,t)D_j w] = 0 \quad \textit{in} \quad Q_r.$$

Then we have that

$$\fint_{Q_\rho} |w - (w)_{Q_\rho}|^2 \,dx\,dt \leq c\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_r} |w - (w)_{Q_r}|^2 \,dx\,dt \qquad (0 < \rho \leq r),$$

where $c = c(n,\lambda,\Lambda)$ and $\alpha = \alpha(n,\lambda,\Lambda) \in (0,1]$.

With Lemma 2.2 and the energy estimate, we obtain the following lemma.

Lemma 2.3. *Suppose that* $k \in \{2, 3, \cdots, n\}$. *For* g_k *in* (2.7)*, we have that*

$$\fint_{Q_\tau(y,s)} \left|g_k - (g_k)_{Q_\tau(y,s)}\right|^2 \, dxdt \le c \left(\frac{\tau}{\rho}\right)^{2\alpha} \fint_{Q_\rho(y,s)} \left|g_k - (g_k)_{Q_\rho(y,s)}\right|^2 \, dxdt, \tag{2.12}$$

and

$$\int_{Q_\tau(y,s)} |Dg_k|^2 \, dxdt \le \frac{c}{(\rho - \tau)^2} \int_{Q_\rho(y,s)} \left|g_k - \zeta^k\right|^2 \, dxdt \qquad (\zeta^k \in \mathbb{R}), \tag{2.13}$$

for any $Q_\rho(y, s) \subset Q_{2r}(x_0, t_0)$ *and* $0 < \tau < \rho$.

Proof. Let $k \in \{2, 3, \cdots, n\}$ be an arbitrary integer. The estimate (2.12) follows by applying Lemma 2.2 and (2.10) to (2.11).

Next, we choose a cut-off function $\eta \in C_c^\infty(Q_\rho(y, s))$ with

$$0 \le \eta \le 1, \qquad \eta = 1 \text{ on } Q_\tau(y, s), \qquad |D\eta| \le \frac{c}{\rho - \tau} \qquad \text{and} \qquad |\partial_t \eta| \le \frac{c}{(\rho - \tau)^2}. \tag{2.14}$$

We have from (2.11) that

$$\partial_t \left(g_k - \zeta^k\right) - D_i \left[\hat{a}_{ij}(x, t) D_j \left(g_k - \zeta^k\right)\right] = 0 \quad \text{in} \quad Q_{2r}(x_0, t_0). \tag{2.15}$$

Since $\eta \in C_c^\infty(Q_\rho(y, s))$ and $Q_\rho(y, s) \subset Q_{2r}(x_0, t_0)$, we test the above equation by $[g_k - \zeta^k]\eta^2$ to find that

$$\begin{aligned} 0 = &\int_{Q_\rho(y,s)} \left[\partial_t \left\{\frac{\left([g_k - \zeta^k]\eta\right)^2}{2}\right\} - [g_k - \zeta^k]^2 \eta \, \partial_t \eta\right] dxdt \\ &+ \int_{Q_\rho(y,s)} \hat{a}_{ij}(x, t) D_j g_k D_i \left\{[g_k - \zeta^k]\eta^2\right\} dxdt. \end{aligned}$$

Since $\eta \in C_c^\infty(Q_\rho(y, s))$, one can check that $\int_{Q_\rho(y,s)} \partial_t \left\{\left([g_k - \zeta^k]\eta\right)^2\right\} dxdt \ge 0$. So by (2.10),

$$\begin{aligned} \lambda \int_{Q_\rho(y,s)} |Dg_k|^2 \eta^2 \, dxdt &\le \int_{Q_\rho(y,s)} \left[\partial_t \left\{\frac{\left([g_k - \zeta^k]\eta\right)^2}{2}\right\} + \hat{a}_{ij}(x, t) D_j g_k D_i g_k \, \eta^2\right] dxdt \\ &= \int_{Q_\rho(y,s)} \left\{[g_k - \zeta^k]^2 \eta \, \partial_t \eta - \hat{a}_{ij}(x, t) D_j g_k \, [g_k - \zeta^k] 2\eta D_i \eta\right\} dxdt \\ &\le c \int_{Q_\rho(y,s)} \left\{\left|g_k - \zeta^k\right|^2 |\eta| \, |\partial_t \eta| + |Dg_k| \left|g_k - \zeta^k\right| |\eta| \, |D\eta|\right\} dxdt. \end{aligned}$$

By Young's inequality,

$$\int_{Q_\rho(y,s)} |Dg_k|^2 \eta^2 \, dxdt \le c \int_{Q_\rho(y,s)} \left|g_k - \zeta^k\right|^2 \left\{|D\eta|^2 + |\eta| \, |\partial_t \eta|\right\} dxdt.$$

So (2.13) follows from (2.14).

Now, we extend g_t from $Q^+_{2r}(x_0,t_0)$ to $Q_{2r}(x_0,t_0)$ and obtain some estimates on the extended function of g_t. Since g is a weak solution of (2.4), one can use the difference quotient method to show that $g_t = \partial_t g$ is a weak solution of

$$(g_t)_t - D_i\left[a_{ij}(x,t)D_j(g_t)\right] = \partial_t\left[g_t - D_i\left\{a^i(Dg,x^1)\right\}\right] = 0 \quad \text{in} \quad Q^+_{2r}(x_0,t_0). \tag{2.16}$$

In view of (2.4), one can check that $g_t = 0$ on $T_{2r}(x'_0,t_0)$. So

$$\text{let } g_{n+1} \text{ be the odd extension of } g_t \text{ from } Q^+_{2r}(x_0,t_0) \text{ to } Q_{2r}(x_0,t_0) \tag{2.17}$$

defined as

$$g_{n+1}(x^1,x') = \begin{cases} g_t(x^1,x') & \text{in } Q^+_{2r}(x_0,t_0), \\ -g_t(-x^1,x') & \text{in } Q_{2r}(x_0,t_0)\setminus Q^+_{2r}(x_0,t_0). \end{cases}$$

So we find from (2.9) and (2.16) that

$$\partial_t g_{n+1} - D_i\left[\hat{a}_{ij}(x,t)D_j g_{n+1}\right] = 0 \quad \text{in} \quad Q_{2r}(x_0,t_0). \tag{2.18}$$

Then we have the following energy estimate for g_{n+1} in (2.18).

Lemma 2.4. *If g_{n+1} is a weak solution of* (2.18)*, then we have that*

$$\int_{Q_\tau(y,s)} |Dg_{n+1}|^2\ dxdt \le \frac{c}{(\rho-\tau)^2}\int_{Q_\rho(y,s)} |g_{n+1}|^2\ dxdt \qquad (0<\tau<\rho),$$

for any $Q_\rho(y,s)\subset Q_{2r}(x_0,t_0)$.

Proof. Choose a cut-off function $\eta\in C^\infty_c(Q_\rho(y,s))$ with

$$0\le\eta\le 1,\quad \eta=1 \text{ in } Q_\tau(y,s),\quad |D\eta|\le\frac{c}{\rho-\tau} \quad\text{and}\quad |\partial_t\eta|\le\frac{c}{(\rho-\tau)^2}. \tag{2.19}$$

We test (2.18) by $\varphi = \eta^2 g_{n+1}$ to find that

$$0 = \int_{Q_\rho(y,s)} \left[\partial_t(g_{n+1})\,g_{n+1}\eta^2 + \hat{a}_{ij}(x,t)D_j g_{n+1}D_i\left(\eta^2 g_{n+1}\right)\right]\ dxdt. \tag{2.20}$$

Then a direct calculation gives that

$$\begin{aligned}
\lambda\int_{Q_\rho(y,s)} |Dg_{n+1}|^2\,\eta^2 dxdt &\le \int_{Q_\rho(y,s)} \hat{a}_{ij}(x,t)D_j(g_{n+1})\,\eta^2 D_i(g_{n+1})\ dxdt \\
&= \int_{Q_\rho(y,s)} \hat{a}_{ij}(x,t)D_j(g_{n+1})\left[D_i\left(\eta^2 g_{n+1}\right) - 2\eta D_i\eta\, g_{n+1}\right]\ dxdt \\
&= -\int_{Q_\rho(y,s)} \left[\partial_t(g_{n+1})\,\eta^2 g_{n+1} + \hat{a}_{ij}(x,t)D_j(g_{n+1})\,2\eta D_i\eta\, g_{n+1}\right]\ dxdt.
\end{aligned}$$

From the fact that $\eta\in C^\infty_c(Q_\rho(y,s))$, we get

$$\begin{aligned}
\int_{Q_\rho(y,s)} \partial_t(g_{n+1})\,\eta^2 g_{n+1}\ dxdt &= \int_{Q_\rho(y,s)} \frac{\partial_t\left(g^2_{n+1}\eta^2\right)}{2} - |g_{n+1}|^2\,\eta\eta_t\ dxdt \\
&\ge -\int_{Q_\rho(y,s)} |g_{n+1}|^2\,\eta\eta_t\ dxdt.
\end{aligned}$$

By combining the above two equalities and applying the elliptic condition (2.6), we get

$$\int_{Q_\rho(y,s)} |Dg_{n+1}|^2 \eta^2 dxdt \le c \int_{Q_\rho(y,s)} |g_{n+1}|^2 \left(|D\eta|^2 + |\eta\eta_t|\right) dxdt.$$

So the lemma follows from the choice of the cut-off function η in (2.19).

Since the nonlinearity $a(\xi, x^1)$ depends on x^1-variable, we obtain an excess decay estimate in terms of $a^1(Dg, x^1)$ instead of $D_1 g$. Let g_1 be the even extension of $a^1(Dg, x^1)$ from $Q^+_{2r}(x_0, t_0)$ to $Q_{2r}(x_0, t_0)$ defined as

$$\begin{cases} g_1(x^1, x', t) = a^1(Dg(x^1, x', t), x^1) & \text{in } Q^+_{2r}(x_0, t_0) \\ g_1(x^1, x', t) = a^1(Dg(-x^1, x', t), -x^1) & \text{in } Q_{2r}(x_0, t_0) \setminus Q^+_{2r}(x_0, t_0). \end{cases} \tag{2.21}$$

So from (2.7), (2.17) and (2.21), we have following extensions from $Q^+_{2r}(x_0, t_0)$ to $Q_{2r}(x_0, t_0)$:

$$\begin{cases} g_1 \text{ is the even extension of } a^1(Dg, x^1), \\ g_k \ (k \in \{2, 3, \cdots, n\}) \text{ is the odd extension of } D_k g - \gamma^k, \\ g_{n+1} \text{ is the odd extension of } g_t. \end{cases} \tag{2.22}$$

Then we define $G : Q_{2r}(x_0, t_0) \to \mathbb{R}^n$ as

$$G = (g_1, g_2, \cdots, g_n). \tag{2.23}$$

The desired excess-decay estimate will be obtained with the function G in (2.23).

With Lemma 2.4, we estimate g_{n+1} by using the function G in (2.23).

Lemma 2.5. *For g_{n+1} in* (2.17)*, we have that*

$$\int_{Q_\tau(y,s)} |g_{n+1}|^2 \ dxdt \le \frac{c}{(\rho - \tau)^2} \int_{Q_\rho(y,s)} |G - \zeta|^2 \ dxdt \qquad (0 < \tau < \rho),$$

for any $Q_\rho(y, s) \subset Q_{2r}(x_0, t_0)$ and $\zeta = \left(\zeta^1, \zeta^2, \cdots, \zeta^n\right) \in \mathbb{R}^n$.

Proof. Let $d_0 = \tau$ and $d_\infty = \rho$. Let

$$d_m = d_0 + \sum_{l=1}^{m} \frac{\rho - \tau}{2^l} \qquad \text{and} \qquad e_{m-1} = \frac{d_{m-1} + d_m}{2} \qquad (m = 1, 2, 3, \cdots).$$

Choose a cut-off function $\eta \in C^\infty_c\left(Q_{e_m}(x_0, t_0)\right)$ with

$$0 \le \eta \le 1, \qquad \eta = 1 \ \text{ in } \ Q_{d_m}(x_0, t_0), \qquad |D\eta| \le \frac{c\, 2^m}{r - \rho} \qquad \text{and} \qquad |\partial_t \eta| \le \frac{c\, 4^m}{(r - \rho)^2}. \tag{2.24}$$

Since $g_t = 0$ on $T_{2r}(x'_0, t_0)$, we test (2.4) by $\eta^2 g_t$ to find that

$$\int_{Q^+_{e_m}(y,s)} g_t \left[\eta^2 g_t\right] dxdt = -\int_{Q^+_{e_m}(y,s)} \left\langle a(Dg, x^1), D\left[\eta^2 g_t\right]\right\rangle dxdt. \tag{2.25}$$

Since $\eta \in C_c^\infty(Q_{e_m}(x_0,t_0))$ and $g_t = 0$ on $T_{2r}(x_0', t_0)$, one can easily check that

$$\int_{Q^+_{e_m}(y,s)} a^1\left(Dg, x^1\right) D_1\left[\eta^2 g_t\right] dxdt = \int_{Q^+_{e_m}(y,s)} \left[a^1(Dg,x^1) - \zeta^1\right] D_1\left[\eta^2 g_t\right] dxdt.$$

Then for $\kappa > 0$, Young's inequality implies that

$$\begin{aligned} &\left|\int_{Q^+_{e_m}(y,s)} a^1(Dg,x^1) D_1[\eta^2 g_t]\, dxdt\right| \\ &\quad \le \int_{Q^+_{e_m}(y,s)} \left[\kappa\,|Dg_t|^2\,\eta^2 + \frac{\eta^2|g_t|^2}{48} + c\left(\frac{\eta^2}{\kappa} + |D\eta|^2\right)\left|a^1(Dg,x^1) - \zeta^1\right|^2\right] dxdt. \end{aligned} \tag{2.26}$$

By using integration by parts, for any $k \in \{2,3,\cdots,n\}$ we have that

$$\int_{Q^+_{e_m}(y,s)} a^k\left(Dg, x^1\right) D_k\left[\eta^2 g_t\right]\, dxdt = -\int_{Q^+_{e_m}(y,s)} D_k\left[a^k(Dg,x^1)\right] \eta^2 g_t\, dxdt.$$

Then Young's inequality implies that

$$\begin{aligned} &\sum_{k=2}^n \left|\int_{Q^+_{e_m}(y,s)} a^k(Dg,x^1) D_k\left[\eta^2 g_t\right] dxdt\right| \\ &\quad \le \int_{Q^+_{e_m}(y,s)} \left[\frac{\eta^2|g_t|^2}{48} + c\sum_{k=2}^n \left|D_k\left[a^k(Dg,x^1)\right]\right|^2 \eta^2\right] dxdt. \end{aligned}$$

By (2.7), g_k is the odd extension of $D_k g - \gamma^k$ from $Q^+_{2r}(x_0,t_0)$ to $Q_{2r}(x_0,t_0)$, which implies

$$\begin{aligned} \sum_{k=2}^n \int_{Q^+_{e_m}(y,s)} \left|D_k\left[a^k(Dg,x^1)\right]\right|^2 \eta^2\, dxdt &\le c\sum_{k=2}^n \int_{Q^+_{e_m}(y,s)} |DD_k g|^2\,\eta^2\, dxdt \\ &\le c\sum_{k=2}^n \int_{Q^+_{e_m}(y,s)} |Dg_k|^2\,\eta^2\, dxdt. \end{aligned}$$

By combining the above two estimates, we apply $\tau = e_m$ and $\rho = d_{m+1}$ in Lemma 2.3. Then

$$\begin{aligned} &\left|\sum_{k=2}^n \int_{Q^+_{e_m}(y,s)} a^k(Dg,x^1) D_k\left[\eta^2 g_t\right] dxdt\right| \\ &\quad \le \int_{Q^+_{e_m}(y,s)} \frac{\eta^2|g_t|^2}{48}\, dxdt + \int_{Q^+_{d_{m+1}}(y,s)} \frac{c\,4^m}{(\rho-\tau)^2} \sum_{k=2}^n \left|g_k - \zeta^k\right|^2\, dxdt, \end{aligned} \tag{2.27}$$

because $d_{m+1} - e_{m+1} = \dfrac{d_{m+1} - d_m}{2} = \dfrac{\rho-\tau}{2^{m+1}}$. So we obtain from (2.25), (2.26) and (2.27) that

$$\begin{aligned} &\int_{Q^+_{e_m}(y,s)} \eta^2|g_t|^2\, dxdt \\ &\quad \le \int_{Q^+_{e_m}(y,s)} \left[\kappa\,|Dg_t|^2\,\eta^2 + \frac{\eta^2|g_t|^2}{24}\right] dxdt \\ &\qquad + c\int_{Q^+_{d_{m+1}}(y,s)} \left[\left(\frac{\eta^2}{\kappa} + |D\eta|^2\right)\left|a^1(Dg,x^1) - \zeta^1\right|^2 + \frac{4^m}{(\rho-\tau)^2}\sum_{k=2}^n \left|g_k - \zeta^k\right|^2\right] dxdt. \end{aligned}$$

By (2.22), g_1 is the even extension of $a^1\left(Dg, x^1\right)$, g_k ($k \in \{2, 3, \cdots, n\}$) is the odd extension of $D_k g - \gamma^k$ and g_{n+1} is the odd extension of g_t from $Q^+_{2r}(x_0, t_0)$ to $Q_{2r}(x_0, t_0)$. Thus

$$\begin{aligned}
&\int_{Q_{e_m}(y,s)} \eta^2 |g_{n+1}|^2 \, dxdt \\
&\quad \le \int_{Q_{e_m}(y,s)} \left[2\kappa \, |Dg_{n+1}|^2 \, \eta^2 + \frac{\eta^2 |g_{n+1}|^2}{12} \right] dxdt \\
&\quad + c \int_{Q_{d_{m+1}}(y,s)} \left[\left(\frac{\eta^2}{\kappa} + |D\eta|^2 \right) \left| g_1 - \zeta^1 \right|^2 + \frac{4^m}{(\rho-\tau)^2} \sum_{k=2}^{n} \left| g_k - \zeta^k \right|^2 \right] dxdt.
\end{aligned} \tag{2.28}$$

Now, take $\rho = e_m$ and $r = d_{m+1}$ in Lemma 2.4 to find that

$$\int_{Q_{e_m}(y,s)} |Dg_{n+1}|^2 \; dxdt \le \frac{c_1 4^m}{(\rho-\tau)^2} \int_{Q_{d_{m+1}}(y,s)} |g_{n+1}|^2 \; dxdt. \tag{2.29}$$

Take κ so that $\dfrac{c_1 \kappa 4^m}{(r-\rho)^2} = \dfrac{1}{48}$. By combining (2.24), (2.28) and (2.29), we have

$$\int_{Q_{d_m}(y,s)} |g_{n+1}|^2 \, dxdt \le \int_{Q_{d_{m+1}}(y,s)} \frac{|g_{n+1}|^2}{8} + \frac{4^m c \, |G-\zeta|^2}{(\rho-\tau)^2} \, dxdt. \tag{2.30}$$

Multiply (2.30) by $\dfrac{1}{8^m}$ and sum it from $m = 0$ to ∞. Then we have that

$$\begin{aligned}
&\sum_{m=0}^{\infty} \frac{1}{8^m} \int_{Q_{d_m}(y,s)} |g_{n+1}|^2 \, dxdt \\
&\quad \le \sum_{m=0}^{\infty} \frac{1}{8^{m+1}} \int_{Q_{d_{m+1}}(y,s)} |g_{n+1}|^2 \, dxdt + \sum_{m=0}^{\infty} \frac{c}{2^m (\rho-\tau)^2} \int_{Q_\rho(y,s)} |G-\zeta|^2 \, dxdt.
\end{aligned} \tag{2.31}$$

Thus from (2.31) and the fact that $d_0 = \tau$, we have

$$\int_{Q_\tau(y,s)} |g_{n+1}|^2 \, dxdt = \int_{Q_{d_0}(y,s)} |g_{n+1}|^2 \, dxdt \le \frac{c}{(\rho-\tau)^2} \int_{Q_\rho(y,s)} |G-\zeta|^2 \, dxdt,$$

which finishes the proof.

To obtain the desired excess-decay estimate on $G = (g_1, \cdots, g_n)$, we will use Poincaré's inequality. In Lemma 2.3 and Lemma 2.4, the derivatives $Dg_2, \cdots, Dg_n$ and Dg_{n+1} were obtained. So it only remains to obtain the following estimate on Dg_1.

Lemma 2.6. *For g_1 in* (2.22)*, $Dg_1 \in L^2(Q_r(x_0, t_0))$ exists with the estimate*

$$|Dg_1| \le c \left(\sum_{2 \le k \le n} |Dg_k| + |g_{n+1}| \right) \in L^2\left(Q_r(x_0, t_0)\right).$$

Proof. We discover from Lemma 2.3 and Lemma 2.5 that

$$Dg_k \in L^2\left(Q_r(x_0,t_0)\right) \quad \text{and} \quad g_{n+1} \in L^2\left(Q_r(x_0,t_0)\right),$$

for any $k \in \{2,3,\cdots,n\}$. It follows from (2.22) that

$$DD_{x'}g \in L^2\left(Q_r^+(x_0,t_0)\right) \quad \text{and} \quad g_t \in L^2\left(Q_r^+(x_0,t_0)\right). \tag{2.32}$$

Since g is a weak solution of (2.4) and the nonlinearities $a(\xi,x^1)$ are independent of x^k-variable for any $k \in \{2,3,\cdots,n\}$, we have from (2.32) that

$$\begin{cases} D_1[a^1(Dg,x^1)] = g_t - \sum_{k=2}^{n} D_k\left[a^k(Dg,x^1)\right] = g_t - \sum_{k=2}^{n} a_{kj}(x,t)D_{jk}g \in L^2\left(Q_r^+(x_0,t_0)\right) \\ D_{x'}[a^1(Dg,x^1)] = a_{1j}(x,t)D_jD_{x'}g \in L^2\left(Q_r^+(x_0,t_0)\right). \end{cases}$$

From (2.6), $a_{ij}(x,t)$ is uniformly elliptic. So we find from (2.22) that

$$|D[a^1(Dg,x^1)]| \le c\left(|DD_{x'}g| + |g_t|\right) \le c\left(\sum_{2\le k\le n} |Dg_k| + |g_{n+1}|\right) \in L^2\left(Q_r^+(x_0,t_0)\right).$$

By (2.22), g_1 is the even extension of $a^1(Dg,x^1)$, g_k ($k \in \{2,3,\cdots,n\}$) is the odd extension of $D_kg - \gamma^k$ and g_{n+1} is the odd extension of g_t from $Q_{2r}^+(x_0,t_0)$ to $Q_{2r}(x_0,t_0)$. So the lemma follows by extending the above estimate from $Q_r^+(x_0,t_0)$ to $Q_r(x_0,t_0)$.

We obtain the following excess-decay estimate and L^∞-estimate of g_{n+1}.

Lemma 2.7. *For the odd extension g_{n+1} of g_t in* (2.22)*, we have that*

$$\fint_{Q_\tau(y,s)} |g_{n+1} - (g_{n+1})_{Q_\tau(y,s)}|^2\,dxdt \le c\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_\rho(y,s)} |g_{n+1} - (g_{n+1})_{Q_\rho(y,s)}|^2\,dxdt,$$

and

$$\|g_{n+1}\|^2_{L^\infty\left(Q_{\frac{\rho}{2}}(y,s)\right)} \le c\fint_{Q_\rho(y,s)} |g_{n+1}|^2\,dxdt. \tag{2.33}$$

for any $Q_\rho(y,s) \subset Q_r(x_0,t_0)$ and $0 < \tau \le \rho$ where $\alpha = \alpha(n,\lambda,\Lambda) \in (0,1]$.

Proof. From (2.18), g_{n+1} is a weak solution of

$$\partial_t g_{n+1} - D_i\left[a_{ij}(x,t)D_jg_{n+1}\right] = 0 \quad \text{in} \quad Q_r(x_0,t_0). \tag{2.34}$$

By using (2.6) and applying Lemma 2.2 to (2.34), we find that

$$\fint_{Q_\tau(y,s)} |g_{n+1} - (g_{n+1})_{Q_\tau(y,s)}|^2\,dxdt \le c\left(\frac{\rho}{r}\right)^{2\alpha} \fint_{Q_\rho(y,s)} |g_{n+1} - (g_{n+1})_{Q_\rho(y,s)}|^2\,dxdt, \tag{2.35}$$

for any $Q_\rho(y,s) \subset Q_r(x_0,t_0)$ and $0 < \tau \le \rho$. The L^∞-estimate of g_{n+1} in (2.33) follows by applying Campanato type embedding to the excess-decay estimate (2.35).

Recall the definition of G in (2.22) and (2.23). In view of Lemma 2.4 and Lemma 2.6, one can estimate DG and $\partial_t G$ as

$$\begin{cases} |DG| \le c\left\{\sum_{1\le k\le n} |Dg_k|\right\} \le c\left\{\sum_{2\le k\le n} |Dg_k| + |g_{n+1}|\right\}, \\ |\partial_t G| \le c\,|Dg_{n+1}|, \end{cases} \tag{2.36}$$

which implies that

$$\fint_{Q_\rho(y,s)} \left\{|DG|^2 + \rho^2 |\partial_t G|^2\right\} dxdt \le c \fint_{Q_\rho(y,s)} \left\{\sum_{2\le k\le n} |Dg_k|^2 + |g_{n+1}|^2 + \rho^2\,|Dg_{n+1}|^2\right\} dxdt, \tag{2.37}$$

for any $Q_\rho(y,s) \subset Q_r(x_0,t_0)$. Here, Dg_k ($k \in \{2,3,\cdots,n\}$), g_{n+1} and Dg_{n+1} were estimated in Lemma 2.3, Lemma 2.5 and Lemma 2.4 respectively. So we use Sobolev type embeddings to have the following reverse Hölder type inequality.

Lemma 2.8. *For G in* (2.22) *and* (2.23)*, we have that*

$$\left(\fint_{Q_{\frac{\rho}{2}}(y,s)} |G-\zeta|^{2^*}\, dxdt\right)^{\frac{1}{2^*}} \le c \fint_{Q_\rho(y,s)} |G-\zeta|\, dxdt \qquad (\zeta \in \mathbb{R}^n),$$

for any $Q_\rho(y,s) \subset Q_r(x_0,t_0)$. Here, $2^* = \dfrac{2(n+1)}{n-1} > 2$ is the Sobolev conjugate for $(n+1)$-dimension.

Proof. Fix any $Q_\rho(y,s) \subset Q_r(x_0,t_0)$. Choose arbitrary constants $\frac{\rho}{2} \le \tau_1 < \tau_2 \le \rho$. Then by the Sobolev type embedding,

$$\left(\fint_{Q_{\tau_1}(y,s)} |G-\zeta|^{2^*}\, dxdt\right)^{\frac{2}{2^*}} \le c \fint_{Q_{\tau_1}(y,s)} \left\{\tau_1^2\,|DG|^2 + \tau_1^4\,|\partial_t G|^2 + |G-\zeta|^2\right\} dxdt.$$

Here, the Sobolev conjugate for $(n+1)$-dimension is denoted as $2^* = \dfrac{2(n+1)}{n-1} > 2$. We have from (2.37) that

$$\fint_{Q_{\tau_1}(y,s)} \left[\tau_1^2\,|DG|^2 + \tau_1^4\,|\partial_t G|^2\right] dxdt \le c\tau_1^2 \fint_{Q_{\tau_1}(y,s)} \left[\sum_{2\le k\le n} |Dg_k|^2 + |g_{n+1}|^2 + \tau_1^2\,|Dg_{n+1}|^2\right] dxdt.$$

Since $\frac{\rho}{2} \le \tau_1 < \tau_2 \le \rho$, we have from (2.13) in Lemma 2.3 and Lemma 2.5 that

$$\begin{aligned} \fint_{Q_{\tau_1}(y,s)} \left[\sum_{2\le k\le n} |Dg_k|^2 + |g_{n+1}|^2\right] dxdt &\le \frac{c}{(\tau_2-\tau_1)^2} \fint_{Q_{\tau_2}(y,s)} \left[\sum_{2\le k\le n} \left|g_k - \zeta^k\right|^2 + |G-\zeta|^2\right] dxdt \\ &\le \frac{c}{(\tau_2-\tau_1)^2} \fint_{Q_{\tau_2}(y,s)} |G-\zeta|^2\, dxdt. \end{aligned}$$

Since $\frac{\rho}{2} \le \tau_1 < \tau_2 \le \rho$, we have from Lemma 2.4 and Lemma 2.5 that

$$\fint_{Q_{\tau_1}(y,s)} |Dg_{n+1}|^2\,dxdt \le \frac{c}{(\tau_2-\tau_1)^2}\fint_{Q_{\frac{\tau_1+\tau_2}{2}}(y,s)} |g_{n+1}|^2\,dxdt \le \frac{c}{(\tau_2-\tau_1)^4}\fint_{Q_{\tau_2}(y,s)} |G-\zeta|^2\,dxdt.$$

Since $\frac{\tau_1^2}{(\tau_2-\tau_1)^2} \ge 1$, by combining the above four estimates, we get

$$\left(\fint_{Q_{\tau_1}(y,s)} |G-\zeta|^{2^*}\,dxdt\right)^{\frac{2}{2^*}} \le \frac{c\,\tau_1^4}{(\tau_2-\tau_1)^4}\fint_{Q_{\tau_2}(y,s)} |G-\zeta|^2\;dxdt.$$

By the interpolation inequality, we get

$$\fint_{Q_{\tau_2}(y,s)} |G-\zeta|^2\;dxdt \le \left(\fint_{Q_{\tau_2}(y,s)} |G-\zeta|^{\frac{2(n+1)}{n-1}}\;dxdt\right)^{\frac{n-1}{n+3}} \left(\fint_{Q_{\tau_2}(y,s)} |G-\zeta|\;dxdt\right)^{\frac{4}{n+3}}.$$

Since $2^* = \frac{2(n+1)}{n-1} > 2$, we obtain from Young's inequality that

$$\begin{aligned}&\left(\fint_{Q_{\tau_1}(y,s)} |G-\zeta|^{2^*}\,dxdt\right)^{\frac{2}{2^*}}\\ &\quad\le \frac{1}{2}\left(\fint_{Q_{\tau_2}(y,s)} |G-\zeta|^{2^*}\,dxdt\right)^{\frac{2}{2^*}} + \frac{c\,\tau_1^{2(n+3)}}{(\tau_2-\tau_1)^{2(n+3)}}\left(\fint_{Q_{\tau_2}(y,s)} |G-\zeta|\;dxdt\right)^2.\end{aligned}$$

Since $\frac{\rho}{2} \le \tau_1 < \tau_2 \le \rho$ were chosen arbitrarily, by [11, Lemma 4.3], we get

$$\left(\fint_{Q_{\frac{\rho}{2}}(y,s)} |G-\zeta|^{2^*}\,dxdt\right)^{\frac{2}{2^*}} \le c\left(\fint_{Q_{\rho}(y,s)} |G-\zeta|\,dxdt\right)^2,$$

and the lemma follows.

By using (2.37), we apply Poincaré's inequality to obtain the desired excess-decay estimate on $G = (g_1, \cdots, g_n)$. We remark that Dg_1, Dg_k $(k \in \{2, \cdots, n\})$, Dg_{n+1} and g_{n+1} were estimated in Lemma 2.6, Lemma 2.3, Lemma 2.4 and Lemma 2.5 respectively.

Lemma 2.9. *For G in* (2.22) *and* (2.23)*, we have that*

$$\fint_{Q_{\tau}(y,s)} |G-(G)_{Q_{\tau}(y,s)}|\;dxdt \le c\left(\frac{\tau}{\rho}\right)^{\alpha}\fint_{Q_{\rho}(y,s)} |G-(G)_{Q_{\rho}(y,s)}|\;dxdt \qquad (0<\tau\le\rho),$$

for any $Q_\rho(y,s) \subset Q_r(x_0,t_0)$ *where* $\alpha = \alpha(n,\lambda,\Lambda) \in (0,1]$.

Proof. Assume that $8\tau \le \rho$, otherwise the lemma holds from the dilation. We claim that

$$\fint_{Q_{\tau}(y,s)} |G-(G)_{Q_{\tau}(y,s)}|^2\,dxdt \le c\left(\frac{\tau}{\rho}\right)^{2\alpha}\fint_{Q_{\frac{\rho}{2}}(y,s)} |G-(G)_{Q_{\rho}(y,s)}|^2\,dxdt. \tag{2.38}$$

From Poincaré's inequality and (2.37), we have that

$$\begin{aligned}\fint_{Q_\tau(y,s)} |G-(G)_{Q_\tau(y,s)}|^2\,dxdt &\le c\fint_{Q_\tau(y,s)} \left\{\tau^2|DG|^2+\tau^4|\partial_t G|^2\right\}\,dxdt.\\ &\le c\tau^2\fint_{Q_\tau(y,s)} \left\{\sum_{2\le k\le n}|Dg_k|^2+|g_{n+1}|^2+\tau^2|Dg_{n+1}|^2\right\}\,dxdt.\end{aligned} \tag{2.39}$$

By taking $\zeta^k=(g_k)_{Q_{2\tau}(y,s)}$ in Lemma 2.3, we get

$$\begin{aligned}&\tau^2\fint_{Q_\tau(y,s)} \left\{\sum_{2\le k\le n}|Dg_k|^2+\tau^2|g_{n+1}|^2\right\}\,dxdt\\ &\le c\fint_{Q_{2\tau}(y,s)} \left\{\sum_{2\le k\le n}\left|g_k-(g_k)_{Q_{2\tau}(y,s)}\right|^2+\tau^4\|g_{n+1}\|^2_{L^\infty(Q_\tau(y,s))}\right\}\,dxdt.\end{aligned}$$

By the assumption $8\tau\le\rho$, we have from (2.12) in Lemma 2.3 that

$$\begin{aligned}\sum_{2\le k\le n}\fint_{Q_{2\tau}(y,s)}\left|g_k-(g_k)_{Q_{2\tau}(y,s)}\right|^2\,dxdt &\le c\left(\frac{\tau}{\rho}\right)^{2\alpha}\sum_{2\le k\le n}\fint_{Q_{\frac{\rho}{2}}(y,s)}\left|g_k-(g_k)_{Q_{\frac{\rho}{2}}(y,s)}\right|^2\,dxdt\\ &\le c\left(\frac{\tau}{\rho}\right)^{2\alpha}\fint_{Q_{\frac{\rho}{2}}(y,s)}\left|G-(G)_{Q_{\frac{\rho}{2}}(y,s)}\right|^2\,dxdt\\ &\le c\left(\frac{\tau}{\rho}\right)^{2\alpha}\fint_{Q_{\frac{\rho}{2}}(y,s)}\left|G-(G)_{Q_\rho(y,s)}\right|^2\,dxdt.\end{aligned}$$

By combining the above two estimates, we get

$$\begin{aligned}&\tau^2\fint_{Q_\tau(y,s)} \left\{\sum_{2\le k\le n}|Dg_k|^2+\tau^2|g_{n+1}|^2\right\}\,dxdt\\ &\le c\left[\left(\frac{\tau}{\rho}\right)^{2\alpha}\fint_{Q_{\frac{\rho}{2}}(y,s)}\left|G-(G)_{Q_\rho(y,s)}\right|^2\,dxdt+\tau^4\|g_{n+1}\|^2_{L^\infty(Q_\tau(y,s))}\right].\end{aligned} \tag{2.40}$$

By the assumption $8\tau\le\rho$, we have from (2.33) in Lemma 2.7 that

$$\tau^4\|g_{n+1}\|^2_{L^\infty(Q_\tau(y,s))}\le\tau^4\|g_{n+1}\|^2_{L^\infty(Q_{\frac{\rho}{8}}(y,s))}\le c\tau^4\fint_{Q_{\frac{\rho}{4}}(y,s)}|g_{n+1}|^2\,dxdt.$$

Also we take $\zeta=(G)_{Q_{\frac{\rho}{2}}(y,s)}$ in Lemma 2.5 to find that

$$\tau^4\fint_{Q_{\frac{\rho}{4}}(y,s)}|g_{n+1}|^2\,dxdt\le\frac{c\tau^4}{\rho^2}\fint_{Q_{\frac{\rho}{2}}(y,s)}\left|G-(G)_{Q_{\frac{\rho}{2}}(y,s)}\right|^2\,dxdt\le\frac{c\tau^4}{\rho^2}\fint_{Q_{\frac{\rho}{2}}(y,s)}\left|G-(G)_{Q_\rho(y,s)}\right|^2\,dxdt.$$

By combining the above two estimates, we get

$$\tau^4\|g_{n+1}\|^2_{L^\infty(Q_\tau(y,s))}\le\frac{c\tau^4}{\rho^2}\fint_{Q_{\frac{\rho}{2}}(y,s)}\left|G-(G)_{Q_\rho(y,s)}\right|^2\,dxdt. \tag{2.41}$$

The claim (2.38) holds from (2.39), (2.40) and (2.41). With Hölder's inequality, the lemma follows from (2.38) and Lemma 2.8 by taking $\zeta=(G)_{Q_\rho(y,s)}$.

3. Comparison estimates

For the comparison estimates on $Q_r(x_0, t_0)$, we handle the interior case $x_0^1 > r$ in Subsection 3.1 and the boundary case $0 \le x_0^1 \le r$ in Subsection 3.2. From [12, Lemma 4.1], the absolute value of measurable nonlinearities $|a(\xi, x^1)|$ is comparable to $|\xi|$. In fact, one can easily modify the proof of [12, Lemma 4.1] to obtain the following result, where the nonlinearities depend on ξ, x and t.

Lemma 3.1. *Suppose that* (1.2)*. For any* $(x, t) \in \mathbb{R}^{n+1}$*, we have that*

$$|\xi| \le c\left[|\xi'| + (2\Lambda)^{-1}|a^1(\xi, x, t)|\right] \le c|\xi| \qquad \left(\xi = (\xi^1, \xi') \in \mathbb{R}^n\right).$$

3.1. Interior comparison estimates

For a weak solution u of

$$u_t - \operatorname{div} a(Du, x^1, x', t) = \mu \quad \text{in} \quad Q_r(x_0, t_0),$$

let v and g be the weak solution of

$$\begin{cases} v_t - \operatorname{div} a(Dv, x^1, x', t) = 0 & \text{in} \quad Q_r(x_0, t_0), \\ v = u & \text{on} \quad \partial_p Q_r(x_0, t_0), \end{cases}$$

and

$$\begin{cases} g_t - \operatorname{div} a(Dg, x^1, x_0', t_0) = 0 & \text{in} \quad Q_{\frac{r}{2}}(x_0, t_0), \\ g = v & \text{on} \quad \partial_p Q_{\frac{r}{2}}(x_0, t_0), \end{cases}$$

where ∂_p denotes the parabolic boundary. By repeating the proof of the comparison estimate for Du and Dv such as in [1, Lemma 4.1] and [3, Lemma 4.1], one can prove that

$$\fint_{Q_r(x_0,t_0)} |Du - Dv|\, dx \le \frac{c|\mu|(Q_r(x_0, t_0))}{r^{n+1}}.$$

By repeating the proof such as in [1, Lemma 4.2], one can prove that

$$\fint_{Q_{\frac{r}{2}}(x_0,t_0)} |Dv - Dg|\, dx \le c\,\omega(r) \fint_{Q_r(x_0,t_0)} |Dv|\, dx.$$

So we obtain that

$$\fint_{Q_{\frac{r}{2}}(x_0,t_0)} |Du - Dg|\, dx \le c\left[\frac{|\mu|(Q_r(x_0, t_0))}{r^{n+1}} + \omega(r) \fint_{Q_r(x_0,t_0)} |Du|\, dx\right]. \tag{3.1}$$

We set

$$\begin{cases} U = (a^1(Du, x^1, x_0', t_0), D_2u, \cdots, D_nu), \\ G = (a^1(Dg, x^1, x_0', t_0), D_2g, \cdots, D_ng). \end{cases} \tag{3.2}$$

From [13, Lemma 4.9], we have that

$$\fint_{Q_\rho(x_0,t_0)} |G - (G)_{Q_\rho(x_0,t_0)}|\, dx \le c\left(\frac{\rho}{r}\right)^\alpha \fint_{Q_{\frac{r}{2}}(x_0,t_0)} |G - (G)_{Q_{\frac{r}{2}}(x_0,t_0)}|\, dx \qquad (0 < 2\rho \le r). \tag{3.3}$$

From Lemma 3.1, we have that $|Du| \le c|U|$. Since $|U - G| \le c|Du - Dg|$, we find from (3.1) that

$$\fint_{Q_{\frac{r}{2}}(x_0,t_0)} |U - G|\,dx \le c\left[\frac{|\mu|(Q_r(x_0,t_0))}{r^{n+1}} + \omega(r)\fint_{Q_r(x_0,t_0)} |Du|\,dx\right]$$
$$\le c\left[\frac{|\mu|(Q_r(x_0,t_0))}{r^{n+1}} + \omega(r)\fint_{Q_r(x_0,t_0)} |U|\,dx\right].$$

So we obtain from (3.3) that

$$\fint_{Q_\rho(x_0,t_0)} |U - (U)_{Q_\rho(x_0,t_0)}|\,dx \le c\left(\frac{\rho}{r}\right)^\alpha \fint_{Q_{\frac{r}{2}}(x_0,t_0)} |U - (U)_{Q_{\frac{r}{2}}(x_0,t_0)}|\,dx$$
$$+ c\left(\frac{r}{\rho}\right)^n \left[\frac{|\mu|(Q_r(x_0,t_0))}{r^{n+1}} + \omega(r)\fint_{Q_r(x_0,t_0)} |U|\,dx\right].$$

One can easily check that

$$\fint_{Q_{\frac{r}{2}}(x_0,t_0)} |U - (U)_{Q_{\frac{r}{2}}(x_0,t_0)}|\,dx$$
$$\le \fint_{Q_{\frac{r}{2}}(x_0,t_0)} |U - (U)_{Q_r(x_0,t_0)}|\,dx + \fint_{Q_{\frac{r}{2}}(x_0,t_0)} |(U)_{Q_r(x_0,t_0)} - (U)_{Q_{\frac{r}{2}}(x_0,t_0)}|\,dx$$
$$\le 2^{n+1}\fint_{Q_r(x_0,t_0)} |U - (U)_{Q_r(x_0,t_0)}|\,dx$$

so that

$$\fint_{Q_\rho(x_0,t_0)} |U - (U)_{Q_\rho(x_0,t_0)}|\,dx \le c\left(\frac{\rho}{r}\right)^\alpha \fint_{Q_r(x_0,t_0)} |U - (U)_{Q_r(x_0,t_0)}|\,dx$$
$$+ c\left(\frac{r}{\rho}\right)^n \left[\frac{|\mu|(Q_r(x_0,t_0))}{r^{n+1}} + \omega(r)\fint_{Q_r(x_0,t_0)} |U|\,dx\right] \tag{3.4}$$

for any $0 < 2\rho \le r$.

3.2. Comparison estimates near the boundary

To handle the boundary case, we assume that

$$0 \le x_0^1 \le r. \tag{3.5}$$

For the boundary data, we assume that

$$D_{x'}\psi \in L^\infty(T_{4r}(x_0', t_0)) \qquad \text{and} \qquad \partial_t\psi \in L^2(T_{4r}(x_0', t_0)).$$

We regard the boundary data ψ as a function in $Q_{4r}^+(x_0,t_0)$ by defining $\psi(x^1, x', t) = \psi(0, x', t)$ for every $0 < x^1 < 4r$. For a weak solution u of

$$\begin{cases} u_t - \operatorname{div} a(Du, x^1, x', t) = \mu & \text{in } Q_{4r}^+(x_0,t_0), \\ u = \psi & \text{on } T_{4r}(x_0', t_0), \end{cases} \tag{3.6}$$

let v, w and g be the weak solution of

$$\begin{cases} v_t - \operatorname{div} a(Dv, x^1, x', t) &= 0 \quad \text{in} \quad Q^+_{4r}(x_0, t_0), \\ v &= u \quad \text{on} \quad \partial_p \left[Q^+_{4r}(x_0, t_0)\right], \end{cases} \tag{3.7}$$

$$\begin{cases} w_t - \operatorname{div} a(Dw, x^1, x', t) &= 0 \quad \text{in} \quad Q^+_{4r}(x_0, t_0), \\ w &= v - \psi + D_{x'}\psi(x'_0, t_0) \cdot x' \quad \text{on} \quad \partial_p \left[Q^+_{4r}(x_0, t_0)\right], \end{cases} \tag{3.8}$$

and

$$\begin{cases} g_t - \operatorname{div} a(Dg, x^1, x'_0, t_0) &= 0 \quad \text{in} \quad Q^+_{3r}(x_0, t_0), \\ g &= w \quad \text{on} \quad \partial_p \left[Q^+_{3r}(x_0, t_0)\right]. \end{cases} \tag{3.9}$$

We have from (3.6) and (3.7) that $v = u = \psi$ on $T_{4r}(x'_0, t_0)$. So from (3.7) and (3.8), we have that

$$w = D_{x'}\psi(x'_0, t_0) \cdot x' \ \text{ on } \ T_{4r}(x'_0, t_0). \tag{3.10}$$

By following the proof of [3, Lemma 4.1] (although [3] considered p-Laplace type equations), we obtain the comparison estimate for Du and Dv to our problems. The proof for Lemma 3.2 is similar to that of [3, Lemma 4.1], but we give the proof for the convenience of the readers.

Lemma 3.2. *Under the assumption* (3.5)*, we have that*

$$\fint_{Q^+_{4r}(x_0,t_0)} |Du - Dv| \, dx \le \frac{c\,|\mu|\left(Q^+_{4r}(x_0, t_0)\right)}{r^{n+1}}.$$

Proof. We first claim that

$$\sup_{\tau \in (t_0 - r^2, t_0)} \int_{K_{4r}(x_0)} |u(x, \tau) - v(x, \tau)| \; dx \le |\mu| \left(Q^+_{4r}(x_0, t_0)\right). \tag{3.11}$$

To prove the claim (3.11), fix $\tau \in (t_0 - r^2, t_0)$. For $m \ge 1$, let $\phi : \mathbb{R} \to [0, 1]$ be a function only depending on t-variable defined as

$$\phi(t) = \begin{cases} 0 & \text{if} \quad t \ge \tau, \\ m\,(\tau - t) & \text{if} \quad \tau - \frac{1}{m} \le t < \tau, \\ 1 & \text{if} \quad t < \tau - \frac{1}{m}. \end{cases} \tag{3.12}$$

Here, we remark that we will let $m \to \infty$ later. We also define

$$\eta_{1,\epsilon} = \pm \min\left\{1, \frac{(u - v)_\pm}{\epsilon}\right\} \phi \qquad (\epsilon > 0),$$

which implies that

$$D\eta_{1,\epsilon} = \frac{1}{\epsilon}\, D\,(u - v)\, \chi_{0<(u-v)_\pm<\epsilon}\, \phi. \tag{3.13}$$

Now, we test (3.6) and (3.7) by $\eta_{1,\epsilon}$ to find that

$$\begin{aligned} \int_{Q^+_{4r}(x_0,t_0)} \partial_t\,(u - v)\, \eta_{1,\epsilon}\, dxdt + \int_{Q^+_{4r}(x_0,t_0)} \langle a(Du, x, t) - a(Dv, x, t), D\eta_{1,\epsilon} \rangle \; dxdt \\ = \int_{Q^+_{4r}(x_0,t_0)} \eta_{1,\epsilon}\, d\mu. \end{aligned} \tag{3.14}$$

One can check that

$$\partial_t (u-v)\, \eta_{1,\epsilon} = \pm \partial_t (u-v)\, \min\left\{1, \frac{(u-v)_\pm}{\epsilon}\right\} \;= \partial_t \left[\int_0^{(u-v)_\pm} \min\left\{1, \frac{s}{\epsilon}\right\} ds\right].$$

Since $\phi = 0$ on $K_{4r} \times \{t_0\}$ and $u = v$ on $K_{4r} \times \left\{t_0 - r^2\right\}$, integration by parts gives

$$\begin{aligned}
&\int_{Q^+_{4r}(x_0,t_0)} \partial_t (u-v)\, \eta_{1,\epsilon}\, dxdt \\
&\quad = \int_{Q^+_{4r}(x_0,t_0)} \partial_t \left[\int_0^{(u-v)_\pm} \min\left\{1, \frac{s}{\epsilon}\right\} ds\right] \phi\, dxdt \\
&\quad = \int_{Q^+_{4r}(x_0,t_0)} \int_0^{(u-v)_\pm} \min\left\{1, \frac{s}{\epsilon}\right\} ds\, \partial_t \{-\phi(t)\}\, dxdt.
\end{aligned} \tag{3.15}$$

We obtain from (3.13) and (3.14) that

$$\begin{aligned}
&\int_{Q^+_{4r}(x_0,t_0)} \int_0^{(u-v)_\pm} \min\left\{1, \frac{s}{\epsilon}\right\} ds\, \partial_t \{-\phi(t)\}\, dxdt \\
&\qquad + \frac{1}{\epsilon} \int_{Q^+_{4r}(x_0,t_0)} \left\langle a(Du,x) - a(Dv,x), D(u-v)\, \chi_{0<(u-v)_\pm<\epsilon} \phi \right\rangle dxdt \\
&\quad = \int_{Q^+_{4r}(x_0,t_0)} \eta_{1,\epsilon}\, d\mu.
\end{aligned} \tag{3.16}$$

By Lebesgue dominated convergence theorem, we get

$$\begin{aligned}
&\int_{Q^+_{4r}(x_0,t_0)} \int_0^{(u-v)_\pm} \min\left\{1, \frac{s}{\epsilon}\right\} ds\, \partial_t \{-\phi(t)\}\, dxdt \\
&\qquad \xrightarrow{\epsilon \to 0} \int_{Q^+_{4r}(x_0,t_0)} (u-v)_\pm \;\, \partial_t \{-\phi(t)\}\, dxdt.
\end{aligned}$$

On the other hand, by ellipticity condition (1.2), we have that

$$0 \le \frac{1}{\epsilon} \int_{Q^+_{4r}(x_0,t_0)} \left\langle a(Du,x,t) - a(Dv,x,t), D(u-v)\, \chi_{0<(u-v)_\pm<\epsilon} \phi \right\rangle dxdt.$$

Since $0 \le \eta_{1,\epsilon} \le 1$, we find that

$$\int_{Q^+_{4r}(x_0,t_0)} \eta_{1,\epsilon}\, d\mu \le |\mu| \left(Q^+_{4r}(x_0,t_0)\right).$$

So we obtain from (3.16) that

$$\int_{Q^+_{4r}(x_0,t_0)} (u-v)_\pm \;\, \partial_t \{-\phi(t)\}\, dxdt \le |\mu| \left(Q^+_{4r}(x_0,t_0)\right).$$

By letting $m \to \infty$ for ϕ, we find that

$$\int_{K_{4r}(x_0)} (u(x,\tau) - v(x,\tau))_\pm \, dx \le |\mu| \left(Q^+_{4r}(x_0,t_0)\right).$$

Since $\tau \in (t_0 - r^2, t_0)$ was chosen arbitrarily, this proves the claim (3.11).

Since ϕ is non-increasing, we have that $\partial_t \{-\phi(t)\} \ge 0$, which implies that

$$\int_{Q^+_{4r}(x_0,t_0)} \int_0^{(u-v)_\pm} \min\left\{1, \frac{s}{\epsilon}\right\} ds \, \partial_t \{-\phi(t)\} \, dxdt \ge 0 \qquad (\epsilon > 0).$$

So it follows from (3.16) that

$$\frac{1}{\epsilon} \int_{Q^+_{4r}(x_0,t_0)} \left\langle a(Du,x,t) - a(Dv,x,t), D(u-v)\chi_{0<(u-v)_\pm<\epsilon}\phi \right\rangle dxdt \le \int_{Q^+_{4r}(x_0,t_0)} \eta_{1,\epsilon} \, d\mu.$$

Since $0 \le \eta_{1,\epsilon} \le 1$, we also obtain from (3.13) that

$$\int_{Q^+_{4r}(x_0,t_0)} \left\langle a(Du,x,t) - a(Dv,x,t), D\eta_{1,\epsilon} \right\rangle dxdt \le |\mu| \left(Q^+_{4r}(x_0,t_0)\right). \tag{3.17}$$

We next claim that

$$\int_{Q^+_{4r}(x_0,t_0)} \frac{|Du - Dv|^2}{(\beta + |u-v|)^\nu} \le \frac{c\beta^{1-\nu}}{\nu - 1} \, |\mu| \left(Q^+_{4r}(x_0,t_0)\right) \tag{3.18}$$

for $\beta > 0$ and $\nu > 1$. To this end, for $\epsilon > 0$, we test (3.6) and (3.7) by

$$\eta_{2,\epsilon} = \frac{\eta_{1,\epsilon}}{(\beta + (u-v)_\pm)^{\nu-1}}, \tag{3.19}$$

which implies that

$$\begin{aligned} &\int_{Q^+_{4r}(x_0,t_0)} \partial_t (u-v)\, \eta_{2,\epsilon} \, dxdt + \int_{Q^+_{4r}(x_0,t_0)} \left\langle a(Du,x,t) - a(Dv,x,t), D\eta_{2,\epsilon} \right\rangle dxdt \\ &\quad = \int_{Q^+_{4r}(x_0,t_0)} \eta_{2,\epsilon} \, d\mu. \end{aligned} \tag{3.20}$$

By the same reasoning for (3.15), we get

$$\int_{Q^+_{4r}(x_0,t_0)} \partial_t (u-v)\, \eta_{2,\epsilon} \, dxdt = \int_{Q^+_{4r}(x_0,t_0)} \int_0^{(u-v)_\pm} \frac{\min\{1, s/\epsilon\}}{(\beta + s)^{\nu-1}} ds \, \partial_t \{-\phi(t)\} \, dxdt.$$

Since ϕ is non-increasing, we have that $\partial_t \{-\phi(t)\} \ge 0$. So the above equality and (3.11) give that

$$\begin{aligned} \sup_{\epsilon>0} \int_{Q^+_{4r}(x_0,t_0)} \partial_t (u-v)\, \eta_{2,\epsilon} \, dxdt &\le \beta^{1-\nu} \sup_{\tau \in (t_0 - r^2, t_0)} \int_{K_{4r}(x_0)} |u(x,\tau) - v(x,\tau)| \, dx \\ &\le \beta^{1-\nu} |\mu| \left(Q^+_{4r}(x_0,t_0)\right). \end{aligned}$$

One can compute that

$$\begin{aligned}
&\int_{Q_{4r}^{+}(x_0,t_0)} \left\langle a(Du,x,t)-a(Dv,x,t), D\eta_{2,\epsilon}\right\rangle \, dxdt \\
&\quad = \int_{Q_{4r}^{+}(x_0,t_0)} \left\langle a(Du,x,t)-a(Dv,x,t), D\eta_{1,\epsilon}\right\rangle \frac{1}{(\beta+(u-v)_{\pm})^{\nu-1}} \, dxdt \\
&\qquad + (1-\nu)\int_{Q_{4r}^{+}(x_0,t_0)} \left\langle a(Du,x,t)-a(Dv,x,t), D\,(u-v)_{\pm}\right\rangle \frac{\eta_{1,\epsilon}}{(\beta+(u-v)_{\pm})^{\nu}} \, dxdt.
\end{aligned}$$

Here, we have from (3.17) that

$$\begin{aligned}
&\int_{Q_{4r}^{+}(x_0,t_0)} \left\langle a(Du,x,t)-a(Dv,x,t), D\eta_{1,\epsilon}\right\rangle \frac{1}{(\beta+(u-v)_{\pm})^{\nu-1}} \, dxdt \\
&\quad \le \beta^{1-\nu}\int_{Q_{4r}^{+}(x_0,t_0)} \left\langle a(Du,x,t)-a(Dv,x,t), D\eta_{1,\epsilon}\right\rangle \, dxdt \\
&\quad \le \beta^{1-\nu}\,|\mu|\left(Q_{4r}^{+}(x_0,t_0)\right)
\end{aligned}$$

and

$$\left|\int_{Q_{4r}^{+}(x_0,t_0)} \eta_{2,\epsilon}\, d\mu\right| \le \beta^{1-\nu}\,|\mu|\left(Q_{4r}^{+}(x_0,t_0)\right).$$

With the above four estimates, we find from (3.20) that

$$\begin{aligned}
&(\nu-1)\int_{Q_{4r}^{+}(x_0,t_0)} \frac{\left\langle a(Du,x,t)-a(Dv,x,t), D\,(u-v)_{\pm}\right\rangle}{(\beta+(u-v)_{\pm})^{\nu-1}}\eta_{1,\epsilon}\, dxdt \\
&\quad \le 3\beta^{1-\nu}\,|\mu|\left(Q_{4r}^{+}(x_0,t_0)\right).
\end{aligned} \tag{3.21}$$

By the definition of $\eta_{1,\epsilon}$, one can see that

$$\begin{aligned}
&\int_{Q_{4r}^{+}(x_0,t_0)} \frac{\left\langle a(Du,x,t)-a(Dv,x,t), D\,(u-v)_{\pm}\right\rangle}{(\beta+(u-v)_{\pm})^{\nu-1}}\eta_{1,\epsilon}\, dxdt \\
&\quad = \int_{Q_{4r}^{+}(x_0,t_0)} \frac{\left\langle a(Du,x,t)-a(Dv,x,t), D\,(u-v)_{\pm}\right\rangle}{(\beta+(u-v)_{\pm})^{\nu-1}} \pm \min\left\{1, \frac{(u-v)_{\pm}}{\epsilon}\right\}\phi\, dxdt \\
&\quad = \int_{Q_{4r}^{+}(x_0,t_0)} \frac{\left\langle a(Du,x,t)-a(Dv,x,t), D\,(u-v)\right\rangle}{(\beta+|u-v|)^{\nu-1}} \min\left\{1, \frac{(u-v)_{\pm}}{\epsilon}\right\}\phi\, dxdt,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\int_{Q_{4r}^{+}(x_0,t_0)} \frac{\left\langle a(Du,x,t)-a(Dv,x,t), D\,(u-v)_{\pm}\right\rangle}{(\beta+(u-v)_{\pm})^{\nu-1}}\eta_{1,\epsilon}\, dxdt \\
&\quad \xrightarrow{\epsilon\to 0} \int_{Q_{4r}^{+}(x_0,t_0)} \frac{\left\langle a(Du,x,t)-a(Dv,x,t), D\,(u-v)\right\rangle}{(\beta+|u-v|)^{\nu-1}}\phi\, dxdt.
\end{aligned}$$

By letting $\tau \to t_0$ and $m \to \infty$, the claim (3.18) follows from (3.12) and (3.21).

Choose $\beta = \left(\fint_{Q^+_{4r}(x_0,t_0)} |u-v|^{\frac{n+1}{n}} \, dxdt \right)^{\frac{n}{n+1}}$ and $\nu = \frac{n+1}{n}$. Then by the paraoblic Sobolev embedding (see for instance [14, Chapter 1, Proposition 3.1]), we get

$$\beta \le c(n,q) \left[\fint_{Q^+_{4r}(x_0,t_0)} |Du - Dv| \, dxdt \left(\sup_{\tau \in (t_0 - r^2, t_0)} \int_{K_{4r}(x_0)} |u(x,\tau) - v(x,\tau)| \, dx \right)^{\frac{1}{n}} \right]^{\frac{n}{n+1}}.$$

It follows from (3.11) that

$$\beta \le c \left[|\mu| \left(Q^+_{4r}(x_0,t_0) \right) \right]^{\frac{1}{n+1}} \left(\fint_{Q^+_{4r}(x_0,t_0)} |Du - Dv| \; dxdt \right)^{\frac{n}{n+1}}. \tag{3.22}$$

By Hölder's inequaltiy, (3.17) and (3.22), we obtain that

$$\begin{aligned}
\fint_{Q^+_{4r}(x_0,t_0)} |Du - Dv| \, dxdt &= \fint_{Q^+_{4r}(x_0,t_0)} \frac{|Du - Dv|}{(\beta + |u-v|)^{\frac{\nu}{2}}} \; (\beta + |u-v|)^{\frac{\nu}{2}} \; dxdt \\
&\le \left[\fint_{Q^+_{4r}(x_0,t_0)} \frac{|Du - Dv|^2}{(\beta + |u-v|)^{\nu}} \, dxdt \right]^{\frac{1}{2}} \left[\fint_{Q^+_{4r}(x_0,t_0)} (\beta + |u-v|)^{\nu} \; dxdt \right]^{\frac{1}{2}} \\
&\le c \left(\frac{|\mu| \left(Q^+_{4r}(x_0,t_0) \right)}{\left| Q^+_{4r}(x_0,t_0) \right|} \beta^{1-\nu} \right)^{\frac{1}{2}} \beta^{\frac{\nu}{2}} \\
&\le c \left[\frac{\left\{ |\mu| \left(Q^+_{4r}(x_0,t_0) \right) \right\}^{\frac{n+2}{n+1}}}{\left| Q^+_{4r}(x_0,t_0) \right|} \left(\fint_{Q^+_{4r}(x_0,t_0)} |Du - Dv| \; dxdt \right)^{\frac{n}{n+1}} \right]^{\frac{1}{2}}.
\end{aligned}$$

Since $\left| Q^+_{4r}(x_0,t_0) \right| \ge c \, r^{n+2}$, the lemma follows.

We also prove the comparison estimate between Dv and Dw as follows.

Lemma 3.3. *Under the assumption* (3.5)*, we have that*

$$\fint_{Q^+_{4r}(x_0,t_0)} |Dv - Dw|^2 \, dxdt \le c \left[\left(\operatorname*{osc}_{T_{4r}(x_0',t_0)} D_{x'}\psi \right)^2 + r^2 \fint_{Q^+_{4r}(x_0,t_0)} |\partial_t \psi|^2 \, dxdt \right].$$

Proof. With $v - w + \psi - D_{x'}\psi(x_0', t_0) \cdot x'$, test (3.7) and (3.8). Fix $\tau \in (t_0 - 16r^2, t_0)$. Then

$$\begin{aligned}
0 = &\int_{t_0 - 16r^2}^{\tau} \int_{K^+_{4r}(x_0)} \partial_t (v - w) \left(v - w + \psi - D_{x'}\psi(x_0', t_0) \cdot x' \right) \; dxdt \\
&+ \int_{t_0 - 16r^2}^{\tau} \int_{K^+_{4r}(x_0)} \langle a(Dv, x^1, x', t) - a(Dw, x^1, x', t), Dv - Dw + D_x\psi - D_x\psi(x_0', t_0) \rangle \, dxdt.
\end{aligned}$$

Recall that $\psi(x^1, x', t) = \psi(x', t)$. It follows from (1.2) and Young's inequality that

$$\begin{aligned}
&\int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} \langle a(Dv, x^1, x', t) - a(Dw, x^1, x', t), Dv - Dw + D_x\psi - D_x\psi(x_0', t_0)\rangle\, dxdt \\
&\geq \frac{\lambda}{2}\int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} |Dv - Dw|^2\, dxdt - c\int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} |D_x\psi - D_x\psi(x_0', t_0)|^2\, dxdt \\
&\geq \frac{\lambda}{2}\int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} |Dv - Dw|^2\, dxdt - c\left\{\tau - (t_0 - 16r^2)\right\}\left|K^+_{4r}(x_0)\right|\left(\operatorname*{osc}_{T_{4r}(x_0', t_0)} D_{x'}\psi\right)^2.
\end{aligned}$$

By a direct calculation,

$$\begin{aligned}
&\int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} \partial_t\,(v - w)\,(v - w + \psi - D_{x'}\psi(x_0', t_0)\cdot x')\; dxdt \\
&= \int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} \partial_t\left\{\frac{\left(v - w + \psi - D_{x'}\psi(x_0', t_0)\cdot x'\right)^2}{2}\right\} dxdt \\
&\quad - \int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} \partial_t\,\psi\,\{v - w + \psi - D_{x'}\psi(x_0', t_0)\cdot x'\}\; dxdt.
\end{aligned}$$

From Young's inequality, we get that

$$\begin{aligned}
&\int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} \partial_t\,(v - w)\,(v - w + \psi - D_{x'}\psi(x_0', t_0)\cdot x')\; dxdt \\
&\geq \int_{K^+_{4r}(x_0)} \frac{\left\{v(x,\tau) - w(x,\tau) + \psi(x,\tau) - D_{x'}\psi(x_0', t_0)\cdot x'\right\}^2}{2}\, dx \\
&\quad - \int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} \frac{\left\{v(x,t) - w(x,t) + \psi(x,t) - D_{x'}\psi(x_0', t_0)\cdot x'\right\}^2}{4\,\{\tau - (t_0 - 16r^2)\}}\, dxdt \\
&\quad - c\left\{\tau - (t_0 - 16r^2)\right\}\int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} |\partial_t\,\psi|^2\, dxdt.
\end{aligned}$$

Thus we find that

$$\begin{aligned}
&\frac{\lambda}{2}\int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} |Dv - Dw|^2\, dxdt + \int_{K^+_{4r}(x_0)} \frac{\left\{v(x,\tau) - w(x,\tau) + \psi(x,\tau) - D_{x'}\psi(x_0', t_0)\cdot x'\right\}^2}{2}\, dx \\
&\leq \int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} \frac{\left\{v(x,t) - w(x,t) + \psi(x,t) - D_{x'}\psi(x_0', t_0)\cdot x'\right\}^2}{4\,\{\tau - (t_0 - 16r^2)\}}\, dxdt \\
&\quad + c\left\{\tau - (t_0 - 16r^2)\right\}\left[\left|K^+_{4r}(x_0)\right|\left(\operatorname*{osc}_{T_{4r}(x_0', t_0)} D_{x'}\psi\right)^2 + \int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} |\partial_t\,\psi|^2\, dxdt\right],
\end{aligned}$$

where

$$\int_{t_0-16r^2}^{\tau}\int_{K^+_{4r}(x_0)} \frac{\left\{v(x,t)-w(x,t)+\psi(x,t)-D_{x'}\psi(x_0',t_0)\cdot x'\right\}^2}{4\left\{\tau-(t_0-16r^2)\right\}}\,dxdt$$
$$\le \sup_{\tau\in(t_0-16r^2,t_0)}\int_{K^+_{4r}(x_0)} \frac{\left\{v(x,\tau)-w(x,\tau)+\psi(x,\tau)-D_{x'}\psi(x_0',t_0)\cdot x'\right\}^2}{4}\,dx.$$

Since $\tau\in(t_0-16r^2,t_0)$ was arbitrary chosen, we find that

$$\int_{Q^+_{4r}(x_0,t_0)}|Dv-Dw|^2\,dxdt \le cr^2\left[\left|K^+_{4r}(x_0)\right|\left(\operatorname*{osc}_{T_{4r}(x_0',t_0)} D_{x'}\psi\right)^2+\int_{Q^+_{4r}(x_0,t_0)}|\partial_t\psi|^2\,dxdt\right],$$

which proves the lemma.

We use the following reverse Hölder type inequality for comparing Dw and Dg.

Lemma 3.4. *Under the assumption* (3.5)*, we have that*

$$\left(\fint_{Q^+_{3r}(x_0,t_0)}|Dw|^2\,dxdt\right)^{\frac{1}{2}} \le c\left(\fint_{Q^+_{4r}(x_0,t_0)}|Dw|+|D_{x'}\psi(x_0',t_0)|\,dxdt\right).$$

Proof. We let

$$\gamma'=D_{x'}\psi(x_0',t_0) \qquad\text{and}\qquad \gamma=(0,\gamma')=(0,D_{x'}\psi(x_0',t_0)).$$

We obtain from (3.5), (3.6) and (3.7) that

$$v=u=\psi \quad\text{on}\quad T_{4r}(x_0',t_0). \tag{3.23}$$

It follows from (3.10) and (3.23) that

$$w-\gamma'\cdot x'=v-\psi=0 \quad\text{on}\quad T_{4r}(x_0',t_0).$$

Define $\hat{w}$ as the zero extension of $w-\gamma'\cdot x'$ from $Q^+_{4r}(x_0,t_0)$ to $Q_{4r}(x_0,t_0)$. Then

$$\hat{w}=\begin{cases} w-\gamma'\cdot x' & \text{in } Q^+_{4r}(x_0,t_0),\\ 0 & \text{in } Q_{4r}(x_0,t_0)\setminus Q^+_{4r}(x_0,t_0). \end{cases} \tag{3.24}$$

Let $2_*=\frac{2n}{n+2}$. Then by dividing into two cases (1) $Q_{2\rho}(y,s)\subset\mathbb{R}^{n+1}_+$ and (2) $Q_{2\rho}(y,s)\not\subset\mathbb{R}^{n+1}_+$, we prove the following assertion that

$$\left(\fint_{Q_\rho(y,s)}|D\hat{w}|^2\,dxdt\right)^{\frac{2_*}{2}} \le c\fint_{Q_{3\rho}(y,s)}|D\hat{w}|^{2_*}+|\gamma|^{2_*}\,dxdt \tag{3.25}$$

for any $Q_{3\rho}(y,s)\subset\subset Q_{4r}(x_0,t_0)$.

Choose $Q_{2\rho}(y,s)\subset\subset Q_{4r}(x_0,t_0)$. First, suppose that $Q_{2\rho}(y,s)\subset\mathbb{R}^{n+1}_+$. Then $Q_{2\rho}(y,s)\subset Q^+_{4r}(x_0,t_0)$. Fix $\rho\le r_1<r_2\le 2\rho$. Let $\eta\in C^\infty_c(Q_{r_2}(y,s))$ be a cut-off function with

$$0\le\eta\le1,\qquad \eta=1 \text{ on } Q_{r_1}(y,s),\qquad |D\eta|\le\frac{c}{r_2-r_1}\qquad\text{and}\qquad |\partial_t\eta|\le\frac{c}{(r_2-r_1)^2}. \tag{3.26}$$

Fix $\tau \in \left(s-r_1^2, s\right)$. Take a test function $\left\{w-(w)_{Q_{r_1}(y,s)}\right\}\eta^2$ for (3.8) to find that

$$\begin{aligned} 0 &= \int_{s-r_2^2}^{\tau}\int_{K_{r_2}(y)} \partial_t w \left\{w-(w)_{Q_{r_1}(y,s)}\right\}\eta^2\,dxdt \\ &\quad + \int_{s-r_2^2}^{\tau}\int_{K_{r_2}(y)} \left\langle a(Dw,x,t), D\left[\left\{w-(w)_{Q_{r_1}(y,s)}\right\}\eta^2\right]\right\rangle dxdt. \end{aligned} \tag{3.27}$$

One can check that

$$\begin{aligned} &\int_{s-r_2^2}^{\tau}\int_{K_{r_2}(y)} \partial_t w \left\{w-(w)_{Q_{r_1}(y,s)}\right\}\eta^2\,dxdt \\ &\quad = \int_{s-r_2^2}^{\tau}\int_{K_{r_2}(y)} \frac{1}{2}\cdot\partial_t\left[\left\{w-(w)_{Q_{r_1}(y,s)}\right\}^2\eta^2\right] - \left\{w-(w)_{Q_{r_1}(y,s)}\right\}^2 2\eta\,\partial_t\eta\,dxdt, \end{aligned}$$

which implies that

$$\begin{aligned} &\int_{K_{r_2}(y)} \frac{1}{2}\left[\left\{w(x,\tau)-(w)_{Q_{r_1}(y,s)}\right\}\eta(x,\tau)\right]^2 dx \\ &\quad \le \int_{s-r_2^2}^{\tau}\int_{K_{r_2}(y)} \left[\partial_t w\left\{w-(w)_{Q_{r_1}(y,s)}\right\}\eta^2 + c\left|w-(w)_{Q_{r_1}(y,s)}\right|^2|\eta|\,|\partial_t\eta|\right]dxdt. \end{aligned} \tag{3.28}$$

In view of (3.27) and (3.28), we apply the ellipticity condition (1.2) to find that

$$\begin{aligned} &\lambda\int_{s-r_2^2}^{\tau}\int_{K_{r_2}(y)} |Dw|^2\eta^2\,dxdt + \int_{K_{r_2}(y)} \frac{1}{2}\left[\left\{w(x,\tau)-(w)_{Q_{r_1}(y,s)}\right\}\eta(x,\tau)\right]^2 dx \\ &\le \int_{s-r_2^2}^{\tau}\int_{K_{r_2}(y)} \langle a(Dw,x,t),\eta^2 Dw\rangle + \partial_t w\left\{w-(w)_{Q_{r_1}(y,s)}\right\}\eta^2 + c\left|w-(w)_{Q_{r_1}(y,s)}\right|^2|\eta|\,|\partial_t\eta|\,dxdt \\ &\le c\int_{s-r_2^2}^{\tau}\int_{K_{r_2}(y)} \left\{|Dw|\left|w-(w)_{Q_{r_1}(y,s)}\right||\eta|\,|D\eta| + \left|w-(w)_{Q_{r_1}(y,s)}\right|^2|\eta|\,|\partial_t\eta|\right\}dxdt. \end{aligned}$$

First, apply Young's inequality. Then (3.26) gives that

$$\begin{aligned} &\frac{\lambda}{2}\int_{s-r_1^2}^{\tau}\int_{K_{r_1}(y)} |Dw|^2\,dxdt + \int_{K_{r_1}(y,s)} \frac{1}{2}\left[w(x,\tau)-(w)_{Q_{r_1}(y,s)}\right]^2 dx \\ &\quad \le \frac{c}{(r_2-r_1)^2}\int_{s-r_2^2}^{\tau}\int_{K_{r_2}(y)} \left|w-(w)_{Q_{r_1}(y,s)}\right|^2 dxdt \\ &\quad \le \frac{c}{(r_2-r_1)^2}\int_{Q_{r_2}(y,s)} \left|w-(w)_{Q_{r_2}(y,s)}\right|^2 dxdt, \end{aligned} \tag{3.29}$$

where we used that

$$\begin{aligned} \int_{s-r_2^2}^{\tau}\int_{K_{r_2}(y)} \left|w-(w)_{Q_{r_1}(y,s)}\right|^2 dxdt &\le \int_{Q_{r_2}(y,s)} \left|w-(w)_{Q_{r_1}(y,s)}\right|^2 dxdt \\ &\le c\int_{Q_{r_2}(y,s)} \left|w-(w)_{Q_{r_2}(y,s)}\right|^2 dxdt, \end{aligned}$$

which holds from that $\tau \in \left(s - r_1^2, s\right)$ and $\rho \le r_1 < r_2 \le 2\rho$.

Since $\tau \in \left(s - r_1^2, s\right)$ was arbitrary chosen, we find from (3.28) and (3.29) that

$$\begin{aligned}
&\int_{Q_{r_1}(y,s)} |Dw|^2\, dxdt + \sup_{\tau\in\left(s-r_1^2,s\right)} \int_{K_{r_1}(y)} \left[w(x,\tau) - (w)_{Q_{r_1}(y,s)}\right]^2 dx \\
&\quad \le \frac{c}{(r_2-r_1)^2} \int_{Q_{r_2}(y,s)} \left|w - (w)_{Q_{r_2}(y,s)}\right|^2 dxdt,
\end{aligned}$$

for any $\rho \le r_1 < r_2 \le 2\rho$. By the parabolic Sobolev embedding (see for instance [14, Chapter 1, Proposition 3.1]), we get

$$\begin{aligned}
&\int_{Q_{r_2}(y,s)} \left|w - (w)_{Q_{r_2}(y,s)}\right|^2 dxdt \\
&\quad \le c\left(\int_{Q_{r_2}(y,s)} |Dw|^{\frac{2n}{n+2}}\, dxdt\right)\left(\sup_{\tau\in\left(s-r_2^2,s\right)} \int_{K_{r_2}(y)} \left[w(x,\tau) - (w)_{Q_{r_2}(y,s)}\right]^2 dx\right)^{\frac{2}{n+2}}.
\end{aligned}$$

So one can use Young's inequality to find that

$$\begin{aligned}
&\int_{Q_{r_1}(y,s)} |Dw|^2\, dxdt + \sup_{\tau\in\left(s-r_1^2,s\right)} \int_{K_{r_1}(y)} \left[w(x,\tau) - (w)_{Q_{r_1}(y,s)}\right]^2 dx \\
&\quad \le \frac{1}{2} \sup_{\tau\in\left(s-r_2^2,s\right)} \int_{K_{r_2}(y)} \left[w(x,\tau) - (w)_{Q_{r_2}(y,s)}\right]^2 dx + c\left[\frac{1}{(r_2-r_1)^2} \int_{Q_{r_2}(y,s)} |Dw|^{\frac{2n}{n+2}}\, dxdt\right]^{\frac{n+2}{n}}.
\end{aligned}$$

Let $g(\theta) = \int_{Q_\theta(y,s)} |Dw|^2\, dxdt + \sup_{\tau\in(s-\theta,s)} \int_{K_\theta(y)} \left[w(x,\tau) - (w)_{Q_\theta(y,s)}\right]^2 dx$. Then

$$g(r_1) \le \frac{1}{2}\, g(r_2) + \frac{c}{(r_2-r_1)^{\frac{2(n+2)}{n}}} \left(\int_{Q_{2\rho}(y,s)} |Dw|^{\frac{2n}{n+2}}\, dxdt\right)^{\frac{n+2}{n}}.$$

Since $\rho \le r_1 < r_2 \le 2\rho$ were arbitrary chosen, we obtain from [11, Lemma 4.3] that

$$g(\rho) \le \frac{c}{\rho^{\frac{2(n+2)}{n}}} \left(\int_{Q_{2\rho}(y,s)} |Dw|^{\frac{2n}{n+2}}\, dxdt\right)^{\frac{n+2}{n}},$$

which implies that

$$\begin{aligned}
&\int_{Q_\rho(y,s)} |Dw|^2\, dxdt + \sup_{\tau\in\left(s-\rho^2,s\right)} \int_{K_\rho(y)} \left[w(x,\tau) - (w)_{Q_\rho(y,s)}\right]^2 dx \\
&\quad \le \frac{c}{\rho^{\frac{2(n+2)}{n}}} \left(\int_{Q_{2\rho}(y,s)} |Dw|^{\frac{2n}{n+2}}\, dxdt\right)^{\frac{n+2}{n}}.
\end{aligned}$$

Thus

$$\fint_{Q_\rho(y,s)} |Dw|^2\, dx \le c\left(\fint_{Q_{2\rho}(y,s)} |Dw|^{2_*}\, dx\right)^{\frac{2}{2_*}}.$$

Recall from (3.24) that $\hat{w} = w - \gamma' \cdot x'$ in $Q_{2\rho}(y,s)$, which implies that

$$\fint_{Q_\rho(y,s)} |D\hat{w} + \gamma|^2 dx \le c \left(\fint_{Q_{2\rho}(y,s)} |D\hat{w} + \gamma|^{2_*} \, dx \right)^{\frac{2}{2_*}} .$$

So the assertion (3.25) holds for the case $Q_{2\rho}(y,s) \subset \mathbb{R}^{n+1}_+$.

Now, suppose that $Q_{2\rho}(y,s) \not\subset \mathbb{R}^{n+1}_+$. Then by the fact that $Q_{3\rho}(y,s) \subset Q_{4r}(x_0,t_0)$,

$$|Q_{3\rho}(y,s) \cap [Q_{4r}(x_0,t_0) \setminus \mathbb{R}^{n+1}_+]| \ge c\rho^n \ge c|Q_{3\rho}(y,s)|. \tag{3.30}$$

Fix $\rho \le r_1 < r_2 \le 2\rho$. Let $\eta \in C_c^\infty(Q_{r_2}(y,s))$ be a cut-off function with

$$0 \le \eta \le 1, \qquad \eta = 1 \text{ on } Q_{r_1}(y,s), \qquad |D\eta| \le \frac{c}{r_2 - r_1} \qquad \text{and} \qquad |\partial_t \eta| \le \frac{c}{(r_2 - r_1)^2}. \tag{3.31}$$

Since $Q_{3\rho}(y,s) \subset Q_{4r}(x_0,t_0)$, it follows from (3.31) that $\eta \in C_c^\infty(Q_{4r}(x_0,t_0))$. We also have from (3.24) that $\hat{w} = 0$ in $Q_{4r}(x_0,t_0) \setminus \mathbb{R}^{n+1}_+$. So we discover that

$$\hat{w}\eta^2 \in W_0^{1,2}\left(Q^+_{4r}(x_0,t_0)\right).$$

Fix $\tau \in \left(s - r_1^2, s\right)$. Take the test function $\hat{w}\eta^2 \in W_0^{1,2}\left(Q^+_{4r}(x_0,t_0)\right)$ for (3.8) to find that

$$0 = \int_{s-r_2^2}^{\tau} \int_{K^+_{r_2}(y)} \partial_t w \, [\hat{w}\eta^2] \, dxdt + \int_{s-r_2^2}^{\tau} \int_{K^+_{r_2}(y)} \langle a(Dw,x,t), D[\hat{w}\eta^2] \rangle \, dxdt,$$

where we used (3.31) and that $Q_{3\rho}(y,s) \subset Q_{4r}(x_0,t_0)$. By a direct calculation,

$$\begin{aligned} &\int_{s-r_2^2}^{\tau} \int_{K^+_{r_2}(y)} \partial_t \hat{w} \, [\hat{w}\eta^2] \, dxdt + \int_{s-r_2^2}^{\tau} \int_{K^+_{r_2}(y)} \langle a(Dw,x,t) - a(\gamma,x,t), \eta^2 D\hat{w} \rangle \, dxdt \\ &\quad = -\int_{s-r_2^2}^{\tau} \int_{K^+_{r_2}(y)} \left\{ \langle a(Dw,x,t) - a(\gamma,x,t), \hat{w} 2\eta D\eta \rangle + \langle a(\gamma,x,t), \eta^2 D\hat{w} + \hat{w} 2\eta D\eta \rangle \right\} dxdt. \end{aligned}$$

From (3.24), we have that $D\hat{w} = Dw - \gamma$ in $\mathbb{R}^{n+1}_+$. So (1.2) gives that

$$\begin{aligned} &\int_{s-r_2^2}^{\tau} \int_{K^+_{r_2}(y)} \partial_t \left\{ \frac{(\hat{w}\eta)^2}{2} \right\} + \lambda |D\hat{w}|^2 \eta^2 \, dxdt \\ &\quad \le c \int_{s-r_2^2}^{\tau} \int_{K^+_{r_2}(y)} \left\{ |\hat{w}|^2 |\eta| \, |\partial_t \eta| + |\eta||D\hat{w}||\hat{w}||D\eta| + |\gamma| \left(|\eta|^2 |D\hat{w}| + |\hat{w}||\eta||D\eta| \right) \right\} dxdt. \end{aligned}$$

Since $\eta \in C_c^\infty(Q_{r_2}(y,s))$, we have from Young's inequality that

$$\begin{aligned} &\int_{K^+_{r_2}(y)} \frac{|\hat{w}(x,\tau)\,\eta(x,\tau)|^2}{2} \, dx + \lambda \int_{s-r_2^2}^{\tau} \int_{K^+_{r_2}(y)} |D\hat{w}|^2 \eta^2 dxdt \\ &\quad \le c \int_{Q^+_{r_2}(y,s)} \left\{ |\hat{w}|^2 \left(|D\eta|^2 + |\eta| \, |\partial_t \eta| \right) + |\gamma|^2 \eta^2 \right\} dxdt. \end{aligned} \tag{3.32}$$

By (3.24), $\hat{w} = 0$ in $Q_{4r}(x_0, t_0) \setminus \mathbb{R}^{n+1}_+$. Since $Q_{2\rho}(y, s) \subset Q_{4r}(x_0, t_0)$, we have that

$$\hat{w} = 0 \text{ in } Q_{2\rho}(y, s) \setminus \mathbb{R}^{n+1}_+,$$

and it follows from (3.32) that

$$\begin{aligned}
&\int_{K_{r_2}(y)} \frac{|\hat{w}(x,\tau)\,\eta(x,\tau)|^2}{2}\,dx + \int_{s-r_2^2}^{\tau}\int_{K_{r_2}(y)} |D\hat{w}|^2\eta^2\,dxdt \\
&\quad \le c\int_{Q_{r_2}(y,s)} \left\{|\hat{w}|^2\left(|D\eta|^2 + |\eta|\,|\partial_t\eta|\right) + |\gamma|^2\eta^2\right\} dxdt.
\end{aligned}$$

Since $\tau \in \left(s - r_1^2, s\right)$ was arbitrary chosen, we have from (3.31) that

$$\begin{aligned}
&\sup_{\tau\in\left(s-r_1^2,s\right)} \int_{K_{r_1}(y)} |\hat{w}(x,\tau)|^2\,dx + \int_{Q_{r_1}(y,s)} |D\hat{w}|^2\,dxdt \\
&\quad \le c\left[\frac{1}{(r_2-r_1)^2}\int_{Q_{r_2}(y,s)} |\hat{w}|^2 dxdt + |\gamma|^2\right].
\end{aligned} \tag{3.33}$$

From (3.30) and that $\hat{w} = 0$ in $Q_{4r}(x_0, t_0) \setminus \mathbb{R}^{n+1}_+$, we have the Sobolev-Poincaré type inequality in [15, Theorem 3.16] to get

$$\begin{aligned}
\fint_{Q_{r_2}(y,s)} |\hat{w}|^2 dx &= \fint_{Q_{r_2}(y,s)} |\hat{w}(x,\tau)|^{\frac{2n}{n+2}}|\hat{w}(x,\tau)|^{\frac{4}{n+2}}dxdt \\
&= \fint_{s-r_2^2}^{s}\left(\fint_{K_{r_2}(y)} |\hat{w}(x,\tau)|^2\,dx\right)^{\frac{n}{n+2}}\left(\fint_{K_{r_2}(y)} |\hat{w}(x,\tau)|^2 dx\right)^{\frac{2}{n+2}} dt \\
&\le c\left(\fint_{Q_{r_2}(y)} |D\hat{w}(x,\tau)|^{\frac{2n}{n+2}}\,dx\right)\left(\sup_{\tau\in\left(s-r_2^2,s\right)}\fint_{K_{r_2}(y)} |\hat{w}(x,\tau)|^2\,dx\right)^{\frac{2}{n+2}}.
\end{aligned} \tag{3.34}$$

So with Young's inequality, one can use the above two inequalities to find that

$$\begin{aligned}
&\sup_{\tau\in\left(s-r_1^2,s\right)} \int_{K_{r_1}(y)} |\hat{w}(x,\tau)|^2\,dx + \int_{Q_{r_1}(y,s)} |D\hat{w}|^2\,dxdt \\
&\quad \le \frac{1}{2}\sup_{\tau\in\left(s-r_2^2,s\right)} \int_{K_{r_2}(y)} |\hat{w}(x,\tau)|^2\,dx + c\left[\frac{1}{(r_2-r_1)^{\frac{2(n+2)}{n}}}\left(\int_{Q_{r_2}(y,s)} |D\hat{w}|^{\frac{2n}{n+2}}\,dxdt\right)^{\frac{n+2}{n}} + |\gamma|^2\right].
\end{aligned}$$

Let $g(\theta) = \sup_{\tau\in(s-\theta,s)} \int_{K_\theta(y)} [\hat{w}(x,\tau)]^2\,dx + \int_{Q_\theta(y,s)} |D\hat{w}|^2\,dxdt$. Then

$$g(r_1) \le \frac{1}{2}g(r_2) + c\left[\frac{1}{(r_2-r_1)^{\frac{2(n+2)}{n}}}\left(\int_{Q_{3\rho}(y,s)} |D\hat{w}|^{\frac{2n}{n+2}}\,dxdt\right)^{\frac{n+2}{n}} + |\gamma|^2\right].$$

Since $\rho \le r_1 < r_2 \le 2\rho$ were arbitrary chosen, we obtain from [11, Lemma 4.3] that

$$g(\rho) \le c\left[\frac{1}{\rho^{\frac{2(n+2)}{n}}}\left(\int_{Q_{3\rho}(y,s)} |D\hat{w}|^{\frac{2n}{n+2}}\,dxdt\right)^{\frac{n+2}{n}} + |\gamma|^2\right],$$

which implies that

$$\int_{Q_\rho(y,s)} |D\hat{w}|^2\,dxdt + \sup_{\tau\in(s-\rho^2,s)} \int_{K_\rho(y)} |\hat{w}(x,\tau)|^2\,dx$$
$$\le c\left[\frac{1}{\rho^{\frac{2(n+2)}{n}}}\left(\int_{Q_{3\rho}(y,s)} |D\hat{w}|^{\frac{2n}{n+2}}\,dxdt\right)^{\frac{n+2}{n}} + |\gamma|^2\right].$$

So (3.25) holds for the case $Q_{2\rho}(y,s) \not\subset \mathbb{R}^{n+1}_+$.

By dividing into two cases (1) $Q_{2\rho}(y,s) \subset \mathbb{R}^{n+1}_+$ and (2) $Q_{2\rho}(y,s) \not\subset \mathbb{R}^{n+1}_+$, we have the assertion (3.25). Since $Q_{3\rho}(y,s) \subset\subset Q_{4r}(x_0,t_0)$ in (3.25) was arbitrary chosen, by applying [16, Lemma 3.1] for $s = |\gamma|$ and $\chi_0 = \frac{2}{2_*} > 1$ (with a suitable covering argument because the size is 3ρ in the right-hand side of (3.25) not 2ρ), we have that

$$\left(\fint_{Q_{3r}(x_0,t_0)} |D\hat{w}|^2 dx\right)^{\frac{1}{2}} \le c\fint_{Q_{4r}(x_0,t_0)} |D\hat{w}| + |\gamma|\,dx.$$

Since $\gamma = (0, D_{x'}\psi(x_0', t_0))$, the lemma follows from (3.5) and (3.24).

In Lemma 3.4, we obtained the reverse Hölder type inequality. So we can obtain the following comparison estimate for Dw and Dg.

Lemma 3.5. *Under the assumption* (3.5)*, we have that*

$$\fint_{Q^+_{3r}(x_0,t_0)} |Dw - Dg|\,dxdt \le c\,\omega(3r)\fint_{Q^+_{4r}(x_0,t_0)} |Dw| + |D_{x'}\psi(x_0',t_0)|\,dxdt.$$

Proof. By using $w - g$, test (3.8) and (3.9). Then we get

$$\fint_{Q^+_{3r}(x_0,t_0)} \langle a(Dw, x^1, x_0', t_0) - a(Dg, x^1, x_0', t_0), Dw - Dg\rangle\,dxdt$$
$$\le \fint_{Q^+_{3r}(x_0,t_0)} \langle a(Dw, x^1, x_0', t_0) - a(Dw, x^1, x', t), Dw - Dg\rangle\,dxdt.$$

We obtain from (1.2) and (1.3) that

$$\fint_{Q^+_{3r}(x_0,t_0)} |Dw - Dg|^2\,dxdt \le c\,\omega(3r)\fint_{Q^+_{3r}(x_0,t_0)} |Dw||Dw - Dg|\,dxdt.$$

With Young's inequality and Lemma 3.4, we get that

$$\fint_{Q^+_{3r}(x_0,t_0)} |Dw - Dg|^2\,dxdt \le c[\omega(3r)]^2\fint_{Q^+_{3r}(x_0,t_0)} |Dw|^2\,dxdt$$
$$\le c[\omega(3r)]^2\left(\fint_{Q^+_{4r}(x_0,t_0)} |Dw| + |D_{x'}\psi(x_0',t_0)|\,dxdt\right)^2.$$

From Hölder's inequality, the lemma follows.

With Lemma 3.2, Lemma 3.3 and Lemma 3.5, the comparison estimates for Du and Dg will be obtained. We now have the comparison estimate Lemma 3.6 and the excess decay estimate Lemma 2.9, so the remaining proof is similar to the elliptic case in [5]. However, for the sake of the completeness, we give a detailed proof.

Lemma 3.6. *Under the assumption* (3.5)*, we have that*

$$\fint_{Q^+_{3r}(x_0,t_0)} |Du - Dg|\,dxdt \le c\left[\frac{|\mu|(Q^+_{4r}(x_0,t_0))}{r^{n+1}} + \operatorname*{osc}_{T_{4r}(x_0',t_0)} D_{x'}\psi + r^2 \fint_{Q^+_{4r}(x_0,t_0)} |\partial_t\psi|^2\,dxdt\right]$$
$$+ c\,\omega(4r)\fint_{Q^+_{4r}(x_0,t_0)} |Du| + |D_{x'}\psi(x_0',t_0)|\,dxdt.$$

Proof. By Lemma 3.5,

$$\fint_{Q^+_{3r}(x_0,t_0)} |Dw - Dg|\,dxdt \le c\,\omega(3r)\fint_{Q^+_{4r}(x_0,t_0)} |Dw| + |D_{x'}\psi(x_0',t_0)|\,dxdt$$
$$\le c\,\omega(3r)\fint_{Q^+_{4r}(x_0,t_0)} |Du| + |Du - Dw| + |D_{x'}\psi(x_0',t_0)|\,dxdt.$$

In view of Lemma 3.2 and Lemma 3.3,

$$\fint_{Q^+_{4r}(x_0,t_0)} |Du - Dw|\,dxdt \le \fint_{Q^+_{4r}(x_0,t_0)} |Du - Dv| + |Dv - Dw|\,dxdt$$
$$\le c\left[\frac{|\mu|(Q^+_{4r}(x_0,t_0))}{r^{n+1}} + \operatorname*{osc}_{T_{4r}(x_0',t_0)} D_{x'}\psi + r^2\fint_{Q^+_{4r}(x_0,t_0)} |\partial_t\psi|^2\,dxdt\right].$$

From the above two estimates, the lemma follows.

Recall the definition of G in (3.2), which is an extension of $(a^1(Dg, x^1, x_0', t_0), D_2 g, \cdots, D_n g)$. In Lemma 3.6, we obtained an excess decay estimate of G. For an extension $U : Q_{4r}(x_0,t_0) \to \mathbb{R}^{n+1}$ of $(a^1(Du, x^1, x_0', t_0), D_2u - D_2\psi, \cdots, D_nu - D_n\psi)$ which is defined as

$$U = (u_1, \cdots, u_n), \tag{3.35}$$

where

$$\begin{cases} u_1 \text{ is the even extension of } a^1(Du, x^1, x_0', t_0) \text{ from } Q^+_{4r}(x_0,t_0) \text{ to } Q_{4r}(x_0,t_0), \\ u_k\ (k \in \{2,\cdots,n\}) \text{ is the odd extension of } D_ku - D_k\psi \text{ from } Q^+_{4r}(x_0,t_0) \text{ to } Q_{4r}(x_0,t_0), \end{cases} \tag{3.36}$$

we derive an excess decay estimate in the following lemma.

Lemma 3.7. *Under the assumption* (3.5)*, we have that*

$$\fint_{Q_\rho(x_0,t_0)} |U - (U)_{Q_\rho(x_0,t_0)}|\,dxdt$$
$$\le c\left[\left(\frac{\rho}{r}\right)^\alpha \fint_{Q_{4r}(x_0,t_0)} |U - (U)_{Q_{4r}(x_0,t_0)}|\,dxdt + \omega(4r)\left(\frac{r}{\rho}\right)^n \fint_{Q_{4r}(x_0,t_0)} |U| + |D_{x'}\psi(x_0',t_0)|\,dxdt\right]$$
$$+ c\left(\frac{r}{\rho}\right)^n\left[\frac{|\mu|(Q^+_{4r}(x_0,t_0))}{r^{n+1}} + \operatorname*{osc}_{T_{4r}(x_0',t_0)} D_{x'}\psi + r^2\fint_{Q^+_{4r}(x_0,t_0)} |\partial_t\psi|^2\,dxdt\right]$$

for any $0 < 2\rho \le r$.

Proof. We set $G : Q_{3r}(x_0,t_0) \to \mathbb{R}^n$ as

$$G = (g_1, \cdots, g_n), \tag{3.37}$$

where

$$\begin{cases} g_1 \text{ is the even extension of } a^1(Dg, x^1, x_0', t_0) \text{ from } Q^+_{3r}(x_0,t_0) \text{ to } Q_{3r}(x_0,t_0), \\ g_k\ (k \in \{2,\cdots,n\}) \text{ is the odd extension of } D_k g - D_k\psi(x_0',t_0) \text{ from } Q^+_{3r}(x_0,t_0) \text{ to } Q_{3r}(x_0,t_0). \end{cases}$$

We have from (3.5) and Lemma 2.9 that

$$\fint_{Q_\rho(x_0,t_0)} |G - (G)_{Q_\rho(x_0,t_0)}|\, dxdt \le c\left(\frac{\rho}{r}\right)^\alpha \fint_{Q_{2r}(x_0,t_0)} |G - (G)_{Q_{2r}(x_0,t_0)}|\, dxdt \tag{3.38}$$

for any $0 < 2\rho \le r$. In view of (3.36) and (3.37), we discover that

$$|u_k - g_k| \le |D_k u - D_k g| + |D_k\psi - D_k\psi(x_0',t_0)| \le |Du - Dg| + \operatorname*{osc}_{T_{3r}(x_0',t_0)} D_{x'}\psi \text{ in } Q^+_{3r}(x_0,t_0),$$

for any $k \in \{2,3,\cdots,n\}$. Since $0 \le x_0^1 \le r$, we have from (3.36) and (3.37) that

$$\begin{aligned} \fint_{Q_{3r}(x_0,t_0)} |u_k - g_k|\, dxdt &\le 2\fint_{Q^+_{3r}(x_0,t_0)} |u_k - g_k|\, dxdt \\ &\le 2\fint_{Q^+_{3r}(x_0,t_0)} |Du - Dg|\, dxdt + 2\operatorname*{osc}_{T_{3r}(x_0',t_0)} D_{x'}\psi \end{aligned} \tag{3.39}$$

for any $k \in \{2,3,\cdots,n\}$. From (3.36) and (3.37), we find that

$$|u_1 - g_1| \le |a^1(Du, x^1, x_0', t_0) - a^1(Dg, x^1, x_0', t_0)| \le c|Du - Dg| \quad \text{in} \quad Q^+_{3r}(x_0,t_0),$$

which implies that

$$\fint_{Q_{3r}(x_0,t_0)} |u_1 - g_1|\, dxdt \le 2\fint_{Q^+_{3r}(x_0,t_0)} |u_1 - g_1|\, dxdt \le 2\fint_{Q^+_{3r}(x_0,t_0)} |Du - Dg|\, dxdt, \tag{3.40}$$

because $0 \le x_0^1 \le r$ in (3.40). For U and G in (3.35) and (3.37), we have from (3.39) and (3.40) that

$$\begin{aligned} \fint_{Q_{3r}(x_0,t_0)} |U - G|\, dxdt &= \fint_{Q_{3r}(x_0,t_0)} |(u_1,\cdots,u_n) - (g_1,\cdots,g_n)|\, dxdt \\ &\le c\left[\fint_{Q^+_{3r}(x_0,t_0)} |Du - Dg|\, dxdt + \operatorname*{osc}_{T_{3r}(x_0',t_0)} D_{x'}\psi\right]. \end{aligned}$$

On the other hand, Lemma 3.6 gives that

$$\begin{aligned} \fint_{Q^+_{3r}(x_0,t_0)} |Du - Dg|\, dxdt \le{}& c\left[\frac{|\mu|(Q^+_{4r}(x_0,t_0))}{r^{n+1}} + \operatorname*{osc}_{T_{4r}(x_0',t_0)} D_{x'}\psi + r^2\fint_{Q^+_{4r}(x_0,t_0)} |\partial_t\psi|^2\, dxdt\right] \\ &+ c\,\omega(4r)\fint_{Q^+_{4r}(x_0,t_0)} \left(|Du| + |D_{x'}\psi(x_0',t_0)|\right)\, dxdt. \end{aligned}$$

We find from Lemma 3.1 that $|Du| \le c|U|$. So by combining the above two estimates,

$$\begin{aligned} \fint_{Q_{3r}(x_0,t_0)} |U - G| \, dxdt \le c \left[\frac{|\mu|(Q^+_{4r}(x_0,t_0))}{r^{n+1}} + \operatorname*{osc}_{T_{4r}(x_0',t_0)} D_{x'}\psi + r^2 \fint_{Q^+_{4r}(x_0,t_0)} |\partial_t \psi|^2 \, dxdt \right] \\ + c\,\omega(4r) \fint_{Q_{4r}(x_0,t_0)} |U| + |D_{x'}\psi(x_0',t_0)| \, dxdt. \end{aligned} \tag{3.41}$$

From (3.38) and (3.41), we have that

$$\begin{aligned} &\fint_{Q_\rho(x_0,t_0)} |U - (U)_{Q_\rho(x_0,t_0)}| \, dxdt \\ &\le c \left[\left(\frac{\rho}{r}\right)^\alpha \fint_{Q_{2r}(x_0,t_0)} |U - (U)_{Q_{2r}(x_0,t_0)}| \, dxdt + \left(\frac{r}{\rho}\right)^n \omega(4r) \fint_{Q_{4r}(x_0,t_0)} |U| + |D_{x'}\psi(x_0',t_0)| \, dxdt \right] \\ &\quad + c \left(\frac{r}{\rho}\right)^n \left[\frac{|\mu|(Q^+_{4r}(x_0,t_0))}{r^{n+1}} + \operatorname*{osc}_{T_{4r}(x_0',t_0)} D_{x'}\psi + r^2 \fint_{Q^+_{4r}(x_0,t_0)} |\partial_t \psi|^2 \, dxdt \right], \end{aligned}$$

for any $0 < 2\rho \le r$. One can easily check that

$$\begin{aligned} &\fint_{Q_{2r}(x_0,t_0)} |U - (U)_{Q_{2r}(x_0,t_0)}| \, dxdt \\ &\quad \le \fint_{Q_{2r}(x_0,t_0)} |U - (U)_{Q_{4r}(x_0,t_0)}| \, dxdt + \fint_{Q_{2r}(x_0,t_0)} |(U)_{Q_{4r}(x_0,t_0)} - (U)_{Q_{2r}(x_0,t_0)}| \, dxdt \\ &\quad \le 2^{n+1} \fint_{Q_{4r}(x_0,t_0)} |U - (U)_{Q_{4r}(x_0,t_0)}| \, dxdt, \end{aligned}$$

and the lemma follows.

4. Pointwise Riesz potential estimates

The remaining proof is similar to the elliptic case [5], but we give a detailed proof for the completeness. For a weak solution u of (1.7), define $U : Q_{2R} \to \mathbb{R}^n$ as

$$U = (u_1, u_2, \cdots, u_n), \tag{4.1}$$

where

$$\begin{cases} u_1 \text{ is the even extension of } a^1(Du, x^1, x_0', t_0) \text{ from } Q^+_{2R} \text{ to } Q_{2R}, \\ u_k \ (k \in \{2,3,\cdots,n\}) \text{ is the odd extension of } D_k u - D_k \psi \text{ from } Q^+_{2R} \text{ to } Q_{2R}. \end{cases} \tag{4.2}$$

Lemma 4.1. *For any $(x_0,t_0) \in \overline{Q^+_R}$ and $0 < \rho \le 4r \le R$, we have that*

$$\begin{aligned} &\fint_{Q_\rho(x_0,t_0)} |U - (U)_{Q_\rho(x_0,t_0)}| \, dxdt \\ &\le c \left[\left(\frac{\rho}{r}\right)^\alpha \fint_{Q_{4r}(x_0,t_0)} |U - (U)_{Q_{4r}(x_0,t_0)}| \, dxdt + \omega(4r) \left(\frac{r}{\rho}\right)^n \fint_{Q_{4r}(x_0,t_0)} |U| + |D_{x'}\psi(x_0',t_0)| \, dxdt \right] \\ &\quad + c \left(\frac{r}{\rho}\right)^n \left[\frac{|\mu|(Q^+_{4r}(x_0,t_0))}{r^{n+1}} + \operatorname*{osc}_{T_{4r}(x_0',t_0)} D_{x'}\psi + r^2 \fint_{Q^+_{4r}(x_0,t_0)} |\partial_t \psi|^2 \, dxdt \right]. \end{aligned}$$

Proof. If $r < 2\rho \le 8r$, then one can directly check that

$$\begin{aligned}
&\fint_{Q_\rho(x_0,t_0)} |U - (U)_{Q_\rho(x_0,t_0)}|\, dxdt \\
&\quad \le \fint_{Q_\rho(x_0,t_0)} |U - (U)_{Q_{4r}(x_0,t_0)}| + |(U)_{Q_{4r}(x_0,t_0)} - (U)_{Q_\rho(x_0,t_0)}|\, dxdt \\
&\quad \le 2\left(\frac{4r}{\rho}\right)^n \fint_{Q_{4r}(x_0,t_0)} |U - (U)_{Q_{4r}(x_0,t_0)}|\, dxdt \\
&\quad \le c\left(\frac{\rho}{r}\right)^\alpha \fint_{Q_{4r}(x_0,t_0)} |U - (U)_{Q_{4r}(x_0,t_0)}|\, dxdt,
\end{aligned}$$

and the lemma follows.

We now suppose that $0 < 2\rho \le r$. Assume that $x_0^1 > r$, which is the interior case. Then $Q_r(x_0,t_0) \subset \mathbb{R}^{n+1}_+$. The definition of U in (3.2) and (4.1) are different, but the value of $U - (U)_{Q_\rho(x_0,t_0)}$ in $Q_\rho(x_0,t_0)$ and $U - (U)_{Q_r(x_0,t_0)}$ in $Q_r(x_0,t_0)$ from (3.2) and (4.1) differs by at most $\underset{T_{4r}(x_0',t_0)}{\operatorname{osc}} D_{x'}\psi$. Also the value of U from (3.2) and (4.1) differs by at most $|D_{x'}\psi| \le |D_{x'}\psi(x_0',t_0)| + \underset{T_{4r}(x_0',t_0)}{\operatorname{osc}} D_{x'}\psi$ in $Q_{4r}(x_0,t_0)$. So we find from (3.4) that

$$\begin{aligned}
&\fint_{Q_\rho(x_0,t_0)} |U - (U)_{Q_\rho(x_0,t_0)}|\, dxdt \\
&\quad \le c\left[\left(\frac{\rho}{r}\right)^\alpha \fint_{Q_r(x_0,t_0)} |U - (U)_{Q_r(x_0,t_0)}|\, dxdt + \omega(r)\left(\frac{r}{\rho}\right)^n \fint_{Q_r(x_0,t_0)} |U| + |D_{x'}\psi(x_0',t_0)|\, dxdt\right] \\
&\qquad + c\left(\frac{r}{\rho}\right)^n \left[\frac{|\mu|(Q_r(x_0,t_0))}{r^{n+1}} + \underset{T_r(x_0',t_0)}{\operatorname{osc}} D_{x'}\psi + r^2 \fint_{Q_r(x_0,t_0)} |\partial_t\psi|^2\, dxdt\right].
\end{aligned}$$

So the lemma holds when $x_0^1 > r$.

On the other hand, if $0 \le x_0^1 \le r$, Lemma 3.7 implies that

$$\begin{aligned}
&\fint_{Q_\rho(x_0,t_0)} |U - (U)_{Q_\rho(x_0,t_0)}|\, dxdt \\
&\quad \le c\left[\left(\frac{\rho}{r}\right)^\alpha \fint_{Q_{4r}(x_0,t_0)} |U - (U)_{Q_{4r}(x_0,t_0)}|\, dxdt + \omega(4r)\left(\frac{r}{\rho}\right)^n \fint_{Q_{4r}(x_0,t_0)} |U| + |D_{x'}\psi(x_0',t_0)|\, dxdt\right] \\
&\qquad + c\left(\frac{r}{\rho}\right)^n \left[\frac{|\mu|(Q^+_{4r}(x_0,t_0))}{r^{n+1}} + \underset{T_{4r}(x_0',t_0)}{\operatorname{osc}} D_{x'}\psi + r^2 \fint_{Q^+_{4r}(x_0,t_0)} |\partial_t\psi|^2\, dxdt\right],
\end{aligned}$$

where U in Lemma 3.7 was defined in (3.35) which is same to that in (4.1). So the lemma holds when $0 \le x_0^1 \le r$.

Now, we prove Theorem 1.1.

Proof of Theorem 1.1. Let $x_0 = (x_0^1, x_0', t_0) \in \overline{Q_R^+}$ be a Lebesgue point of Du. From Lemma 4.1, we get that

$$
\begin{aligned}
&\fint_{Q_{\rho_1}(x_0,t_0)} |U-(U)_{Q_{\rho_1}(x_0,t_0)}|\,dxdt \\
&\le c_2\left[\left(\frac{\rho_1}{\rho_2}\right)^{\alpha}\fint_{Q_{\rho_2}(x_0,t_0)} |U-(U)_{Q_{\rho_2}(x_0,t_0)}|\,dxdt+\omega(\rho_2)\left(\frac{\rho_2}{\rho_1}\right)^{n}\fint_{Q_{\rho_2}(x_0,t_0)} |U|+|D_{x'}\psi(x_0',t_0)|\,dxdt\right] \\
&\quad + c\left(\frac{\rho_2}{\rho_1}\right)^{n}\left[\frac{|\mu|(Q^+_{\rho_2}(x_0,t_0))}{r^{n+1}}+\operatorname*{osc}_{T_{\rho_2}(x_0',t_0)} D_{x'}\psi+\rho_2^2\fint_{Q^+_{\rho_2}(x_0,t_0)}|\partial_t\psi|^2\,dxdt\right],
\end{aligned}
$$

for any $0<\rho_1\le\rho_2\le R$. Choose $\delta=\delta(n,\lambda,\Lambda)\in(0,1/2]$ satisfying that

$$c_2\delta^{\alpha}\le\frac{1}{16}, \tag{4.3}$$

which implies that

$$
\begin{aligned}
&\fint_{Q_{\delta\rho}(x_0,t_0)} |U-(U)_{Q_{\delta\rho}(x_0,t_0)}|\,dxdt \\
&\quad\le \frac{1}{16}\fint_{Q_{\rho}(x_0,t_0)} |U-(U)_{Q_{\rho}(x_0,t_0)}|\,dxdt+c_3\left[\omega(\rho)\fint_{Q_{\rho}(x_0,t_0)}|U|+|D_{x'}\psi(x_0',t_0)|\,dxdt\right] \\
&\quad\quad + c_3\left[\frac{|\mu|(Q^+_{\rho}(x_0,t_0))}{\rho^{n+1}}+\operatorname*{osc}_{T_{\rho}(x_0',t_0)} D_{x'}\psi+\rho^2\fint_{Q^+_{\rho}(x_0,t_0)}|\partial_t\psi|^2\,dxdt\right],
\end{aligned} \tag{4.4}
$$

for any $0<\rho\le R$. We choose the constant $c_1=c_1(n,\lambda,\Lambda)\ge 1$ in (1.6) as

$$c_1=\max\left\{\frac{16\delta^{-2n}c_3}{\log 2},1\right\}.$$

Then from the assumption (1.6) in Theorem 1.1, one can check that

$$\frac{\delta^{-2n}c_3}{\log 2}\int_0^{2r}\frac{\omega(\rho)\,d\rho}{\rho}\le\frac{1}{16}\quad\text{for some}\quad r\in(0,R]. \tag{4.5}$$

For $i=1,2,\cdots$, let $r_i=\delta^i r$, $Q_i=Q_{r_i}(x_0,t_0)$, $Q_i^+=Q^+_{r_i}(x_0,t_0)$, $T_{r_i}=T_{r_i}(x_0',t_0)$, $E_i=\fint_{Q_i}|U-(U)_{Q_i}|\,dxdt$, $F_i=\left|\fint_{Q_i}U\,dxdt\right|$ and

$$\nu_i=\frac{|\mu|(Q_i^+)}{r_i^{n-1}}+\operatorname*{osc}_{T_{r_i}}D_{x'}\psi+\omega(r_i)|D_{x'}\psi(x_0',t_0)|+r_i^2\fint_{Q_i^+}|\partial_t\psi|^2\,dxdt. \tag{4.6}$$

Choose $\rho=r_i$ in (4.4). Then we get that

$$E_{i+1}\le\frac{E_i}{16}+c_3\left[\omega(r_i)\fint_{Q_i}|U|\,dxdt+\nu_i\right]\qquad(i=0,1,2,\cdots).$$

Here, we obtain that

$$E_{i+1} \le \left[\frac{1}{16} + c_3\omega(r_i)\right] E_i + c_3[\omega(r_i)F_i + \nu_i] \qquad (i = 0, 1, 2, \cdots), \tag{4.7}$$

because of that

$$⨍_{Q_i} |U|\, dxdt \le ⨍_{Q_i} |U - (U)_{Q_i}|\, dxdt + |(U)_{Q_i}| \le E_i + F_i \qquad (i = 0, 1, 2, \cdots).$$

Since $\delta = \delta(n, \lambda, \Lambda) \in (0, 1/2]$, one can directly check that

$$\sum_{j=0}^{\infty} \omega(r_j) \le \left[\frac{1}{\log 2}\int_r^{2r} \frac{\omega(\rho)\, d\rho}{\rho} + \sum_{j=0}^{\infty} \frac{1}{\log(1/\delta)} \int_{r_{j+1}}^{r_j} \frac{\omega(\rho)\, d\rho}{\rho}\right] \le \frac{1}{\log 2}\int_0^{2r} \frac{\omega(\rho)\, d\rho}{\rho}.$$

From (4.5), we get

$$\delta^{-2n} c_3 \sum_{j=0}^{\infty} \omega(r_j) \le \frac{1}{16}, \tag{4.8}$$

which implies that $c_3\omega(r_i) \le 1/16$ for $i = 0, 1, 2, \cdots$. So from (4.7), we have that

$$E_{j+1} \le \frac{E_j}{8} + c_3[\omega(r_j)F_j + \nu_j] \qquad (j = 0, 1, 2, \cdots).$$

Sum the above inequality over $j \in \{0, 1, \cdots, i-1\}$. Then we get

$$\sum_{j=1}^{i} E_i \le \frac{E_0}{4} + 2\sum_{j=0}^{i-1} c_3[\omega(r_j)F_j + \nu_j] \qquad (i = 1, 2, 3, \cdots). \tag{4.9}$$

To simplify the computation, define $\nu = 4\delta^{-n} c_3 \sum_{j=0}^{\infty} \nu_j$. Then we have from (4.6) and (4.8) that

$$\begin{aligned}\nu &= 4\delta^{-n} c_3 \sum_{j=0}^{\infty} \left[\frac{|\mu|(Q_j^+)}{r_j^{n-1}} + \operatorname*{osc}_{T_{r_j}(x_0', t_0)} D_{x'}\psi + \omega(r_j)|D_{x'}\psi(x_0', t_0)| + r_j^2 ⨍_{Q_j^+} |\partial_t\psi|^2\, dxdt\right] \\ &\le c\left[\int_0^{2r} \left(\frac{|\mu|(Q_\rho^+(x_0, t_0))}{\rho^{n+1}} + \operatorname*{osc}_{T_\rho(x_0', t_0)} D_{x'}\psi + \rho^2 ⨍_{Q_\rho^+(x_0,t_0)} |\partial_t\psi|^2\, dxdt\right)\frac{d\rho}{\rho} + |D_{x'}\psi(x_0', t_0)|\right].\end{aligned}$$

We claim that

$$F_i \le 4\delta^{-n} ⨍_{Q_0} |U|\, dxdt + \nu \qquad (i = 0, 1, 2, \cdots). \tag{4.10}$$

To this end, since $\delta \in (0, 1/2]$ and $F_0 = \left|⨍_{Q_0} U\, dxdt\right|$, the claim follows holds when $i = 0$. To use induction, we next assume that (4.10) holds when $0, 1, \cdots, i$, which means that

$$F_0, F_1, \cdots, F_i \le 4\delta^{-n} ⨍_{Q_0} |U|\, dxdt + \nu. \tag{4.11}$$

By a direct computation,

$$F_{i+1} = \left| \fint_{Q_{i+1}} U\, dxdt \right| \le \left| \fint_{Q_0} U\, dxdt \right| + \sum_{j=0}^{i} \left| \fint_{Q_{j+1}} U\, dxdt - \fint_{Q_j} U\, dxdt \right|,$$

which implies that

$$F_{i+1} \le F_0 + \sum_{j=0}^{i} \delta^{-n} \fint_{Q_j} |U - (U)_{Q_j}|\, dxdt.$$

By using (4.9), we obtain that

$$F_{i+1} \le F_0 + \delta^{-n} \sum_{j=0}^{i} E_j \le F_0 + \delta^{-n} \left[\frac{5E_0}{4} + 2 \sum_{j=0}^{i} c_3[\omega(r_j)F_j + \nu_j] \right].$$

We discover from (4.8) and (4.11) that

$$\begin{aligned} 2\delta^{-n} \sum_{j=0}^{i} c_3[\omega(r_j)F_j + \nu_j] &\le 2\delta^{-n} c_3 \left[\sum_{j=0}^{i} \omega(r_j) \left(4\delta^{-n} \fint_{Q_0} |U|\, dxdt + \nu \right) + \sum_{j=0}^{i} \nu_j \right] \\ &\le \frac{1}{2} \left[\fint_{Q_0} |U|\, dxdt + \nu \right] + \frac{\nu}{2}, \end{aligned}$$

which implies that

$$F_{i+1} \le F_0 + \frac{5\delta^{-n} E_0}{4} + \frac{1}{2} \left[\fint_{Q_0} |U|\, dxdt + \nu \right].$$

Since $F_0 = \left| \fint_{Q_0} U\, dxdt \right|$, we have that $F_0 \le \fint_{Q_0} |U|\, dxdt$. Also we have that

$$\frac{5\delta^{-n} E_0}{4} = \frac{5\delta^{-n}}{4} \fint_{Q_0} |U - (U)_{Q_0}|\, dxdt \le \frac{5\delta^{-n}}{2} \fint_{Q_0} |U|\, dxdt.$$

Thus

$$F_{i+1} \le 4\delta^{-n} \fint_{Q_0} |U|\, dxdt + \nu,$$

and the claim (4.10) holds when $i + 1$. So by an induction, the claim (4.10) holds for $i = 0, 1, 2, \cdots$.

We have from (4.9) that

$$\fint_{Q_i} |U|\, dxdt \le \fint_{Q_i} |U - (U)_{Q_i}|\, dxdt + |(U)_{Q_i}| \le E_i + F_i \qquad (i = 0, 1, 2, \cdots),$$

which implies that

$$\fint_{Q_i} |U|\, dxdt \le \frac{E_0}{4} + 2 \sum_{j=0}^{i} c_3[\omega(r_j)F_j + \nu_j] + F_i \qquad (i = 0, 1, 2, \cdots). \tag{4.12}$$

By (4.8) and (4.10), we get that

$$\sum_{j=0}^{i} c_3[\omega(r_j)F_j + \nu_j] \le c\sum_{j=0}^{i} c_3\left[\omega(r_j)\left(4\delta^{-n}\fint_{Q_0}|U|\,dxdt + \nu\right) + \nu_j\right] \le c\left[\fint_{Q_0}|U|\,dxdt + \nu\right].$$

So we find from (4.10) that

$$\sum_{j=0}^{i} c_3[\omega(r_j)F_j + \nu_j] + F_i \le c\left[\fint_{Q_0}|U|\,dxdt + \nu\right].$$

In view of (4.12), we get that

$$\fint_{Q_i}|U|\,dxdt \le c\left[\fint_{Q_0}|U|\,dxdt + \nu\right] \qquad (i = 0, 1, 2, \cdots).$$

By the definition $Q_i = Q_{\delta^i r}(x_0, t_0)$,

$$\fint_{Q_{\delta^i r}(x_0,t_0)}|U|\,dxdt \le c\left[\fint_{Q_r(x_0,t_0)}|U|\,dxdt + \nu\right] \qquad (i = 0, 1, 2, \cdots). \tag{4.13}$$

With the estimate (4.13), it is ready to estimate Du. Recall from (4.1) and Lemma 3.1 that

$$|Du| \le c(|U| + |D_{x'}\psi|) \le c(|Du| + |D_{x'}\psi|) \quad \text{in} \quad Q_{2R}^+. \tag{4.14}$$

Since $x_0 \in \overline{Q_R^+}$, we find from (4.14) that

$$\fint_{Q^+_{\delta^i r}(x_0,t_0)}|Du|\,dxdt \le c\left[\fint_{Q_{\delta^i r}(x_0,t_0)}|U|\,dxdt + \sup_{T_{\delta^i r}(x_0',t_0)}|D_{x'}\psi|\right] \qquad (i = 0, 1, 2, \cdots).$$

Since $x_0 \in \overline{Q_R^+}$ and U is an extension defined in (4.2), we have from (4.14) that

$$\fint_{Q_r(x_0,t_0)}|U|\,dxdt \le c\fint_{Q_r^+(x_0,t_0)}|U|\,dxdt \le c\left[\fint_{Q_r^+(x_0,t_0)}|Du|\,dxdt + \sup_{T_r(x_0',t_0)}|D_{x'}\psi|\right].$$

By using the above two estimates, we have from (4.13) that

$$\fint_{Q^+_{\delta^i r}(x_0,t_0)}|Du|\,dxdt \le c\left[\fint_{Q_r^+(x_0,t_0)}|Du|\,dxdt + \nu + \sup_{T_r(x_0',t_0)}|D_{x'}\psi|\right] \qquad (i = 0, 1, 2, \cdots),$$

where we used that $\sup_{T_{\delta^i r}(x_0',t_0)}|D_{x'}\psi| \le \sup_{T_r(x_0',t_0)}|D_{x'}\psi|$ for $i = 0, 1, 2, \cdots$. So from the following computation

$$\begin{aligned}\sup_{T_r(x_0',t_0)}|D_{x'}\psi| &\le |D_{x'}\psi(x_0', t_0)| + \operatorname*{osc}_{T_r(x_0',t_0)} D_{x'}\psi \\ &\le |D_{x'}\psi(x_0', t_0)| + \frac{1}{\log 2}\int_r^{2r} \operatorname*{osc}_{T_\rho(x_0',t_0)} D_{x'}\psi\,\frac{d\rho}{\rho} \le |D_{x'}\psi(x_0', t_0)| + c\nu,\end{aligned}$$

we obtain that

$$\fint_{Q^+_{\delta^i r}(x_0,t_0)} |Du|\, dxdt \le c\left[\fint_{Q^+_r(x_0,t_0)} |Du|\, dxdt + |D_{x'}\psi(x_0', t_0)| + \nu\right] \qquad (i = 0, 1, 2, \cdots). \tag{4.15}$$

From the assumption, (x_0, t_0) is a Lebesgue point of Du, which implies that

$$|Du(x_0, t_0)| = \lim_{i\to\infty}\fint_{Q^+_{\delta^i r}(x_0,t_0)} |Du|\, dxdt \le c\left[\fint_{Q_r(x_0,t_0)} |Du|\, dxdt + |D_{x'}\psi(x_0', t_0)| + \nu\right].$$

From the definition of ν, we showed that

$$\nu \le c\left[\int_0^{2r}\left(\frac{|\mu|(Q^+_\rho(x_0, t_0))}{\rho^{n+1}} + \operatorname*{osc}_{T_\rho(x_0',t_0)} D_{x'}\psi + \rho^2\fint_{Q^+_\rho(x_0,t_0)} |\partial_t\psi|^2\, dxdt\right)\frac{d\rho}{\rho} + |D_{x'}\psi(x_0', t_0)|\right].$$

So we find that Theorem 1.1 holds.

5. SOLA (Solutions Obtained by Limit of Approximations)

We explain just a sketch proof for Corollary 1.3 because the proof related to SOLA appears in many papers, say [1, Section 5.2], [17, Section 4.3] and [4, Remark 7]. Suppose that $u_h \to u$ in $L^2\left(-4R^2, 0\,;\, W^{1,1}(K^+_{2R})\right)$ and $L^\infty \ni \mu_h \rightharpoonup \mu$ in the sense of measures satisfying

$$\limsup_{h\to\infty} |\mu_h|(Q^+_\rho(x_0, t_0)) \le |\mu|(\lfloor Q^+_\rho(x_0, t_0)\rfloor) \text{ for any } Q^+_\rho(x_0, t_0) \subset Q^+_{2R},$$

for every $Q^+_\rho(x_0, t_0)$, where $\lfloor Q\rfloor$ denotes the parabolic closure of Q. We return to (4.15) in the proof of Theorem 1.1 for the size $r/2$ instead of r. For $i \in \{0, 1, 2, \cdots\}$, replace u and μ with u_h and μ_h, respectively. Then we find that

$$\begin{aligned}\fint_{Q_{\delta^i r/2}(x_0,t_0)} |Du_h|\, dxdt &\le c\left[\fint_{Q_{r/2}(x_0,t_0)} |Du_h|\, dxdt + |D_{x'}\psi(x_0', t_0)|\right]\\ &\quad + c\int_0^r\left(\frac{|\mu_h|(Q^+_\rho(x_0, t_0))}{\rho^{n+1}} + \operatorname*{osc}_{T_\rho(x_0',t_0)} D_{x'}\psi + \fint_{Q^+_\rho(x_0,t_0)} |\partial_t\psi|^2\, dxdt\right)\frac{d\rho}{\rho}\end{aligned}$$

for $i = 0, 1, 2, \cdots$. By sending $h \to \infty$, we find that

$$\begin{aligned}\fint_{Q_{\delta^i r/2}(x_0,t_0)} |Du|\, dxdt &\le c\left[\fint_{Q_{r/2}(x_0,t_0)} |Du|\, dxdt + |D_{x'}\psi(x_0', t_0)|\right]\\ &\quad + c\int_0^r\left(\frac{|\mu|\left(\lfloor Q^+_\rho(x_0, t_0)\rfloor\right)}{\rho^{n+1}} + \operatorname*{osc}_{T_\rho(x_0',t_0)} D_{x'}\psi + \fint_{Q^+_\rho(x_0,t_0)} |\partial_t\psi|^2\, dxdt\right)\frac{d\rho}{\rho}\end{aligned}$$

for $i = 0, 1, 2, \cdots$, which implies that

$$\begin{aligned}\fint_{Q_{\delta^i r/2}(x_0,t_0)} |Du|\, dxdt &\le c\left[\fint_{Q_r(x_0,t_0)} |Du|\, dxdt + |D_{x'}\psi(x_0', t_0)|\right]\\ &\quad + c\int_0^{2r}\left(\frac{|\mu|(Q^+_\rho(x_0, t_0))}{\rho^{n+1}} + \operatorname*{osc}_{T_\rho(x_0',t_0)} D_{x'}\psi + \fint_{Q^+_\rho(x_0,t_0)} |\partial_t\psi|^2\, dxdt\right)\frac{d\rho}{\rho}\end{aligned}$$

for $i = 0, 1, 2, \cdots$, because $\mu\big|_{\mathbb{R}^{n+1}\setminus Q^+_{2R}} = 0$. So Corollary 1.3 follows, because (x_0, t_0) is a Lebesgue point of Du.

Author contributions

Ho-Sik Lee : Writing-Original draft preparation; Youchan Kim : Writing-Original draft preparation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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