Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 17(1): 41–60.
DOI: 10.3934/cam.2025003
Received: 02 May 2024
Revised: 14 November 2024
Accepted: 07 January 2025
Published: 20 January 2025

Research article

Quasilinear parabolic variational-hemivariational inequalities in $\mathbb{R}^N \times (0, \tau)$ under bilateral constraints

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Abstract: In this paper, we considered quasilinear variational-hemivariational inequalities in the unbounded cylindrical domain $\mathbb{Q} = \mathbb{R}^N \times (0, \tau)$ of the form: Find $u \in K \subset X$ with $u(\cdot, 0) = 0$ satisfying

$$\langle u_t - \operatorname{div} A(x, t, \nabla u), v - u \rangle + \int_{\mathbb{Q}} a(x, t) j^o(x, t, u; v - u)\, dxdt \geq 0, \ \forall v \in K,$$

where $K \subset X$ represents the bilateral constraints in $X = L^p(0, \tau; D^{1,p}(\mathbb{R}^N))$ with $D^{1,p}(\mathbb{R}^N)$ denoting the Beppo-Levi space (or homogeneous Sobolev space), and $j^o(x, t, s; \varrho)$ denoting Clarke's generalized directional derivative of the locally Lipschitz function $s \mapsto j(x, t, s)$ at s in the direction ϱ. The main goal and the novelty of this paper was to prove existence results without assuming coercivity conditions on the time-dependent elliptic operators involved, and without supposing the existence of sub-supersolutions. Further difficulties arise in the treatment of the problem under consideration due to the lack of compact embedding of $D^{1,p}(\mathbb{R}^N))$ into Lebesgue spaces $L^\sigma(\mathbb{R}^N)$, and the fact that the constraint K has an empty interior, which prevents us from applying recent results on evolutionary variational inequalities. Instead our approach was based on an appropriately designed penalty technique and the use of weighted Lebesgue spaces as well as multi-valued pseudomontone operator theory.

Keywords: parabolic variational-hemivariational inequality; multi-valued variational inequality; bilateral obstacle; Beppo-Levi space; penalty approximation
Mathematics Subject Classification: 35K55, 35K86, 35K90, 47J20, 47J35

1. Introduction

Let $\mathbb{Q} = \mathbb{R}^N \times (0, \tau)$ be the unbounded space-time cylindrical domain, and let $V = D^{1,p}(\mathbb{R}^N)$ be the homogeneous Sobolev space (also called the Beppo-Levi space), which is the completion of $C_c^\infty(\mathbb{R}^N)$

(the space of infinitely differentiable functions with compact support in $\mathbb{R}^N$) with respect to the norm

$$\|u\|_V = \Big(\int_{\mathbb{R}^N} |\nabla u|^p \, dx \Big)^{\frac{1}{p}}.$$

For the range $1 < p < N$, the Beppo-Levi space V is a Banach space which is separable, reflexive, and even uniformly convex, see [1] and [2, Theorem 12.2.3]. Due to the Gagliardo-Nirenberg-Sobolev inequality, the Beppo-Levi space V is continuously embedded into $L^{p^*}(\mathbb{R}^N)$ with

$$p^* = \frac{Np}{N-p} \text{ denoting the critical Sobolev exponent.}$$

Thus V can be characterized as

$$V = \Big\{ v \in L^{p^*}(\mathbb{R}^N) : \int_{\mathbb{R}^N} |\nabla u|^p < \infty \Big\}.$$

Apparently, $V \subset W^{1,p}_{\text{loc}}(\mathbb{R}^N)$, where $W^{1,p}_{\text{loc}}(\mathbb{R}^N)$ stands for the local Sobolev space on $\mathbb{R}^N$, but the Sobolev space $W^{1,p}(\mathbb{R}^N)$ is a strict subspace of V, which can easily be verified by simple examples.

Let $X = L^p(0,\tau;V)$ be the Banach-valued Lebesgue space with its dual space $X^* = L^{p'}(0,\tau;V^*)$, where p' is the Hölder conjugate, that is, $\frac{1}{p} + \frac{1}{p'} = 1$, and assume throughout that $2 \le p < N$. In this paper, we consider the parabolic variational-hemivariational inequalities in the unbounded cylindrical domain $\mathbb{Q}$ of the form: Find $u \in K \subset X$ with $u(\cdot,0) = 0$ such that

$$\langle u_t - \operatorname{div} A(x,t,\nabla u), v-u \rangle + \int_{\mathbb{Q}} a(x,t) j^o(x,t,u;v-u)\, dxdt \ge 0, \ \forall v \in K, \tag{1.1}$$

where K denotes the closed convex subset of X representing the bilateral constraint given by

$$K = \{ v \in X \ : \ \phi(x,t) \le v(x,t) \le \psi(x,t) \text{ for a.a. } (x,t) \in \mathbb{Q} \} \tag{1.2}$$

with ϕ, $\psi \in X$, and $\langle \cdot,\cdot \rangle$ denoting the duality pairing between X and X^*. The time derivative $u_t := \frac{\mathrm{d}u(\cdot,t)}{\mathrm{d}t}$ is understood as the distributional time-derivative of the Banach-valued function $u\colon (0,\tau) \to V$, and $-\operatorname{div} A(\cdot,\cdot,\nabla)$ is a quasilinear elliptic operator of monotone type. The coefficient $a : \mathbb{Q} \to \mathbb{R}_+$ is supposed to decay like $\sim |x|^{-(N+\alpha)}$ with $\alpha > 0$, and $j^o(x,t,s;\varrho)$ denotes Clarke's generalized directional derivative of the locally Lipschitz function $s \mapsto j(x,t,s)$ at s in the direction ϱ defined by [3, Chap. 2]:

$$j^o(x,t,s;\varrho) = \limsup_{y \to s,\ r \downarrow 0} \frac{j(x,t,y+r\varrho) - j(x,t,y)}{r},$$

where $(x,t) \mapsto j(x,t,s)$ is measurable in $\mathbb{Q}$ for all $s \in \mathbb{R}$ and $s \mapsto j(x,t,s)$ is locally Lipschitz for a.a. $(x,t) \in \mathbb{Q}$. Clarke's generalized gradient of $s \mapsto j(x,t,s)$, denoted by $s \mapsto \partial j(x,t,s)$ and defined by

$$\partial j(x,t,s) := \{ \eta \in \mathbb{R} : j^o(x,t,s;\varrho) \ge \eta\varrho, \ \forall\, \varrho \in \mathbb{R} \},$$

gives rise to the following associated multi-valued parabolic bilateral variational inequality: Find $u \in K$ with $u(\cdot,0) = 0$, and $\eta(x,t) \in \partial j(x,t,u(x,t))$ such that

$$\langle u_t - \operatorname{div} A(x,t,\nabla u), v-u \rangle + \int_{\mathbb{Q}} a\eta(v-u)\, dxdt \ge 0, \ \forall v \in K. \tag{1.3}$$

By using the definition of Clarke's gradient $s \mapsto \partial j(x, t, s)$, one readily observes that any solution of (1.3) is also a solution of the original parabolic variational-hemivariational inequality (1.1). As will be seen later, under the conditions we impose on the data, the reverse holds true as well.

Let F be the multi-valued Nemytskij operator generated by Clarke's gradient, that is, $F(u)(x, t) = \partial j(x, t, u(x, t))$, and let

$$\mathcal{A}u = -\operatorname{div} A(x, t, \nabla u).$$

Then the multi-valued parabolic variational inequality (1.3) is equivalent to: Find $u \in K$ with $u(\cdot, 0) = 0$ such that

$$0 \in u_t + \mathcal{A}u + aF(u) + \partial I_K(u) \quad \text{in } X^*, \tag{1.4}$$

where I_K is the indicator function related to K with ∂I_K denoting its subdifferential in the sense of convex analysis. Note that if $s \mapsto j(x, t, s)$ is continuously differentiable, then $\partial j(x, t, s) = \frac{\partial j}{\partial s}(x, t, s)$ is single-valued, $j^o(x, t, s; \varrho) = \frac{\partial j}{\partial s}(x, t, s)\,\varrho$, and (1.3) reduces to a single-valued parabolic bilateral variational inequality, which in this case is related to the work in [4–7].

Existence results for evolutionary variational inequalities and systems of them on bounded cylindrical domains and under either coercivity conditions on the operator $\mathcal{A} + aF : X \to 2^{X^*}$ or the existence of appropriately defined sub- and supersolutions were obtained, e.g., in [8–12]. In comparison with its elliptic counterpart, in the treatment of evolutionary variational inequalities of the form (1.4), an additional difficulty arises due to the subdifferential of the indicator function ∂I_K in (1.4). This is because no growth condition can be assumed on ∂I_K, and thus, in general, there is no growth estimate of the time derivative u_t in the dual space X^* available, which would be needed for proving the existence of solutions. Usually, this lack is compensated in the treatment of evolutionary variational inequalities by requiring that K admits a nonempty interior, that is, $\operatorname{int}(K) \neq \emptyset$, see, e.g., [13], [14, Chap. 32], and [15]. Namely, if $\operatorname{int}(K) \neq \emptyset$, then Rockafellar's theorem about the sums of maximal monotone operators may be applied, which allows one to study evolutionary variational inequalities by implementation of arguments and results for elliptic variational inequalities to evolutionary variational inequalities. Unfortunately, the interior of the constraint K we are dealing with is empty, i.e., $\operatorname{int}(K) = \emptyset$, and therefore a similar approach as for elliptic variational inequalities cannot be applied. Instead, we are going to deal with this difficulty by using an appropriately designed penalty technique, which also will enable us to handle the lack of coercivity of the operator $\mathcal{A} + aF : X \to 2^{X^*}$. A further difficulty arises due to the unboundedness of the space domain $\mathbb{R}^N$, and hence the lack of compact embedding $V \hookrightarrow\hookrightarrow L^r(\mathbb{R}^N)$ which will be resolved by working in weighted Lebesgue spaces.

Finally, we mention that existence results for general parabolic variational inequalities in bounded as well as unbounded cylindrical domains of the form

$$u \in X \cap K : 0 \in u_t + \mathcal{A}u + a\mathcal{F}(u) + \partial I_K \quad \text{in } X^*, \tag{1.5}$$

where the lower-order term $\mathcal{F} : X \to 2^{X^*}$ is a general, multi-valued, upper semicontinuous operator, can be found as part of the recent monograph [16], see also the relevant references therein. Even though, in [16], the constraint K satisfies $\operatorname{int}(K) = \emptyset$, general existence results and a detailed study of the quality of the solution set has been obtained under either certain coercivity assumptions on the (possibly) multi-valued operator $\mathcal{A} + a\mathcal{F} : X \to 2^{X^*}$, or the existence of appropriately defined sub- and supersolutions. As for the application of variational-hemivariational inequalities, we refer to the monographs [17–20].

The main goal and the novelty of this paper is to prove existence results without assuming coercivity conditions on the operator $\mathcal{A} + aF : X \to 2^{X^*}$, and without supposing the existence of sub- and supersolutions. Moreover, as mentioned above, additional difficulties arise due to the lack of compact embedding of $V = D^{1,p}(\mathbb{R}^N)$ into Lebesgue spaces $L^\sigma(\mathbb{R}^N)$, and the fact that the domain K of ∂I_K has an empty interior. Our approach is based on an appropriately designed penalty technique and the use of weighted Lebesgue spaces as well as multi-valued pseudomontone operator theory.

The paper is organized as follows. In Section 2, we present hypotheses and the main tools that are needed in the sequel, and provide some examples. In Section 3, we prove our main results.

2. Hypotheses and preliminaries

Throughout this paper, we assume $2 \le p < N$, and the following notations will be used: For any $\sigma \in (1, \infty)$, its Hölder conjugate is denoted by σ', i.e., $1/\sigma + 1/\sigma' = 1$, the $L^\sigma(\mathbb{R}^N)$-norm is denoted by $\|\cdot\|_\sigma$, and the $L^\sigma(\mathbb{Q})$-norm is denoted by $\|\cdot\|_{\mathbb{Q},\sigma}$. For normed linear spaces W and Z, $W \hookrightarrow Z$ denotes the continuous embedding, and $W \hookrightarrow\hookrightarrow Z$ stands for the compact embedding of W into Z.

Let us introduce the function space Y by

$$Y = \{u \in X \, : \, u_t \in X^*\}.$$

Since the Beppo-Levi space is separable, reflexive, and even uniformly convex, it follows that the Lebesgue space $X = L^p(0, \tau; V)$ and Y are separable and uniformly convex, and thus reflexive Banach spaces (see, e.g., Zeidler [21, Proposition 23.2, Proposition 23.7]) equipped with the norms

$$\|u\|_Y = \|u\|_X + \|u_t\|_{X^*},$$

where $\|u\|_X$ and $\|u\|_{X^*}$ are defined by

$$\|u\|_X = \left(\int_0^\tau \|u(\cdot, t)\|_V^p \, \mathrm{d}t\right)^{\frac{1}{p}}, \quad \|u\|_{X^*} = \left(\int_0^\tau \|u(\cdot, t)\|_{V^*}^{p'} \, \mathrm{d}t\right)^{\frac{1}{p'}},$$

with $\|\cdot\|_V$ and $\|\cdot\|_{V^*}$ being the norms in $V = D^{1,p}(\mathbb{R}^N)$ and V^*, respectively.

We assume the following conditions on the coefficient a, and on the vector field $A : \mathbb{Q} \times \mathbb{R}^N \to \mathbb{R}^N$ of the quasilinear elliptic operator $\mathcal{A} = -\operatorname{div} A(x, t, \nabla)$.

(Ha) The function $a : \mathbb{Q} \to \mathbb{R}_+$ is measurable and satisfies the decay for some positive constants c_a, α :

$$0 \le a(x, t) \le c_a w(x), \quad \text{for a.a. } (x, t) \in \mathbb{Q}, \text{ with } w(x) = \frac{1}{1 + |x|^{N+\alpha}}. \tag{2.1}$$

The vector field $A : \mathbb{Q} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, that is, $(x, t) \mapsto A(x, t, \xi)$ is measurable in $\mathbb{Q}$ for all $\xi \in \mathbb{R}^N$, and $\xi \mapsto A(x, t, \xi)$ is continuous in $\mathbb{R}^N$ for a.a. $(x, t) \in \mathbb{Q}$ and fulfills the following structure conditions for a.a. $(x, t) \in \mathbb{Q}$ and for all ξ, $\hat{\xi} \in \mathbb{R}^N$:

(A1) $|A(x, t, \xi)| \le c_0|\xi|^{p-1} + k_0(x, t), \quad c_0 > 0, \ k_0 \in L^{p'}(\mathbb{Q})$;

(A2) $(A(x, t, \xi) - A(x, t, \hat{\xi}))(\xi - \hat{\xi}) > 0, \ \forall \xi, \hat{\xi}$ with $\xi \ne \hat{\xi}$;

(A3) $A(x,t,\xi)\xi \geq \nu|\xi|^p - k_1(x,t), \quad \nu > 0, \;\; k_1 \in L^1(\mathbb{Q})$.

With the weight function w given by (2.1), we introduce the weighted Lebesgue spaces $L^q(\mathbb{R}^N, w)$ and $L^q(\mathbb{Q}, w)$ as follows:

$$L^q(\mathbb{R}^N, w) = \Big\{u \in L^0(\mathbb{R}^N) : \int_{\mathbb{R}^N} w|u|^q \, dx < \infty\Big\}$$

with norm

$$\|u\|_{q,w} = \Big(\int_{\mathbb{R}^N} w|u|^q \, dx\Big)^{\frac{1}{q}},$$

and

$$L^q(\mathbb{Q}, w) = \Big\{u \in L^0(\mathbb{Q}) : \int_{\mathbb{Q}} w|u|^q \, dxdt = \int_0^\tau \Big(\int_{\mathbb{R}^N} w|u|^q \, dx\Big)dt < \infty\Big\}$$

with norm

$$\|u\|_{\mathbb{Q},q,w} = \Big(\int_{\mathbb{Q}} w|u|^q \, dxdt\Big)^{\frac{1}{q}},$$

where $L^0(\mathbb{R}^N)$ and $L^0(\mathbb{Q})$ denote the space of real-valued measurable functions on $\mathbb{R}^N$ and $\mathbb{Q}$, respectively. The weighted Lebesgue spaces $L^q(\mathbb{R}^N, w)$ and $L^q(\mathbb{Q}, w)$ are separable and reflexive Banach spaces for $1 < q < \infty$.

The function $j : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ is supposed to satisfy:

(Hj) $(x,t) \mapsto j(x,t,s)$ is measurable in $\mathbb{Q}$ for all $s \in \mathbb{R}$, $s \mapsto j(x,t,s)$ is locally Lipschitz in $\mathbb{R}$ for a.a. $(x,t) \in \mathbb{Q}$, and Clarke's generalized gradient satisfies the following growth:

$$\sup\{|\eta| : \eta \in \partial j(x,t,s)\} \leq k(x,t) + c_j|s|^{p-1}, \tag{2.2}$$

for a.a $(x,t) \in \mathbb{Q}$, and for all $s \in \mathbb{R}$, where $c_j > 0$ and $k \in L^{p'}(\mathbb{Q}, w)$.

As for the functions ϕ and ψ of the bilateral constraint K, we assume:

(Hψ) The function $\psi \colon \mathbb{Q} \to \mathbb{R}$ of K is supposed to satisfy: $\psi \in Y$, $\psi(\cdot, 0) \geq 0$ in $\mathbb{R}^N$, and

$$\langle \psi_t + \mathcal{A}\psi, \varphi\rangle \geq 0 \;\text{ for all } \varphi \in X \text{ with } \varphi \geq 0.$$

(Hϕ) The function $\phi \colon \mathbb{Q} \to \mathbb{R}$ of K is supposed to satisfy: $\phi \in Y$, $\phi(\cdot, 0) \leq 0$ in $\mathbb{R}^N$, and

$$\langle \phi_t + \mathcal{A}\phi, \varphi\rangle \leq 0 \;\text{ for all } \varphi \in X \text{ with } \varphi \geq 0.$$

Remark 2.1. *A few remarks regarding the hypotheses are in order.*

(i) $\mathcal{A} = -\Delta_p$, *where* Δ_p *is the p-Laplacian with* $A(x,t,\xi) = |\xi|^{p-2}\xi$, *is a special case.*

(ii) Hypotheses (Hψ*) and (H*ϕ*) imply that* $\phi \leq 0 \leq \psi$ *in* $\mathbb{Q}$*. Moreover, (H*ψ*) and (H*ϕ*) do not imply that* ψ *is a supersolution and* ϕ *is a subsolution for the variational inequality (1.3).*

(iii) *Clearly the bilateral constraint $K \subset X$ has an empty interior and satisfies the following lattice condition:*

$$K \vee K \subset K \quad \text{and} \quad K \wedge K \subset K, \tag{2.3}$$

where

$$K \vee K = \{v \vee w : w, v \in K\} \ \text{ with } \ v \vee w = \max\{v, w\},$$

$$K \wedge K = \{v \wedge w : w, v \in K\} \ \text{ with } \ v \wedge w = \min\{v, w\}.$$

(iv) *We are going to provide examples for ψ and ϕ that satisfy hypotheses (Hψ) and (Hϕ), and provide an example showing that the operator $\mathcal{A} + aF : X \to 2^{X^*}$ is, in general, noncoercive.*

Lemma 2.2. *The weight function w given by (2.1) belongs to $L^r(\mathbb{R}^N)$ for $1 \leq r \leq \infty$.*

Proof. In fact, clearly $w \in L^\infty(\mathbb{R}^N)$, and using spherical coordinates, we get for any $r \in [1, \infty)$:

$$\begin{aligned}
\int_{\mathbb{R}^N} w^r\, dx &= \int_{|x|<1} w^r\, dx + \int_{|x|\geq 1} w^r\, dx \\
&\leq |B(0,1)| + c \int_1^\infty \Big(\frac{1}{1+\varrho^{N+\alpha}}\Big)^r \varrho^{N-1}\, d\varrho \\
&\leq c + c \int_1^\infty \varrho^{-(N+\alpha)r+N-1}\, d\varrho < \infty,
\end{aligned}$$

since $-(N+\alpha)r + N < 0$. Here, $|B(0,1)|$ denotes the Lebesgue measure of the unit ball $B(0,1)$ in $\mathbb{R}^N$. □

Corollary 2.3. *If $1 < q < r < \infty$, then $L^r(\mathbb{R}^N, w) \hookrightarrow L^q(\mathbb{R}^N, w)$, and $L^r(\mathbb{Q}, w) \hookrightarrow L^q(\mathbb{Q}, w)$.*

Proof. We only show $L^r(\mathbb{R}^N, w) \hookrightarrow L^q(\mathbb{R}^N, w)$, since the proof for $L^r(\mathbb{Q}, w) \hookrightarrow L^q(\mathbb{Q}, w)$ follows in the same way. Using the Hölder inequality, we get:

$$\begin{aligned}
\int_{\mathbb{R}^N} w|u|^q\, dx &= \int_{\mathbb{R}^N} w^{\frac{q}{r}} |u|^q w^{1-\frac{q}{r}}\, dx \\
&\leq \Big(\int_{\mathbb{R}^N} w|u|^r\, dx\Big)^{\frac{q}{r}} \Big(\int_{\mathbb{R}^N} w\, dx\Big)^{\frac{r-q}{r}},
\end{aligned}$$

which yields

$$\|u\|_{q,w} \leq c\|u\|_{r,w}, \quad \text{where } c = \Big(\int_{\mathbb{R}^N} w\, dx\Big)^{\frac{r-q}{rq}}.$$

□

Lemma 2.4. *$V \hookrightarrow L^q(\mathbb{R}^N, w)$ for $1 \leq q \leq p^*$.*

Proof. Let $u \in V = D^{1,p}(\mathbb{R}^N)$, and then $u \in L^{p^*}(\mathbb{R}^N)$, and thus with Lemma 2.2 we get for any $q \in [1, p^*)$:

$$\int_{\mathbb{R}^N} w|u|^q\, dx \leq \|w\|_{\frac{p^*}{p^*-q}} \|u\|_{p^*}^q \leq c\|w\|_{\frac{p^*}{p^*-q}} \|u\|_V^q,$$

that is,

$$\|u\|_{q,w} \leq c\|w\|_{\frac{p^*}{p^*-q}}^{\frac{1}{q}} \|u\|_V,$$

and for $q = p^*$, we have

$$\|u\|_{p^*,w}^{p^*} = \int_{\mathbb{R}^N} w|u|^{p^*}\, dx \le \|u\|_{p^*}^{p^*} \le c\|u\|_V^{p^*},$$

which shows that $i_w : V \to L^q(\mathbb{R}^N, w)$ is linear and continuous for $q \in [1, p^*]$. □

For the following embedding result, we refer to [16, Lemma 6.1].

Lemma 2.5. *The embedding* $V \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N, w)$ *is compact for* $1 \le q < p^*$*, that is, the embedding operator* $i_w : V \to L^q(\mathbb{R}^N, w)$ *defined by* $u \mapsto i_w u = u$ *is linear and compact.*

From Lemma 2.5, it follows that, in particular, $V \hookrightarrow\hookrightarrow L^2(\mathbb{R}^N, w)$. Therefore, identifying the Hilbert space $L^2(\mathbb{R}^N, w)$ with its dual, we have the following evolution triple $(V, L^2(\mathbb{R}^N, w), V^*)$ with the embeddings

$$V \overset{i_w}{\hookrightarrow} L^2(\mathbb{R}^N, w) \overset{i_w^*}{\hookrightarrow} V^*$$

being dense and compact, where $i_w : V \to L^q(\mathbb{R}^N, w)$ is the embedding operator of V into $L^q(\mathbb{R}^N, w)$, and $i_w^* : L^{q'}(\mathbb{R}^N, w) \to V^*$ is its adjoint operator defined by

$$v \in L^{q'}(\mathbb{R}^N, w) : \quad \langle i_w^* v, \varphi\rangle = \int_{\mathbb{R}^N} wv\varphi\, dx, \quad \forall\, \varphi \in V.$$

The spaces $X = L^p(0, \tau; V)$ and Y introduced in the preceding section along with the evolution triple $(V, L^2(\mathbb{R}^N, w), V^*)$ yield the following result.

Lemma 2.6. *The following holds true:*

(i) Continuous embedding: $Y \hookrightarrow C([0, \tau]; L^2(\mathbb{R}^N, w))$*;*

(ii) If $u \in Y$*, then the following integration by parts formula is valid:*

$$\int_0^\tau \langle u_t(\cdot, t), u(\cdot, t)\rangle\, \mathrm{d}t = \frac{1}{2}\left(\|u(\cdot, \tau)\|_{2,w}^2 - \|u(\cdot, 0)\|_{2,w}^2\right);$$

(iii) If $u \in Y$*, then it holds that*

$$\int_0^\tau \langle u_t(\cdot, t), u(\cdot, t)^+\rangle\, \mathrm{d}t = \frac{1}{2}\left(\|u(\cdot, \tau)^+\|_{2,w}^2 - \|u(\cdot, 0)^+\|_{2,w}^2\right),$$

where $s^+ = \max\{s, 0\}$.

Proof. (i) and (ii) are immediate consequences of Proposition 23.23 in Zeidler [21].

(iii) In a similar way as in the proof of Lemma 2.146 in Carl-Le-Motreanu [22], one obtains this formula by regularization and density arguments. □

Lemma 2.7. *The following embeddings hold:*

$$X \hookrightarrow L^p(\mathbb{Q}, w), \quad Y \hookrightarrow\hookrightarrow L^p(\mathbb{Q}, w).$$

Proof. The continuous embedding $X \hookrightarrow L^p(\mathbb{Q}, w)$ is an immediate consequence of $V \hookrightarrow L^p(\mathbb{R}^N, w)$. Since V is even compactly embedded into $L^p(\mathbb{R}^N, w)$, that is, $V \hookrightarrow\hookrightarrow L^p(\mathbb{R}^N, w)$ and $L^p(\mathbb{R}^N, w) \hookrightarrow L^{p'}(\mathbb{R}^N, w) \hookrightarrow V^*$ (note that $2 \le p < N$), we finally get $V \hookrightarrow\hookrightarrow L^p(\mathbb{R}^N, w) \hookrightarrow V^*$. Hence, we may apply the Lions-Aubin Theorem (see, e.g., Carl-Le [16, Theorem 2.52]), which results in $Y \hookrightarrow\hookrightarrow L^p(0, \tau; L^p(\mathbb{R}^N, w)) = L^p(\mathbb{Q}, w)$. □

With the coefficient a satisfying (Ha), we define the operator $i_a^* : L^{p'}(\mathbb{Q}, w) \to X^*$ as follows:

$$\eta \in L^{p'}(\mathbb{Q}, w) : \quad \langle i_a^* \eta, \varphi \rangle = \int_{\mathbb{Q}} a\eta\varphi \, dxdt, \quad \forall\, \varphi \in X. \tag{2.4}$$

Lemma 2.8. *The operator $i_a^* : L^{p'}(\mathbb{Q}, w) \to X^*$ is linear and continuous. Analogously, $i_w^* : L^{p'}(\mathbb{Q}, w) \to X^*$ is linear and continuous, where i_w^* is defined as in (2.4) with a replaced by w.*

Proof. For any $\eta \in L^{p'}(\mathbb{Q}, w)$, using the Hölder inequality and Lemma 2.7, we have the following estimate:

$$\begin{aligned} |\langle i_a^* \eta, \varphi \rangle| &\le \int_{\mathbb{Q}} a\, |\eta|\, |\varphi| \, dxdt \le c_a \int_{\mathbb{Q}} w\, |\eta|\, |\varphi| \, dxdt \\ &\le c_a \int_{\mathbb{Q}} w^{\frac{1}{p'}}\, |\eta|\, w^{\frac{1}{p}} |\varphi| \, dxdt \le c_a \|\eta\|_{\mathbb{Q},p',w} \|\varphi\|_{\mathbb{Q},p,w} \\ &\le c\, \|\eta\|_{\mathbb{Q},p',w} \|\varphi\|_X, \quad \forall\, \varphi \in X. \end{aligned}$$

As the linearity of i_a^* is obvious, the above estimate shows that i_a^* is bounded. The proof for i_w^* follows the same line. □

Lemma 2.9. *Let $j : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ satisfy hypothesis (Hj). Then the multi-valued function $f(x, t, s) := \partial j(x, t, s)$ has the following properties:*

(f1) *$f : \mathbb{Q} \times \mathbb{R} \to \mathcal{K}(\mathbb{R}) \subset 2^{\mathbb{R}} \setminus \{\emptyset\}$ is graph measurable on $\mathbb{Q} \times \mathbb{R}$, and for a.e. $(x, t) \in \mathbb{Q}$, the multi-valued function $f(x, t, \cdot) : \mathbb{R} \to 2^{\mathbb{R}} \setminus \{\emptyset\}$ is upper semicontinuous, where $\mathcal{K}(\mathbb{R})$ denotes the set of all closed intervals of $\mathbb{R}$.*

(f2) *f satisfies the growth condition*

$$\sup\{|\eta| : \eta \in f(x, t, s)\} \le k(x, t) + c_j |s|^{p-1}, \tag{2.5}$$

for a.a $(x, t) \in \mathbb{Q}$, and for all $s \in \mathbb{R}$, where $c_j > 0$ and $k \in L^{p'}(\mathbb{Q}, w)$.

Proof. Property (f2) follows immediately from (2.2) of (Hj). The measurability of $(x, t) \mapsto j(x, t, s)$ and the local Lipschitz continuity of $s \mapsto j(x, t, s)$ imply that for each ϱ, the function $(x, t, s) \mapsto j^o(x, t, s; \varrho)$ is measurable on $\mathbb{Q} \times \mathbb{R}$ with respect to the measure $\mathcal{L}(\mathbb{Q}) \times \mathcal{B}(\mathbb{R})$, as a countable limit superior of measurable functions. Hence the functions $(x, t, s) \mapsto j^o(x, t, s; 1)$ and $(x, t, s) \mapsto j^o(x, t, s; -1)$ are measurable on $\mathbb{Q} \times \mathbb{R}$ with respect to the measure $\mathcal{L}(\mathbb{Q}) \times \mathcal{B}(\mathbb{R})$. Here $\mathcal{L}(\mathbb{Q})$ is the family of Lebesgue measurable subsets of $\mathbb{Q}$, and $\mathcal{B}(\mathbb{R})$ is the σ-algebra of the Borel sets of $\mathbb{R}$.

From the definition of $\partial j(x, t, s)$ and the positive homogeneity of the mapping $\varrho \mapsto j^o(x, t, s; \varrho)$ (see Clarke's calculus [3, Chap. 2]), we see that for almost all $(x, t) \in \mathbb{Q}$ and all $s \in \mathbb{R}$, we have

$$\partial j(x, t, s) = [-j^o(x, t, s; -1), j^o(x, t, s; 1)].$$

Thus

$$\begin{aligned}\mathrm{Gr}(f) &= \{(x,t,s,\eta)\in\mathbb{Q}\times\mathbb{R}\times\mathbb{R} : \eta\in\partial j(x,t,s)\}\\ &= \{(x,t,s,\eta)\in\mathbb{Q}\times\mathbb{R}\times\mathbb{R} : -j^o(x,t,s;-1)\le\eta\le j^o(x,t,s;1)\}\\ &= \{(x,t,s,\eta)\in\mathbb{Q}\times\mathbb{R}\times\mathbb{R} : \eta\ge -j^o(x,t,s;-1)\}\\ &\quad \cap\{(x,s,t,\eta)\in\mathbb{Q}\times\mathbb{R}\times\mathbb{R} : \eta\le j^o(x,t,s;1)\}.\end{aligned}$$

Since $(x,t,s)\mapsto j^o(x,t,s;1)$ and $(x,t,s)\mapsto j^o(x,t,s;-1)$ are measurable, it follows that $\mathrm{Gr}(f)$ belongs to $[\mathcal{L}(\mathbb{Q})\times\mathcal{B}(\mathbb{R})]\times\mathcal{B}(\mathbb{R})$, that is, f is graph measurable on $\mathbb{Q}\times\mathbb{R}$.

Let us show that $f(x,t,\cdot):\mathbb{R}\to 2^{\mathbb{R}}\setminus\{\emptyset\}$ is upper semicontinuous for a.e. $(x,t)\in\mathbb{Q}$. For a.e. $(x,t)\in\mathbb{Q}$, the functions $s\mapsto j^o(x,t,s;\pm1)$ are upper semicontinuous on $\mathbb{R}$ by the properties of j^o, see [3, Chap. 2]. Let $s_0\in\mathbb{R}$ and U be an open neighborhood of $\partial j(x,t,s_0)$. Then there exists $\varepsilon>0$ such that

$$(-j^o(x,t,s_0;-1)-\varepsilon,\, j^o(x,t,s_0;1)+\varepsilon)\subset U.$$

From the upper semicontinuity of the (single-valued) functions $s\mapsto j^o(x,t,s;\pm1)$ at s_0, there exists an open neighborhood O of s_0 such that

$$\begin{cases} j^o(x,t,s;1) < j^o(x,t,s_0;1)+\varepsilon, \text{ and}\\ j^o(x,t,s;-1) < j^o(x,t,s_0;-1)+\varepsilon, \ \forall s\in O.\end{cases}$$

Hence, for all $s\in O$,

$$\begin{aligned}\partial j(x,t,s) &= [-j^o(x,t,s;-1), j^o(x,t,s;1)]\\ &\subset (-j^o(x,t,s_0;-1)-\varepsilon, j^o(x,t,s_0;1)+\varepsilon)\\ &\subset U.\end{aligned}$$

This shows the upper semicontinuity of f at s_0. □

Let $L^0(\mathbb{Q})$ be the space of all measurable functions defined on $\mathbb{Q}$. By hypothesis (f1), the multi-valued function f is superpositionally measurable which allows us to introduce the multi-valued Nemytskij operator $F: L^0(\mathbb{Q})\to 2^{L^0(\mathbb{Q})}\setminus\{\emptyset\}$ associated with the multi-valued function f by

$$F(u)=\{\eta:\mathbb{Q}\to\mathbb{R} : \eta\in L^0(\mathbb{Q}) \text{ and } \eta(x,t)\in f(x,t,u(x,t)) \text{ for a.a. } (x,t)\in\mathbb{Q}\}. \tag{2.6}$$

Due to the measurability of $(x,t)\mapsto f(x,t,u(x,t))$ on $\mathbb{Q}$, we have $F(u)\neq\emptyset$ for any $u\in L^0(\mathbb{Q})$, and thus $F(u)$ is well-defined, and the growth condition (2.2) of hypothesis (f2) implies that the multi-valued Nemytskij operator

$$F: L^p(\mathbb{Q},w)\to 2^{L^{p'}(\mathbb{Q},w)}$$

is well-defined.

Lemma 2.10. *Assume (Ha), and let $g:\mathbb{Q}\times\mathbb{R}\to\mathbb{R}$ be a (single-valued) Carathéodory function that satisfies the growth condition*

$$|g(x,t,s)|\le k(x,t)+c_g|s|^{p-1}, \quad \forall\ s\in\mathbb{R}, \text{ and for a.e. } (x,t)\in\mathbb{Q},$$

where c_g is some positive constant and $k\in L^{p'}(\mathbb{Q},w)$. Let $G(u)(x)=g(x,t,u(x,t))$ denote its Nemytskij operator. Then $G: L^p(\mathbb{Q},w)\to L^{p'}(\mathbb{Q},w)$ is continuous and bounded. Moreover, the operators $aG: Y\to X^*$ and $wG: Y\to X^*$, defined by

$$aG=i_a^*\circ G\circ i_w \quad \text{and} \quad wG=i_w^*\circ G\circ i_w, \tag{2.7}$$

are bounded and completely continuous.

Proof. In view of the growth condition on g, by standard arguments on Nemytskij operators, it follows that $G : L^p(\mathbb{Q}, w) \to L^{p'}(\mathbb{Q}, w)$ is continuous and bounded. The compact embedding $i_w : Y \hookrightarrow\hookrightarrow L^p(\mathbb{Q}, w)$ due to Lemma 2.7 implies that $G \circ i_w : Y \to L^{p'}(\mathbb{Q}, w)$ is bounded and completely continuous. Finally, $i_a^* : L^{p'}(\mathbb{Q}, w) \to X^*$ and $i_w^* : L^{p'}(\mathbb{Q}, w) \to X^*$ are bounded and continuous, which completes the proof. □

By means of the functions ϕ and ψ of the bilateral constraint K, we are introducing the operator P defined by

$$\langle P(u), \varphi\rangle = \int_{\mathbb{Q}} w\Big([u-\psi)^+]^{p-1} - [(\phi - u)^+]^{p-1}\Big)\varphi\, dxdt, \quad u, \varphi \in X. \tag{2.8}$$

Let $b : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ be given by

$$b(x, t, s) = [(s - \psi(x, t))^+]^{p-1} - [(\phi(x, t) - s)^+]^{p-1}, \tag{2.9}$$

which can equivalently be characterized by

$$b(x,t,s) = \begin{cases} (s-\psi(x,t))^{p-1} & \text{if} \quad s > \psi(x,t), \\ 0 & \text{if} \quad \phi(x,t) \le s \le \psi(x,t), \\ -(\phi(x,t) - s)^{p-1} & \text{if} \quad s < \phi(x,t). \end{cases}$$

One readily verifies that $b : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, which satisfies the following growth condition:

$$|b(x, t, s)| \le \beta(x, t) + c_b|s|^{p-1}, \quad \forall (x, t, s) \in \mathbb{Q} \times \mathbb{R},\ c_b > 0, \tag{2.10}$$

where $\beta(x, t) = c\big(|\psi(x, t)|^{p-1} + |\phi(x, t)|^{p-1}\big)$ with some positive constant c, and thus $\beta \in L^{p'}(\mathbb{Q}, w)$, since $\phi, \psi \in X \hookrightarrow L^p(\mathbb{Q}, w)$. Therefore, b fulfills qualitatively the same regularity and growth conditions like g in Lemma 2.10, and hence the Nemytskij operator B associated with b through $B(u)(x, t) = b(x, t, u(x, t))$ yields a continuous and bounded mapping from $L^p(\mathbb{Q}, w)$ to $L^{p'}(\mathbb{Q}, w)$. Moreover, $s \mapsto b(x, t, s)$ is monotone nondecreasing.

In view of (2.8), the operator P can be characterized as $P = wB$ or

$$P = i_w^* \circ B \circ i_w : X \to X^*, \tag{2.11}$$

with the embeddings $i_w : X \to L^p(\mathbb{Q}, w)$ and $i_w^* : L^{p'}(\mathbb{Q}, w) \to X^*$ (see Lemma 2.8), and thus $P : X \to X^*$ is bounded and continuous.

Lemma 2.11. *The operator $P = wB : X \to X^*$ defined by (2.8) (resp. (2.11)) is a penalty operator associated with K, that is, $P : X \to X^*$ is a bounded, hemicontinuous (even continuous), and monotone operator, which satisfies*

$$P(u) = 0 \iff u \in K.$$

Moreover, $P = wB : Y \to X^*$ is bounded and completely continuous.

Proof. $P : X \to X^*$ is bounded and continuous, hence hemicontinuous, and also monotone due to the monotonicity of $s \mapsto b(x, t, s)$. Therefore, it only remains to show

$$P(u) = 0 \iff u \in K.$$

If $u \in K$, then by the definition of the function b, we have $b(x,t,u) = 0$, and thus $P(u) = 0$. To show the converse, let $P(u) = 0$, that is, $\langle P(u), \varphi \rangle = 0$ for all $\varphi \in X$. Using the special test function $\varphi = (u-\psi)^+ \in X$, we get

$$0 = \langle P(u), (u-\psi)^+ \rangle = \int_{\mathbb{Q}} w[(u-\psi)^+]^p \, dxdt,$$

which implies $(u-\psi)^+ = 0$, i.e., $u \leq \psi$ a.e. in $\mathbb{Q}$. Testing $\langle P(u), \varphi \rangle = 0$ with $\varphi = (\phi - u)^+ \in X$ yields

$$0 = \langle P(u), (\phi-u)^+ \rangle = -\int_{\mathbb{Q}} w[(\phi-u)^+]^p \, dxdt,$$

which implies $(\phi-u)^+ = 0$, i.e., $\phi \leq u$ a.e. in $\mathbb{Q}$. Finally, since b fulfills qualitatively the same regularity and growth conditions like g in Lemma 2.10, we may apply Lemma 2.10 which proves that $P = wB : Y \to X^*$ is also completely continuous. □

Lemma 2.12. *The penalty operator $P\colon X \to X^*$ satisfies the inequality*

$$\langle P(u), (u-\psi)^+ - (\phi-u)^+ \rangle \geq d \, \|P(u)\|_{X^*} \left(\|(u-\psi)^+\|_{\mathbb{Q},p,w} + \|(\phi-u)^+\|_{\mathbb{Q},p,w} \right)$$

with $d > 0$.

Proof. From (2.8), we get

$$\begin{aligned} \langle P(u), (u-\psi)^+ - (\phi-u)^+ \rangle &= \int_{\mathbb{Q}} w \left([u-\psi)^+]^{p-1} - [(\phi-u)^+]^{p-1} \right) \times \\ &\quad ((u-\psi)^+ - (\phi-u)^+) \, dxdt \\ &= \int_{\mathbb{Q}} w \left([u-\psi)^+]^p + [(\phi-u)^+]^p \right) dxdt, \end{aligned}$$

that is

$$\langle P(u), (u-\psi)^+ - (\phi-u)^+ \rangle = \|(u-\psi)^+\|^p_{\mathbb{Q},p,w} + \|(\phi-u)^+\|^p_{\mathbb{Q},p,w}. \tag{2.12}$$

Applying the Hölder inequality and $X \hookrightarrow L^p(\mathbb{Q}, w)$, we estimate

$$\begin{aligned} |\langle P(u), \varphi \rangle| &\leq \int_{\mathbb{Q}} w \left([u-\psi)^+]^{p-1} + [(\phi-u)^+]^{p-1} \right) |\varphi| \, dxdt \\ &\leq \int_{\mathbb{Q}} w^{\frac{1}{p'}} \left([u-\psi)^+]^{p-1} + [(\phi-u)^+]^{p-1} \right) w^{\frac{1}{p}} |\varphi| \, dxdt \\ &\leq \left(\|(u-\psi)^+\|^{p-1}_{\mathbb{Q},p,w} + \|(\phi-u)^+\|^{p-1}_{\mathbb{Q},p,w} \right) \|\varphi\|_{\mathbb{Q},p,w} \\ &\leq c \left(\|(u-\psi)^+\|^{p-1}_{\mathbb{Q},p,w} + \|(\phi-u)^+\|^{p-1}_{\mathbb{Q},p,w} \right) \|\varphi\|_X, \end{aligned}$$

where c is some positive constant, which yields

$$\|P(u)\|_{X^*} \leq c \left(\|(u-\psi)^+\|^{p-1}_{\mathbb{Q},p,w} + \|(\phi-u)^+\|^{p-1}_{\mathbb{Q},p,w} \right). \tag{2.13}$$

Using the elementary inequality $r^p + s^p \geq \frac{1}{2}(r^{p-1} + s^{p-1})(r+s)$ for any real numbers $r \geq 0$ and $s \geq 0$, from (2.12) and (2.13), we get for some positive constant d independent of u, ψ, and ϕ:

$$\langle P(u), (u-\psi)^+ - (\phi-u)^+ \rangle \geq d \, \|P(u)\|_{X^*} \left(\|(u-\psi)^+\|_{\mathbb{Q},p,w} + \|(\phi-u)^+\|_{\mathbb{Q},p,w} \right),$$

which completes the proof. □

Lemma 2.13. *For any $u \in Y$ with $u(\cdot,0) = 0$, it holds that*

$$\langle u_t + \mathcal{A}u, (u-\psi)^+ - (\phi - u)^+\rangle \geq 0.$$

Proof. We use hypotheses (Hψ) and (Hϕ) and note that $u-\psi \in Y$ and $\phi - u \in Y$, as well as $(u-\psi)^+(x,0) = 0$ and $(\phi-u)^+(x,0) = 0$, which by applying the integration by parts formula (see Lemma 2.6) yields

$$\langle (u-\psi)_t, (u-\psi)^+\rangle = \frac{1}{2}\|(u-\psi)^+(\cdot,\tau)\|_{2,w}^2, \tag{2.14}$$

$$\langle (\phi-u)_t, (\phi-u)^+\rangle = \frac{1}{2}\|(\phi-u)^+(\cdot,\tau)\|_{2,w}^2. \tag{2.15}$$

With (2.14), we get the following estimate by taking into account (Hψ) and the fact that $\mathcal{A}: X \to X^*$ is a bounded, continuous and strictly monotone operator:

$$\begin{aligned}&\langle u_t + \mathcal{A}u - (\psi_t + \mathcal{A}\psi), (u-\psi)^+\rangle\\ &= \langle (u-\psi)_t, (u-\psi)^+\rangle + \langle \mathcal{A}u - \mathcal{A}\psi, (u-\psi)^+\rangle \geq 0,\end{aligned}$$

which results in

$$\langle u_t + \mathcal{A}u, (u-\psi)^+\rangle \geq \langle \psi_t + \mathcal{A}\psi, (u-\psi)^+\rangle \geq 0. \tag{2.16}$$

Similarly with (Hϕ) and (2.15), we get

$$\begin{aligned}&\langle u_t + \mathcal{A}u - (\phi_t + \mathcal{A}\phi), -(\phi-u)^+\rangle\\ &= \langle (u-\phi)_t, -(\phi-u)^+\rangle + \langle \mathcal{A}u - \mathcal{A}\phi, -(\phi-u)^+\rangle \geq 0,\end{aligned}$$

which yields

$$\langle u_t + \mathcal{A}u, -(\phi-u)^+\rangle \geq \langle \phi_t + \mathcal{A}\phi, -(\phi-u)^+\rangle \geq 0, \tag{2.17}$$

and thus (2.16) and (2.17) complete the proof. □

Example 2.14. *Let $p = 2$ and $N = 6$, which gives $p^* = 3$ and $p^{*\prime} = \frac{3}{2}$. Let $\mathcal{A} = -\Delta$. Then ψ, $\phi : \mathbb{Q} \to \mathbb{R}$ with $\mathbb{Q} = \mathbb{R}^6 \times (0,\tau)$, given by*

$$\psi(x,t) = \left(2\tau - t + |x|^2\right)^{-2} \quad \text{and} \quad \phi(x,t) = -\psi(x,t),$$

satisfy hypotheses (Hψ) and (Hϕ) with $X = L^2(0,\tau;V)$, $X^* = L^2(0,\tau;V^*)$, and $V = D^{1,2}(\mathbb{R}^6)$.

Further, let $j : \mathbb{Q}\times\mathbb{R} \to \mathbb{R}$ and $a : \mathbb{Q} \to \mathbb{R}$ be given by

$$j(x,t,s) = -\frac{c_j}{2}s^2 + k(x,t)s, \quad a(x,t) = w(x) = \frac{1}{1+|x|^{6+\alpha}}, \tag{2.18}$$

where $c_j > 0$ and $k \in L^2(\mathbb{Q},w)$. Then $\partial j(x,t,s) = -c_j s + k(x,t)$ is single-valued and satisfies (Hj), and a satisfies (Ha). Next let us verify that the operator $-\Delta + wF : X \to X^*$ fails to be coercive on $X = L^2(0,\tau;V)$ with $f(x,t,s) = \partial j(x,t,s) = -c_j s + k(x,t)$ and w as in (2.18).

$$\langle -\Delta u + wF(u), u\rangle = \int_{\mathbb{Q}} |\nabla u|^2\,dxdt - \int_{\mathbb{Q}} (c_j w|u|^2 - wku)\,dxdt \leq \|u\|_X^2 - c_j\|u\|_{\mathbb{Q},2,w}^2 + \|k\|_{\mathbb{Q},2,w}\|u\|_{\mathbb{Q},2,w}.$$

Let $u_0 \neq 0$, $u_0 \in X$, and using $u = \lambda u_0$, we obtain

$$\frac{1}{\|u\|_X}\langle -\Delta u + wF(u), u\rangle \leq \frac{\lambda}{\|u_0\|_X}\left(\|u_0\|_X^2 - c_j\|u_0\|_{\mathbb{Q},2,w}^2\right) + \frac{1}{\|u_0\|_X}\|k\|_{\mathbb{Q},2,w}\|u_0\|_{\mathbb{Q},2,w}.$$

If $c_j \geq \frac{\|u_0\|_X^2}{\|u_0\|_{\mathbb{Q},2,w}^2}$, then the right-hand side of the last inequality is bounded above for all $\lambda \geq 0$, which proves that $-\Delta + wF : X \to X^*$ is not coercive.

3. Main result

Let us assume conditions (Ha), (A1)–(A3), (Hj), (Hψ), and (Hϕ) throughout this section. Our goal is to prove an existence result for the parabolic variational-hemivariational inequality (1.1) via the existence of solutions for the multi-valued parabolic variational inequality (1.3), respectively (1.4). For this purpose, first we are going to reformulate the multi-valued parabolic variational inequality (1.3), respectively (1.4), and recall the multi-valued function $f(x,t,s) = \partial j(x,t,s)$ (see Lemma 2.9) along with its multi-valued Nemytskij operator F defined by (2.6). Since

$$V \hookrightarrow L^2(\mathbb{R}^N, w) \hookrightarrow V^*$$

forms an evolution triple, the initial value $u(\cdot,0) = 0$ is well-defined. Let $Lu := u_t$ be the time derivative operator with domain $D(L)$ given by

$$D(L) = \{u \in Y : u(\cdot,0) = 0\}.$$

Then by using [14, Proposition 32.10], we have the following result.

Lemma 3.1. *The operator $L\colon D(L) \to X^*$ is densely defined, closed, and maximal monotone.*

Now we can reformulate the parabolic variational inequality (1.3) as follows: Find $u \in D(L) \cap K$ and $\eta \in F(u) \subset L^{p'}(\mathbb{Q}, w)$ such that

$$\langle Lu + \mathcal{A}u, v - u\rangle + \int_{\mathbb{Q}} a\eta(v-u)\, dxdt \geq 0 \quad \text{for all } v \in K, \tag{3.1}$$

which is equivalent with

$$\langle Lu + \mathcal{A}u + i_a^*\eta, v - u\rangle \geq 0 \quad \text{for all } v \in K. \tag{3.2}$$

Lemma 3.2. *If $u \in D(L) \cap K$ is a solution of (3.1) (resp. (3.2)), then u is a solution of the original parabolic variational-hemivariational inequality (1.1).*

Proof. Let u be a solution of (3.1), that is, $u \in Y$, $u(\cdot,0) = 0$, and there is $\eta \in L^{p'}(\mathbb{Q}, w)$ with $\eta(x,t) \in f(x,t,u(x,t)) = \partial j(x,t,u(x,t))$ such that inequality (3.1) holds true. By the definition of Clarke's generalized gradient, we get for any $v \in K$ (note: $a(x,t) \geq 0$):

$$a(x,t) j^o(x,t,u(x,t); v(x,t) - u(x,t)) \geq a(x,t)\eta(x,t)(v(x,t) - u(x,t)), \tag{3.3}$$

with $(x,t) \mapsto a(x,t) j^o(x,t,u(x,t); v(x,t) - u(x,t))$ being a measurable function. From Clarke's calculus, we have

$$j^o(x,t,u(x,t);\varrho) = \max\{\eta(x,t)\varrho : \eta(x,t) \in \partial j(x,t,u(x,t))\}, \quad \forall \varrho \in \mathbb{R},$$

which in view of the growth condition (Hj) implies that

$$(x,t) \mapsto a(x,t) j^o(x,t,u(x,t); v(x,t) - u(x,t))$$

is integrable. Hence, from (3.3), we get

$$\int_{\mathbb{Q}} a j^o(x,t,u; v-u)\, dxdt \geq \int_{\mathbb{Q}} a\eta(v-u) dxdt, \quad \forall v \in K.$$

The last inequality along with (3.1) proves that u is a solution of (1.1). □

Remark 3.3. *We note that under the conditions we assume throughout this section, the reverse statement of Lemma 3.1 holds true, that is, any solution of (1.1) is a solution of (3.1) (resp. (3.2)). The proof of the reverse statement, which is more involved, makes use of the lattice property of the constraint K as given in (2.3) and can be carried out by appropriately adapting the idea of the proof of [16, Theorem 5.4].*

The existence proof for (3.1) (resp. (3.2)) is based on an appropriately designed penalty approach and makes use of an abstract existence result for multi-valued evolution equations of the form

$$u \in D(L) : h \in Lu + Tu \quad \text{in } X^* \tag{3.4}$$

where $h \in X^*$, and $T : X \to 2^{X^*}$ is a multi-valued pseudomonotone operator with respect to the graph norm topology of the domain $D(L)$ (shortly: pseudomonotone w.r.t. $D(L)$), which is defined below. To this end, let $D(L)$ be equipped with its graph norm $\|u\|_{D(L)} = \|u\|_X + \|Lu\|_{X^*}$.

Definition 3.4. *Let $L : D(L) \subset X \to X^*$ be a linear, closed, densely defined, and maximal monotone operator. The operator $T : X \to 2^{X^*}$ is called pseudomonotone w.r.t. D(L) if the following conditions are satisfied.*

(i) The set Tu is nonempty, bounded, closed, and convex for all $u \in X$.

(ii) T is upper semicontinuous from each finite dimensional subspace of X to the weak topology of X^*.

(iii) If $(u_n) \subset D(L)$ with $u_n \rightharpoonup u$ in X, $Lu_n \rightharpoonup Lu$ in X^*, $u_n^* \in Tu_n$ with $u_n^* \rightharpoonup u^*$ in X^*, and $\limsup\langle u_n^*, u_n - u\rangle \le 0$, then $u^* \in Tu$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$.

Definition 3.5. *The operator $T : X \to 2^{X^*}$ is called coercive iff either its domain $D(T)$ is bounded or $D(T)$ is unbounded and*

$$\frac{\inf\{\langle v^*, v\rangle : v^* \in Tv\}}{\|v\|_X} \to +\infty \ \textit{as} \ \|v\|_X \to \infty, \ v \in D(T).$$

The following surjectivity result can be found, e.g., in [16, Theorem 2.56].

Theorem 3.6. *Let X be a real reflexive, strictly convex Banach space with dual space X^*, and let $L : D(L) \subset X \to X^*$ be a linear, closed, densely defined, and maximal monotone operator. If the multi-valued operator $T : X \to 2^{X^*}$ is pseudomonotone w.r.t. D(L), bounded, and coercive, then $L + T$ is surjective, i.e., $(L + T)(D(L)) = X^*$.*

Our main result reads as follows.

Theorem 3.7. *Under hypotheses (Ha), (A1)–(A3), (Hj), (Hψ), and (Hϕ), the parabolic variational-hemivariational inequality (1.1) admits at least one solution.*

Before proving Theorem 3.7, we provide some auxiliary results.

Lemma 3.8. *The multi-valued differential operator $\mathcal{A} + aF + \lambda P : X \to 2^{X^*}$ is bounded and pseudomonotone w.r.t. D(L), where the multi-valued operator F is given by (2.6) and the penalty operator is defined by (2.8). Moreover, $\mathcal{A} + aF + \lambda P : X \to 2^{X^*}$ is coercive for large $\lambda > 0$.*

Proof. First, we recall that $X = L^p(0,\tau;V)$ is separable and uniformly convex, and thus a reflexive Banach space. Clearly, the operator $\mathcal{A} : X \to X^*$ is bounded, continuous, and strictly monotone, which implies that $\mathcal{A} : X \to X^*$ is pseudomonotone in the usual sense (see, e.g., Zeidler [14, Proposition 27.6 (a)]), and thus, in particular, pseudomonotone w.r.t. $D(L)$. The penalty operator P introduced in (2.8) is given by $P = wB$, where B denotes the Nemytskij operator generated by the function $b : \mathbb{Q} \times \mathbb{R} \to \mathbb{R}$. In view of Lemma 2.11, we have that $P : Y \to X^*$ is bounded and completely continuous, which readily implies that $P : X \to X^*$ is pseudomonotone w.r.t. $D(L)$. Next, consider the multi-valued operator F defined by (2.6) generated by the multi-valued function $f(x,t,s) = \partial j(x,t,s)$ with $F : L^p(\mathbb{Q},w) \to 2^{L^{p'}(\mathbb{Q},w)}$ being a bounded mapping. We have $i_a^* : L^{p'}(\mathbb{Q},w) \hookrightarrow X^*$, and in view of Lemma 2.7, it holds that $Y \hookrightarrow\hookrightarrow X$. Moreover, since f satisfies (f1) and (f2) (see Lemma 2.9), we may use [16, Lemma 3.16], which is easily seen to be applicable with only slight modifications, and thus $aF : X \to 2^{X^*}$ is pseudomonotone w.r.t. $D(L)$. Finally, as the sum of the operators that are pseudomonotone w.r.t. $D(L)$ is again pseudomonotone w.r.t. $D(L)$, we see that $\mathcal{A} + aF + \lambda P : X \to 2^{X^*}$ is bounded and pseudomonotone w.r.t. $D(L)$. Let us show next that $\mathcal{A} + aF + \lambda P : X \to 2^{X^*}$ is coercive for large $\lambda > 0$, that is,

$$\lim_{\|u\|_X \to \infty} \left[\inf_{\eta \in F(u)} \frac{\langle \mathcal{A}u + a\eta + \lambda P(u), u\rangle}{\|u\|_X} \right] = +\infty.$$

With (A3), we get the inequality

$$\langle \mathcal{A}u, u\rangle = \int_{\mathbb{Q}} A(x,t,\nabla u)\nabla u \, dxdt \geq \nu \|u\|_X^p - \|k_1\|_{\mathbb{Q},1}. \tag{3.5}$$

From the definition of the penalty operator in (2.8) along with the function b defined by (2.9), we arrive at the inequality

$$\langle P(u), u\rangle = \int_{\mathbb{Q}} wb(x,t,u)u \, dxdt \geq c_1 \|u\|_{\mathbb{Q},p,w}^p - c_2, \tag{3.6}$$

for some positive constants c_1 and c_2, and with the growth condition (2.2) on $s \mapsto f(x,t,s) = \partial j(x,t,s)$, we get for $\eta \in F(u)$:

$$\begin{aligned} \langle a\eta, u\rangle &= \int_{\mathbb{Q}} a\eta u \, dxdt \geq -\int_{\mathbb{Q}} a|\eta||u| \, dxdt \\ &\geq -c_a c_j \|u\|_{\mathbb{Q},p,w}^p - c_a \|k\|_{\mathbb{Q},p',w} \|u\|_{\mathbb{Q},p,w}. \end{aligned} \tag{3.7}$$

From (3.5)–(3.7), it follows that

$$\inf_{\eta \in F(u)} \langle \mathcal{A}u + a\eta + \lambda P(u), u\rangle \geq \nu \|u\|_X^p + (\lambda c_1 - c_a c_j)\|u\|_{\mathbb{Q},p,w}^p - c_a \|k\|_{\mathbb{Q},p',w}\|u\|_{\mathbb{Q},p,w} - \lambda c_2$$

which implies the coercivity of the operator $\mathcal{A} + aF + \lambda P : X \to 2^{X^*}$ provided that λ is large enough such that $\lambda c_1 - c_a c_j \geq 0$, that is, $\lambda \geq \frac{c_a c_j}{c_1}$. □

Now we are ready to prove our main result.

Proof of Theorem 3.7

Let $\varepsilon > 0$ be arbitrarily given, and consider the multi-valued penalty equation

$$u \in D(L) : 0 \in Lu + \mathcal{A}u + aF(u) + \lambda P(u) + \frac{1}{\varepsilon}P(u) \quad \text{in } X^*, \tag{3.8}$$

where $\lambda > 0$ is large enough to ensure coercivity of $\mathcal{A} + aF + \lambda P : X \to 2^{X^*}$ according to Lemma 3.8, which infers that $\mathcal{A} + aF + (\lambda + \frac{1}{\varepsilon})P : X \to 2^{X^*}$ is coercive as well for any ε and enjoys the same properties. With $T = \mathcal{A} + aF + (\lambda + \frac{1}{\varepsilon})P : X \to 2^{X^*}$, we may apply the existence result due to Theorem 3.6, and it follows that (3.8) admits at least one solution u_ε, that is, there is $\eta_\varepsilon \in F(u_\varepsilon)$ such that

$$u_\varepsilon \in D(L) : Lu_\varepsilon + \mathcal{A}u_\varepsilon + a\eta_\varepsilon + \lambda P(u_\varepsilon) + \frac{1}{\varepsilon}P(u_\varepsilon) = 0 \quad \text{in } X^*.$$

Let (ε_n) be such that $\varepsilon_n \searrow 0$ and select a sequence of corresponding penalty solutions $(u_{\varepsilon_n}) := (u_n)$ of (3.8), that is, there is $\eta_n \in F(u_n)$ such that

$$u_n \in D(L) : Lu_n + \mathcal{A}u_n + a\eta_n + \lambda P(u_n) + \frac{1}{\varepsilon_n}P(u_n) = 0 \quad \text{in } X^*. \tag{3.9}$$

Testing (3.9) with $\varphi = u_n$, and taking the monotonicity of the penalty operator P into account as well as $P(0) = 0$, we get (note: $Lu = u_t$ and $u(\cdot, 0) = 0$)

$$\langle \mathcal{A}u_n + a\eta_n + \lambda P(u_n), u_n\rangle = -\frac{1}{\varepsilon_n}\langle P(u_n), u_n\rangle - \langle u_{nt}, u_n\rangle \le 0.$$

Since $\mathcal{A} + aF + \lambda P : X \to 2^{X^*}$ is coercive, the last inequality implies that $(\|u_n\|_X)$ is bounded, and therefore, $(\mathcal{A}u_n)$, $(a\eta_n) \subset (aF(u_n))$, and $P(u_n)$ are bounded in X^*. Consider next the sequence $(\frac{1}{\varepsilon_n}P(u_n))$. By Lemma 2.12, we have

$$\langle P(u_n), (u_n - \psi)^+ - (\phi - u_n)^+\rangle \ge d\,\|P(u_n)\|_{X^*}\left(\|(u_n - \psi)^+\|_{\mathbb{Q},p,w} + \|(\phi - u_n)^+\|_{\mathbb{Q},p,w}\right). \tag{3.10}$$

Testing the penalty equation (3.9) with $\varphi = (u_n - \psi)^+ - (\phi - u_n)^+$, we obtain the following equation:

$$\begin{aligned}&\langle u_{nt} + \mathcal{A}u_n, (u_n - \psi)^+ - (\phi - u_n)^+\rangle \\ &+ \left\langle a\eta_n + \lambda P(u_n) + \frac{1}{\varepsilon_n}P(u_n), (u_n - \psi)^+ - (\phi - u_n)^+\right\rangle = 0.\end{aligned} \tag{3.11}$$

Using Lemma 2.13, from (3.11), we obtain

$$\begin{aligned}\left\langle \frac{1}{\varepsilon_n}P(u_n), (u_n - \psi)^+ - (\phi - u_n)^+\right\rangle \le &- \langle a\eta_n, (u_n - \psi)^+ - (\phi - u_n)^+\rangle \\ &- \langle \lambda P(u_n), (u_n - \psi)^+ - (\phi - u_n)^+\rangle.\end{aligned} \tag{3.12}$$

Let us estimate the first term on the right-hand side of (3.12) (note: $\eta_n \in F(u_n)$):

$$
\begin{aligned}
|\langle a\eta_n), (u_n-\psi)^+ - (\phi-u_n)^+\rangle| &\le c_a \int_{\mathbb{Q}} w|\eta_n|[(u_n-\psi)^+ + (\phi-u_n)^+]\,dxdt \\
&\le c_a \int_{\mathbb{Q}} w^{\frac{1}{p'}}|\eta_n|\left(w^{\frac{1}{p}}(u_n-\psi)^+ + w^{\frac{1}{p}}(\phi-u_n)^+\right)dxdt \\
&\le c_a\|\eta_n\|_{\mathbb{Q},p',w}\left(\|(u_n-\psi)^+\|_{\mathbb{Q},p,w} + \|(\phi-u_n)^+\|_{\mathbb{Q},p,w}\right) \\
&\le c\left(\|(u_n-\psi)^+\|_{\mathbb{Q},p,w} + \|(\phi-u_n)^+\|_{\mathbb{Q},p,w}\right),
\end{aligned}
$$

where we have taken into account the boundedness of the sequence (η_n) in $L^{p'}(\mathbb{Q}, w)$. Similarly, we have for the second term on the right-hand side of (3.12):

$$
|\langle \lambda P(u_n), (u_n-\psi)^+ - (\phi-u_n)^+\rangle| \le c\left(\|(u_n-\psi)^+\|_{\mathbb{Q},p,w} + \|(\phi-u_n)^+\|_{\mathbb{Q},p,w}\right).
$$

Hence, from (3.12), it follows that

$$
\left\langle \frac{1}{\varepsilon_n}P(u_n), (u_n-\psi)^+ - (\phi-u_n)^+\right\rangle \le c\left(\|(u_n-\psi)^+\|_{\mathbb{Q},p,w} + \|(\phi-u_n)^+\|_{\mathbb{Q},p,w}\right),
$$

which in view of Lemma 2.12 yields

$$
\begin{aligned}
&\frac{d}{\varepsilon_n}\|P(u_n)\|_{X^*}\left(\|(u_n-\psi)^+\|_{\mathbb{Q},p,w} + \|(\phi-u_n)^+\|_{\mathbb{Q},p,w}\right) \\
&\le c\left(\|(u_n-\psi)^+\|_{\mathbb{Q},p,w} + \|(\phi-u_n)^+\|_{\mathbb{Q},p,w}\right).
\end{aligned}
$$

We see from the last inequality that

$$
\frac{1}{\varepsilon_n}\|P(u_n)\|_{X^*} \le \frac{c}{d}, \quad \forall\, \varepsilon_n. \tag{3.13}
$$

From the penalty equations (3.9), we have

$$
u_{nt} = -\left(-\Delta_p u_n + a\eta_n + \lambda P(u_n) + \frac{1}{\varepsilon_n}P(u_n)\right),
$$

which in view of (3.13) shows that (u_{nt}) is bounded in X^*. Hence it follows that (u_n) is bounded in Y. Thus there exists a subsequence (again denoted by (u_n)) such that

$$
u_n \rightharpoonup u \quad \text{in } X \quad \text{and} \quad u_{nt} \rightharpoonup u_t \quad \text{in } X^* \tag{3.14}
$$

as $n \to \infty$ and $\varepsilon_n \searrow 0$. Since $D(L)$ is closed in Y and convex, it is weakly closed, and therefore the limit $u \in D(L)$. Next, let us show that the weak limit u belongs to K, which is equivalent to $P(u) = 0$. From (3.13), it follows that $P(u_n) \to 0$ in X^*. Since the penalty operator $P\colon X \to X^*$ is monotone, we get $\langle P(v) - P(u_n), v - u_n\rangle \ge 0$ for all $v \in X$ and for all n, which by passing to the limit as $n \to \infty$ yields

$$
\langle P(v), v - u\rangle \ge 0 \quad \text{for all } v \in X.
$$

In particular, the last inequality holds for $v = u + \delta\varphi$ for any $\delta > 0$ and $\varphi \in X$, that is,

$$\langle P(u + \delta\varphi), \varphi\rangle \geq 0 \quad \text{for all } \varphi \in X.$$

Passing to the limit as $\delta \searrow 0$, we get

$$\langle P(u), \varphi\rangle \geq 0 \quad \text{for all } \varphi \in X,$$

which implies $P(u) = 0$, that is, $u \in K$. Testing the penalty equation (3.9) by $\varphi = u - u_n$ and using the monotonicity of P as well as $P(u) = 0$, we obtain

$$\langle Lu_n + \mathcal{A}u_n + a\eta_n, u - u_n\rangle = (\lambda + \frac{1}{\varepsilon_n})\langle P(u) - P(u_n), u - u_n\rangle.$$

Applying $\langle Lu - Lu_n, u - u_n\rangle \geq 0$, the last inequality yields

$$\begin{aligned}\langle \mathcal{A}u_n + a\eta_n, u - u_n\rangle &\geq -\langle Lu, u - u_n\rangle + \left(\lambda + \frac{1}{\varepsilon_n}\right)\langle P(u) - P(u_n), u - u_n\rangle \\ &\geq -\langle Lu, u - u_n\rangle,\end{aligned}$$

which results in

$$\langle \mathcal{A}u_n + a\eta_n, u_n - u\rangle \leq \langle Lu, u - u_n\rangle.$$

Set $u_n^* = \mathcal{A}u_n + a\eta_n \in \mathcal{A}u_n + aF(u_n)$, and then with (3.14) and passing to the lim sup in the last inequality, we see that

$$\limsup_{n\to\infty}\langle \mathcal{A}u_n + a\eta_n, u_n - u\rangle \leq 0. \tag{3.15}$$

As $(\|u_n\|_X)$ is bounded, it follows that $(\|u_n^*\|_{X^*})$ is bounded, and thus there is some subsequence of (u_n^*) (again denoted by (u_n^*)) such that $u_n^* \rightharpoonup u^*$ in X^*. Recall that the operator $\mathcal{A} + aF : X \to 2^{X^*}$ is pseudomonotone w.r.t. $D(L)$, and thus by taking into account (3.15), we get

$$u^* \in \mathcal{A}u + aF(u) \quad \text{and} \quad \langle u_n^*, u_n\rangle \to \langle u^*, u\rangle, \tag{3.16}$$

where $u^* = \mathcal{A}u + a\eta$ with $\eta \in F(u)$. Testing the penalty equation (3.9) by $\varphi = v - u_n$ with $v \in K$ and noting $P(v) = 0$, we have

$$\langle Lu_n + \mathcal{A}u_n + a\eta_n, v - u_n\rangle = \left(\lambda + \frac{1}{\varepsilon_n}\right)\langle P(v) - P(u_n), v - u_n\rangle.$$

Using the identity

$$\langle Lu_n, v - u_n\rangle = \langle Lu, v - u_n\rangle - \langle Lu_n - Lu, u_n - u\rangle + \langle Lu_n - Lu, v - u\rangle,$$

we obtain

$$\begin{aligned}\langle Lu + \mathcal{A}u_n + a\eta_n, v - u_n\rangle &= \left(\lambda + \frac{1}{\varepsilon_n}\right)\langle P(v) - P(u_n), v - u_n\rangle \\ &\quad + \langle Lu_n - Lu, u_n - u\rangle - \langle Lu_n - Lu, v - u\rangle,\end{aligned}$$

which by using the monotonicity of the operators L and P results in

$$\langle Lu + \mathcal{A}u_n + a\eta_n, v - u_n\rangle \geq -\langle Lu_n - Lu, v - u\rangle. \tag{3.17}$$

With $u_n^* = \mathcal{A}u_n + a\eta_n \rightharpoonup u^* = \mathcal{A}u + a\eta$, (3.14) and (3.16), from (3.17), we obtain by passing to the limit as $n \to \infty$:

$$\langle Lu + \mathcal{A}u + a\eta, v - u\rangle \geq 0, \quad \forall v \in K,$$

where $\eta \in F(u)$, which proves that u is a solution of the multi-valued parabolic variational inequality (3.1) (resp. (3.2)). Hence, by applying Lemma 3.2, we see that u is a solution of the original parabolic variational-hemivariational inequality (1.1) under the bilateral constraint K, completing the proof. □

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflict of interest.

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