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Research article

Long-time dynamics for a coupled system modeling the oscillations of suspension bridges

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Abstract: This paper is concerned with a coupled system modeling the oscillations of suspension bridges, which consists of a beam equation and a viscoelastic string equation. We first transformed the original initial-boundary value problem into an equivalent one in the history space framework. Then we obtained the global well-posedness and regularity of mild solutions by using the semigroup theory. In addition, we employed the perturbed energy method to establish a stabilizability estimate. By verifying the gradient property and quasi-stability of the corresponding dynamical system, we derived the existence of a global attractor with finite fractal dimension.

Keywords: coupled beam-string system; global well-posedness; global attractors; gradient dynamical system; quasi-stability
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1. Introduction

Suspension bridge refers to a bridge with cables as the main load-bearing component of a superstructure, which is composed of suspension cable, a cable tower, hanger, bridge deck, etc. Compared with other types of bridges, suspension bridges have the advantages of saving materials, being light weight, and having a large span, but they also have poor stiffness and are prone to deflections and oscillations under vehicle and wind loads. In the past, a number of destructive and unexpected events occurred, see, e.g., the Tacoma Narrows Bridge [1–3] and the London Millennium Bridge [4, 5]. It can be said that the development experience of suspension bridges is a history of fighting against deformations and oscillations. Therefore, it is very necessary to study deformations and oscillations of suspension bridges.

In this paper, we study the following coupled beam-string system modeling the small amplitude oscillations of suspension bridges:

$$\begin{cases} m_1 u_{tt} + K u_{xxxx} + \mu_1 u_t + \Phi(u-v) + f_1(u,v) = h_1(x), \quad x \in (0,l), \ t > 0, \\ m_2 v_{tt} - G(0) v_{xx} + \displaystyle\int_{-\infty}^{t} g(t-\tau) v_{xx}(\tau)\, \mathrm{d}\tau + \mu_2 v_t - \Phi(u-v) \\ \ + f_2(u,v) = h_2(x), \quad x \in (0,l), \ t > 0, \end{cases} \tag{1.1}$$

with mixed boundary conditions consisting of simply supported and Dirichlet boundary conditions

$$\begin{cases} u(0,t) = u(l,t) = u_{xx}(0,t) = u_{xx}(l,t) = 0, \quad t > 0, \\ v(0,t) = v(l,t) = 0, \quad t \in \mathbb{R}, \end{cases} \tag{1.2}$$

and initial conditions

$$\begin{cases} u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \quad x \in (0,l), \\ v(x,t) = v_0(x,t), \ v_t(x,0) = v_1(x), \quad x \in (0,l), \ t \le 0. \end{cases} \tag{1.3}$$

Here, the two unknown functions u and v measure the vertical displacements of the bridge deck and the suspension cable, respectively. The function $v_0 : (0,l) \times (-\infty, 0] \to \mathbb{R}$ is a prescribed past history, and $v_1(x) := \partial_t v_0(x,t)|_{t=0}$. The function $\Phi(s) := \lambda s$ or λs^+ represents the restraining force experienced by both the bridge deck and the suspension cable as transmitted through the hangers, thereby producing the coupling between the two, $s^+ := \max\{0, s\}$, and $\lambda > 0$ is the stiffness coefficient of the hangers connecting the bridge deck to the suspension cable. The terms $\mu_1 u_t$ and $\mu_2 u_t$ denote the weak damping, and $\mu_1, \mu_2 > 0$ are the damping coefficients. The quantities $m_1, m_2 > 0$ are the masses per unit length of the bridge deck and the suspension cable, respectively. The constant K is the flexural rigidity of the bridge deck, and $G(0) > 0$ accounts for the tensile strength of the suspension cable with viscoelasticity, where $G'(t) := -g(t) \le 0$ and $\lim_{t\to\infty} G(t) > 0$. In addition, $h_1, h_2 \in L^2(\Omega)$ stand for the external forces acting on the bridge deck and the suspension cable, respectively. The memory kernel g and the nonlinear source terms f_1, f_2 will be specified later. For more details about the physical background of this kind of model, we refer the reader to [6, 7].

Deformations and oscillations of suspension bridges have enjoyed growing attention. Beams and rods have been used to simulate deformations and oscillations of suspension bridges. In this respect, we refer the reader to [6–20] and the references therein. Meanwhile, various plate equations have been employed to model deformations and oscillations of suspension bridges, see, e.g., [1, 21–29] and the references therein. These works are very interesting and make us better understand the behavior of suspension bridges to a certain extent. Next, in order to explain the motivation of this paper, we restrict our attention to beam and rod models considering the roles of the hangers.

Lazer and McKenna [15] investigated the following beam equation:

$$u_{tt} + K u_{xxxx} + \lambda u^+ = \sin \frac{\pi x}{l} (S + \varepsilon h(t)),$$

where λu^+ denotes the restoring force provided by the hangers, S is a large constant, ε is a small parameter, and $h(t)$ is a periodic function. They obtained the existence of multiple periodic solutions. McKenna and Walter [19] considered a beam equation of the form

$$u_{tt} + K u_{xxxx} + \lambda u^+ = 1 + \varepsilon h, \tag{1.4}$$

where $h = h(x,t)$ is a periodic external force. For certain ranges of λ, they also derived the existence of multiple periodic solutions. In the case where $K = 1$ and $\varepsilon h = 0$ (namely εh is ignored), McKenna and Walter [20] dealt with travelling wave solutions to Eq. (1.4). McKenna [17] suggested a model that treats the cross section of the suspension bridge as a rod suspended from springs at both sides, and is free to move vertically and to rotate about its center of gravity. [17, 18] simulated the sudden transition from vertical to torsional oscillations, which was the crucial event in the Tacoma Narrows Bridge collapse. Arioli and Gazzola [9] suggested a multiple rods model for the oscillations of suspension bridges, and used the stability of a fixed point of a suitable Poincaré map to provide a new explanation for the sudden appearance of torsional oscillations.

The above works made full use of the bridge deck to study deformations and oscillations of suspension bridges. Considering the suspension cable into the whole mechanical structure, Lazer and McKenna [7] proposed the following coupled beam-string model:

$$\begin{cases} m_1 u_{tt} + K u_{xxxx} + \mu_1 u_t + \lambda(u-v)^+ = h_1(x), \\ m_2 v_{tt} - G v_{xx} + \mu_2 u_t - \lambda(u-v)^+ = \varepsilon h_2, \end{cases}$$

and established the existence and multiplicity of periodic solutions. Ahmed and Harbi [6] investigated several variations of the model

$$\begin{cases} m_1 u_{tt} + K u_{xxxx} + \Phi(u-v) = h_1, \\ m_2 v_{tt} - G v_{xx} - \Phi(u-v) = h_2. \end{cases}$$

They dealt with the stability and dynamic behavior of the solutions, and provided simulation results and physical interpretations. Dell'Oro et al. [13] studied the linear system

$$\begin{cases} u_{tt} + u_{xxxx} + \mu_1 u_t + \lambda(u-v) + \omega(u_t - v_t) = 0, \\ \varepsilon v_{tt} - v_{xx} - \lambda(u-v) - \omega(u_t - v_t) = 0, \end{cases}$$

and discussed the decay of the solution semigroup in dependence of the nonnegative parameters λ and ω.

An attractor is an effective way to describe the long-time dynamics of solutions to nonlinear evolution equations. In autonomous infinite-dimensional dynamical systems, the existence of a global attractor can be derived by verifying the existence of a bounded absorbing set and the compactness of the semigroup. The commonly used compactness criterions mainly include: uniform compactness [30], asymptotic compactness [31, 32], asymptotic smoothness [33], and Condition (C) [34], which could be chosen according to the characteristics of the problem under consideration. Ma and Zhong [16] studied a coupled beam-string system:

$$\begin{cases} u_{tt} + K u_{xxxx} + \mu_1 u_t + \lambda(u-v)^+ + f_1(u) = h_1(x), \\ v_{tt} - G v_{xx} + \mu_2 v_t - \lambda(u-v)^+ + f_2(v) = h_2(x). \end{cases}$$

They obtained the global existence and uniqueness of solutions, and derived the existence of a global attractor by verifying the existence of a bounded absorbing set and Condition (C). Taking into account the midplane stretching of the bridge deck due to its elongation, Bochicchio et al. [12] handled the following system:

$$\begin{cases} u_{tt} + u_{xxxx} + \left(p - \int_0^l u_x^2 \, \mathrm{d}x\right) u_{xx} + u_t + \lambda(u-v)^+ = h_1(x), \\ v_{tt} - G v_{xx} + v_t - \lambda(u-v)^+ = h_2(x). \end{cases}$$

They proved the existence of a global attractor with optimal regularity by verifying the existence of a bounded absorbing set and asymptotic compactness, and gave its characterization in terms of the corresponding stationary problem. Subsequently, Aouadi [8] studied the following system with fractional damping:

$$\begin{cases} m_1 u_{tt} + K u_{xxxx} + \left(p - \epsilon \int_0^l u_x^2 \, \mathrm{d}x\right) u_{xx} + (-\partial_{xx})^r u_t + \lambda(u-v)^+ + f_1(u,v) = 0, \\ m_2 v_{tt} - G v_{xx} + (-\partial_{xx})^r v_t - \lambda(u-v)^+ + f_2(u,v) = 0, \end{cases}$$

where $0 < \epsilon \leq 1$ is a perturbed parameter, and $0 < r < 1$ is a fractional exponent. They obtained the global well-posedness and regularity of mild solutions, and derived the existence of a global attractor with finite fractal dimension by verifying the existence of a bounded absorbing set and asymptotic smoothness. Moreover, they analyzed the upper-semicontinuity of global attractors in terms of ε and r, respectively.

The purpose of the present paper is to discuss the long-time dynamics for system (1.1). The main features are summarized as follows.

First of all, from the perspective of the restraining force Φ, system (1.1) actually includes two models. The physical meaning of the nonlinear case $\Phi(s) := \lambda s^+$ is obvious, while the linear case $\Phi(s) := \lambda s$ is what is desirable for engineering structures (see [6]). We are able to handle the two cases simultaneously.

Second, our model is more realistic. Modern suspension cables are generally made of high-strength steel wires with multiple strands. As is well known, alloy materials are not absolute elastic solids, and their internal structure has a certain degree of viscoelasticity. In the vibration process or under high temperature, the viscoelasticity of alloy materials is dominant compared with the elasticity. Therefore, for the reality of the model, we consider the viscoelasticity of the suspension cable in the oscillations of suspension bridges. Regarding the suspension cable as the string, we can understand the appearance of Eq. $(1.1)_2$, and this type of viscoelastic equation has been widely investigated (see, e.g., [35–43] and the references therein).

Last but not least, the present paper aims to employ the gradient property and quasi-stability of the dynamical system discussed by Chueshov and Lasiecka [44] to handle the existence of a global attractor with finite fractal dimension for system (1.1). In the gradient dynamical systems, the quasi-stability can conveniently induce the asymptotic smoothness, which further allows us to readily obtain the existence of the global attractor. In this process, without additional efforts, the topological structure and finite fractal dimension of the global attractor can be obtained along with the existence. Moreover, it is unnecessary to verify the existence of a bounded absorbing set.

The rest of this paper is organized as follows. In Section 2, we first transform (1.1)–(1.3) into an equivalent problem. Moreover, we display some notations and assumptions on the memory kernel g and the nonlinear source terms f_1 and f_2. Finally, we state the main results of our paper. In Section 3, we prove the global well-posedness and regularity of mild solutions, namely, our first main result. In Section 4, we prove the existence of a global attractor with finite fractal dimension, namely, the other main result.

2. Preliminaries and main results

2.1. Reformulation of the problem

Since the results of this paper are independent of the coefficients in system (1.1), we take all the coefficients as 1 except for $G(0) = 1 + \int_0^\infty g(t)\,\mathrm{d}t$ for the sake of convenience.

Regarding the evolution equations with memory like $(1.1)_2$, to obtain a solution semigroup, the so-called history space framework suggests to introduce an auxiliary variable as an additional component of the phase space so that the problem under consideration could be turned into an autonomous system. The pioneering idea goes back to Dafermos [37]. Here, also following [35, 36, 39, 42, 43], we define an auxiliary variable

$$w := w^t(x,\tau) = v(x,t) - v(x,t-\tau), \quad x \in \Omega,\ \tau > 0,\ t \geq 0. \tag{2.1}$$

Then the memory term in $(1.1)_2$ can be written in the form

$$\begin{aligned}\int_{-\infty}^t g(t-\tau)v_{xx}(\tau)\,\mathrm{d}\tau &= \int_0^\infty g(\tau)v_{xx}(t-\tau)\,\mathrm{d}\tau \\ &= (G(0)-1)v_{xx} - \int_0^\infty g(\tau)w_{xx}(\tau)\,\mathrm{d}\tau.\end{aligned}$$

Consequently, (1.1) is transformed into the following equivalent system:

$$\begin{cases} u_{tt} + u_{xxxx} + u_t + \Phi(u-v) + f_1(u,v) = h_1(x), \quad x \in (0,l),\ t > 0, \\ v_{tt} - v_{xx} - \displaystyle\int_0^\infty g(\tau)w_{xx}(\tau)\,\mathrm{d}\tau + v_t - \Phi(u-v) + f_2(u,v) = h_2(x), \quad x \in (0,l),\ t > 0, \\ w_t = v_t - w_\tau, \quad x \in (0,l),\ \tau > 0,\ t > 0, \end{cases} \tag{2.2}$$

with boundary conditions

$$\begin{cases} u(0,t) = u(l,t) = u_{xx}(0,t) = u_{xx}(l,t) = 0, \quad t > 0, \\ v(0,t) = v(l,t) = 0, \quad t > 0, \\ w^t(0,\tau) = w^t(l,\tau) = 0, \quad \tau > 0,\ t > 0, \end{cases} \tag{2.3}$$

and initial conditions

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \quad x \in (0,l), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x), \quad x \in (0,l), \\ w^0(x,\tau) = w_0(x,\tau), \quad x \in (0,l),\ \tau > 0, \end{cases} \tag{2.4}$$

where

$$\begin{aligned} &v_0(x) := v_0(x,0), \quad x \in (0,l), \\ &w_0(x,\tau) := v_0(x,0) - v_0(x,-\tau), \quad x \in (0,l),\ \tau > 0. \end{aligned}$$

2.2. Notations and assumptions

Throughout the paper, for the sake of simplicity, we denote

$$\|\cdot\|_p := \|\cdot\|_{L^p(0,l)}, \quad \|\cdot\| := \|\cdot\|_2.$$

Moreover, $\langle\cdot,\cdot\rangle$ stands for the L^2-inner product, $\langle\cdot,\cdot\rangle_*$ denotes a duality pairing between a space and its dual space, C represents a generic positive constant that may be different even in the same formula, $C(\cdot,\cdots,\cdot)$ stands for a positive constant depending on the quantities appearing in the parentheses, and $\mathfrak{C}_1$, $\mathfrak{C}_2$ represent the embedding constants for inequalities

$$\|u\| \le \mathfrak{C}_1\|u_x\|, \quad \|u\| \le \mathfrak{C}_2\|u_{xx}\|.$$

Now we give the following assumptions on the memory kernel g.

(A_1): $g \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$, $g(t) \ge 0$, and $g'(t) \le 0$ for all $t \in [0,\infty)$, and

$$k := \int_0^\infty g(t)\,\mathrm{d}t.$$

Concerning the nonlinear source terms f_1 and f_2, we adopt the assumptions in [8] with a slight modification.

(A_2): There exist a function $F \in C^2(\mathbb{R}^2)$ and two constants $a > 0$, $p > 1$ such that $\nabla F = (f_1, f_2)$ and

$$|\nabla f_i(u,v)| \le a\left(|u|^{p-1} + |v|^{p-1} + 1\right), \quad u, v \in \mathbb{R}. \tag{2.5}$$

Moreover, there exist constants

$$0 \le \eta < \min\left\{\frac{1}{\mathfrak{C}_1^2}, \frac{1}{\mathfrak{C}_2^2}\right\}$$

and $b > 0$ such that

$$F(u,v) \ge -\frac{\eta}{2}\left(u^2 + v^2\right) - b \tag{2.6}$$

and

$$\nabla F(u,v)\cdot(u,v) \ge -\eta\left(u^2 + v^2\right) - b. \tag{2.7}$$

2.3. *Statement of main results*

We introduce a weighted L^2-space

$$L_g := L_g^2(\mathbb{R}^+; H_0^1(0,l)) = \left\{ w : \mathbb{R}^+ \to H_0^1(0,l) \,\middle|\, \int_0^\infty g(\tau)\|w_x(\tau)\|^2\,\mathrm{d}\tau < \infty \right\},$$

which is a Hilbert space endowed with the inner product

$$\langle w, \xi\rangle_g := \int_0^\infty g(\tau)\langle w_x(\tau), \xi_x(\tau)\rangle\,\mathrm{d}\tau$$

and the norm

$$\|w\|_g^2 := \int_0^\infty g(\tau)\|w_x(\tau)\|^2\,\mathrm{d}\tau.$$

In order to exhibit our main results, we define the phase space

$$Y := \left(H^2(0,l) \cap H_0^1(0,l)\right) \times H_0^1(0,l) \times L^2(0,l) \times L^2(0,l) \times L_g$$

with the norm

$$\|(u,v,\varphi,\phi,w)\|_Y^2 := \|u_{xx}\|^2 + \|v_x\|^2 + \|\varphi\|^2 + \|\phi\|^2 + \|w\|_g^2.$$

The main results of our paper are stated as follows.

Theorem 2.1 (Global well-posedness of mild solutions). *If* $(u_0, v_0, u_1, v_1, w_0) \in Y$, *then problem* (2.2)–(2.4) *admits a unique mild solution* $(u, v, u_t, v_t, w) \in C([0,\infty); Y)$ *depending continuously on the initial data. If* $(u_0, v_0, u_1, v_1, w_0) \in D(\mathcal{L})$, *then the mild solution has higher regularity* $(u, v, u_t, v_t, w) \in C([0,\infty); D(\mathcal{L}))$.

In Theorem 2.1, $D(\mathcal{L})$ will be stated in detail in Section 3.

Define an operator $S(t) : Y \to Y$ by

$$S(t)y_0 := (u(t), v(t), u_t(t), v_t(t), w^t), \quad y_0 := (u_0, v_0, u_1, v_1, w_0).$$

Then it is easy to see from Theorem 2.1 that $\{S(t)\}_{t\geq 0}$ is a C^0-semigroup generated by problem (2.2)–(2.4).

Theorem 2.2 (Existence of global attractors). *In addition to the assumptions of Theorem 2.1, suppose that there exists a constant* $\rho > 0$ *such that* $g'(t) + \rho g(t) \leq 0$ *for all* $t \in [0,\infty)$. *Then the dynamical system* $(Y, S(t))$ *corresponding to problem* (2.2)–(2.4) *possesses a compact global attractor* $A = \mathcal{M}^z(\mathcal{N})$ *with finite fractal dimension, where* $\mathcal{N}$ *is the set of stationary points of the dynamical system* $(Y, S(t))$, *that is,*

$$\mathcal{N} := \left\{ (u, v, 0, 0, 0) \,\middle|\, \begin{cases} u_{xxxx} + \Phi(u-v) + f_1(u,v) = h_1, \\ -v_{xx} - \Phi(u-v) + f_2(u,v) = h_2 \end{cases} \right\},$$

and $\mathcal{M}^z(\mathcal{N})$ *is an unstable manifold emanating from the set* $\mathcal{N}$ *as a set of all* $y_0 \in Y$ *such that there exists a full trajectory* $\{z(t) | t \in \mathbb{R}\}$ *with the properties* $z(0) = y_0$ *and* $\lim_{t\to-\infty} \mathrm{dist}_Y(z(t), \mathcal{N}) = 0$.

Remark 2.3 (Finite Hausdorff dimension). *From Theorem 2.2, we readily see that the global attractor for problem* (2.2)–(2.4) *has finite Hausdorff dimension. In fact, the Hausdorff dimension does not exceed the fractal one (see, e.g., [44, Section 7.3]).*

Remark 2.4 (Extensions of main results). *In order to more realistically show the beam-string model for the oscillations of suspension bridges, we restrict our attention to the one-dimensional case. Here we would like to mention that Theorems 2.1 and 2.2 can be extended to the higher-dimensional case by adjusting the growth exponent* p *in* (A_2). *In addition, even if the restraining force* $\Phi(u-v)$ *is replaced by a nonlocal one* $a(x)\Phi(u-v)$ *in system* (2.2), *Theorems 2.1 and 2.2 remain valid, provided the function* $a(x) \geq 0$ *is bounded measurable.*

3. Proof of Theorem 2.1

As in [36, 39, 40, 42, 43], we consider a linear operator $\mathcal{T} : D(\mathcal{T}) \subset L_g \to L_g$ given by $\mathcal{T}w := -w_\tau$, which is the infinitesimal generator of a C^0-semigroup, where domain

$$D(\mathcal{T}) := \{w \in L_g | \mathcal{T}w \in L_g, \ w(0) = 0\}.$$

In this section, we denote $y(t) := (u(t), v(t), \varphi(t), \phi(t), w^t)$ with $\varphi(t) := u_t(t)$ and $\phi(t) := v_t(t)$. Then problem (2.2)–(2.4) can be rewritten as the following equivalent Cauchy problem:

$$\frac{\mathrm{d}}{\mathrm{d}t} y = \mathcal{L}y + \mathcal{F}(y), \quad t > 0, \tag{3.1}$$

$$y(0) = y_0, \tag{3.2}$$

where the linear operator $\mathcal{L} : D(\mathcal{L}) \subset Y \to Y$ is defined by

$$\mathcal{L}y := \begin{bmatrix} \varphi \\ \phi \\ -u_{xxxx} - \varphi \\ v_{xx} + \displaystyle\int_0^\infty g(\tau) w_{xx}(\tau)\, \mathrm{d}\tau - \phi \\ \mathcal{T}w + \phi \end{bmatrix}$$

with domain

$$D(\mathcal{L}) := \Big\{ y \in Y \Big| \varphi \in H^2(0,l) \cap H_0^1(0,l),\ \phi \in H_0^1(0,l),\ u_{xxxx} \in L^2(0,l),\\ v_{xx} + \int_0^\infty g(\tau) w_{xx}(\tau)\, \mathrm{d}\tau \in L^2(0,l),\ w \in D(\mathcal{T}) \Big\},$$

and $\mathcal{F} : Y \to Y$ is defined by

$$\mathcal{F}(y) := \begin{bmatrix} 0 \\ 0 \\ -\Phi(u-v) - f_1(u,v) + h_1 \\ \Phi(u-v) - f_2(u,v) + h_2 \\ 0 \end{bmatrix}. \tag{3.3}$$

Definition 3.1 (Mild solutions). *If $\mathcal{L}$ is the infinitesimal generator of a C^0-semigroup of contractions $e^{t\mathcal{L}}$ on Y, and $y_0 \in Y$, then the function $y \in C([0,T]; Y)$ given by*

$$y(t) = e^{t\mathcal{L}} y_0 + \int_0^t e^{(t-\tau)\mathcal{L}} \mathcal{F}(y(\tau))\, \mathrm{d}\tau \tag{3.4}$$

is called a mild solution to problem (3.1)–(3.2) *on* $[0,T)$. *Here, T is the maximum existence time of the solution.*

Remark 3.2 (Relationship between mild and weak solutions). *According to [45], any mild solution to problem* (3.1)–(3.2) *is also a weak solution, i.e., a solution satisfies* (3.1) *in the sense of distribution. The concepts of these two solutions are equivalent when $\mathcal{F} \equiv 0$.*

In the sequel we shall apply the abstract results [46, and Theorems 6.1.4 and 6.1.5] to prove local existence and uniqueness of mild solutions to problem (3.1)–(3.2). To this end, we first verify the conditions for these two abstract results. Thus we need to demonstrate that $\mathcal{L}$ is the infinitesimal generator of a C^0-semigroup of contractions $e^{t\mathcal{L}}$ on Y, and $\mathcal{F}$ is locally Lipschitz.

Lemma 3.3. *The operator $\mathcal{L}$ is the infinitesimal generator of a C^0-semigroup of contractions $e^{t\mathcal{L}}$ on Y.*

Proof. A direct calculation yields

$$\langle \mathcal{L}y, y \rangle_Y = -\|\varphi\|^2 - \|\phi\|^2 + \langle \mathcal{T}w, w \rangle_g \tag{3.5}$$

for all $y \in D(\mathcal{L})$. For the third term on the right-hand side of (3.5), we have

$$\langle \mathcal{T} w, w\rangle_g = -\frac{1}{2}\int_0^\infty \frac{\partial}{\partial\tau}\left(g(\tau)\|w_x(\tau)\|^2\right) \mathrm{d}\tau + \frac{1}{2}\int_0^\infty g'(\tau)\|w_x(\tau)\|^2 \,\mathrm{d}\tau.$$

In view of (2.1), we have $\lim_{\tau\to 0}\|w_x(\tau)\|^2 = 0$. From ($A_1$) we are in a position to get $\lim_{\tau\to\infty} g(\tau) = 0$. Hence

$$\langle \mathcal{T} w, w\rangle_g = \frac{1}{2}\int_0^\infty g'(\tau)\|w_x(\tau)\|^2 \,\mathrm{d}\tau.$$

Again from (A_1) we have

$$\langle \mathcal{T} w, w\rangle_g \le 0. \tag{3.6}$$

Thus we infer from (3.5) and (3.6) that $\langle \mathcal{L}y, y\rangle_Y \le 0$, which shows that $\mathcal{L}$ is dissipative in Y.

Next we prove that $\mathcal{L}$ is maximal. To achieve this, it suffices to show that, for any $\hat{y} = (\hat{u}, \hat{v}, \hat{\varphi}, \hat{\phi}, \hat{w}) \in Y$, there exists a solution $y \in D(\mathcal{L})$ to $(I - \mathcal{L})y = \hat{y}$, i.e.,

$$\begin{cases} u - \varphi = \hat{u}, \\ v - \phi = \hat{v}, \\ 2\varphi + u_{xxxx} = \hat{\varphi}, \\ 2\phi - v_{xx} - \displaystyle\int_0^\infty g(\tau)w_{xx}(\tau)\,\mathrm{d}\tau = \hat{\phi}, \\ w - \mathcal{T} w - \phi = \hat{w}. \end{cases} \tag{3.7}$$

Multiplying $(3.7)_5$ by e^τ and integrating over $[0, \tau]$, we deduce from $w(0) = 0$ that

$$w(\tau) = \left(1 - e^{-\tau}\right)\phi + \int_0^\tau e^{s-\tau}\hat{w}(s)\,\mathrm{d}s. \tag{3.8}$$

Using $(3.7)_1$ in $(3.7)_3$ and substituting (3.8) and $(3.7)_2$ into $(3.7)_4$, we obtain

$$\begin{cases} 2u + u_{xxxx} = \vartheta_1, \\ 2v - m v_{xx} = \vartheta_2, \end{cases} \tag{3.9}$$

where

$$\vartheta_1 := \hat{\varphi} + 2\hat{u} \in L^2(0, l),$$

$$\vartheta_2 := \hat{\phi} + 2\hat{v} - \int_0^\infty g(\tau)\left(1 - e^{-\tau}\right)\mathrm{d}\tau\hat{v}_{xx} + \int_0^\infty\int_0^\tau g(\tau)e^{s-\tau}\hat{w}_{xx}(s)\,\mathrm{d}s\mathrm{d}\tau$$

and

$$m := 1 + \int_0^\infty g(\tau)\left(1 - e^{-\tau}\right)\mathrm{d}\tau.$$

In order to ensure that the Lax-Milgram theorem can be applied to show the existence of $y \in D(\mathcal{L})$, we now prove $\vartheta_2 \in H^{-1}(0, l)$, where $H^{-1}(0, l)$ is the dual space of $H_0^1(0, l)$. Indeed, for any $\xi \in H_0^1(0, l)$ with $\|\xi_x\| \le 1$, we have

$$\left|\left\langle \int_0^\infty\int_0^\tau g(\tau)e^{s-\tau}\hat{w}_{xx}(s)\,\mathrm{d}s\mathrm{d}\tau, \xi\right\rangle_*\right| = \left|\int_0^\infty\int_0^\tau g(\tau)e^{s-\tau}\langle\hat{w}_x(s), \xi_x\rangle_*\,\mathrm{d}s\mathrm{d}\tau\right|$$

$$\leq \int_0^\infty \int_0^\tau g(\tau) e^{s-\tau} \|\hat{w}_x(s)\| \, \mathrm{d}s \mathrm{d}\tau$$
$$= \int_0^\infty \int_s^\infty g(\tau) e^{s-\tau} \|\hat{w}_x(s)\| \, \mathrm{d}\tau \mathrm{d}s.$$

Due to the fact that

$$\int_s^\infty g(\tau) e^{s-\tau} \, \mathrm{d}\tau \leq g(s),$$

we further deduce from Schwarz's inequality and (A_1) that

$$\left| \left\langle \int_0^\infty \int_0^\tau g(\tau) e^{s-\tau} \hat{w}_{xx}(s) \, \mathrm{d}s \mathrm{d}\tau, \xi \right\rangle_* \right| \leq \int_0^\infty g(s) \|\hat{w}_x(s)\| \, \mathrm{d}s \tag{3.10}$$
$$\leq k^{\frac{1}{2}} \|\hat{w}\|_g$$
$$< \infty.$$

This implies that

$$\int_0^\infty \int_0^\tau g(\tau) e^{s-\tau} \hat{w}_{xx}(s) \, \mathrm{d}s \mathrm{d}\tau \in H^{-1}(0, l).$$

Hence $\vartheta_2 \in H^{-1}(0, l)$.

To treat system (3.9), we consider a variational problem:

$$\mathcal{B}((u, v), (\bar{u}, \bar{v})) = \mathcal{G}((\bar{u}, \bar{v})),$$

where the bilinear form $\mathcal{B} : \left(\left(H^2(0, l) \cap H_0^1(0, l)\right) \times H_0^1(0, l)\right)^2 \to \mathbb{R}$ is defined by

$$\mathcal{B}((u, v), (\bar{u}, \bar{v})) := 2 \int_0^l (u\bar{u} + v\bar{v}) \, \mathrm{d}x + \int_0^l u_{xx} \bar{u}_{xx} \, \mathrm{d}x + m \int_0^l v_x \bar{v}_x \, \mathrm{d}x,$$

and the linear form $\mathcal{G} : \left(H^2(0, l) \cap H_0^1(0, l)\right) \times H_0^1(0, l) \to \mathbb{R}$ is defined by

$$\mathcal{G}(\bar{u}, \bar{v}) := \int_0^l \vartheta_1 \bar{u} \, \mathrm{d}x + \int_0^l \vartheta_2 \bar{v} \, \mathrm{d}x.$$

It is easy to check that $\mathcal{B}$ is continuous and coercive. Moreover, $\mathcal{G}$ is continuous. Therefore, according to the Lax-Milgram theorem, we infer that system (3.9) admits a unique weak solution $(u, v) \in \left(H^2(0, l) \cap H_0^1(0, l)\right) \times H_0^1(0, l)$. From $(3.7)_1$–$(3.7)_4$ we further have $\varphi \in H^2(0, l) \cap H_0^1(0, l)$, $\phi \in H_0^1(0, l)$, $u_{xxxx} \in L^2(0, l)$, and

$$v_{xx} + \int_0^\infty g(\tau) w_{xx}(\tau) \, \mathrm{d}\tau \in L^2(0, l).$$

In view of (3.8), it follows that

$$\|w_x(\tau)\|^2 \leq 2\|\phi_x\|^2 + 2 \int_0^\tau e^{s-\tau} \|\hat{w}_x(s)\|^2 \, \mathrm{d}s.$$

Hence, by the arguments similar to the proof of (3.10), we can derive

$$\begin{aligned}\int_0^{\infty} g(\tau)\|w_x(\tau)\|^2 \,\mathrm{d}\tau &\le 2k\|\phi_x\|^2 + 2\int_0^{\infty}\int_0^{\tau} g(\tau)e^{s-\tau}\|\hat{w}_x(s)\|^2 \,\mathrm{d}s\mathrm{d}\tau \\ &\le 2k\|\phi_x\|^2 + 2\|\hat{w}\|_g^2 \\ &< \infty,\end{aligned}$$

which means $w \in L_g$. Thus we infer from $(3.7)_5$ that $\mathcal{T}w \in L_g$, and so $w \in D(\mathcal{T})$. As a result, we have demonstrated that there exists $y \in D(\mathcal{L})$ satisfying $(I - \mathcal{L})y = \hat{y}$. Then this lemma immediately follows from the Lumer-Phillips theorem [46].

Concerning the operator $\mathcal{F}$ given by (3.3), we have the following conclusion.

Lemma 3.4. *The operator $\mathcal{F}$ is locally Lipschitz.*

Proof. Let $\|y\|_Y, \|\bar{y}\|_Y \le R$, where $\|\bar{y}\|_Y^2 := \|\bar{u}_{xx}\|^2 + \|\bar{v}_x\|^2 + \|\bar{\varphi}\|^2 + \|\bar{\phi}\|^2 + \|\bar{w}\|_g^2$, $\bar{y} := (\bar{u}, \bar{v}, \bar{\varphi}, \bar{\phi}, \bar{w})$, and $R > 0$. Then

$$\begin{aligned}\|\mathcal{F}(\bar{y}) - \mathcal{F}(y)\|_Y^2 &= \int_0^l |\Phi(\bar{u},\bar{v}) - \Phi(u,v) + f_1(\bar{u},\bar{v}) - f_1(u,v)|^2 \,\mathrm{d}x \\ &\quad + \int_0^l |\Phi(\bar{u},\bar{v}) - \Phi(u,v) - f_2(\bar{u},\bar{v}) + f_2(u,v)|^2 \,\mathrm{d}x \\ &\le 2\Big(2\int_0^l |\Phi(\bar{u},\bar{v}) - \Phi(u,v)|^2 \,\mathrm{d}x + \int_0^l |f_1(\bar{u},\bar{v}) - f_1(u,v)|^2 \,\mathrm{d}x \\ &\quad + \int_0^l |f_2(\bar{u},\bar{v}) - f_2(u,v)|^2 \,\mathrm{d}x\Big).\end{aligned} \tag{3.11}$$

We claim that

$$|\Phi(\bar{u} - \bar{v}) - \Phi(u - v)|^2 \le |(\bar{u} - u) - (\bar{v} - v)|^2. \tag{3.12}$$

Indeed, in the case where $\Phi(s) := s^+$, we have

$$\begin{aligned}|\Phi(\bar{u} - \bar{v}) - \Phi(u - v)|^2 &\le |((\bar{u} - u) - (\bar{v} - v))^+|^2 \\ &\le |(\bar{u} - u) - (\bar{v} - v)|^2.\end{aligned}$$

In the case where $\Phi(s) := s$, (3.12) remains valid.

By virtue of (2.5) in (A_2), there exist constants $0 < \theta_i < 1$ $(i = 1, 2)$ such that

$$\begin{aligned}|f_i(\bar{u},\bar{v}) - f_i(u,v)|^2 &= |\nabla f_i(\theta_i(\bar{u},\bar{v}) + (1-\theta_i)(u,v))|^2|(\bar{u},\bar{v}) - (u,v)|^2 \\ &\le C\left(|u|^{2p-2} + |v|^{2p-2} + |\bar{u}|^{2p-2} + |\bar{v}|^{2p-2} + 1\right)\left(|\bar{u} - u|^2 + |\bar{v} - v|^2\right).\end{aligned} \tag{3.13}$$

Inserting (3.12) and (3.13) into (3.11), we get

$$\|\mathcal{F}(\bar{y}) - \mathcal{F}(y)\|_Y^2 \le C\int_0^l \left(|u|^{2p-2} + |v|^{2p-2} + |\bar{u}|^{2p-2} + |\bar{v}|^{2p-2} + 1\right)\left(|\bar{u} - u|^2 + |\bar{v} - v|^2\right) \mathrm{d}x.$$

Hence, by applying Hölder's inequality with $(p-1)/p + 1/p = 1$, we have

$$\|\mathcal{F}(\bar{y}) - \mathcal{F}(y)\|_Y^2 \le C\left(\|u\|_{2p}^{2p-2} + \|v\|_{2p}^{2p-2} + \|\bar{u}\|_{2p}^{2p-2} + \|\bar{v}\|_{2p}^{2p-2} + 1\right)\left(\|\bar{u} - u\|_{2p}^2 + \|\bar{v} - v\|_{2p}^2\right).$$

Using the Sobolev inequalities for the embeddings $\left(H^2(0,l) \cap H_0^1(0,l)\right) \hookrightarrow L^{2p}(0,l)$ and $H_0^1(0,l) \hookrightarrow L^{2p}(0,l)$, we can obtain

$$\begin{aligned} \|\mathcal{F}(\bar{y}) - \mathcal{F}(y)\|_Y^2 \le\ & C\left(\|u_{xx}\|^{2p-2} + \|v_x\|^{2p-2} + \|\bar{u}_{xx}\|^{2p-2} + \|\bar{v}_x\|^{2p-2} + 1\right) \\ & \cdot \left(\|\bar{u}_{xx} - u_{xx}\|^2 + \|\bar{v}_x - v_x\|^2\right). \end{aligned} \tag{3.14}$$

Therefore,

$$\begin{aligned} \|\mathcal{F}(\bar{y}) - \mathcal{F}(y)\|_Y^2 &\le C\left(\|y\|_Y^{2p-2} + \|\bar{y}\|_Y^{2p-2} + 1\right)\|\bar{y} - y\|_Y^2 \\ &\le C(R)\|\bar{y} - y\|_Y^2. \end{aligned}$$

The proof of this lemma is complete.

Now we define the total energy function associated with problem (2.2)–(2.4):

$$\begin{aligned} E(t) :=\ & \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|v_t(t)\|^2 + \frac{1}{2}\|u_{xx}(t)\|^2 + \frac{1}{2}\|v_x(t)\|^2 + \frac{1}{2}\|w^t\|_g^2 + \frac{1}{2}\|\Phi(u(t) - v(t))\|^2 \\ & + \int_0^l F(u(t), v(t))\,\mathrm{d}x - \int_0^l (h_1 u(t) + h_2 v(t))\,\mathrm{d}x, \quad t \in [0,T). \end{aligned} \tag{3.15}$$

The following lemma provides the properties of $E(t)$.

Lemma 3.5. *Let $y \in C([0,T); D(\mathcal{L}))$ be a mild solution to problem* (2.2)–(2.4) *with $y_0 \in D(\mathcal{L})$. Then $E(t)$ is non-increasing for all $t \in [0,T)$, and*

$$E'(t) = -\|u_t(t)\|^2 - \|v_t(t)\|^2 + \frac{1}{2}\int_0^\infty g'(\tau)\|w_x^t(\tau)\|^2\,\mathrm{d}\tau. \tag{3.16}$$

Moreover, there exist constants $M_i > 0$ $(i = 1,2,3,4)$ such that for all $t \in [0,T)$,

$$\begin{aligned} E(t) \ge\ & \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|v_t(t)\|^2 + M_1\|u_{xx}(t)\|^2 + M_2\|v_x(t)\|^2 + \frac{1}{2}\|w^t\|_g^2 \\ & + \frac{1}{2}\|\Phi(u(t) - v(t))\|^2 - M_3\|h_1\|^2 - M_4\|h_2\|^2 - bl. \end{aligned} \tag{3.17}$$

Proof. We multiply $(2.2)_1$ by $u_t(t)$ and $(2.2)_2$ by $v_t(t)$, respectively. Then, by integrating by parts over $(0,l)$ and using $(2.2)_3$, it can be seen that (3.16) holds.

For the third term on the right-hand side of (3.16), we see from (A_1) that

$$\int_0^\infty g'(\tau)\|w_x^t(\tau)\|^2\,\mathrm{d}\tau \le 0.$$

Hence $E'(t) \le 0$, which means that $E(t)$ is non-increasing for all $t \in [0,T)$.

Next we prove (3.17). Concerning the seventh term on the right-hand side of (3.15), it follows from (2.6) in (A_2) that

$$\begin{aligned}\int_0^l F(u(t), v(t))\,\mathrm{d}x &\geq -\frac{\eta}{2}\left(\|u(t)\|^2 + \|v(t)\|^2\right) - bl \\ &\geq -\frac{\eta\mathfrak{C}_2^2}{2}\|u_{xx}(t)\|^2 - \frac{\eta\mathfrak{C}_1^2}{2}\|v_x(t)\|^2 - bl. \end{aligned} \tag{3.18}$$

For the last term on the right-hand side of (3.15), we deduce from Schwarz's inequality and Cauchy's inequalities with $\epsilon_1, \epsilon_2 > 0$ that

$$\begin{aligned} -\int_0^l (h_1 u(t) + h_2 v(t))\,\mathrm{d}x &\geq -\|h_1\|\|u(t)\| - \|h_2\|\|v(t)\| \\ &\geq -\epsilon_1\mathfrak{C}_2^2\|u_{xx}(t)\|^2 - \frac{1}{4\epsilon_1}\|h_1\|^2 - \epsilon_2\mathfrak{C}_1^2\|v_x(t)\|^2 - \frac{1}{4\epsilon_2}\|h_2\|^2. \end{aligned} \tag{3.19}$$

Consequently, by choosing sufficiently small ϵ_1 and ϵ_2 such that

$$\begin{aligned} M_1 &:= \frac{1}{2} - \frac{\eta\mathfrak{C}_2^2}{2} - \epsilon_1\mathfrak{C}_2^2 > 0, \\ M_2 &:= \frac{1}{2} - \frac{\eta\mathfrak{C}_1^2}{2} - \epsilon_2\mathfrak{C}_1^2 > 0, \\ M_3 &:= \frac{1}{4\epsilon_1}, \quad M_4 := \frac{1}{4\epsilon_2}, \end{aligned}$$

estimate (3.17) follows from (3.15), (3.18), and (3.19).

Proof of Theorem 2.1. According to [46, Theorem 6.1.4], and Lemmas 3.3 and 3.4, problem (3.1)–(3.2) with $y_0 \in Y$ admits a unique mild solution $y \in C([0,T); Y)$. In addition, we learn from [46, Theorem 6.1.5] that if $y_0 \in D(\mathcal{L})$, then the mild solution has higher regularity $y \in C([0,T); D(\mathcal{L}))$.

For the solution with higher regularity, we infer from Lemma 3.5 that $E(t) \leq E(0)$ for all $t \in [0,T)$. Hence we conclude from (3.17) in Lemma 3.5 that

$$\|y(t)\|_Y^2 \leq C\left(E(0) + M_3\|h_1\|^2 + M_4\|h_2\|^2 + bl\right) \tag{3.20}$$

for all $t \in [0,T)$. Thus, by the continuation principle, we have $T = \infty$, i.e., the solution is global. By the density arguments [47–49], estimate (3.20) still holds for the mild solution $y \in C([0,T); Y)$ to problem (3.1)–(3.2), and so the mild solution is also global.

Next we prove the continuous dependence of solutions on the initial data. Suppose that y and $\bar{y}$ are two mild solutions to problem (3.1)–(3.2) with the initial data $y_0, \bar{y}_0 \in Y$, respectively. From (3.4) we have

$$\|\bar{y}(t) - y(t)\|_Y \leq \|e^{t\mathcal{L}}(\bar{y}_0 - y_0)\|_Y + \int_0^t \|e^{(t-\tau)\mathcal{L}}(\mathcal{F}(\bar{y}(\tau)) - \mathcal{F}(y(\tau)))\|_Y\,\mathrm{d}\tau.$$

By Lemma 3.4 we can get

$$\|\bar{y}(t) - y(t)\|_Y \leq \|\bar{y}_0 - y_0\|_Y + C\int_0^t \|\bar{y}(\tau) - y(\tau)\|_Y\,\mathrm{d}\tau,$$

which together with Gronwall's inequality gives

$$\|\bar{y}(t) - y(t)\|_Y \leq \left(1 + Cte^{Ct}\right) \|\bar{y}_0 - y_0\|_Y \tag{3.21}$$

for all $t \in [0, T_0]$ with any $T_0 > 0$. Thus

$$\|\bar{y}(t) - y(t)\|_Y \leq C(T_0)\|\bar{y}_0 - y_0\|_Y.$$

The proof of Theorem 2.1 is finished.

By the density arguments, we have the following corollary, which will be used in the next section.

Corollary 3.6. *Lemma 3.5 remains valid for the mild solution $y \in C([0, \infty); Y)$ to problem* (2.2)–(2.4).

4. Proof of Theorem 2.2

4.1. Technical approach

We shall employ the gradient property and quasi-stability of the dynamical system to perform the proof of Theorem 2.2. For the convenience of the reader, we first introduce several definitions and properties on the gradient dynamical system and the quasi-stability in [44], which will play a crucial role in the proof of Theorem 2.2.

Generally, in terms of [44, Definition 7.5.3], a gradient dynamical system is defined as follows.

Definition 4.1. *A dynamical system $(Y, S(t))$ is said to be gradient if there exists a strict Lyapunov functional L for $(Y, S(t))$ on the whole phase space Y, that is,*

(i) the function $t \mapsto L(S(t)y_0)$ is non-increasing for any $y_0 \in Y$;
(ii) the equation $L(S(t)y_0) = L(y_0)$ for all $t > 0$ and some $y_0 \in Y$ implies that $S(t)y_0 = y_0$ for all $t > 0$.

Under appropriate conditions, the existence and structure of global attractors for a gradient and asymptotically smooth dynamical system can be provided by [44, Corollary 7.5.7], namely, the following theorem.

Theorem 4.2. *Assume that $(Y, S(t))$ is a gradient and asymptotically smooth dynamical system, and its Lyapunov functional $L(\chi)$ is bounded from above on any bounded subset of Y. In addition, assume that the set $L_R := \{\chi \in Y | L(\chi) \leq R\}$ is bounded for every R. If the set $\mathcal{N}$ of stationary points of $(Y, S(t))$, that is,*

$$\mathcal{N} := \{v \in Y | S(t)v = v \text{ for all } t \geq 0\},$$

is bounded, then $(Y, S(t))$ possesses a compact global attractor $A = \mathcal{M}^u(\mathcal{N})$.

In order to better introduce the quasi-stability of a dynamical system, we display the following assumptions.

(A): Let U, V, and W be three reflexive Banach spaces with U compactly embedded in V. We endow the space $Y := U \times V \times W$ with the norm

$$\|(u(t), u_t(t), v(t))\|_Y^2 := \|(u(t)\|_U^2 + \|u_t(t)\|_V^2 + \|v(t)\|_W^2, \quad (u(t), u_t(t), v(t)) \in Y.$$

We assume that $(Y, S(t))$ is a dynamical system on Y with the evolution operator of the form

$$S(t)y_0 := (u(t), u_t(t), v(t)), \quad y_0 := (u_0, u_1, v_0) \in Y,$$

where

$$u \in C(\mathbb{R}^+; U) \cap C^1(\mathbb{R}^+; V), \quad v \in C(\mathbb{R}^+; W).$$

The definition of the quasi-stability of a dynamical system is given by [44, Definition 7.9.2], namely, the following definition.

Definition 4.3. *The dynamical system* $(Y, S(t))$ *satisfying* (A) *is said to be quasi-stable on a set* $B \subset Y$ *if there exist a compact seminorm* $n_U(\cdot)$ *on the space* U *and non-negative functions* $\varsigma_i(t)$ $(i = 1, 2, 3)$ *such that*

(i) $\varsigma_1(t)$ *and* $\varsigma_3(t)$ *are locally bounded on* $[0, \infty)$*;*
(ii) $\varsigma_2 \in L^1(\mathbb{R}^+)$ *and* $\lim_{t\to\infty} \varsigma_2(t) = 0$*;*
(iii) the following relations,

$$\|S(t)\bar{y}_0 - S(t)y_0\|_Y^2 \le \varsigma_1(t)\|\bar{y}_0 - y_0\|_Y^2 \tag{4.1}$$

and

$$\|S(t)\bar{y}_0 - S(t)y_0\|_Y^2 \le \varsigma_2(t)\|\bar{y}_0 - y_0\|_Y^2 + \varsigma_3(t) \sup_{0<s<t} (n_U(\bar{u}(s) - u(s)))^2, \tag{4.2}$$

hold for every $y_0, \bar{y}_0 \in B$ *and* $t > 0$*, where* $S(t)\bar{y}_0 := (\bar{u}(t), \bar{u}_t(t), \bar{v}(t))$ *and* $\bar{y}_0 := (\bar{u}_0, \bar{u}_1, \bar{v}_0)$*.*

A quasi-stable dynamical system possesses the following properties from [44, Proposition 7.9.4, and Theorem 7.9.6].

Proposition 4.4. *Let* (A) *be fulfilled. Assume that the dynamical system* $(Y, S(t))$ *is quasi-stable on every bounded positively invariant set* $B \subset Y$*. Then* $(Y, S(t))$ *is asymptotically smooth.*

Theorem 4.5. *Let* (A) *be fulfilled. Assume that the dynamical system* $(Y, S(t))$ *possesses a compact global attractor* A *and is quasi-stable on* A*. Then* A *has finite fractal dimension.*

4.2. Our proof

In order to verify that the dynamical system $(Y, S(t))$ corresponding to problem (2.2)–(2.4) is gradient, we need to seek a strict Lyapunov functional L in terms of Definition 4.1.

Lemma 4.6 (Gradient property)**.** *Under the assumptions of Theorem 2.1, the dynamical system* $(Y, S(t))$ *corresponding to problem* (2.2)–(2.4) *is gradient.*

Proof. For any $y \in Y$, we take $L(S(t)y)$ as $E(t)$. Then we see from Lemma 3.5 that $L(S(t)y)$ is non-increasing.

Let $L(S(t)y) = L(y)$ for all $t > 0$ and some $y \in Y$. Then, by Corollary 3.6, we can integrate (3.16) with respect to t from 0 to t to reach

$$E(t) + \int_0^t \left(\|u_s(s)\|^2 + \|v_s(s)\|^2\right) \mathrm{d}s - \frac{1}{2}\int_0^t \int_0^\infty g'(\tau)\|w_x^s(\tau)\|^2 \,\mathrm{d}\tau \,\mathrm{d}s = E(0).$$

Consequently,

$$\int_0^t \left(\|u_s(s)\|^2 + \|v_s(s)\|^2\right) \mathrm{d}s - \frac{1}{2}\int_0^t \int_0^\infty g'(\tau)\|w_x^s(\tau)\|^2 \,\mathrm{d}\tau \,\mathrm{d}s = 0,$$

which together with (A_1) gives

$$\int_0^t \left(\|u_s(s)\|^2 + \|v_s(s)\|^2\right) \mathrm{d}s \le 0.$$

Thus $u_t(t) = 0$ and $v_t(t) = 0$ for all $t \ge 0$, which implies that $u(t) = u_0$ and $v(t) = v_0$ for all $t \ge 0$. From (2.1) we further get $w^t = 0$ for all $t \ge 0$. Hence $(u(t), v(t), u_t(t), v_t(t), w^t) = (u_0, v_0, 0, 0, 0)$ for all $t \ge 0$, i.e., $S(t)y = y$ for all $t \ge 0$. By Definition 4.1 we easily see that $(Y, S(t))$ is gradient.

To show that the dynamical system $(Y, S(t))$ corresponding to problem (2.2)–(2.4) is quasi-stable, we first use the perturbed energy method employed by [8, 12, 27, 39, 49], with some modifications, to establish the following stabilizability estimate.

Lemma 4.7 (Stabilizability estimate)**.** *Under the assumptions of Theorem 2.2, for a given bounded set $B \subset Y$, there exist constants $\alpha, \beta > 0$ and $\sigma > 0$ depending on B such that*

$$\|S(t)\bar{y}_0 - S(t)y_0\|_Y^2 \le \alpha e^{-\beta t}\|\bar{y}_0 - y_0\|_Y^2 + \sigma \int_0^t e^{-\beta(t-s)} \left(\|\bar{u}(s) - u(s)\|_{2p}^2 + \|\bar{v}(s) - v(s)\|_{2p}^2\right) \mathrm{d}s \tag{4.3}$$

for every $y_0, \bar{y}_0 \in B$ and $t > 0$, where $S(t)\bar{y}_0 := (\bar{u}(t), \bar{v}(t), \bar{u}_t(t), \bar{v}_t(t), \bar{w}^t)$ and $\bar{y}_0 := (\bar{u}_0, \bar{v}_0, \bar{u}_1, \bar{v}_1, \bar{w}_0)$.

Proof. Set $\tilde{u} := \bar{u} - u$, $\tilde{v} := \bar{v} - v$, and $\tilde{w} := \bar{w} - w$. Then, by Remark 3.2, we know that for any $\xi_1 \in H^2(0, l) \cap H_0^1(0, l)$, $\xi_2 \in H_0^1(0, l)$, $\xi_3 \in L_g$, and a.e. $t > 0$, $(\tilde{u}, \tilde{v}, \tilde{u}_t, \tilde{v}_t, \tilde{w})$ satisfies

$$\begin{cases} \langle \tilde{u}_{tt}(t), \xi_1 \rangle_* + \langle \tilde{u}_{xx}(t), \xi_{1xx} \rangle + \langle \tilde{u}_t(t), \xi_1 \rangle + \langle \Phi(\bar{u}(t) - \bar{v}(t)) - \Phi(u(t) - v(t)), \xi_1 \rangle \\ \quad + \langle f_1(\bar{u}(t), \bar{v}(t)) - f_1(u(t), v(t)), \xi_1 \rangle = 0, \\ \langle \tilde{v}_{tt}(t), \xi_2 \rangle_* + \langle \tilde{v}_x(t), \xi_{2x} \rangle + \langle \tilde{w}^t, \xi_2 \rangle_g + \langle \tilde{v}_t(t), \xi_2 \rangle - \langle \Phi(\bar{u}(t) - \bar{v}(t)) - \Phi(u(t) - v(t)), \xi_2 \rangle \\ \quad + \langle f_2(\bar{u}(t), \bar{v}(t)) - f_2(u(t), v(t)), \xi_2 \rangle = 0, \\ \langle \tilde{w}_t^t, \xi_3 \rangle_g = \langle \tilde{v}_t(t), \xi_3 \rangle_g - (\tilde{w}_\tau^t, \xi_3 \rangle_g, \end{cases} \tag{4.4}$$

with

$$\tilde{u}(0) = \bar{u}_0 - u_0, \quad \tilde{v}(0) = \bar{v}_0 - v_0, \quad \tilde{w}^0 = \bar{w}_0 - w_0.$$

We write a part of the total energy function as

$$\widetilde{E}(t) := \|\tilde{u}_{xx}(t)\|^2 + \|\tilde{v}_x(t)\|^2 + \|\tilde{u}_t(t)\|^2 + \|\tilde{v}_t(t)\|^2 + \|\tilde{w}^t\|_g^2. \tag{4.5}$$

Meanwhile, we perform a suitable modification of $\widetilde{E}(t)$ as follows:

$$\Psi(t) := \widetilde{E}(t) + \varepsilon\psi(t), \tag{4.6}$$

where

$$\psi(t) := \langle \tilde{u}(t), \tilde{u}_t(t) \rangle + \langle \tilde{v}(t), \tilde{v}_t(t) \rangle,$$

and $\varepsilon > 0$ is a constant to be determined later.

We first claim that there exist two constants $\gamma_1, \gamma_2 > 0$, depending on ε, such that

$$\gamma_1 \widetilde{E}(t) \le \Psi(t) \le \gamma_2 \widetilde{E}(t). \tag{4.7}$$

To confirm this, we deduce from Schwarz's and Cauchy's inequalities that

$$\begin{aligned}|\psi(t)| &\le \|\tilde{u}(t)\|\|\tilde{u}_t(t)\| + \|\tilde{v}(t)\|\|\tilde{v}_t(t)\|\\ &\le \frac{1}{2}\|\tilde{u}(t)\|^2 + \frac{1}{2}\|\tilde{v}(t)\|^2 + \frac{1}{2}\|\tilde{u}_t(t)\|^2 + \frac{1}{2}\|\tilde{v}_t(t)\|^2\\ &\le \frac{\mathfrak{C}_2^2}{2}\|\tilde{u}_{xx}(t)\|^2 + \frac{\mathfrak{C}_1^2}{2}\|\tilde{v}_x(t)\|^2 + \frac{1}{2}\|\tilde{u}_t(t)\|^2 + \frac{1}{2}\|\tilde{v}_t(t)\|^2.\end{aligned}$$

Combining this inequality with (4.5), we infer that there exists a constant $Q > 0$ such that $|\psi(t)| \le Q\widetilde{E}(t)$, which together with (4.6) gives

$$(1 - \varepsilon Q)\widetilde{E}(t) \le \Psi(t) \le (1 + \varepsilon Q)\widetilde{E}(t).$$

Thus assertion (4.7) is demonstrated, and $\gamma_1 > 0$ will be guaranteed by the selection of ε later.

Next we claim that there exist constants $\gamma_3 > 0$ and $\gamma_4 > 0$ depending on B such that

$$\Psi'(t) \le -\gamma_3\widetilde{E}(t) + \gamma_4\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right). \tag{4.8}$$

Indeed, by (4.4), the arguments similar to the proof of (3.16), and the density arguments, we have

$$\widetilde{E}'(t) = -2\|\tilde{u}_t(t)\|^2 - 2\|\tilde{v}_t(t)\|^2 + \sum_{i=1}^{4} I_i, \tag{4.9}$$

where

$$I_1 := \int_0^\infty g'(\tau)\|\tilde{w}_x^t(\tau)\|^2\,\mathrm{d}\tau,$$

$$I_2 := -2\int_0^l (\Phi(\bar{u}(t) - \bar{v}(t)) - \Phi(u(t) - v(t)))(\tilde{u}_t(t) - \tilde{v}_t(t))\,\mathrm{d}x,$$

$$I_3 := -2\int_0^l (f_1(\bar{u}(t), \bar{v}(t)) - f_1(u(t), v(t)))\tilde{u}_t(t)\,\mathrm{d}x,$$

and

$$I_4 := -2\int_0^l (f_2(\bar{u}(t), \bar{v}(t)) - f_2(u(t), v(t)))\tilde{v}_t(t)\,\mathrm{d}x.$$

Concerning the term I_1, we deduce from the assumption $g'(t) + \rho g(t) \le 0$ that

$$I_1 \le -\rho\|\tilde{w}^t\|_g^2. \tag{4.10}$$

For the term I_2, we deduce from Schwarz's and Minkowski's inequalities and (3.12) that there exists a constant $\epsilon_1 > 0$ to be determined such that

$$\begin{aligned}I_2 &\le 2\|\tilde{u}(t) - \tilde{v}(t)\|(\|\tilde{u}_t(t)\| + \|\tilde{v}_t(t)\|)\\ &= \left(\frac{2}{(2\epsilon_1)^{\frac{1}{2}}}\|\tilde{u}(t) - \tilde{v}(t)\|\right)\left((2\epsilon_1)^{\frac{1}{2}}(\|\tilde{u}_t(t)\| + \|\tilde{v}_t(t)\|)\right).\end{aligned}$$

From Cauchy's inequality and $L^{2p}(0,l) \subset L^2(0,l)$ it follows that

$$\begin{aligned} I_2 &\le \frac{1}{\epsilon_1}(\|\tilde{u}(t) - \tilde{v}(t)\|)^2 + \epsilon_1(\|\tilde{u}_t(t)\| + \|\tilde{v}_t(t)\|)^2 \\ &\le C(\epsilon_1)\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right) + 2\epsilon_1\left(\|\tilde{u}_t(t)\|^2 + \|\tilde{v}_t(t)\|^2\right). \end{aligned} \tag{4.11}$$

Consequently, by taking $\epsilon_1 = 1/4$, we get

$$I_2 \le C\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right) + \frac{1}{2}\left(\|\tilde{u}_t(t)\|^2 + \|\tilde{v}_t(t)\|^2\right). \tag{4.12}$$

For the term I_3, it follows from Schwarz's and Cauchy's inequalities and the analogous arguments in the proof of (3.14) that

$$\begin{aligned} I_3 &\le 2\|f_1(\bar{u}(t), \bar{v}(t)) - f_1(u(t), v(t))\|\|\tilde{u}_t(t)\| \\ &\le 2\|f_1(\bar{u}(t), \bar{v}(t)) - f_1(u(t), v(t))\|^2 + \frac{1}{2}\|\tilde{u}_t(t)\|^2 \\ &\le C(B)\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right) + \frac{1}{2}\|\tilde{u}_t(t)\|^2. \end{aligned} \tag{4.13}$$

Likewise,

$$I_4 \le C(B)\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right) + \frac{1}{2}\|\tilde{v}_t(t)\|^2. \tag{4.14}$$

As a consequence, inserting (4.10)–(4.14) into (4.9), we arrive at

$$\widetilde{E}'(t) \le C(B)\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right) - \|\tilde{u}_t(t)\|^2 - \|\tilde{v}_t(t)\|^2 - \rho\|\tilde{w}^t\|_g^2. \tag{4.15}$$

We now turn to the estimates on $\psi'(t)$. Note that

$$\psi'(t) = \|\tilde{u}_t(t)\|^2 + \|\tilde{v}_t(t)\|^2 + \langle \tilde{u}_{tt}(t), \tilde{u}(t)\rangle_* + \langle \tilde{v}_{tt}(t), \tilde{v}(t)\rangle_*.$$

Taking $\xi_1 = \tilde{u}(t)$ in $(4.4)_1$ and $\xi_2 = \tilde{v}(t)$ in $(4.4)_2$, and adding the two results, we further obtain

$$\psi'(t) = \|\tilde{u}_t(t)\|^2 + \|\tilde{v}_t(t)\|^2 - \|\tilde{u}_{xx}(t)\|^2 - \|\tilde{v}_x(t)\|^2 + \sum_{i=5}^{9} I_i,$$

where

$$I_5 := -\langle \tilde{w}^t, \tilde{v}(t)\rangle_g,$$

$$I_6 := -\langle \tilde{u}_t(t), \tilde{u}(t)\rangle - \langle \tilde{v}_t(t), \tilde{v}(t)\rangle,$$

$$I_7 := -\int_0^l (\Phi(\bar{u}(t) - \bar{v}(t)) - \Phi(u(t) - v(t)))(\tilde{u}(t) - \tilde{v}(t))\,\mathrm{d}x,$$

$$I_8 := -\int_0^l (f_1(\bar{u}(t), \bar{v}(t)) - f_1(u(t), v(t)))\tilde{u}(t)\,\mathrm{d}x,$$

and

$$I_9 := -\int_0^l (f_2(\bar{u}(t), \bar{v}(t)) - f_2(u(t), v(t)))\tilde{v}(t)\,\mathrm{d}x.$$

Thus there exists a constant $0 < \theta < 1$ such that

$$\begin{aligned}\psi'(t) = &-\theta\widetilde{E}(t) + (1+\theta)\|\tilde{u}_t(t)\|^2 + (1+\theta)\|\tilde{v}_t(t)\|^2 - (1-\theta)\|\tilde{u}_{xx}(t)\|^2 \\ &- (1-\theta)\|\tilde{v}_x(t)\|^2 + \theta\|\tilde{w}^t\|_g^2 + \sum_{i=5}^{9} I_i.\end{aligned} \tag{4.16}$$

From Schwarz's and Cauchy's inequalities it follows that there exist constants ϵ_i $(i = 2, 3, 4)$ to be determined later such that

$$\begin{aligned}I_5 &\le \int_0^\infty g(\tau)\|\tilde{w}_x^t(\tau)\|\|\tilde{v}_x(t)\|\,\mathrm{d}\tau \\ &= \int_0^\infty \left(\frac{1}{(2\epsilon_2)^{\frac{1}{2}}} g^{\frac{1}{2}}(\tau)\|\tilde{w}_x^t(\tau)\|\right)\left((2\epsilon_2)^{\frac{1}{2}} g^{\frac{1}{2}}(\tau)\|\tilde{v}_x(t)\|\right)\mathrm{d}\tau \\ &\le \epsilon_2 k\|\tilde{v}_x(t)\|^2 + \frac{1}{4\epsilon_2}\|\tilde{w}^t\|_g^2\end{aligned} \tag{4.17}$$

and

$$\begin{aligned}I_6 &\le \|\tilde{u}(t)\|\|\tilde{u}_t(t)\| + \|\tilde{v}(t)\|\|\tilde{v}_t(t)\| \\ &= \left((2\epsilon_3)^{\frac{1}{2}}\|\tilde{u}(t)\|\right)\left(\frac{1}{(2\epsilon_3)^{\frac{1}{2}}}\|\tilde{u}_t(t)\|\right) + \left((2\epsilon_4)^{\frac{1}{2}}\|\tilde{v}(t)\|\right)\left(\frac{1}{(2\epsilon_4)^{\frac{1}{2}}}\|\tilde{v}_t(t)\|\right) \\ &\le \epsilon_3\mathfrak{C}_2^2\|\tilde{u}_{xx}(t)\|^2 + \frac{1}{4\epsilon_3}\|\tilde{u}_t(t)\|^2 + \epsilon_4\mathfrak{C}_1^2\|\tilde{v}_x(t)\|^2 + \frac{1}{4\epsilon_4}\|\tilde{v}_t(t)\|^2.\end{aligned} \tag{4.18}$$

By the arguments similar to the proof of (4.11), we infer that there exists a constant $\epsilon_5 > 0$ to be determined later such that

$$I_7 \le C(\epsilon_5)\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right) + 2\epsilon_5\mathfrak{C}_2^2\|\tilde{u}_{xx}(t)\|^2 + 2\epsilon_5\mathfrak{C}_1^2\|\tilde{v}_x(t)\|^2. \tag{4.19}$$

Moreover, by the arguments similar to the proof of (4.13), we conclude that there exist two constants $\epsilon_6, \epsilon_7 > 0$ to be determined later such that

$$I_8 \le C(B, \epsilon_6)\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right) + \epsilon_6\mathfrak{C}_2^2\|\tilde{u}_{xx}(t)\|^2 \tag{4.20}$$

and

$$I_9 \le C(B, \epsilon_7)\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right) + \epsilon_7\mathfrak{C}_1^2\|\tilde{v}_x(t)\|^2. \tag{4.21}$$

Substituting (4.17)–(4.21) into (4.16), we obtain

$$\begin{aligned}\psi'(t) \le &-\theta\widetilde{E}(t) + \left(1+\theta+\frac{1}{4\epsilon_3}\right)\|\tilde{u}_t(t)\|^2 + \left(1+\theta+\frac{1}{4\epsilon_4}\right)\|\tilde{v}_t(t)\|^2 \\ &- \left((1-\theta) - \epsilon_3\mathfrak{C}_2^2 - 2\epsilon_5\mathfrak{C}_2^2 - \epsilon_6\mathfrak{C}_2^2\right)\|\tilde{u}_{xx}(t)\|^2 \\ &- \left((1-\theta) - \epsilon_2 k - \epsilon_4\mathfrak{C}_1^2 - 2\epsilon_5\mathfrak{C}_1^2 - \epsilon_7\mathfrak{C}_1^2\right)\|\tilde{v}_x(t)\|^2 \\ &+ \left(\theta + \frac{1}{4\epsilon_2}\right)\|\tilde{w}^t\|_g^2 + C(B, \epsilon_5, \epsilon_6, \epsilon_7)\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right).\end{aligned}$$

Choosing sufficiently small ϵ_i $(i = 2, 3, \cdots, 7)$ such that

$$(1-\theta) - \epsilon_3\mathfrak{C}_2^2 - 2\epsilon_5\mathfrak{C}_2^2 - \epsilon_6\mathfrak{C}_2^2 > 0$$

and

$$(1-\theta) - \epsilon_2 k - \epsilon_4\mathfrak{C}_1^2 - 2\epsilon_5\mathfrak{C}_1^2 - \epsilon_7\mathfrak{C}_1^2 > 0,$$

we arrive at

$$\begin{aligned}\psi'(t) \le & -\theta\widetilde{E}(t) + C(B)\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right) + \left(1+\theta+\frac{1}{4\epsilon_3}\right)\|\tilde{u}_t(t)\|^2 \\ & + \left(1+\theta+\frac{1}{4\epsilon_4}\right)\|\tilde{v}_t(t)\|^2 + \left(\theta+\frac{1}{4\epsilon_2}\right)\|\tilde{w}^t\|_g^2.\end{aligned} \tag{4.22}$$

Therefore, from (4.6), (4.15), and (4.22), we deduce that

$$\begin{aligned}\Psi'(t) \le & -\varepsilon\theta\widetilde{E}(t) + C(B,\varepsilon)\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right) - \left(1-\varepsilon\left(1+\theta+\frac{1}{4\epsilon_3}\right)\right)\|\tilde{u}_t(t)\|^2 \\ & - \left(1-\varepsilon\left(1+\theta+\frac{1}{4\epsilon_4}\right)\right)\|\tilde{v}_t(t)\|^2 - \left(\rho-\varepsilon\left(\theta+\frac{1}{4\epsilon_2}\right)\right)\|\tilde{w}^t\|_g^2.\end{aligned} \tag{4.23}$$

For fixed ϵ_i $(i = 2, 3, 4)$, we choose

$$\varepsilon < \min\left\{\frac{1}{Q}, \frac{4\epsilon_3}{4(1+\theta)\epsilon_3+1}, \frac{4\epsilon_4}{4(1+\theta)\epsilon_4+1}, \frac{4\rho\epsilon_2}{4\theta\epsilon_2+1}\right\}$$

such that the last three terms on the right-hand side of (4.23) are non-positive and could be neglected. Thus assertion (4.8) is proved. Here, $\varepsilon < 1/Q$ ensures $\gamma_1 > 0$ in assertion (4.7).

By assertion (4.8) and the second inequality in assertion (4.7), we get

$$\Psi'(t) \le -\frac{\gamma_3}{\gamma_2}\Psi(t) + \gamma_4\left(\|\tilde{u}(t)\|_{2p}^2 + \|\tilde{v}(t)\|_{2p}^2\right).$$

Hence

$$\Psi(t) \le \Psi(0)e^{-\beta t} + \gamma_4\int_0^t e^{-\beta(t-s)}\left(\|\tilde{u}(s)\|_{2p}^2 + \|\tilde{v}(s)\|_{2p}^2\right)\mathrm{d}s, \tag{4.24}$$

where $\beta = \gamma_3/\gamma_2$. Again by the second inequality in assertion (4.7), we have $\Psi(0) \le \gamma_2\widetilde{E}(0)$, which combined with (4.24) and the first inequality in assertion (4.7) yields

$$\widetilde{E}(t) \le \alpha\widetilde{E}(0)e^{-\beta t} + \sigma\int_0^t e^{-\beta(t-s)}\left(\|\tilde{u}(s)\|_{2p}^2 + \|\tilde{v}(s)\|_{2p}^2\right)\mathrm{d}s,$$

where $\alpha = \gamma_2/\gamma_1$ and $\sigma = \gamma_4/\gamma_1$. Thus (4.3) follows from (4.5) immediately.

Lemma 4.8 (Quasi-stability)**.** *Under the assumptions of Theorem 2.2, the dynamical system* $(Y, S(t))$ *corresponding to problem* (2.2)–(2.4) *is quasi-stable on any bounded positively invariant set* $B \subset Y$.

Proof. Let $U = \left(H^2(0,l) \cap H_0^1(0,l)\right) \times H_0^1(0,l)$, $V = L^2(0,l) \times L^2(0,l)$, and $W = L_g$. Then Theorem 2.1 implies that the dynamical system $(Y, S(t))$ satisfies (A). From (3.21) we get (4.1). Moreover, by taking

$$n_U^2(\tilde{u}, \tilde{v}) = \|\tilde{u}\|_{2p}^2 + \|\tilde{v}\|_{2p}^2, \quad \varsigma_2(t) = \alpha e^{-\beta t}, \quad \varsigma_3(t) = \sigma \int_0^t e^{-\beta(t-s)}\,\mathrm{d}s,$$

we conclude from Lemma 4.7 that (4.2) holds. Thus, by Definition 4.3, the proof of Lemma 4.8 is finished.

Proof of Theorem 2.2. Since L is defined as E given by (3.15), we easily see that $L(y)$ is bounded from above on any bounded subset of Y. For the mild solution (u, v, u_t, v_t, w) to problem (2.2)–(2.4) such that $L(y_0) \le R$, we conclude from Lemma 3.5, Corollary 3.6, and $h_1, h_2 \in L^2(\Omega)$ that

$$C\|(u(t), v(t), u_t(t), v_t(t), w^t)\|_Y^2 - C \le R,$$

i.e., $\|(u(t), v(t), u_t(t), v_t(t), w^t)\|_Y^2 \le C$. Thus L_R is bounded.

For the stationary solution $(u, v, 0, 0, 0)$ to problem (2.2)–(2.4), we have

$$\|u_{xx}\|^2 + \|v_x\|^2 + \int_0^l \Phi(u-v)(u-v)\,\mathrm{d}x = I_1 + I_2, \tag{4.25}$$

where

$$I_1 := -\int_0^l (f_1(u,v)u + f_2(u,v)v)\,\mathrm{d}x$$

and

$$I_2 := \int_0^l (h_1 u + h_2 v)\,\mathrm{d}x.$$

It follows from (2.7) in (A_2) that

$$\begin{aligned} I_1 &\le \eta\left(\|u\|^2 + \|v\|^2\right) + bl \\ &\le \eta\mathfrak{C}_2^2\|u_{xx}\|^2 + \eta\mathfrak{C}_1^2\|v_x\|^2 + bl. \end{aligned} \tag{4.26}$$

Moreover, from Schwarz's and Cauchy's inequalities, we deduce that there exist two constants $\epsilon_1, \epsilon_2 > 0$ to be determined such that

$$\begin{aligned} I_2 &\le \|u\|\|h_1\| + \|v\|\|h_2\| \\ &= \left((2\epsilon_1)^{\frac{1}{2}}\|u\|\right)\left(\frac{1}{(2\epsilon_1)^{\frac{1}{2}}}\|h_1\|\right) + \left((2\epsilon_2)^{\frac{1}{2}}\|v\|\right)\left(\frac{1}{(2\epsilon_2)^{\frac{1}{2}}}\|h_2\|\right) \\ &\le \epsilon_1\mathfrak{C}_2^2\|u_{xx}\|^2 + \frac{1}{4\epsilon_1}\|h_1\|^2 + \epsilon_2\mathfrak{C}_1^2\|v_x\|^2 + \frac{1}{4\epsilon_2}\|h_2\|^2. \end{aligned} \tag{4.27}$$

Consequently, by substituting (4.26) and (4.27) into (4.25) and observing

$$\int_0^l \Phi(u-v)(u-v)\,\mathrm{d}x \ge 0,$$

we obtain

$$\left(1 - \eta\mathfrak{C}_2^2 - \epsilon_1\mathfrak{C}_2^2\right)\|u_{xx}\|^2 + \left(1 - \eta\mathfrak{C}_1^2 - \epsilon_2\mathfrak{C}_1^2\right)\|v_x\|^2 \le \frac{1}{4\epsilon_1}\|h_1\|^2 + \frac{1}{4\epsilon_2}\|h_2\|^2 + bl.$$

Choosing sufficiently small ϵ_1 and ϵ_2 such that

$$1 - \eta\mathfrak{C}_2^2 - \epsilon_1\mathfrak{C}_2^2 > 0$$

and

$$1 - \eta\mathfrak{C}_1^2 - \epsilon_2\mathfrak{C}_1^2 > 0,$$

we further derive

$$\|u_{xx}\|^2 + \|v_x\|^2 \leq C.$$

Hence $\mathcal{N}$ is bounded. From Lemma 4.8 and Proposition 4.4, it is easy to see that $(Y, S(t))$ is asymptotically smooth. By Theorem 4.2 and Lemma 4.6, it is obvious that $(Y, S(t))$ possesses a compact global attractor $A = \mathcal{M}^z(\mathcal{N})$. Finally, we conclude from Lemma 4.8 and Theorem 4.5 that A has finite fractal dimension.

5. Conclusions

In this paper, we considered the initial-boundary value problem for a coupled beam-string system modeling the small amplitude oscillations of suspension bridges, namely, (1.1)–(1.3). In order to handle the long-time dynamics for problem (1.1)–(1.3), we transformed problem (1.1)–(1.3) into the equivalent problem (2.2)–(2.4) in the history space framework. We first used the semigroup theory to obtain the global well-posedness and regularity of mild solutions to problem (2.2)–(2.4), namely, Theorem 2.1. In addition, by exploiting the properties of the total energy function, we obtained the gradient property of the dynamical system $(Y, S(t))$ corresponding to problem (2.2)–(2.4). By employing the perturbed energy method, we established a stabilizability estimate, which enabled us to get the quasi-stability of the dynamical system $(Y, S(t))$ corresponding to problem (2.2)–(2.4). Based on the gradient property and quasi-stability of the dynamical system $(Y, S(t))$, we derived the existence of a global attractor with finite fractal dimension, namely, Theorem 2.2.

In the future, we will focus on the study on other qualitative properties of problem (1.1)–(1.3).

Author contributions

Yang Liu: Investigation, Methodology, Writing-review & editing, Funding acquisition; Xiao Long: Investigation, Methodology, Writing-original draft; Li Zhang: Investigation. All authors have read and approved the final version of the manuscript for publication.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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