Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 17(1): 1–14.
DOI: 10.3934/cam.2025001
Received: 16 May 2024
Revised: 21 November 2024
Accepted: 07 January 2025
Published: 14 January 2025

Research article

On one boundary control problem for a pseudo-parabolic equation in a two-dimensional domain

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Abstract: In this paper, we consider a boundary control problem associated with a non-homogeneous pseudo-parabolic type equation in a bounded two-dimensional domain. In the part of the bound of the given region, the value of the solution is given, and it is required to find control to get the average value of the solution. The initial-boundary problem is solved by the Fourier method, and the control problem under consideration is analyzed with the Volterra integral equation of the second kind. The control function is found using the Laplace transform method and proved to be admissible.

Keywords: pseudo-parabolic equation; initial-boundary problem; Volterra integral equation; admissible control; Laplace transform
Mathematics Subject Classification: 35K70, 35K05

1. Introduction. Statement of the Problem

In this paper, we consider the following pseudo-parabolic equation in the rectangular domain $\Omega = (0, l_1) \times (0, l_2)$:

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{\partial}{\partial x}\Big(p(x)\frac{\partial u(x,y,t)}{\partial x}\Big) + \frac{\partial^2}{\partial t\partial x}\Big(p(x)\frac{\partial u(x,y,t)}{\partial x}\Big) + \frac{\partial}{\partial y}\Big(q(y)\frac{\partial u(x,y,t)}{\partial y}\Big)$$
$$+ \frac{\partial^2}{\partial t\partial y}\Big(q(y)\frac{\partial u(x,y,t)}{\partial y}\Big), \quad (x,y,t) \in \Omega_T := \Omega \times (0,\infty), \tag{1.1}$$

with Dirichlet boundary conditions

$$u(0,y,t) = \varphi(y)\,\nu(t), \quad u(l_1,y,t) = 0, \quad t \ge 0, \tag{1.2}$$

$$u(x,0,t) = 0, \quad u(x,l_2,t) = 0, \tag{1.3}$$

where $v(t)$ is the control function and $\varphi(y)$ is a given function, and the initial value condition

$$u(x, y, 0) = 0, \quad 0 \le x \le l_1, \quad 0 \le y \le l_2. \tag{1.4}$$

It is called that the control function $v(t) \in W_2^1(\mathbb{R}_+)$ is *admissible* if it fulfills the conditions $v(0) = 0$ and $|v(t)| \le 1$ on the half-line $t \ge 0$.

Suppose that the functions $p(x) \in C^2(\Omega)$ and $q(y) \in C^1(\Omega)$ satisfy the conditions

$$p(x) > 0, \quad p'(x) \le 0, \quad q(y) > 0, \quad 0 \le x \le l_1, \quad 0 \le y \le l_2.$$

It is required that the given functions $\varphi(y) \in W_2^2(\Omega)$ and $\psi(y) \in L^2(\Omega)$ satisfy the following conditions:

$$\varphi(0) = \varphi(l_2) = 0, \quad \varphi_n \cdot \psi_n \ge 0, \quad n = 1, 2, \cdots,$$

where φ_n and ψ_n are the Fourier coefficients of functions $\varphi(y)$ and $\psi(y)$, respectively.

Pseudo-parabolic equations are characterized by the occurrence of a time derivative appearing in the highest order term, which describes various important physical processes. We know that models of the theory of incompressible fluids with memory can be described by equations of pseudo-parabolic type [1].

Control Problem. S*uppose that the function $\phi(t)$ is given and let $\psi \in L^2(\Omega)$. Then we find the control function $v(t)$ from the condition*

$$\int_0^{l_1} \int_0^{l_2} \psi(y)\, u(x, y, t)\, dy\, dx = \phi(t), \quad t \ge 0, \tag{1.5}$$

where $u(x, y, t)$ is a solution of the mixed problem (1.1)-(1.4).

In [2], the control problem for the parabolic equation with the Neumann–Robin boundary condition was solved using the Laplace operator, and the optimal time for reaching the given temperature in the bounded domain was found. The boundary control problem for the heat transfer equation with the Robin boundary condition was studied in [3], and a mathematical model of the heating process of a cylindrical domain was developed.

In [4], the control problem related to the nonhomogeneous parabolic equation with Dirichlet boundary condition in a bounded one-dimensional domain was considered, and the optimal estimate of the minimum time required to reach a given temperature of a thin rod was found. In [5], the boundary control problem with the Neumann boundary condition for the heat transfer equation in a one-dimensional domain was studied, and an estimate of the minimum time for heating a thin rod was obtained.

The initial-boundary problem for a class of finite degenerate semilinear parabolic equations with a single potential term was studied in [6]. Also, the local existence and uniqueness of the weak solution were determined by applying the Galerkin method and the Banach invariance theorem. The initial-boundary value problems for nonlinear parabolic systems with power-type source terms are considered in [7].

Control problems for the infinite-dimensional case were studied by Egorov [8], who generalized Pontryagin's maximum principle to a class of equations in Banach space, and the proof of a bang-bang principle was shown in the particular conditions. The control problem for a linear parabolic type equation in a one-dimensional domain with a Robin boundary condition was studied by Fattorini and Russell [9]. In [10], an estimate of Carleman type for the one-dimensional heat equation was proved.

The stability, uniqueness, and existence of solutions of some classical problems for the pseudo-parabolic equation were studied in [11]. In [12], the point control problems for pseudo-parabolic and parabolic type equations are considered. In [13], some problems related to distributed parameter impulse control problems for systems were studied.

In [14], the control problem associated with a pseudo-parabolic type equation in a one-dimensional domain was studied, and the existence of an admissible control was proved using the Laplace transform method. Some boundary control problems for the pseudo-parabolic equation can be seen in [15]. An initial-boundary value problem for a pseudo-parabolic equation with singular potential was considered by Lian, et al. [16], and global existence and blow-up of solutions were studied. In [17], a class of semi-linear pseudo-parabolic equations was considered, and the invariance, global existence, non-existence, and asymptotic behavior of some sets with initial energy were proved by introducing a family of potential wells.

In this work, the boundary control problem associated with the pseudo-parabolic type equation is considered. Our main goal is to prove that there is a control function. The boundary control problem studied in this work is reduced to the Volterra integral equation of t the second type using the separation of variables method. The existence of a solution to this integral equation can also be proved using the Laplace transform method. In Section 2, the continuity of the kernel of the integral equation on the half-line $t \geq 0$ is proved. In Section 3, we prove the existence of an admissible control function and derive the required value for it.

2. Main integral equation

In this section, we consider the reduction of the control problem to the Volterra integral equation of the second kind. For this we first need the following spectral problem:

$$\frac{\partial}{\partial x}\Big(p(x)\frac{\partial v(x,y)}{\partial x}\Big)+\frac{\partial}{\partial y}\Big(q(y)\frac{\partial v(x,y)}{\partial y}\Big)=-\lambda\, v(x,y),$$

where λ is a constant to be determined later, and with boundary conditions

$$v(0,y,t) \;=\; 0, \quad v(l_1,y,t) \;=\; 0, \quad 0\leq y\leq l_2,$$

and

$$v(x,0,t) \;=\; 0, \quad v(x,l_2,t) \;=\; 0, \quad 0\leq x\leq l_1.$$

As we know, the above spectral problem is self-adjoint in $L^2(\Omega)$ and there exists a sequence of eigenvalues $\{\lambda_{mn}\}$ so that $0<\lambda_{11}\leq ...\leq\lambda_{mn}\to\infty,\ m,n\to\infty$. The corresponding eigenfunction v_{mn} forms a complete orthonormal system $\{v_{mn}\}$ in $L^2(\Omega)$, and these eigenfunctions belong to $C(\bar{\Omega})$(see [18,19]).

Let the eigenfunction v_{mn} is $v_{mn}(x,y)=\vartheta_m(x)\,\omega_n(y)$, and the eigenfunctions $\vartheta_m(x)$, $\omega_n(y)$ are solutions of the following spectral problems

$$\begin{cases}\frac{d}{dx}(p(x)\frac{d\vartheta_m(x)}{dx})=-\mu_m\,\vartheta_m(x), \quad 0<x<l_1,\\ \vartheta_m(0)=\vartheta_m(l_1)=0, \quad 0\leq x\leq l_1,\end{cases} \tag{2.1}$$

and

$$\begin{cases}\frac{d}{dy}(q(y)\frac{d\omega_n(y)}{dy})=-\nu_n\,\omega_n(y), \quad 0<y<l_2,\\ \omega_n(0)=\omega_n(l_2)=0, \quad 0\leq y\leq l_2,\end{cases} \tag{2.2}$$

where μ_m and ν_n are eigenvalues of spectral problems (2.1) and (2.2), respectively. We denote the eigenvalues λ_{mn} by $\lambda_{mn} = \mu_m + \nu_n$, $m, n = 1, 2, \cdots$.

A solution to the initial-boundary problem (1.1) - (1.4) is the function $u(x, y, t)$, which is expressed as follows:

$$u(x, y, t) = \nu(t)\,\varphi(y)\,\frac{l_1 - x}{l_1} - w(x, y, t), \tag{2.3}$$

where the function $w(x, y, t)$ with the regularity $w(x, y, t) \in C^{2,2,1}_{x,y,t}(\Omega_T) \cap C(\bar{\Omega}_T)$ is the solution to the mixed problem

$$\begin{aligned}
&\frac{\partial w}{\partial t} - \frac{\partial^2}{\partial t \partial x}\Big(p(x)\frac{\partial w}{\partial x}\Big) - \frac{\partial}{\partial x}\Big(p(x)\frac{\partial w}{\partial x}\Big) - \frac{\partial^2}{\partial t \partial y}\Big(q(y)\frac{\partial w}{\partial y}\Big) - \frac{\partial}{\partial y}\Big(q(y)\frac{\partial w}{\partial y}\Big)\\
&= \nu(t)\Big(\frac{\varphi(y)\,p'(x)}{l_1} - \frac{l_1 - x}{l_1}\,\frac{\partial}{\partial y}(q(y)\,\varphi'(y))\Big)\\
&+ \nu'(t)\Big(\varphi(y)\,\frac{l_1 - x}{l_1} + \frac{\varphi(y)\,p'(x)}{l_1} - \frac{l_1 - x}{l_1}\,\frac{\partial}{\partial y}(q(y)\,\varphi'(y))\Big),
\end{aligned}$$

with initial-boundary value conditions

$$w(x, y, t)\,|_{\partial\Omega} = 0, \quad w(x, y, 0) = 0.$$

We set

$$\beta_{mn} = (\lambda_{mn}\,a_{mn} - b_{mn} + c_{mn})\,\gamma_{mn}, \tag{2.4}$$

where the coefficients a_{mn}, b_{mn}, c_{mn} and γ_{mn} are as follows:

$$a_{mn} = \int_0^{l_1}\int_0^{l_2} \varphi(y)\,\frac{l_1 - x}{l_1}\,v_{mn}(x, y)\,dy\,dx, \tag{2.5}$$

$$b_{mn} = \int_0^{l_1}\int_0^{l_2} \frac{\varphi(y)\,p'(x)}{l_1}\,v_{mn}(x, y)\,dy\,dx, \tag{2.6}$$

$$c_{mn} = \int_0^{l_1}\int_0^{l_2} \frac{l_1 - x}{l_1}\,(q(y)\,\varphi'(y))'\,v_{mn}(x, y)\,dy\,dx, \tag{2.7}$$

and

$$\gamma_{mn} = \int_0^{l_1}\int_0^{l_2} \psi(y)\,v_{mn}(x, y)\,dy\,dx. \tag{2.8}$$

Thus, we obtain (see [18])

$$w(x, y, t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{b_{mn} - c_{mn}}{1 + \lambda_{mn}}\Big(\int_0^t e^{-\rho_{mn}(t-s)}\,\nu(s)\,ds\Big)v_{mn}(x, y)$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}+b_{mn}-c_{mn}}{1+\lambda_{mn}}\Big(\int_0^t e^{-\rho_{mn}(t-s)}\,\nu'(s)\,ds\Big)v_{mn}(x,y), \tag{2.9}$$

where $\rho_{mn}=\frac{\lambda_{mn}}{1+\lambda_{mn}}<1$.

Using (2.3) and (2.9), we obtain the solution of the mixed problem (1.1)–(1.4):

$$u(x,y,t)=\frac{l_1-x}{l_1}\varphi(y)\,\nu(t)-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_{mn}-c_{mn}}{1+\lambda_{mn}}\Big(\int_0^t e^{-\rho_{mn}(t-s)}\,\nu(s)\,ds\Big)v_{mn}(x,y)$$
$$-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}+b_{mn}-c_{mn}}{1+\lambda_{mn}}\Big(\int_0^t e^{-\rho_{mn}(t-s)}\,\nu'(s)\,ds\Big)v_{mn}(x,y). \tag{2.10}$$

From (2.10) and the condition (1.5), we can write

$$\begin{aligned}\phi(t)&=\int_0^{l_1}\int_0^{l_2}\psi(y)\,u(x,y,t)\,dy\,dx\\&=\nu(t)\int_0^{l_1}\int_0^{l_2}\psi(y)\,\varphi(y)\,\frac{l_1-x}{l_1}\,dy\,dx\\&-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(b_{mn}-c_{mn})\,\gamma_{mn}}{1+\lambda_{mn}}\int_0^t e^{-\rho_{mn}(t-s)}\,\nu(s)\,ds\\&-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(a_{mn}+b_{mn}-c_{mn})\,\gamma_{mn}}{1+\lambda_{mn}}\int_0^t e^{-\rho_{mn}(t-s)}\,\nu'(s)\,ds,\end{aligned}$$

where γ_{mn} is defined by (2.8).

According to the properties of the function $\nu(t)$, we have

$$\phi(t)=\nu(t)\int_0^{l_1}\int_0^{l_2}\psi(y)\,\varphi(y)\,\frac{l_1-x}{l_1}\,dy\,dx-\nu(t)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(a_{mn}+b_{mn}-c_{mn})\,\gamma_{mn}}{1+\lambda_{mn}}$$
$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(a_{mn}\,\lambda_{mn}-b_{mn}+c_{mn})\,\gamma_{mn}}{(1+\lambda_{mn})^2}\int_0^t e^{-\rho_{mn}(t-s)}\,\nu(s)\,ds.$$

Note that

$$\int_0^{l_1}\int_0^{l_2}\psi(y)\,\varphi(y)\,\frac{l_1-x}{l_1}\,dy\,dx\;=\;\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{mn}\,\gamma_{mn}. \tag{2.11}$$

Using (2.11), we can write

$$\phi(t)=\nu(t)\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(a_{mn}\,\lambda_{mn}-b_{mn}+c_{mn})\,\gamma_{mn}}{1+\lambda_{mn}}$$

$$+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(a_{mn}\,\lambda_{mn}-b_{mn}+c_{mn})\,\gamma_{mn}}{(1+\lambda_{mn})^2}\int_0^t e^{-\rho_{mn}(t-s)}\,\nu(s)\,ds.$$

We set

$$B(t) \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Lambda_{mn}\,e^{-\rho_{mn}\,t},\quad t>0, \tag{2.12}$$

and

$$\alpha \;=\; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\beta_{mn}}{1+\lambda_{mn}}, \tag{2.13}$$

where β_{mn} is defined by (2.4), and Λ_{mn} is as follows:

$$\Lambda_{mn}=\frac{\beta_{mn}}{(1+\lambda_{mn})^2},\quad m,n=1,2,\cdots. \tag{2.14}$$

Thus, we have the following Volterra integral equation of the second kind:

$$\alpha\,\nu(t)+\int_0^t B(t-s)\,\nu(s)\,ds=\phi(t),\quad t>0. \tag{2.15}$$

Lemma 1. *The following estimate is valid:*

$$0\le\beta_{mn}\le C\,\varphi_n\,\psi_n,\quad m,n=1,2,\cdots,$$

where $C=const>0$ and β_{mn} is defined by (2.4).

Proof. **Step 1.** Using (2.1), (2.5), and the formula for integration by parts, we write

$$\begin{aligned}
a_{mn}\,\mu_m &= \mu_m\int_0^{l_1}\int_0^{l_2}\varphi(y)\,\frac{l_1-x}{l_1}\,\vartheta_m(x)\,\omega_n(y)\,dy\,dx\\
&= -\varphi_n\int_0^{l_1}\frac{l_1-x}{l_1}\,\frac{d}{dx}\Big(p(x)\frac{d\vartheta_m(x)}{dx}\Big)dx\\
&= \varphi_n\,p(0)\,\vartheta_m'(0)-\varphi_n\int_0^{l_1}\frac{p(x)}{l_1}\,\vartheta_m'(x)dx\\
&= \varphi_n\,p(0)\,\vartheta_m'(0)+\varphi_n\int_0^{l_1}\frac{p'(x)}{l_1}\,\vartheta_m(x)dx\\
&= \varphi_n\,p(0)\,\vartheta_m'(0)+b_{mn},
\end{aligned}$$

where b_{mn} is defined by (2.6).

Then, by (2.2) and (2.5), we have

$$\begin{aligned} a_{mn}\,\nu_n &= \nu_n \int_0^{l_1}\int_0^{l_2} \varphi(y)\,\frac{l_1-x}{l_1}\;\vartheta_m(x)\,\omega_n(y)\,dy\,dx \\ &= -\int_0^{l_1} \frac{l_1-x}{l_1}\vartheta_m(x)\,dx \int_0^{l_2} \varphi(y)\,\frac{d}{dy}(q(y)\frac{d\omega_n}{dy})\,dy \\ &= \int_0^{l_1} \frac{l_1-x}{l_1}\vartheta_m(x)\,dx \int_0^{l_2} \varphi'(y)\,q(y)\frac{d\omega_n}{dy}\,dy \\ &= -\int_0^{l_1} \frac{l_1-x}{l_1}\vartheta_m(x)\,dx \int_0^{l_2} [q(y)\,\varphi'(y)]'\,\omega_n(y)\,dy \\ &= -c_{mn}, \end{aligned}$$

where c_{mn} is defined by (2.7).

Then we have the following equality:

$$a_{mn}(\mu_m+\nu_n) - b_{mn} + c_{mn} = \varphi_n\,p(0)\,\vartheta'_m(0), \tag{2.16}$$

where $\mu_m+\nu_n=\lambda_{mn}$.

Step 2. As we know, the following inequality holds for the eigenfunctions of problem (2.1) (see [14])

$$\vartheta'_m(0)\int_0^{l_1}\vartheta_m(\tau)d\tau \geq 0, \quad m=1,2,\cdots. \tag{2.17}$$

Step 3. By (2.4), (2.16), and (2.17), we obtain

$$\begin{aligned} \beta_{mn} &= (a_{mn}\,\lambda_{mn} - b_{mn} + c_{mn})\,\gamma_{mn} \\ &= \varphi_n\,p(0)\,\vartheta'_m(0)\int_0^{l_1}\vartheta_m(x)dx\int_0^{l_2}\psi(y)\,\omega_n(y)dy \\ &= \varphi_n\,\psi_n\,p(0)\,\vartheta'_m(0)\int_0^{l_1}\vartheta_m(x)dx. \end{aligned} \tag{2.18}$$

If the function $p(x)\in C^1(\Omega)$, we can write the following estimate (see [20])

$$\max_{0\leq x\leq l_1}|\vartheta'_m(x)| \leq C_1\,\mu_m^{1/2}.$$

Therefore,

$$|\vartheta'_m(0)| \leq C_1\,\mu_m^{1/2}, \quad |\vartheta'_m(l_1)| \leq C_1\,\mu_m^{1/2}, \tag{2.19}$$

where $C_1 = \text{const} > 0$.

Then, by integrating the equation (2.1) from 0 to l_1, we obtain

$$p(l_1)\vartheta'_m(l_1) - p(0)\,\vartheta'_m(0) = -\mu_m \int_0^{l_1} \vartheta_m(x)\,dx. \tag{2.20}$$

According to (2.19) and (2.20), we have the estimate

$$\left|\vartheta'_m(0) \int_0^{l_1} \vartheta_m(x)\,dx\right| \leq \left|\frac{\vartheta'_m(0)}{\mu_m}\Big(p(l_1)\,\vartheta'_m(l_1) - p(0)\,\vartheta'_m(0)\Big)\right| \leq C_1.$$

Thus, we obtain the required estimate

$$0 \leq \beta_{mn} \leq C\,\varphi_n\,\psi_n.$$

Proposition 1. *Assume that* $\varphi, \psi \in L^2(\Omega)$*. Then, the kernel* $B(t)$ *of the integral equation* (2.15) *is continuous on the half-line* $t \geq 0$.

Proof. According to Lemma 1 and (2.12), we have the estimate

$$\begin{aligned} 0 < B(t) \; &= \; \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Lambda_{mn}\, e^{-\rho_{mn}\, t} \\ &\leq \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \Lambda_{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\beta_{mn}}{(1+\lambda_{mn})^2} \\ &\leq C \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{\varphi_n\,\psi_n}{(1+\lambda_{mn})^2}, \end{aligned}$$

where $C = \text{const} > 0$.

3. Main result

In this section, we consider the existence of a solution to the Volterra integral equation of the second kind, that is, the existence of an admissible control function.

For any $M > 0$, we denote $W(M)$ the set of functions $\phi \in W_2^1(-\infty, +\infty)$, which satisfying the following conditions

$$\|\phi\|_{W_2^1(\mathbb{R}_+)} \leq M, \quad \phi(t) = 0 \quad \text{for} \quad t \leq 0.$$

Now, we present the main theorem for proving the existence of admissible control.

Theorem 1. *There exists* $M > 0$ *such that, for any function* $\phi \in W(M)$*, the equation* (2.15) *has a solution* $v(t)$ *meeting the condition* $|v(t)| \leq 1$.

We rewrite the Volterra integral equation (2.15) as follows:

$$\phi(t) \;=\; \alpha\, \nu(t) + \int_0^t B(t-s)\,\nu(s)\,ds, \quad t>0.$$

It is known that we can write the Laplace transform of the function $\nu(t)$ as follows:

$$\widetilde{\nu}(p) = \int_0^\infty e^{-pt}\,\nu(t)\,dt, \tag{3.1}$$

where $p=\sigma+i\zeta$, $\sigma>0$, $\zeta\in\mathbb{R}$.

Then, applying the Laplace transform to the integral equation (2.15), we obtain

$$\begin{aligned}\widetilde{\phi}(p) &= \alpha \int_0^\infty e^{-pt}\,\nu(t)\,dt + \int_0^\infty e^{-pt}\int_0^t B(t-s)\,\nu(s)ds\,dt\\ &= \alpha\,\widetilde{\nu}(p) + \widetilde{B}(p)\,\widetilde{\nu}(p),\end{aligned}$$

where α is defined by (2.13).

Then we can write

$$\widetilde{\nu}(p) = \frac{\widetilde{\phi}(p)}{\alpha+\widetilde{B}(p)},$$

and

$$\nu(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \frac{\widetilde{\phi}(p)}{\alpha+\widetilde{B}(p)}\,e^{pt}dp = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\phi}(\sigma+i\zeta)}{\alpha+\widetilde{B}(\sigma+i\zeta)}\,e^{(\sigma+i\zeta)t}\,d\zeta. \tag{3.2}$$

Lemma 2. *The following estimate*

$$|\alpha+\widetilde{B}(\sigma+i\,\zeta)|\ge\alpha, \quad \sigma>0, \quad \zeta\in\mathbb{R},$$

is valid, where $\alpha = const > 0$ is defined by (2.13).

Proof. According to Lemma 1 and (2.13), we can write

$$\alpha \;=\; \sum_{m=1}^\infty\sum_{n=1}^\infty \frac{\beta_{mn}}{1+\lambda_{mn}} \le \text{const}\sum_{m=1}^\infty\sum_{n=1}^\infty\frac{\varphi_n\,\psi_n}{1+\lambda_{mn}}.$$

It is known that we can write the Laplace transform of the function $B(t)$ as follows:

$$\begin{aligned}\widetilde{B}(p) &= \int_0^\infty B(t)\,e^{-pt}\,dt\\ &= \sum_{m=1}^\infty\sum_{n=1}^\infty \Lambda_{mn}\int_0^\infty e^{-(p+\rho_{mn})t}\,dt\end{aligned}$$

$$= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda_{mn}}{p+\rho_{mn}},$$

where function $B(t)$ is defined by (2.12). Then we can write

$$\begin{aligned}\alpha+\widetilde{B}(\sigma+i\zeta) &= \alpha+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda_{mn}}{\sigma+\rho_{mn}+i\zeta}\\ &= \alpha+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda_{mn}\,(\sigma+\rho_{mn})}{(\sigma+\rho_{mn})^2+\zeta^2} - i\zeta\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda_{mn}}{(\sigma+\rho_{mn})^2+\zeta^2}\\ &= \mathrm{Re}(\alpha+\widetilde{B}(\sigma+i\zeta))+i\,\mathrm{Im}(\alpha+\widetilde{B}(\sigma+i\zeta)),\end{aligned}$$

where

$$\mathrm{Re}(\alpha+\widetilde{B}(\sigma+i\zeta)) = \alpha+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda_{mn}\,(\sigma+\rho_{mn})}{(\sigma+\rho_{mn})^2+\zeta^2},$$

and

$$\mathrm{Im}(\alpha+\widetilde{B}(\sigma+i\zeta)) = -\zeta\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda_{mn}}{(\sigma+\rho_{mn})^2+\zeta^2}.$$

We can see that the following inequality holds:

$$(\sigma+\rho_{mn})^2+\zeta^2 \le ((\sigma+\rho_{mn})^2+1)(1+\zeta^2).$$

As a result, we obtain

$$\frac{1}{(\sigma+\rho_{mn})^2+\zeta^2} \ge \frac{1}{1+\zeta^2}\,\frac{1}{(\sigma+\rho_{mn})^2+1}. \tag{3.3}$$

Then, using the inequality (3.3), we can obtain the following assumptions:

$$\begin{aligned}|\mathrm{Re}(\alpha+\widetilde{B}(\sigma+i\zeta))| &= \alpha+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda_{mn}\,(\sigma+\rho_{mn})}{(\sigma+\rho_{mn})^2+\zeta^2}\\ &\ge \alpha+\frac{1}{1+\zeta^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda_{mn}\,(\sigma+\rho_{mn})}{(\sigma+\rho_{mn})^2+1} \qquad (3.4)\\ &= \alpha+\frac{C_{1,\sigma}}{1+\zeta^2}, \qquad (3.5)\end{aligned}$$

and

$$\begin{aligned}|\mathrm{Im}(\alpha+\widetilde{B}(\sigma+i\zeta))| &= |\zeta|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda_{mn}}{(\sigma+\rho_{mn})^2+\zeta^2}\\ &\ge \frac{|\zeta|}{1+\zeta^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda_{mn}}{(\sigma+\rho_{mn})^2+1} = \frac{C_{2,\sigma}\,|\zeta|}{1+\zeta^2}, \qquad (3.6)\end{aligned}$$

where $C_{1,\sigma}$ and $C_{2,\sigma}$ are defined as follows:

$$C_{1,\sigma} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda_{mn}\,(\sigma+\rho_{mn})}{(\sigma+\rho_{mn})^2+1},\quad C_{2,\sigma} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{\Lambda_{mn}}{(\sigma+\rho_{mn})^2+1}.$$

By (3.4) and (3.6), we obtain the required estimate

$$|\alpha + \widetilde{B}(\sigma + i\zeta)| \geq \alpha + \frac{C_\sigma}{\sqrt{1+\zeta^2}} \geq \alpha, \tag{3.7}$$

where $C_\sigma = \min(C_{1,\sigma}, C_{2,\sigma})$ is bounded for all $\sigma > 0$.

Let the Laplace transform of function $\phi(t)$ satisfy the condition

$$\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|\, d\zeta < +\infty.$$

If we proceed to the limit as $\sigma \to 0$ in the equality (3.2), we have

$$v(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\widetilde{\phi}(i\zeta)}{\alpha + \widetilde{B}(i\zeta)}\, e^{i\zeta t} d\zeta. \tag{3.8}$$

Also, to prove Theorem 1, we need the following lemma.

Lemma 3. *Suppose that the function $\phi(t)$ belongs to $W(M)$. Then for the imaginary part of the Laplace transform of function $\phi(t)$ the following estimate holds*

$$\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|\, d\zeta < +\infty.$$

Proof. Using the formula for integration by parts in (3.1), we can write

$$\begin{aligned} \widetilde{\phi}(\sigma + i\zeta) &= \int_0^\infty e^{-(\sigma+i\zeta)t} \phi(t)\, dt \\ &= -\phi(t)\, \frac{e^{-(\sigma+i\zeta)t}}{\sigma + i\zeta}\Bigg|_{t=0}^{t=\infty} + \frac{1}{\sigma + i\zeta} \int_0^\infty e^{-(\sigma+i\zeta)t}\, \phi'(t)\, dt. \end{aligned}$$

Then, we have

$$(\sigma + i\zeta)\, \widetilde{\phi}(\sigma + i\zeta) = \int_0^\infty e^{-(\sigma+i\zeta)t}\, \phi'(t)\, dt,$$

Further, for $\sigma \to 0$ we obtain

$$i\zeta\, \widetilde{\phi}(i\zeta) = \int_0^\infty e^{-i\zeta t}\, \phi'(t)\, dt.$$

Besides

$$\widetilde{\phi}(i\zeta) = \int_0^\infty e^{-i\zeta t}\, \phi(t)\, dt.$$

Thus, we can write the following inequality:

$$\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|^2(1+\zeta^2)\,d\zeta \le C_2\,\|\phi\|^2_{W_2^1(R_+)},$$

where $C_2 = \text{const} > 0$.

By using elementary identities and inequalities, we have

$$\begin{aligned}\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|\,d\zeta &= \int_{-\infty}^{+\infty} \frac{|\widetilde{\phi}(i\zeta)|}{1+\zeta^2}\,d\zeta + \int_{-\infty}^{+\infty} \frac{\zeta^2\,|\widetilde{\phi}(i\zeta)|}{1+\zeta^2}\,d\zeta \\ &\le \Big(\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|^2\,d\zeta\Big)^{1/2}\Big(\int_{-\infty}^{+\infty} \frac{1}{(1+\zeta^2)^2}\,d\zeta\Big)^{1/2} \\ &+ \Big(\int_{-\infty}^{+\infty} \zeta^2\,|\widetilde{\phi}(i\zeta)|^2\,d\zeta\Big)^{1/2}\Big(\int_{-\infty}^{+\infty} \frac{\zeta^2}{(1+\zeta^2)^2}\,d\zeta\Big)^{1/2} \\ &\le C\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|^2(1+\zeta^2)\,d\zeta \le C_2\,\|\phi\|^2_{W_2^1(\mathbb{R}_+)}.\end{aligned}$$

Now we present the proof of Theorem 1.

Proof of Theorem 1. Let us show that $v \in W_2^1(\mathbb{R}_+)$. Indeed, using (3.7) and (3.8), we can write

$$\begin{aligned}\int_{-\infty}^{+\infty} |\widetilde{v}(\zeta)|^2(1+|\zeta|^2)\,d\zeta &= \int_{-\infty}^{+\infty} \left|\frac{\widetilde{\phi}(i\zeta)}{\alpha+\widetilde{B}(i\zeta)}\right|^2(1+|\zeta|^2)\,d\zeta \\ &\le \frac{1}{\alpha^2}\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|^2(1+|\zeta|^2)\,d\zeta = \text{const}\,\|\phi\|^2_{W_2^1(\mathbb{R})}.\end{aligned}$$

Now, we show that the function $v(t)$ satisfies the Lipschitz condition. Actually,

$$|v(t)-v(s)| = \left|\int_s^t v'(\xi)\,d\xi\right| \le \|v'\|_{L^2(\Omega)}\sqrt{t-s}.$$

Using (3.7), (3.8), and Lemma 3, we have the following estimate

$$\begin{aligned}|v(t)| &\le \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|\widetilde{\phi}(i\zeta)|}{|\alpha+\widetilde{B}(i\zeta)|}d\zeta \\ &\le \frac{1}{2\pi\,\alpha}\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\zeta)|\,d\zeta\end{aligned}$$

$$
\begin{aligned}
&\le \frac{C_2}{2\pi\,\alpha}\|\phi\|_{W_2^1(\mathbb{R}_+)} \\
&\le \frac{C_2\,M}{2\pi\,\alpha} = 1,
\end{aligned}
$$

where α is defined by (2.13) and

$$
M = \frac{2\pi\,\alpha}{C_2}, \quad C_2 = \text{const} > 0.
$$

Theorem 1 is proved.

Use of AI tools declaration

The author declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. B. D. Coleman, W. Noll, An Approximation Theorem for Functionals, with Applications in Continuum Mechanics, *Arch. Rational Mech. Anal.*, **6** (1960), 355–370. https://doi.org/10.1007/BF00276168
2. S. Albeverio, Sh. A. Alimov, On one time-optimal control problem associated with the heat exchange process, *Appl. Math. Opt.*, **57** (2008), 58–68. https://doi.org/10.1007/s00245-007-9008-7
3. Sh. A. Alimov, N. M. Komilov, Determining the thermal mode setting parameters based on output data, *Diff Equat*, **58** (2022), 21–35. https://doi.org/10.1134/S0012266122010049
4. F. N. Dekhkonov, E. I. Kuchkorov, On the time-optimal control problem associated with the heating process of a thin rod, *Lobachevskii. J. Math.*, **44** (2023), 1134–1144. https://doi.org/10.1134/S1995080223030101
5. F. N. Dekhkonov, Boundary control problem for the heat transfer equation associated with heating process of a rod, *Bulletin of the Karaganda University. Mathematics Series*, **110** (2023), 63–71. https://doi.org/10.31489/2023m2/63-71
6. H. Xu, Existence and blow-up of solutions for finitely degenerate semilinear parabolic equations with singular potentials, *Commun. Anal. Mech.*, **15** (2023), 132–161. https://doi.org/10.3934/cam.2023008
7. R. Xu, W. Lian, Y. Niu, Global well-posedness of coupled parabolic systems, *Sci. China Math.*, **63** (2020), 321–356. https://doi.org/10.1007/s11425-017-9280-x
8. Yu. V. Egorov, The optimal control in Banach space, *Uspekhi Mat. Nauk*, **18** (1963), 211–213.
9. H. O. Fattorini, D. L. Russell, Exact controllability theorems for linear parabolic equations in one space dimension, *Arch. Rational Mech. Anal.*, **43** (1971), 272–292. https://doi.org/10.1007/BF00250466

10. F. Alabau-Boussouira, P. Cannarsa, G. Fragnelli, Carleman estimates for degenerate parabolic operators with applications to null controllability, *J. Evol. Equ*, **6** (2006), 161–204. https://doi.org/10.1007/s00028-006-0222-6
11. B. D. Coleman, R. J. Duffin, V. J. Mizel, Instability, uniqueness, and nonexistence theorems for the equation on a strip, *Arch. Rational Mech. Anal.*, **19** (1965), 100–116. https://doi.org/10.1007/BF00282277
12. L. W. White, Point control of pseudoparabolic problems, *J. Differ. Equations*, **42** (1981), 366–374. https://doi.org/10.1016/0022-0396(81)90110-8
13. S. I. Lyashko, On the solvability of pseudo-parabolic equations, *Soviet Math. (Iz. VUZ)*, **29** (1985), 99–101.
14. F. Dekhkonov, On a boundary control problem for a pseudo-parabolic equation, *Commun. Anal. Mech.*, **15** (2023), 289–299. http://doi.org/10.3934/cam.2023015
15. F. N. Dekhkonov, On the control problem associated with a pseudo-parabolic type equation in an one-dimensional domain, *Int. J. Appl. Math.*, **37** (2024), 109–118. http://doi.org/10.12732/ijam.v37i1.9
16. W. Lian, J. Wang, R. Xu, Global existence and blow up of solutions for pseudo-parabolic equation with singular potential, *J. Differ. Equations*, **269** (2020), 4914–4959. http://dx.doi.org/10.1016/j.jde.2020.03.047
17. R. Xu, J. Su, Global existence and finite-time blow-up for a class of semilinear pseudo-parabolic equations, *J. Funct. Anal.*, **264** (2013), 2732–2763. http://dx.doi.org/10.1016/j.jfa.2013.03.010
18. A. N. Tikhonov, A. A. Samarsky, *Equations of Mathematical Physics*, Nauka, Moscow, 1966.
19. M. A. Naimark, *Linear differential operators*, Nauka, Moscow, 1962.
20. V. E. Vladykina, Spectral characteristics of the Sturm-Liouville operator under minimal restrictions on smoothness of coefficients, *Moscow Univ. Math. Bull.*, **74** (2019), 235–240. https://doi.org/10.3103/S0027132219060044