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Research article

A Kato-type criterion for the inviscid limit of the nonhomogeneous NS equations with no-slip boundary condition

Shuai Xi[1,2,*]

[1] School of Mathematical Sciences, Shanghai Jiao Tong University, Shanghai 200240, P.R. China

[2] College of Mathematics and Systems Science,Shandong University of Science and Technology, Qingdao 266590, P. R. China

* **Correspondence:** Email: shuaixi@sdust.edu.cn.

Abstract: The objective of this paper is to examine the vanishing viscosity limit of the nonhomogeneous incompressible NS systerm, subject to the no-slip boundary condition. By adopting Kato's approach of constructing an artificial boundary layer [1], within a smooth and bounded domain designated as $\Omega \subseteq \mathbb{R}^2$, we derive a sufficient condition for the convergence to occur uniformly in time within the energy space $L^2(\Omega)$.

Keywords: nonhomogeneous NS system; the vanishing viscosity limit; no-slip boundary condition
Mathematics Subject Classification: 35B40, 35L60

1. Introduction

The motion of an incompressible viscous fluid with variable density in a bounded domain $\Omega \subset \mathbb{R}^2$ with a smooth boundary is governed by the following equations in $[0, T] \times \Omega$:

$$\begin{cases} \text{div } u = 0, \\ \partial_t \rho + \text{div } (\rho u) = 0, \\ \partial_t(\rho u) + \text{div } (\rho u \otimes u) + \nabla p = \epsilon \text{div } \mathbb{S}(\nabla u), \end{cases} \tag{1.1}$$

where $\epsilon > 0$ is the viscosity coefficient, $\rho = \rho(t, x)$ is the density, $u = u(t, x) = (u_1, u_2)^T$ and $p = p(t, x)$ are velocity and pressure, respectively, and $\mathbb{S}(\nabla u)$ is given by

$$\mathbb{S}(\nabla u) := \mu \left(\nabla u + (\nabla u)^T \right).$$

The aim of this paper is to investigate the vanishing viscosity limit of the nonhomogeneous incompressible NS equations (1.1) with the following initial and no-slip boundary conditions:

$$\begin{cases} u = 0, \quad \text{on } \partial\Omega, \\ \rho(0, x) = \rho_0, (\rho u)(0, x) = \rho_0 u_0. \end{cases} \tag{1.2}$$

Formally, by taking $\epsilon = 0$ in the problem (1.1)-(1.2), this transforms into the subsequent nonhomogeneous Euler equations

$$\begin{cases} \text{div} u^E = 0, \\ \partial_t \rho^E + \text{div}(\rho^E u^E) = 0, \\ \partial_t(\rho^E u^E) + \text{div}(\rho^E u^E \otimes u^E) + \nabla p = 0, \end{cases} \tag{1.3}$$

with the initial and impermeable boundary conditions as follows.

$$\begin{cases} u^E \cdot n = 0, \quad \text{on } \partial\Omega, \\ \rho^E(0, x) = \rho_0^E, (\rho^E u^E)(0, x) = \rho_0^E u_0^E, \end{cases} \tag{1.4}$$

which is used to describe the motion of an ideal fluid without viscosity. We shall investigate the convergence, as $\epsilon \to 0$, of weak solutions for the nonhomogeneous NS system with a no-slip boundary condition to the strong solution of the Euler equations with variable density. The exploration of the inviscid limit of the NS systerm has been a persistent challenge in fluid dynamics for a considerable duration. In the groundbreaking work referenced by Prandtl [2], he examined the behavior of fluid flow in proximity to physical boundaries and introduced the concept of the boundary layer. In the presence of physical boundaries, viscous forces act to slow down the fluid near the boundary, resulting in a rapid change in flow direction perpendicular to the boundary. The impact of viscosity is primarily concentrated in a thin layer close to the boundary, known as the boundary layer.

For incompressible viscous fluids, Prandtl simplified the NS equations to derive the equations governing fluid dynamics within the boundary layer, known as the Prandtl equations. Outside this layer, the fluid can be approximated as ideal and unaffected by viscosity, as the influence of viscosity on the flow is minimal. In this region, the motion of the fluid is governed by the Euler equations.

The rigorous justification of Prandtl's boundary layer theory is both theoretically and practically significant but remains a significant challenge. The primary difficulties stem from two main aspects: the well-posedness of the Prandtl equations and the vanishing viscosity limit issue itself. Although numerous intriguing results have been obtained regarding the well-posedness of the Prandtl equations, as referenced in [3–11] and further citations, the meticulous mathematical verification of the vanishing viscosity limit remains limited to certain specific scenarios. Some notable examples include the works referenced in [12–16].

Studying the vanishing viscosity limit for solutions of the NS equations in the energy space is a pivotal approach, pioneered by Kato. In his work [1], Kato introduced the concept of an artificial boundary layer to investigate the behavior of incompressible viscous flows with a no-slip boundary condition as viscosity diminishes. His research findings uncovered that, under specific conditions of energy dissipation within a boundary region, the width of which scales with viscosity, the viscous flow can be accurately approximated by an inviscid flow in the energy space.

Since Kato's initial work, this result has undergone significant improvement. Wang, in [17], relaxed Kato's energy dissipation conditions, allowing for the dissipation to be captured solely through tangential derivatives of the tangential or normal velocity components. This relaxation came at the cost of a slight increase in the size of the boundary region considered.

In another notable study, Kelliher [18] proposed an alternative approach, replacing Kato's energy condition with one based on the vorticity of the flow. This vorticity-based condition provided a different perspective on the vanishing viscosity limit.

More recently, Wang et al. [19] extended the investigation to the Navier boundary conditions, which encompass the non-slip boundary condition as a special case. Their work explored the vanishing viscosity limit for the incompressible NS equations and derived several Kato-type conditions that guarantee the limit holds in the energy space. These conditions provide a deeper understanding of the behavior of viscous flows as they approach the inviscid limit.

In this paper, we delve into the vanishing viscosity limit for solutions of the nonhomogeneous incompressible NS equations (1.1) within the context of the energy space. Although extensive research has been conducted on the nonhomogeneous incompressible NS equations (1.1), as exemplified by the works in [20–27] and their associated references, there is a notable dearth of studies exploring the vanishing viscosity limit problem for (1.1) in a domain with boundaries. Previously, the vanishing viscosity limit for the nonhomogeneous NS system with Navier friction boundary conditions was investigated in [28], where the influence of the boundary layer was significantly weaker compared to the case with a non-slip boundary condition. In our present work, we focus on the vanishing viscosity limit for solutions of the nonhomogeneous incompressible NS equations (1.1) subject to a no-slip boundary condition. By employing Kato's innovative approach of constructing an artificial boundary layer, we derive a sufficient condition for convergence to occur within the energy space. This investigation not only enhances our understanding of the nonhomogeneous NS equations but also provides valuable insights for further exploring complex fluid dynamics problems.

The structure of this paper is outlined as follows. We revisit the existence of the weak solutions to the nonhomogeneous NS equations (1.1) and the strong solution to the nonhomogeneous Euler equations (1.3) in Section 2, subject to a no-slip boundary condition. Following this, we summarize the main finding of Theorem 2.1. And then, we give a crucial relative energy inequality in Section 3, which serves as a fundamental tool in our analysis. Finally, in Section 4, we provide rigorous proof of our main result.

2. Preliminaries and main result

In the subsequent calculations, we employ the notation $o(1)$ to denote a quantity that converges to 0, as ϵ goes to 0. Additionally, $O(1)$ will be used to signify a quantity that is bounded.

For the NS equations, we initially postulate that the lower bounds imposed on μ and η imply that the tensor product

$$\mathbb{S}(\nabla u) : \nabla u = \frac{\mu}{2}(\partial_i u_j + \partial_j u_i)^2$$

constitutes a quadratic form that is strictly positive, relative to $(\partial_i u_j)_{1\le i,j\le 2}$, and there exists a constant $C_0 > 0$ such that for any $u \in H^1(\Omega)$,

$$\int_\Omega \mathbb{S}(\nabla u) : \nabla u dx \geq C_0 \int_\Omega |\nabla u|^2 dx. \tag{2.1}$$

Let us revisit the definition of weak solutions to the nonhomogeneous incompressible NS equations [29]:

Definition 2.1. *For a fixed $T > 0$, we denote $(\rho,\ u)$ as a weak solution to the systerm* (1.1) *for the nonhomogeneous NS equations on* $[0, T]$ *with the no-slip boundary condition. This solution is associated with initial data:*

$$0 < \underline{\rho_0} \leq \rho_0 \leq \overline{\rho_0} < \infty, \ \ \sqrt{\rho_0} u_0 \in L^2(\Omega), \tag{2.2}$$

if:

$$u \in L^2([0, T]; H_0^1(\Omega)), \ \ \sqrt{\rho} u \in C_w([0, T]; L^2(\Omega)),$$

satisfy the systerm (1.1) *in the sense of distributions,*

$$0 < \underline{\rho_0} \leq \rho \leq \overline{\rho_0} < \infty, \tag{2.3}$$

for $(x, t) \in \Omega \times [0, T]$ *and the energy inequality:*

$$\frac{1}{2} \| \ \sqrt{\rho} u \ \|_{L^2(\Omega)} + \epsilon \int_0^\tau \int_\Omega \mathbb{S}(\nabla u) : \nabla u dx dt \leq \frac{1}{2} \| \ \sqrt{\rho_0} u_0 \ \|_{L^2(\Omega)} \tag{2.4}$$

holds for almost all $\tau \in [0, T]$.

Proposition 2.1. *For the initial data* (ρ_0, u_0) *satisfies (2.2), given any fixed* $T > 0$*, there exists a weak solution, defined in Definition 2.1, to the nonhomogeneous NS equations on the time interval* $[0, T]$.

Additionally, the existence of a strong solution to the problem described by equation (1.2) for the nonhomogeneous Euler equations has been established in numerous studies:

Proposition 2.2. *Given that* $\rho_0^E, u_0^E \in H^3(\Omega)$ *satisfy the compatibility conditions of the systerm* (1.2)*, and that* $0 < \underline{\rho_0^E} \leq \rho_0^E \leq \overline{\rho_0^E} < \infty$*, it follows that there exists* $T > 0$ *and a unique solution* (ρ^E, u^E) *to* (1.2) *on the domain* $[0, T] \times \Omega$ *that satisfies*

$$0 < \underline{\rho^E} \leq \rho^E \leq \overline{\rho^E} < \infty$$

and

$$u^E, \ \rho^E \in C(0, T; H^3(\Omega)),$$
$$\partial_t u^E, \ \partial_t \rho^E \in C(0, T; H^2(\Omega)).$$

The principal result of this paper is summarized as follows:

Theorem 2.1. *Consider* (ρ^E, u^E) *as the strong solution of the Euler equations defined on the time interval* $[0, T]$ *and corresponding to the initial conditions* (ρ_0^E, u_0^E) *as specified in Proposition 2.2. Additionally, let* $(\rho^\epsilon, u^\epsilon)$ *represent a weak solution of the nonhomogeneous NS equations on the same time interval* $[0, T]$ *with initial conditions* $(\rho_0^\epsilon, u_0^\epsilon)$ *that fulfill the conditions stated in* (2.2) *for every value of* ϵ *within the range* $(0, 1)$*, as outlined in Proposition 2.1. If*

$$\| \rho_0^\epsilon - \rho_0^E \|_{L^2(\Omega)} + \| u_0^\epsilon - u_0^E \|_{L^2(\Omega)} = o(1), \tag{2.5}$$

we have

$$\sup_{t\in(0,T)}\Big(\|\,\rho^{\epsilon}-\rho^{E}\,\|_{L^2(\Omega)}+\|u^{\epsilon}-u^{E}\|_{L^2(\Omega)}\Big)=o(1),$$

if the subsequent condition is met:

$$\epsilon\,\|\,\partial_\tau u_n^{\epsilon}\,\|^2_{L^2([0,T];L^2(\Omega_\delta))}\to 0,\quad \|\,\nabla\rho^{\epsilon}\,\|_{L^\infty([0,T]\times\Omega_\delta)}=O(1),$$

where u_n^{ϵ} represents the normal components of u^{ϵ}, ∂_τ represents the tangential derivative, $\Omega_\delta := \{x\in\Omega|dist(x,\ \partial\Omega)<\delta\}$, and under the condition that δ approaches 0 as ϵ approaches 0, with the relationship $\epsilon=o(\delta)$.

3. Relative energy inequality

We define the relative energy $\mathcal{D}([\rho,u]|[U])$ of (ρ,u) with respect to U as follows:

$$\mathcal{D}([\rho,u]|[U]) := \frac{1}{2}\|\sqrt{\rho}(u-U)\|^2_{L^2(\Omega)}dx.$$

Furthermore, we will employ the relative energy inequality provided in [29].

Proposition 3.1. *Consider (ρ,u) as a weak solution of the nonhomogeneous NS equations defined on the interval $[0,T]$ corresponding to the initial data (ρ_0,u_0). For any smooth function U that fulfills $U|_{\partial\Omega}=0$, the following relative energy inequality holds:*

$$\mathcal{D}([\rho,u]|[U])(\tau)+\epsilon\int_0^\tau\int_\Omega \mathbb{S}(\nabla u):\nabla u dxdt\le \mathcal{D}_0+\mathcal{L}(\rho,u,U)$$

for almost all $\tau\in(0,T)$, here

$$\mathcal{D}_0=\mathcal{D}([\rho_0,u_0]|[U_0]),\tag{3.1}$$

with U_0 is the initial data of U, and

$$\mathcal{L}(\rho,u,U):=\int_0^\tau\int_\Omega\rho\,(\partial_t U+(u\cdot\nabla)U)\cdot(U-u)dxdt+\epsilon\int_0^\tau\int_\Omega\mathbb{S}(\nabla u):\nabla U dxdt.$$

In the following, we give a simple proof of this Proposition.

Proof. Multiplying $(1.1)_3$ to U and integrating over $[0,T]\times\Omega$, we obtain

$$\begin{aligned}\int_\Omega\rho u\cdot U dx=&\int_\Omega\rho_0u_0\cdot U_0dx\\&+\int_0^\tau\int_\Omega\Big(\rho u\cdot\partial_t U+(\rho u\otimes u):\nabla U-\epsilon\mathbb{S}(\nabla u):\nabla U\Big)dxdt.\end{aligned}\tag{3.2}$$

Similarly, multiplying $(1.1)_2$ by $\frac{1}{2}|U|^2$ and integrating over $[0,T]\times\Omega$, we obtain

$$\int_\Omega\frac{1}{2}\rho|U|^2dx=\int_\Omega\frac{1}{2}\rho_0|U_0|^2dx+\int_0^\tau\int_\Omega\Big(\rho U\cdot\partial_t U+\rho u\cdot\nabla U\cdot U\Big)dxdt.\tag{3.3}$$

Summing up (2.4), (3.2) and (3.3), we have

$$\begin{aligned}&\mathcal{D}([\rho,u]|[U])(\tau)+\epsilon\int_0^\tau\int_\Omega\mathbb{S}(\nabla u):\nabla u dxdt\\&\le\mathcal{D}_0+\int_0^\tau\int_\Omega\rho\,(\partial_t U+(u\cdot\nabla)U)\cdot(U-u)dxdt+\epsilon\int_0^\tau\int_\Omega\mathbb{S}(\nabla u):\nabla U dxdt\end{aligned}$$

4. Proof of the main result

In this section, we aim to prove Theorem 2.1.

Initially, we introduce a Kato-type "fake" boundary layer. Consider $u^E = (u^E_\tau, u^E_n)^T$ as a smooth solution of equation (1.2), as described in Proposition 2.2. Define

$$v = (v_\tau, v_n)^T := \left(u^E_\tau(t, x_\tau, 0) f\left(\frac{x_n}{\delta}\right), \ -\partial_\tau u^E_\tau(t, x_\tau, 0)\int_0^{x_n} f\left(\frac{s}{\delta}\right) ds\right)^T, \tag{4.1}$$

and f satisfies

$$\begin{aligned} & f \in C^\infty[0,\infty), \ f(0) = 1, \ \text{supp}\, f \subset [0,1), \\ & \int_0^1 f(s)ds = 0, \ \|f\|_{L^\infty} < +\infty, \ \|f'\|_{L^\infty} < +\infty. \end{aligned} \tag{4.2}$$

We can obtain that:

$$\begin{aligned} & v_n|_{\partial\Omega} = 0, \ div\ v = 0, \\ & \sup_{t\in(0,T),x\in\Omega} \|v_\tau, \frac{1}{\delta} v_n, \partial_t v, \partial_\tau v_\tau, \delta\partial_n v_\tau\| = O(1), \end{aligned} \tag{4.3}$$

here, v_n designates the component of v that lies in the normal direction, whereas v_τ characterizes its tangential component. Correspondingly, ∂_n and ∂_τ are symbols that signify the normal and tangential derivatives, respectively. For simplicity and convenience, we omit the subscript ϵ.

We have the following estimate.

$$\underline{\rho_0} \le \rho \le \overline{\rho_0}, \tag{4.4}$$

$$\sup_{t\in(0,T)} \| \sqrt{\rho} u \|^2_{L^2(\Omega)} + \epsilon \int_0^\tau \int_\Omega |\nabla u|^2 dxdt \le O(1). \tag{4.5}$$

Let $U = u^E - v$. By using the no-slip boundary condition of U, we obtain

$$\int_\Omega \frac{1}{2}\rho|u - U|^2 dx + \epsilon \int_0^\tau \int_\Omega \mathbb{S}(\nabla u) : \nabla u dxdt \le \mathcal{E}_0 + \mathcal{L}(\rho, u, U) \tag{4.6}$$

with

$$\begin{aligned} & \mathcal{L}(\rho, u, U) \\ & = \int_0^\tau \int_\Omega \rho\left((u\cdot\nabla)u^E\right)\cdot w dxdt + \int_0^\tau \int_\Omega \rho\left(\partial_t U - (u\cdot\nabla)v\right)\cdot w dxdt \\ & + \epsilon \int_0^\tau \int_\Omega \mathbb{S}(\nabla u) : \nabla U dxdt = \sum_{k=1}^3 \mathcal{L}_k, \end{aligned}$$

Here $w = u - U$, we will calculate every $\mathcal{L}_k (1 \le k \le 3)$.

i) For $\mathcal{L}_1$, by using (4.3) and (4.5) , one can obtain

$$
\begin{aligned}
\mathcal{L}_1 &= \int_0^\tau \int_\Omega \rho\left((u\cdot\nabla)u^E\right)\cdot w dxdt \\
&= \int_0^\tau \int_\Omega \rho\left((w\cdot\nabla)u^E\right)\cdot w dxdt + \int_0^\tau \int_\Omega \rho\left((U\cdot\nabla)u^E\right)\cdot w dxdt \\
&\le C\int_0^\tau \int_\Omega \rho|w|^2 dxdt + \int_0^\tau \int_\Omega \rho\left((u^E\cdot\nabla)u^E\right)\cdot w dxdt \\
&\quad - \int_0^\tau \int_\Omega \rho\left((v\cdot\nabla)u^E\right)\cdot w dxdt \\
&\le C\int_0^\tau \int_\Omega \rho|w|^2 dxdt + \int_0^\tau \int_\Omega \rho\left((u^E\cdot\nabla)u^E\right)\cdot w dxdt + o(1).
\end{aligned}
$$

ii) Decompose $\mathcal{L}_2$ into

$$
\mathcal{L}_2 = \int_0^\tau \int_\Omega \rho\partial_t u^E\cdot w dxdt - \int_0^\tau \int_\Omega \rho\partial_t v\cdot w dxdt - \int_0^\tau \int_\Omega \rho(u\cdot\nabla)v\cdot w dxdt. \tag{4.7}
$$

The second term can be estimated by

$$
\begin{aligned}
-\int_0^\tau \int_\Omega \rho\partial_t v\cdot w dxdt \le& C\int_0^\tau \int_\Omega \rho|w|^2 dxdt + C\int_0^\tau \int_\Omega \rho|\partial_t v|^2 dxdt \\
\le& C\int_0^\tau \int_\Omega |w|^2 dxdt + o(1).
\end{aligned}
$$

For the first term, notice that $\rho^E\partial_t u^E + \rho^E(u^E\cdot\nabla)u^E + \nabla p = 0$, we have

$$
\begin{aligned}
&\int_0^\tau \int_\Omega \rho\partial_t u^E\cdot w dxdt + \int_0^\tau \int_\Omega \rho\left((u^E\cdot\nabla)u^E\right)\cdot w dxdt \\
=& \int_0^\tau \int_\Omega \left(\rho-\rho^E\right)\left(\partial_t u^E + (u^E\cdot\nabla)u^E\right)\cdot w dxdt \\
\le& C\int_0^\tau \int_\Omega |w|^2 dxdt + C\int_0^\tau \int_\Omega \left(\rho-\rho^E\right)^2 dxdt.
\end{aligned}
$$

Next, we proceed to analyze the third term of (4.7). For the sake of simplicity, we initially consider the case of a flat boundary. In general, when dealing with a smooth boundary, we can get a flat boundary by applying localization techniques to a curved one. Maintaining generality throughout, we assume that the domain is situated in the upper half-plane, specifically $\Omega = \{(x_1, x_2)|x_1 \in \mathbb{R}, x_2 > 0\}$, with $\{x_2 = 0\}$ representing the boundary.

Notice that

$$
\begin{aligned}
&\rho(u\cdot\nabla)v\cdot w \\
&= \left(\rho u_j\partial_j v_i w_i - \rho u_2\partial_2 v_1 w_1\right) + \rho u_2\partial_2 v_1 w_1 \\
&= \mathcal{T}_1 + \mathcal{T}_2.
\end{aligned} \tag{4.8}
$$

By using (4.3), (4.5), and the Hölder inequality, we have

$$\begin{aligned}\left|\int_0^\tau\int_\Omega \mathcal{T}_1 dxdt\right| &\le \left|\int_0^\tau\int_\Omega \rho w_j w_i \partial_j v_i dxdt\right| + \left|\int_0^\tau\int_\Omega \rho U_j w_i \partial_j v_i dxdt\right| \\ &\le C\int_0^\tau\int_\Omega \rho |w|^2 dxdt + C\int_0^\tau\int_\Omega \rho |U_j \partial_j v_i|^2 dxdt \\ &\le C\int_0^\tau\int_\Omega |w|^2 dxdt + o(1), \end{aligned} \tag{4.9}$$

for $(i, j) \neq (1, 2)$.

Now we will deal with the other term given in (4.8). By using $\partial_t\rho + \operatorname{div}(\rho u) = 0$, we have,

$$\begin{aligned}\int_\Omega \mathcal{T}_2 dx = &\int_\Omega w_1 \partial_1(\rho u_1) v_1 dx - \int_\Omega \rho \partial_2 w_1 u_2 v_1 dx + \int_\Omega w_1 \partial_t \rho v_1 dx \\ =&\mathcal{T}_{21} + \mathcal{T}_{22} + \mathcal{T}_{23}.\end{aligned} \tag{4.10}$$

Using $\partial_t\rho + \nabla\rho \cdot u = 0$, the estimate of $\mathcal{T}_{23}$ is as follows.

$$\begin{aligned}\mathcal{T}_{23} &= -\int_\Omega w_1 \nabla\rho u v_1 dx = -\int_\Omega w_1 \nabla\rho w v_1 dx - \int_\Omega w_1 \nabla\rho U v_1 dx \\ &\le \|\nabla\rho\|_{L^\infty(\Omega_\delta)} \int_\Omega |w|^2 dx + \delta^{\frac{1}{2}} \|\nabla\rho\|_{L^\infty(\Omega_\delta)} \left(\int_\Omega |w|^2 dx\right)^{\frac{1}{2}} \\ &\le C\|\nabla\rho\|_{L^\infty(\Omega_\delta)} \int_\Omega |w|^2 dx + C\delta \|\nabla\rho\|_{L^\infty(\Omega_\delta)}.\end{aligned} \tag{4.11}$$

The estimate for $\mathcal{T}_{21}$ is as follows.

$$\begin{aligned}\mathcal{T}_{21} &= \int_\Omega w_1 \partial_1(\rho u_1) v_1 dx \\ &= \int_\Omega \partial_1 w_1 v_1 \rho w_1 dx + \int_\Omega \partial_1 U_1 v_1 \rho w_1 dx + \int_\Omega v_1 w_1 \partial_1 \rho u_1 dx \\ &\le \frac{1}{2}\int_\Omega \rho v_1 \partial_1 w_1^2 dx + C\delta^{\frac{1}{2}}\|w\|_{L^2(\Omega)} + C\|\nabla\rho\|_{L^\infty(\Omega_\delta)} \int_\Omega |w|^2 dx + C\delta\|\nabla\rho\|_{L^\infty(\Omega_\delta)} \\ &\le -\frac{1}{2}\int_\Omega \partial_1 \rho v_1 w_1^2 dx - \frac{1}{2}\int_\Omega \rho \partial_1 v_1 w_1^2 dx + C\delta^{\frac{1}{2}}\|w\|_{L^2(\Omega)} \\ &\quad + C\|\nabla\rho\|_{L^\infty(\Omega_\delta)} \int_\Omega |w|^2 dx + C\delta\|\nabla\rho\|_{L^\infty(\Omega_\delta)} \\ &\le C\left(1 + |\nabla\rho\|_{L^\infty(\Omega_\delta)}\right) \int_\Omega |w|^2 dx + C\delta\|\nabla\rho\|_{L^\infty(\Omega_\delta)} + o(1).\end{aligned} \tag{4.12}$$

The estimate of $\mathcal{T}_{22}$ will use the following function $\hat{v}$:

$$\hat{v} := \int_{x_2}^{\delta} v_1^2(t, x_1, s) ds,$$

which satisfies,

$$\sup_{t\in(0,T),x\in\Omega} \|\hat{v}, \partial_1 \hat{v}\| = C\delta.$$

For the term $\mathcal{T}_{22}$, since the divergence is free of u, using integration by parts, one has

$$\begin{aligned}\mathcal{T}_{22} &\leq \|\nabla w\|_{L^2(\Omega)} \left(\int_\Omega u_2^2 v_1^2 dx\right)^{\frac{1}{2}} = \|\nabla w\|_{L^2(\Omega)} \left(-\int_\Omega u_2^2 \partial_2 \hat{v} dx\right)^{\frac{1}{2}} \\ &= \|\nabla w\|_{L^2(\Omega)} \left(-2\int_\Omega u_2 \partial_1 u_1 \hat{v} dx\right)^{\frac{1}{2}} \\ &= \|\nabla w\|_{L^2(\Omega)} \left(2\int_\Omega \partial_1 u_2 u_1 \hat{v} dx + 2\int_\Omega u_2 u_1 \partial_1 \hat{v} dx\right)^{\frac{1}{2}}.\end{aligned} \tag{4.13}$$

Using the Poincaré inequality, we obtain

$$\begin{aligned}\mathcal{T}_{22} &\leq C\|\nabla w\|_{L^2(\Omega)} \left(\delta\|\partial_1 u_2\|_{L^2(\Omega)}\|u_1\|_{L^2(\Omega)} + \delta\|u_2\|_{L^2(\Omega)}\|u_1\|_{L^2(\Omega)}\right)^{\frac{1}{2}} \\ &\leq C\|\nabla w\|_{L^2(\Omega)} \left(\delta^2\|\partial_1 u_2\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} + \delta^3\|\nabla u\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} \\ &\leq C\delta\|\nabla w\|_{L^2(\Omega)}\|\partial_1 u_2\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla u\|^{\frac{1}{2}}_{L^2(\Omega)} + C\delta^{\frac{3}{2}}\|\nabla w\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)}.\end{aligned} \tag{4.14}$$

For the first term, by using the Young inequality, we have:

$$\begin{aligned}&\delta\|\nabla w\|_{L^2(\Omega)}\|\partial_1 u_2\|^{\frac{1}{2}}_{L^2(\Omega)}\|\nabla u\|^{\frac{1}{2}}_{L^2(\Omega)} \\ &\leq \frac{\epsilon}{16}\|\nabla w\|^2_{L^2(\Omega)} + \frac{\epsilon}{16}\|\nabla u\|^2_{L^2(\Omega)} + C\delta^4\epsilon^{-3}\|\partial_1 u_2\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.15}$$

For the second term, by using the Young inequality, we have

$$\begin{aligned}\delta^{\frac{3}{2}}\|\nabla w\|_{L^2(\Omega)}\|\nabla u\|_{L^2(\Omega)} &\leq C\delta^{\frac{3}{2}}\left(\|\nabla u\|_{L^2(\Omega)} + \|\nabla U\|_{L^2(\Omega)}\right)\|\nabla u\|_{L^2(\Omega)} \\ &\leq C\delta^{\frac{3}{2}}\|\nabla u\|^2_{L^2(\Omega)} + C\delta\|\nabla u\|_{L^2(\Omega)} \\ &\leq C\delta^{\frac{3}{2}}\|\nabla u\|^2_{L^2(\Omega)} + C\delta^{\frac{1}{2}}.\end{aligned} \tag{4.16}$$

Combining (4.14)-(4.16) with (4.14), we obtain

$$\begin{aligned}\mathcal{T}_{22} \leq &\frac{\epsilon}{16}\|\nabla w\|^2_{L^2(\Omega)} + C\left(\frac{\epsilon}{16} + \delta^{\frac{3}{2}}\right)\|\nabla u\|^2_{L^2(\Omega)} \\ &+ C\delta^4\epsilon^{-3}\|\partial_1 u_2\|^2_{L^2(\Omega)} + C\delta^{\frac{1}{2}}.\end{aligned} \tag{4.17}$$

Plugging (4.12), (4.17), and (4.11) into (4.10), we obtain

$$\begin{aligned}&\int_0^\tau \int_\Omega \mathcal{T}_2 dxdt \\ &\leq C\left(1 + \|\nabla\rho\|_{L^\infty([0,T]\times\Omega_\delta)}\right)\int_0^\tau\int_\Omega |w|^2 dxdt + C\delta\|\nabla\rho\|_{L^\infty([0,T]\times\Omega_\delta)} + C\frac{\epsilon}{\delta} + o(1)\end{aligned} \tag{4.18}$$

$$+ C\delta^4\epsilon^{-3}\int_0^\tau\int_\Omega |\partial_1 u_2|^2 dxdt + C\left(\frac{\epsilon}{4}+\delta^{\frac{3}{2}}\right)\int_0^\tau\int_\Omega |\nabla u|^2 dxdt,$$

where we have used

$$\begin{aligned}
&\epsilon\int_0^\tau\int_\Omega |\nabla w|^2 dxdt \\
\le&\epsilon\int_0^\tau\int_\Omega |\nabla u|^2 dxdt + \epsilon\int_0^\tau\int_\Omega |\nabla v|^2 dxdt + \epsilon\int_0^\tau\int_\Omega |\nabla u^E|^2 dxdt \\
\le&\epsilon\int_0^\tau\int_\Omega |\nabla u|^2 dxdt + C\frac{\epsilon}{\delta} + C\epsilon.
\end{aligned} \tag{4.19}$$

iii) For $\mathcal{L}_3$, we have

$$\begin{aligned}
\mathcal{L}_3 \le& \epsilon\int_0^\tau\int_\Omega \mathbb{S}(\nabla u) : \nabla u^E dxdt + \epsilon\int_0^\tau\int_\Omega \mathbb{S}(\nabla u) : \nabla v dxdt \\
\le& \frac{\epsilon}{8}\int_0^\tau\int_\Omega |\nabla u|^2 dxdt + C\epsilon + \frac{\epsilon}{8}\int_0^\tau\int_\Omega |\nabla u|^2 dxdt + C\frac{\epsilon}{\delta} \\
=& \frac{\epsilon}{4}\int_0^\tau\int_\Omega |\nabla u|^2 dxdt + C\frac{\epsilon}{\delta} + C\epsilon.
\end{aligned}$$

Substituting the estimates of $\mathcal{R}_1$,$\mathcal{R}_2$ and $\mathcal{R}_3$ into (4.6) and noticing that

$$\begin{aligned}
\frac{1}{2}\parallel \rho-\rho^E \parallel_{L^2(\Omega)} \le& \left|\int_0^\tau\int_\Omega \nabla\rho^E\left(\rho-\rho^E\right)\left(u-u^E\right)dxdt\right| \\
\le& C\int_0^\tau\int_\Omega |u-u^E|^2 dxdt + C\int_0^\tau\int_\Omega \left(\rho-\rho^E\right)^2 dxdt,
\end{aligned}$$

for ϵ small enough, we have

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |w(\tau)|^2 dx + \left(\frac{\epsilon}{2} - C\delta^{\frac{3}{2}}\right)\int_0^\tau\int_\Omega |\nabla u|^2 dxdt + \frac{1}{2}\parallel \rho-\rho^E \parallel_{L^2(\Omega)} \\
\le&\frac{1}{2}\int_\Omega |w(0)|^2 dx + C\left(1+\|\nabla\rho\|_{L^\infty([0,T]\times\Omega_\delta)}\right)\int_0^\tau\int_\Omega |w|^2 dxdt \\
&+ C\delta^4\epsilon^{-3}\int_0^\tau\int_\Omega |\partial_\tau u_n|^2 dxdt + C\int_0^\tau\int_\Omega \left(\rho-\rho^E\right)^2 dxdt \\
&+ C\frac{\epsilon}{\delta} + C\delta\|\nabla\rho\|_{L^\infty([0,T]\times\Omega_\delta)} + o(1).
\end{aligned} \tag{4.20}$$

Taking δ satisfying

$$\epsilon = o(\delta) \text{ and } \delta = o(\epsilon^{\frac{2}{3}}),$$

one can get that $\frac{\epsilon}{\delta} \to 0$ as $\epsilon \to 0$.

Thus, from (4.20), we obtain

$$\int_\Omega |w(\tau)|^2 dx + \parallel \rho-\rho^E \parallel_{L^2(\Omega)} \tag{4.21}$$

$$
\begin{aligned}
&\leq \int_\Omega |w(0)|^2 dx + C\left(1 + \|\nabla\rho\|_{L^\infty([0,T]\times\Omega_\delta)}\right) \int_0^\tau\int_\Omega |w|^2 dxdt \\
&\quad + C\int_0^\tau \int_\Omega \left(\rho - \rho^E\right)^2 dxdt + C\delta^4\epsilon^{-3} \int_0^\tau\int_\Omega |\partial_\tau u_n|^2 dx + C\frac{\epsilon}{\delta} + o(1),
\end{aligned}
$$

when ϵ is small.

Using the Gronwall inequality, we have

$$
\begin{aligned}
&\int_\Omega |w(\tau)|^2 dx + \| \rho - \rho^E \|_{L^2(\Omega)} \\
&\quad \leq C\delta^4\epsilon^{-3} \int_0^\tau\int_\Omega |\partial_\tau u_n|^2 dx + C\frac{\epsilon}{\delta} + o(1),
\end{aligned}
\tag{4.22}
$$

If we choose,

$$
\delta^{-1} = \epsilon^{-1} \max\left(\left(\epsilon\|\partial_\tau u_n\|^2_{L^2((0,T)\times\Omega)}\right)^{\frac{1}{5}}, \epsilon^{\frac{1}{4}}\right) =: \epsilon^{-1} C_{\epsilon,\delta},
$$

it is satisfied that

$$
\lim_{\epsilon\to 0} \frac{\epsilon}{\delta} = \lim_{\epsilon\to 0} C_{\epsilon,\delta} = 0,
$$

$$
\lim_{\epsilon\to 0} \frac{\delta}{\epsilon^{\frac{2}{3}}} = \lim_{\epsilon\to 0} \epsilon^{\frac{1}{3}} C^{-1}_{\epsilon,\delta} \leq \lim_{\epsilon\to 0} \epsilon^{\frac{1}{12}} = 0,
$$

$$
\lim_{\epsilon\to 0} \delta^4\epsilon^{-3} \int_0^\tau\int_\Omega |\partial_\tau u_n|^2 = \lim_{\epsilon\to 0} \epsilon\|\partial_\tau u_n\|^2_{L^2([0,T]\times\Omega)} C^{-4}_{\epsilon,\delta} \leq \lim_{\epsilon\to 0} \left(\epsilon\|\partial_\tau u_n\|^2_{L^2((0,T)\times\Omega)}\right)^{\frac{1}{5}} = 0.
$$

That is

$$
\epsilon = o(\delta), \delta = o(\epsilon^{\frac{2}{3}})
$$

and

$$
C\delta^4\epsilon^{-3} \int_0^\tau\int_\Omega |\partial_\tau u_n|^2 dx + C\frac{\epsilon}{\delta} = o(1).
$$

We can conclude that when

$$
\epsilon\|\partial_\tau u_n\|^2_{L^2([0,T]\times\Omega)} \to 0, \ \|\nabla\rho\|_{L^\infty([0,T]\times\Omega)} = O(1) \quad \text{as} \quad \epsilon \to 0,
$$

then

$$
\sup_{t\in[0,T]} \int_\Omega \left|u - u^E\right|^2 dx + \sup_{t\in[0,T]} \| \rho - \rho^E \|_{L^2(\Omega)} \to 0 \quad \text{as} \quad \epsilon \to 0.
$$

Thus, we obtain the assertion given in Theorem 2.1.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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