Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 16(4): 872–895.
DOI: 10.3934/cam.2024038
Received: 13 February 2024
Revised: 30 July 2024
Accepted: 08 November 2024
Published: 26 November 2024

Research article

Brezis Nirenberg type results for local non-local problems under mixed boundary conditions

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Abstract: In this paper, we are concerned with an elliptic problem with mixed Dirichlet and Neumann boundary conditions that involve a mixed operator (i.e., the combination of classical Laplace operator and fractional Laplace operator) and critical nonlinearity. Also, we focus on identifying the optimal constant in the mixed Sobolev inequality, which we show is never achieved. Furthermore, by using variational methods, we provide an existence and nonexistence theory for both linear and superlinear perturbation cases.

Keywords: mixed local-nonlocal operators; mixed type Sobolev inequality; mixed boundary conditions; existence and non-existence results; variational methods
Mathematics Subject Classification: 35A15, 35J25, 35J20

1. Introduction

In this study, we focus on elliptic operators of mixed local and non-local types (mixed Dirichlet and Neuman boundary conditions) concerning possible positive solutions for critical problems and possible optimizers of appropriate mixed Sobolev inequality. We examine the existence of solutions to the perturbed critical problem.

$$\begin{cases} \mathcal{T} w = w^{2^*-1} + \lambda w^p, \quad w > 0 \quad \text{in } \mathcal{O}, \\ w = 0 \ \text{ in } \ U^c, \\ \mathcal{N}_s(w) = 0 \ \text{ in } \ \Pi_2, \\ \frac{\partial w}{\partial \nu} = 0 \ \text{ in } \ \partial\mathcal{O} \cap \overline{\Pi_2}. \end{cases} \tag{P_λ}$$

where $U = (\Omega \cup \Pi_2 \cup (\partial\Omega \cap \overline{\Pi_2}))$ and $\Omega \cup \Pi_2$ is a bounded set with smooth boundary, $n \geq 3$, $1 \leq p < 2^* - 1 = \frac{n+2}{n-2}$ and $\lambda \in \mathbb{R}$. Here, $\mathcal{O} \subseteq \mathbb{R}^n$ is a nonempty open set, Π_1, Π_2 are open subsets of $\mathbb{R}^n \setminus \bar{\mathcal{O}}$ such that $\overline{\Pi_1} \cup \overline{\Pi_2} = \mathbb{R}^n \setminus \bar{O}$, $\Pi_1 \cap \Pi_2 = \emptyset$ and

$$\mathcal{T} = -\Delta + (-\Delta)^s, \text{ for } s \in (0,1). \tag{1.1}$$

An operator that combines classical Laplace operator $-\Delta$ and fractional Laplace operator $(-\Delta)^s$ is referred to as being 'mixed'. The fractional Laplace operator is defined by

$$(-\Delta)^s w(x) = C_{n,s}\ P.V. \int_{\mathbb{R}^n} \frac{w(x) - w(y)}{|x-y|^{n+2s}}\, dy, \quad s \in (0,1).$$

The abbreviation "P.V." denotes the Cauchy principal value and

$$C_{n,s} = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(z_1)}{|z|^{n+2s}}\, dz \right)^{-1}.$$

Without discussing the specific instances of how this non-local operator is useful in the actual situations, and the motivations for investigating issues with them, see [1] and references therein for more details. The study of the mixed operators of the form $\mathcal{L}$ in (1.1) is motivated by a wide range of applications. The numerous applications serve as the motivation for the comprehensive analysis of operator $\mathcal{T}$ in the problem (P_λ), such as the concept of optimum exploration mathematical biology, we refer to [2–6], and other common uses include heat transmission in magnetized plasmas; see [7]. These types of operators are naturally developed in the applied sciences to investigate the changes in physical phenomena that have both local and nonlocal effects. Based on these operators, diffusion patterns change over different time scales, with the lower-order operator taking the lead for long-term patterns and the higher-order operator leading for short-term patterns they arise, for instance, in bi-modal power law distribution processes; see [8]. Further applications arise in the theory of optimal searching and biomathematics; we refer to [2, 3] and the references therein. Also, this type of operator emerges in models derived from combining two distinct random processes in the probability theory; for a comprehensive explanation of such a phenomenon, refer to [9]. Recently, there has been a considerable focus on investigating elliptic problems having mixed-type operator $\mathcal{T}$, as in (P_λ), exhibiting both local and nonlocal behaviour. Let us put some light on literature involving mixed operator $\mathcal{L}$ problems along the existing long list. Biagi, Dipierro, and Valdinoci [10] have provided a comprehensive analysis of a mixed local and non-local elliptic problem. Throughout their study, they proved the existence of solutions to the elliptic problem, explored the maximum principles that govern the behaviour of these solutions, and investigated the interior Sobolev regularity of the solutions.

The investigation of mixed-type operators is a widely studied area that is emerging in diverse fields. This includes scenarios like combining the Lévy stable process, which finds intriguing applications in understanding animal cropping technique, as discussed in [2, 3, 11, 12]. Aizicovici et al. in [13] have studied the nonlinear logistic equation of the superdiffusive type driven by a nonhomogeneous differential operator with Robin boundary condition and existence and multiplicity results of positive solutions to the linearly coupled Hartree systems with critical exponent involving the classical Laplace operator studied by Mao et al. in [14], and also interesting nonlocal Hilfer proportional sequential fractional multi-valued boundary value problems have been studied by [15]. Motivated by the investigation of non-linear problems having critical exponents, in particular those arising from optimizers of the Sobolev inequality, the concerns addressed in this paper are motivated. Their findings contribute to a better understanding of the characteristics of solutions to the following

$$\begin{cases} \mathcal{T}u = g(u), \quad u > 0 \quad \text{in } \mathcal{O}, \\ u = 0 \ \text{ in } \ \mathbb{R}^n \setminus \mathcal{O}. \end{cases} \tag{1.2}$$

Authors have studied various interesting inequalities for mixed operator $\mathcal{L}$ in [16, 17]. Lamao et al. in [18] found out the behavior and properties of solutions to (1.2), specifically focusing on their summability characteristics. We also refer to [19–21] for interested readers, where we point out that [21] contains a study of mixed operator Neumann problem. The non-linear generalization of $\mathcal{L}$ given by $-\Delta_p + (-\Delta_p)^s$ has also started gaining attention for mixed operators. Dipierro et al. in [22] were the first to consider mixed operator problems in the presence of classical as well as non-local Neumann boundary conditions. Their recent article discusses some characteristics and regularity results corresponding with a mixed local and non-local problem, further with certain specific incentives derived from mathematics models and population growth. It is worth commenting that none of these articles studies mixed operator problems under mixed boundary conditions. This made us curious about what happens when we set up a PDE involving mixed operator $\mathcal{L}$ under boundary conditions involving Dirichlet datum in some part and Neumann (local and non-local both) datum in the remaining part. To answer this, we started with (P_λ) and the combination of mixed operator as well as mixed Dirichlet Neumann boundary, which is the striking feature of our paper. In particular, when $\mathcal{T} = -\Delta$ in (P_λ), Grossi [23] studied classical problems with mixed boundary conditions and established solutions under some appropriate assumption on parameter λ. In [24], authors have studied the behaviour of the Sobolev constant with a mixed boundary condition, where the main goal is to examine appropriate emphasis concerning the Neumann boundary of a sequence that minimizes using the standard isoperimetric inequalities. In our study, we define a function space where it contains those functions that are vanishing on some part of $\mathbb{R}^n \setminus \bar{\mathcal{O}}$. Due to this behavior, the best constant depends on some part of $\mathbb{R}^n \setminus \bar{\mathcal{O}}$, but in the Dirichlet boundary case, best constant does not depend on $\mathcal{O}$. For further study, see [24]. Biagi et al. [25] studied existence and non-existence outcomes to (P_λ) type problems under the Dirichlet boundary condition. The motivation for this paper arises from the investigation of nonlinear problems with critical exponents. These problems are modeled as

$$\mathcal{L}u = f(u) + \lambda g(u),$$

under some conditions on u and $\mathcal{L}$ is some operator like (local/nonlocal/mixed). Here $f(u)$ and $g(u)$ represent the critical and subcritical nonlinearities, respectively, and $\lambda \in \mathbb{R}$ is a parameter. With all the above literature as motivation where, many authors have studied the existence/nonexistence and multiplicity of the solutions. We are curious about the critical problem involving mixed local and nonlocal operator $\mathcal{T}$ under the mixed Dirichlet and Neumann boundary conditions. We proved similar results as in [25] pertaining to problem (P_λ) with mixed boundary conditions in the same spirit. There exists no single article at present that studies such a problem with mixed boundary conditions; hence, our article is a breakthrough in this regard. Last but not least, we try to provide a glimpse of problems available in literature, involving Dirichlet Neumann's mixed boundary datum to the readers. Brezis et al. in [26], consider the following Dirichlet boundary value problem:

$$\begin{cases} -\Delta u = u^p + f(x,u), \quad u > 0 \quad \text{in } \Omega, \\ u = 0 \ \text{ in } \ \partial\Omega. \end{cases} \tag{1.3}$$

where Ω is the bounded domain of $\mathbb{R}^n$, $p = 2^* - 1$ $n \geq 3$ and in particular, $f(x, u) = \lambda u$. They have studied the existence of solutions to (1.3) for every $\lambda \in (0, \lambda_1)$, ($\lambda_1$ is the first eigenvalue of $-\Delta$) when $n \geq 4$ and problem (1.3) has no solution if $\lambda \notin (0, \lambda_1)$ and Ω is star-shaped domain. In [27], Valdinoci et al. studied the fractional counterpart of problem (1.3) and its results can be seen as an extension of the classical Brezis-Nirenberg results. Recently, Biagi et al. in [28] have studied the corresponding Sobolev inequality and found the optimal constant, which is never achieved, and they also proved the existence and non-existence of a positive solution to a mixed elliptic problem (P_λ) with the Dirichlet boundary condition using the variational methods. Currently, we have worked on the first principle eigenvalue problem with mixed operators involving mixed boundary conditions in [29]. Using its advantage, we extend the results of [28] in our article. Our optimal constant does not depend on the whole domain as [28]: it only depends on some part of the boundary of the domain. Our findings offer specific advantages over existing results by introducing a novel approach to nonlinear problems with critical exponent, which enhances the understanding of this problem (P_λ) with mixed boundary conditions. These contributions hold substantial potential impact for both theoretical developments and practical applications in related fields. We also established L^∞ result of positive of solutions and maximum principle for problem (P_λ).

First, we looked into the mixed Sobolev inequality for the case of the mixed operators where the domain is divided into two disjoint parts, Π_1 and Π_2. More precisely, suppose $n \geq 3$ and $2^* = \frac{2n}{n-2}$ critical Sobolev exponent. Fix $s \in (0, 1)$. Let O be an open set that may not be bounded. We assume that $w : \mathbb{R}^n \to \mathbb{R}$ that are vanshis outside of U. So, we define the following mixed Sobolev inequality

$$\mathcal{S}_{n,s}(O, \Pi_1)\, \|w\|^2_{L^{2^*}(O)} \leq \|\nabla w\|^2_{L^2(O)} + \iint_Q \frac{|w(x) - w(y)|^2}{|x - y|^{n+2s}}\, dx\, dy. \tag{1.4}$$

Here, constant $\mathcal{S}_{n,s}(O, \Pi_1)$ is taken to be large enough for which inequality (1.4) is satisfied.

We observe that equation (1.4) holds by choosing the constant that is less than or equal to the following classical Sobolev constant (which depends at some portion on the boundary of O)

$$\mathcal{S}(\Pi_2) = \left(\frac{n \left(\frac{\xi_n}{2}\right)^{\frac{1}{n}}}{C^{\frac{n-2}{2n}}} \right)^2, \tag{1.5}$$

where ξ_n is the measure of the unit ball in $\mathbb{R}^n$ and $C = \frac{\Gamma n}{\Gamma(\frac{n}{2}\Gamma(\frac{n}{2}+1)}(\frac{1}{2})^{\frac{3n-2}{2}}$, to further details, we refer to [23, 24]. From [23], we have classical best Sobolev constant $S_n \leq \mathcal{S}(\Pi_2)$. Since from well known the Sobolev inequality:

$$\mathcal{S}(\Pi_2)\, \|w\|^2_{L^{2^*}(O)} \leq \|\nabla w\|^2_{L^2(O)} \leq \|\nabla w\|^2_{L^2(O)} + \iint_Q \frac{|w(x) - w(y)|^2}{|x - y|^{n+2s}}\, dx\, dy = \zeta(w)^2.$$

Also, we can see the largest possible constant in (1.4) certainly satisfies $\mathcal{S}_{n,s}(O, \Pi_1) \geq \mathcal{S}(\Pi_2)$ and $\mathcal{S}(\Pi_2)$ depends on some part of domain O. Our goal is to demonstrate the interaction between $\mathcal{S}_{n,s}(O, \Pi_1)$ and $\mathcal{S}(\Pi_2)$ in the following theorem.

Theorem 1.1. *Suppose $s \in (0, 1)$, $O \subseteq \mathbb{R}^n$ is an open set. Then, the following holds true*

$$\mathcal{S}_{n,s}(O, \Pi_1) = \mathcal{S}(\Pi_2). \tag{1.6}$$

In the following result, we show the optimal constant $\mathcal{S}_{n,s}(\mathcal{O},\Pi_1)$ in (1.4) is never achieved.

Theorem 1.2. *Suppose $\mathcal{O} \subseteq \mathbb{R}^n$ is an open set. Then, $\mathcal{S}_{n,s}(\mathcal{O},\Pi_1)$ in (1.4) is never achieved.*

Now, our following result gives us surety of the nonexistence solutions to this problem (P_λ) for $\lambda \leq 0$ and star-shaped bounded domain.

Theorem 1.3. *Suppose $\lambda \leq 0$ and $\mathcal{O} \subseteq \mathbb{R}^n$ is a star-shaped bounded domain. Then, (P_λ) has no solutions, where $1 \leq p < 2^* - 1$.*

Proof. We follow the same ideas from [Theorem 1.3 in [25]] to complete this proof and also we can easily complete our proof by using Pohozaév identity and borrowed ideas from [Theorem 1.8 in [30]].

Furthermore, analysis of the existence theory to the problem (P_λ) is dependent on λ. Let us quickly review the approach of Grossi et al. [23]. We used concepts from [25] to identify the existence of solutions and studied the properties of the map $\lambda \mapsto \mathcal{S}_{\lambda,s}$ to determine the inequality $\mathcal{S}_{\lambda,s} < \mathcal{S}(\Pi_2)$. In the case of $p = 1$ and for any range of λ, we show problem (P_λ) has no solution. However, (P_λ) has a solution inside an intermediate range of λ. More precisely, denoting by $\lambda_{1,s}$ the first eigenvalue of $(-\Delta)^s$ with mixed Dirichlet-Neumann boundary condition in a bounded open set $\mathcal{O}$ and by λ' the first eigenvalue of $\mathcal{T}$ in $\mathcal{O}$ with mixed Dirichlet-Neumann boundary condition, see [29], we have the following result.

Theorem 1.4. *Let $p = 1$ (linear case) and $\mathcal{O}$ be nonempty bounded open set. There exists $\lambda^* \in [\lambda_{1,s}, \lambda')$ such that the problem (P_λ) possesses at least one solution if $\lambda \in (\lambda^*, \lambda')$. Furthermore, the following are true*

1. *For $\lambda \geq \lambda'$, problem (P_λ) has no solutions.*
2. *Suppose $0 < \lambda \leq \lambda_{1,s}$ and $B = \{w \in L^{2^*}(\mathcal{O}) : \|w\|_{L^{2^*}(\mathcal{O})} \leq \mathcal{S}(\Pi_2)^{\frac{n-2}{4}}\} \subseteq L^{2^*}(\mathcal{O})$. Then (P_λ) have no solutions in B.*

In the case of the superlinear perturbation, the scenarios are very different, and the following outcome is demonstrated using the variational method, which is based on the Mountain pass geometry, see in [31] and to prove it we state the existence result below.

Theorem 1.5. *Suppose $p \in (1, 2^* - 1)$, $n \geq 3$. Then, problem (P_λ) has a non-trivial solution*

1. *If $\alpha_{s,n} > \tau_{p.n}$, for all $\lambda > 0$.*
2. *If $\alpha_{s,n} \leq \tau_{p.n}$, $\lambda > 0$ large enough.*

where,

$$\alpha_{s,n} = \min(2(1-s), n-2), \quad \text{and} \quad \tau_{p.n} = n - \frac{(p+1)(n-2)}{2}. \tag{1.7}$$

The article is organized in the following way : Section 2 presents the functional framework suitable for mixed classical and fractional Laplace operators with mixed Dirichlet and Neumann boundary conditions. In Section 3, we focused on the analysis of mixed-order Sobolev-type inequality and proved the main results 1.1, 1.2. In Section 4, we present the analysis of the critical problem. Next, we provide an analysis of the critical problem in Section 4 and also for both cases $p = 1$ and $1 < p < 2^* - 1$, we complete proof of Theorem 1.4, Theorem 1.5.

2. Function Space and preliminaries

Throughout this section, we set our notations and formulate the functional setting for (P_λ), which shall be useful throughout the paper. More precisely, we define the function spaces to study (P_λ), and also investigated the existence of an optimal constant for some mixed Sobolev-type inequalities.

For every $s \in (0,1)$, we recall the fractional Sobolev spaces; see [32],

$$H^s(\mathbb{R}^n) = \left\{ u \in L^2(\mathbb{R}^n) : \ \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{2}+s}} \in L^2(\mathbb{R}^n \times \mathbb{R}^n) \right\}.$$

We assume that $\mathcal{O} \cup \Pi_2$ is bounded with a smooth boundary. We define the function space $\mathcal{X}^{1,2}_{\mathcal{D}}(\mathcal{O} \cup \Pi_2 \cup (\partial\mathcal{O} \cap \overline{\Pi_2}))$ as the completion of $C_0^\infty(\mathcal{O} \cup \Pi_2 \cup (\partial\mathcal{O} \cap \overline{\Pi_2}))$ equipped with the following norm

$$\zeta(u)^2 = \|\nabla u\|^2_{L^2(\mathcal{O})} + [u]_s^2, \ \ u \in C_0^\infty(\mathcal{O} \cup \Pi_2 \cup (\partial\mathcal{O} \cap \overline{\Pi_2})),$$

where $[u]_s$ is the Gagliardo seminorm of u defined by

$$[u]_s^2 = \left(\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}} \, dxdy \right).$$

The symbol U is used throughout the article instead of $(\mathcal{O} \cup \Pi_2 \cup (\partial\mathcal{O} \cap \overline{\Pi_2}))$ to keep things simple. and $Q = \mathbb{R}^{2n} \setminus (\Omega^c \times \Omega^c)$.

Specifically for $U \neq \mathbb{R}^n$, we have

$$u \equiv 0 \text{ a.e. in } \ U^c = \Pi_1 \cup (\partial\mathcal{O} \cap \overline{\Pi_1}), \text{ for all } u \in \mathcal{X}^{1,2}_{\Pi_1}(U). \tag{2.1}$$

In order to verify equation (2.1), suppose U is bounded, then we can see as

$$\mathcal{X}^{1,2}_U(\mathcal{O}) = \overline{C_0^\infty(U)}^{\|.\|_{H^1(\mathbb{R}^n)}} = \{u \in H^1(\mathbb{R}^n) : \ w|_U \in H_0^1(U), \ w \equiv 0 \ a.e. \text{ in } U^c\}.$$

The Sobolev inequality allows us to deduce the existence of constant $C = \mathcal{S}(\Pi_2) > 0$. Given by the classical Sobolev inequality, we infer the existence of a constant, independent by $\mathcal{O}$, such that

$$\mathcal{S}(\Pi_2)\|w\|^2_{L^{2^*}(\mathcal{O})} \leq \|\nabla w\|^2_{L^2(\mathcal{O})} \leq \zeta(w)^2 \qquad \forall \ u \in C_0^\infty(U). \tag{2.2}$$

In particular, when $\mathcal{O} = \mathbb{R}^n$, we have

$$\mathcal{X}^{1,2}_{\Pi_1}(\mathbb{R}^n) = \{w \in L^{2^*}(\mathbb{R}^n) : \ \nabla w \in L^2(\mathbb{R}^n) \text{ and } \ [w]_s^2 < \infty\}.$$

We now describe a few essential properties of $\mathcal{X}^{1,2}_{\Pi_1}(U)$ space. The proof of the following proposition can be found in the appendix.

Proposition 2.1. *The space $(\mathcal{X}^{1,2}_{\Pi_1}(U), \langle .,. \rangle)$ is a Hilbert space under the following inner product defined by*

$$\langle w, v \rangle = \int_{\mathcal{O}} \nabla w \cdot \nabla v \, dx + \iint_Q \frac{(w(x)-w(y))(v(x)-v(y))}{|x-y|^{n+2s}} \, dxdy.$$

Proposition 2.2. *If $s \in (0,1)$, then for every $w, v \in \mathcal{X}^{1,2}_{\Pi_1}(U)$, it holds*

$$\int_O v\mathcal{T}w\,dx = \int_O \nabla w\cdot\nabla v\,dx + \iint_Q \frac{(w(x)-w(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dxdy$$
$$-\int_{\partial O\cap\overline{\Pi_2}} v\frac{\partial w}{\partial \nu}\,d\sigma - \int_{\Pi_2} v\mathcal{N}_s w\,dx.$$

Proof. By directly using the integrate by parts formula and $w, v \equiv 0$ a.e. in $\Pi_1 \cup (\partial O \cap \overline{\Pi}_1) = U^c$, we can follow Lemma 3.3 of [33] to obtain the conclusion.

Definition 2.3. *We say that $w \in \mathcal{X}^{1,2}_{\Pi_1}(U)$ is a weak solution to the problem (P_λ) if*

$$\int_O \nabla w\cdot\nabla\varphi\,dx + \iint_Q \frac{(w(x)-w(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy = \int_O (w^{2^*-1}+\lambda w^p)\varphi\,dx, \tag{2.3}$$

$\forall\, \varphi \in \mathcal{X}^{1,2}_{\Pi_1}(U)$ *as a test function.*

We can see the definition 2.3 is well defined. Indeed, if $w, v \in \mathcal{X}^{1,2}_{\Pi_1}(U)$, then

$$\begin{aligned}&\left|\int_O \nabla w\cdot\nabla v\,dx + \iint_Q \frac{(w(x)-w(y))\,(v(x)-v(y))}{|x-y|^{n+2s}}\,dx\,dy\right|\\ &\le \int_O |\nabla w\cdot\nabla v|\,dx + \iint_Q \frac{|w(x)-w(y)|\,|v(x)-v(y)|}{|x-y|^{n+2s}}\,dx\,dy\\ &\le \|\nabla w\|_{L^2(O)}\cdot\|\nabla v\|_{L^2(O)} + [w]_s\cdot[v]_s \le 2\zeta(w)\zeta(v) < \infty.\end{aligned}$$

Moreover, since $\mathcal{X}^{1,2}_{\Pi_1}(O) \hookrightarrow L^{2^*}(O)$ and $p < 2^*-1$, using Hölder's inequality. We also have

$$\begin{aligned}&\int_O |(w^{2^*-1}+\lambda w^p)v|\,dx\\ &\le C\|w\|_{L^{2^*}(O)}\cdot\|v\|_{L^{2^*}(O)} + |\lambda|\,\|w\|_{L^{2^*}(O)}\cdot\|v\|_{L^{\frac{2^*}{2^*-p}}(O)} < \infty.\end{aligned}$$

Now, we introduce the functional $\mathcal{J}_\lambda$ associated to (P_λ) i.e.

$$\mathcal{J}_\lambda : \mathcal{X}^{1,2}_{\Pi_1}(U) \to \mathbb{R}, \text{ such that}$$

$$\mathcal{J}_\lambda(w) = \frac{1}{2}\zeta(w)^2 - \frac{1}{2^*}\int_O |w|^{2^*}\,dx - \frac{\lambda}{p+1}\int_O |w|^{p+1}\,dx, \tag{2.4}$$

we can see that functional $\mathcal{J}_\lambda$ is C^1. If w is a critical point of $\mathcal{J}_\lambda$ then we conclude that w is a weak solution to (P_λ). Due to non-compactness, we can not use the standard minimization technique to prove the existence of the solution to (P_λ) since functional $\mathcal{J}_\lambda$ does not satisfy the Palais-Smale $(PS)_c$ condition.

Remark 2.4. *We can see this by the density argument to $C_0^\infty(U)$ in $\mathcal{X}^{1,2}_{\Pi_1}(U)$, we may extend inequality* (2.2) *to every function $w \in \mathcal{X}^{1,2}_{\Pi_1}(U)$, then we obtain*

$$\mathcal{S}(\Pi_2)\|w\|^2_{L^{2^*}(O)} \le \zeta(w)^2 = \|\nabla w\|^2_{L^2(O)} + [w]_s^2.$$

3. Mixed Sobolev type inequality

The mixed Sobolev inequality (2.2) is the subject of further investigation in this section, that is,

$$\mathcal{S}(\Pi_2)\|w\|^2_{L^{2^*}(O)} \leq \zeta(w)^2, \qquad \forall\, w \in C_0^\infty(U),$$

to proving Theorems 1.1 and 1.2. Because of this, our main aim is to figure out the sharp constant in (2.2), namely,

$$\mathcal{S}_{n,s}(O, \Pi_1) = \inf\{\zeta(w)^2 : \, w \in C_0^\infty(U) \cap \mathcal{M}(O)\}, \tag{3.1}$$

where

$$\mathcal{M}(O) = \{w \in L^{2^*}(O) : \, \|w\|_{L^{2^*}(O)} = 1\}.$$

Since, $C_0^\infty(U)$ is *dense* in $\mathcal{X}^{1,2}_{\Pi_1}(U)$ corresponding to the norm $\zeta(\cdot)$, using embedding $\mathcal{X}^{1,2}_{\Pi_1}(U) \hookrightarrow L^2(O)$, we have

$$\mathcal{S}_{n,s}(O, \Pi_1) = \inf\{\zeta(w)^2 : \, w \in \mathcal{X}^{1,2}_{\Pi_1}(U) \cap \mathcal{M}(O)\}.$$

Furthermore, we know the constant $\mathcal{S}_{n,s}(O, \Pi_1)$ is translation invariant.

Let us recall some useful properties of $\mathcal{S}(\Pi_2)$. For a detail study of these properties, see in [23].

Remark 3.1. *We define*

$$\mathcal{S}(\Pi_2) = \inf\{\|\nabla w\|^2_{L^2(O)} : \, w \in C_0^\infty(U) \cap \mathcal{M}(O)\}, \tag{3.2}$$

We observe that the constant $\mathcal{S}(\Pi_2)$, defined by (3.2)*, depends on some part of the boundary of O, under some suitable hypotheses, is achieved. Lions et al. in [24], establish adequate geometric conditions for the domains O and Π_2 that guarantee the achieving of the optimal constant $\mathcal{S}(\Pi_2)$, utilizing symmetrization arguments derived from the classical isoperimetric inequality. We follow from [Theorem 2.1 in [34]] to get the following result and also for more study about the best Sobolev constant which depends on the domain O, see in [23]. We recall the following result (in our notations) that represents the relations between $\mathcal{S}(\Pi_2)$ and $\mathcal{S}_n = \frac{1}{n(n-2)\pi}\left(\frac{\Gamma(n)}{\Gamma(\frac{n}{2})}\right)^{\frac{2}{n}}$, is the classical Sobolev constant see [26].*

Theorem 3.2. *Let $O \subset \mathbb{R}^n$ be a bounded regular domain, then*

$$\mathcal{S}(\Pi_2) \leq 2^{-\frac{1}{n}}\mathcal{S}_n$$

moreover, if Π_2 is smooth and $\mathcal{S}(\Pi_2) < 2^{-\frac{1}{n}}\mathcal{S}_n$, then the constant $\mathcal{S}(\Pi_2)$ is achieved, where $\mathcal{S}_n$ is the classical best Sobolev constant.

Properties of $\mathcal{S}(\Pi_2)$ and $\mathcal{S}_n$:

1. *The best constant $\mathcal{S}(\Pi_2)$ is dependent on domain O. We have the following explicit expression*

$$\mathcal{S}(\Pi_2) = \left(\frac{n\left(\frac{\xi_n}{2}\right)^{\frac{1}{n}}}{C^{\frac{n-2}{2n}}}\right)^2, \tag{3.3}$$

where, $\Gamma(\cdot)$ is the Euler Gamma function.

2. *For any open set $\mathcal{O}$, we have*

$$\mathcal{S}(\Pi_2) = \inf\{\|\nabla w\|^2_{L^2(\mathcal{O})} : w \in \mathcal{D}^{1,2}_0(U) \cap \mathcal{M}(\mathcal{O})\},$$

where, $\mathcal{D}^{1,2}_0(U)$ is the clousre of the space $C^\infty_0(U)$ under the gradient norm $\|\nabla w\|_{L^2(\mathcal{O})}$.

3. *Suppose $\mathcal{O}$ is a bounded set, then $\mathcal{S}_n$ is* never achieved.
4. *In particular, when $U = \mathbb{R}^n$, then $\mathcal{S}_n$ and $\mathcal{S}(\Pi_2)$ are attained by the family of functions*

$$\mathcal{A} = \{\mathcal{H}_{t,x_0}(x) = t^{\frac{2-n}{2}} F((x-x_0)/t) : t > 0, \ x_0 \in \mathbb{R}^n\},$$

where

$$F(z) = c\,(1+|z|^2)^{\frac{2-n}{2}}, \quad c > 0 \text{ such that } \|F\|_{L^{2^*}(\mathbb{R}^n)} = 1.$$

So, we can show the following Theorem 1.1.

Proof of Theorem 1.1. Since, $\zeta(w)^2 \geq \|\nabla w\|^2_{L^2(\mathcal{O})}$, $\forall\, w \in C^\infty_0(U)$ and also $\mathcal{S}_{n,s}(\mathcal{O},\Pi_1)$ is translation-invariance. So, we have

$$\mathcal{S}_{n,s}(\mathcal{O},\Pi_1) \geq \inf\{\|\nabla w\|^2_{L^2(\mathcal{O})} : w \in C^\infty_0(U) \cap \mathcal{M}(\mathcal{O})\} = \mathcal{S}(\Pi_2).$$

Now, to prove the reverse inequality, without loss of generality, we may suppose that $x_0 = 0 \in \mathcal{O}$, and $r > 0$ such that $B_r(0) \subseteq \mathcal{O}$. We see for any $w \in C^\infty_0(U) \cap \mathcal{M}(\mathcal{O})$, there exists $n_0 = n_0(w) \in \mathbb{N}$ such that

$$\text{supp}(w) \subseteq B_{\kappa r}(0) \quad \forall\, \kappa \geq n_0,$$

now set $w_\kappa(x) = \kappa^{\frac{n-2}{2}} w(\kappa x)$, for $\kappa \geq n_0$, we see that

$$\text{supp}(w_\kappa) \subseteq B_r(0) \subseteq \mathcal{O} \quad \text{and} \quad \|w_\kappa\|_{L^{2^*}(\mathcal{O})} = 1.$$

By the definition of $\mathcal{S}_{n,s}(\mathcal{O},\Pi_1)$, we can see

$$\mathcal{S}_{n,s}(\mathcal{O},\Pi_1) \leq \zeta(w_\kappa)^2 = \|\nabla w_\kappa\|^2_{L^2(\mathcal{O})} + [w_\kappa]^2_s = \|\nabla w\|^2_{L^2(\mathcal{O})} + \kappa^{2s-2}[w]^2_s, \ \forall\, n \geq n_0.$$

Then, we have

$$\mathcal{S}_{n,s}(\mathcal{O},\Pi_1) \leq \|\nabla w\|^2_{L^2(\mathcal{O})},$$

as $\kappa \to \infty$, since $0 < s < 1$. As $w \in C^\infty_0(U) \cap \mathcal{M}(\mathcal{O})$ is arbitrary, then using Remark 3.1, we obtain

$$\mathcal{S}_{n,s}(\mathcal{O},\Pi_1) \leq \mathcal{S}(\Pi_2),$$

and we proved the required inequality: $\mathcal{S}_{n,s}(\mathcal{O},\Pi_1) = \mathcal{S}(\Pi_2)$.

We are now able to show Theorem 1.2.

Proof of Theorem 1.2. By contrary argument, if we suppose there exists a nonzero function $w_0 \in \mathcal{X}^{1,2}_{\Pi_1}(U)$ such that $\|w_0\|_{L^{2^*}(\mathcal{O})} = 1$ and $\zeta(w_0)^2 = \|\nabla w_0\|^2_{L^2(\mathcal{O})} + [w_0]^2_s = \mathcal{S}(\Pi_2)$. We have $\mathcal{X}^{1,2}_{\Pi_1}(U) \subseteq \mathcal{D}^{1,2}_0(U)$ by straightforwardness of the fact that $\zeta(w_0) \geq \|\nabla w_0\|^2_{L^2(\mathcal{O})}$, see above properties (2), we have

$$\mathcal{S}(\Pi_2) \leq \|\nabla w_0\|^2_{L^2(\mathcal{O})} \leq \|\nabla w_0\|^2_{L^2(\mathcal{O})} + [w_0]^2_s = \zeta(w_0)^2 = \mathcal{S}(\Pi_2),$$

that implies $[w_0]_s = 0$. Hence, w_0 must be *constant in* $\mathbb{R}^n$. We have a contradiction with the fact $\|w_0\|_{L^{2^*}(\mathcal{O})} = 1$.

Remark 3.3. *Our Theorem 1.2 says that constant* $\mathcal{S}_{n,s}(\mathcal{O},\Pi_1) = \mathcal{S}(\Pi_2)$ *is* never achieved *in the space* $\mathcal{X}^{1,2}_{\Pi_1}(U)$ *but if* $(U = \mathbb{R}^n)$ *we show* $\mathcal{S}(\Pi_2)$ *achieve in a limiting sense. More precisely, if* $\mathcal{A} = \{\mathcal{H}_{\theta,x_0}\}$ *which is defined in* (4)*, we have*

$$\zeta(\mathcal{H}_{\theta,x_0})^2 \to \mathcal{S}(\Pi_2) \quad as\ \theta \to \infty.$$

Indeed,

$$|\mathcal{F}(\theta)| \le \mathcal{D}\ \min\{1, |\theta|^{2-n}\} \qquad and \qquad |\nabla F(\theta)| \le \mathcal{D}\ \min\{|\theta|,\ |\theta|^{1-n}\},$$

for some $\mathcal{D} > 0$*. Therefore (constant* $\mathcal{D}$ *varies in the following calculations)*

$$\begin{aligned}
[F(\theta)]_s^2 &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|F(x+y)-F(x)|^2}{|y|^{n+2s}}\,dx\,dy \\
&\le \iint_{\mathbb{R}^n\times B_1} \left|\int_0^1 \nabla F(x+\theta y)\cdot y\,d\theta\right|^2 \frac{dx\,dy}{|y|^{n+2s}} \\
&\qquad +2\iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_1)} (|F(x+y)|^2+|F(x)|^2)\frac{dx\,dy}{|y|^{n+2s}} \\
&\le \iiint_{\mathbb{R}^n\times B_1\times(0,1)} \min\{|x+\theta y|^2,\ |x+\theta y|^{2(1-n)}\}\,\frac{dx\,dy\,d\theta}{|y|^{n+2s-2}} \\
&\qquad +4\iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_1)} |F(t)|^2\frac{dt\,dy}{|y|^{n+2s}} \\
&\le \iiint_{\mathbb{R}^n\times B_1\times(0,1)} \min\{|t|^2,\ |t|^{2(1-n)}\}\,\frac{dt\,dy\,d\theta}{|y|^{n+2s-2}} \\
&\qquad +\mathcal{D}\iint_{\mathbb{R}^n\times(\mathbb{R}^n\setminus B_1)} \min\{1,|t|^{2(2-n)}\}\frac{dt\,dy}{|y|^{n+2s}}.
\end{aligned}$$

So, $[F(\theta)]_s^2 < +\infty$*. Hence,* $F \in \mathcal{X}^{1,2}_{\Pi_1}(\mathbb{R}^n)$ *and consequently* $\mathcal{H}_{\theta,x_0} \in \mathcal{X}^{1,2}_{\Pi_1}(\mathbb{R}^n)$*,* $\forall\ \theta > 0$ *and* $x_0 \in \mathbb{R}^n$*. Moreover, recalling that*

$$\mathcal{H}_{\theta,x_0}(x) = \theta^{\frac{2-n}{2}} F\Big(\frac{x-x_0}{\theta}\Big) \quad and \quad \|\mathcal{H}_{\theta,x_0}\|_{L^{2^*}(\mathbb{R}^n)} = \|F\|_{L^{2^*}(\mathbb{R}^n)} = 1,$$

follows same idea of the proof of Theorem 1.1 we have

$$\zeta(\mathcal{H}_{\theta,x_0})^2 = \zeta(\mathcal{H}_{\theta,0})^2 = \|\nabla F\|^2_{L^2(\mathbb{R}^n)} + \theta^{2s-2}[F]_s^2.$$

From this, since $F = \mathcal{H}_{1,0}$*. We obtain*

$$\zeta(\mathcal{H}_{\theta,x_0})^2 \to \|\nabla F\|^2_{L^2(\mathbb{R}^n)} = \mathcal{S}(\Pi_2),\ as\ \theta \to \infty.$$

In the following section, we discussed the critical problems involving mixed operator settings with mixed boundary conditions.

4. Study to critical problems with mixed type operator $\mathcal{T}$ critical

We now develop our study for the existence and non-existence of solutions to the problem (P_λ).

Suppose O is a bounded set with smooth boundary, λ is a real parameter, and $1 \leq p < 2^* - 1$. Moreover, we adopt all the definitions and notation of Section s 2, 3. Finally, we introduce nonnegative space

$$\mathcal{X}_+^{1,2}(U) = \{w \in \mathcal{X}_{\Pi_1}^{1,2}(U) : \ w \geq 0 \text{ a.e. in } U\}.$$

We are defining the definition of solution to (P_λ).

Definition 4.1. *We say that a $u \in \mathcal{X}_+^{1,2}(U)$ is a weak solution to (P_λ) if it satisfies the following properties*

1. $|\{x \in U : u(x) > 0\}| > 0$,
2. *For every test function $\varphi \in \mathcal{X}_+^{1,2}(U)$ we have*

$$\begin{aligned}&\int_O \nabla u \cdot \nabla \varphi dx + \iint_Q \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} dxdy \\ &= \int_O \left(u^{2^*-1} + \lambda u^p\right) \varphi \, dx.\end{aligned}$$

To study the existence of solutions to (P_λ), first we prove the solution is non-negative and bounded by the following theorem.

Theorem 4.2. *Let $w_0 \in \mathcal{X}_+^{1,2}(U)$ be the solution to (P_λ) and $\lambda \in \mathbb{R}$, $1 \leq p < 2^* - 1$. Then, the following facts hold:*

1. $w_0 \in L^\infty(O)$,
2. *if $\lambda \geq 0$, then $w_0 > 0$ a.e. in U.*

Proof. We can easily prove $w_0 \in L^\infty(O)$ by using [Theorem 4.1] [28] and also easily it can be followed from well know result Moser's iteration method; see [Theorem 1.1 in [35]]. By the fact that just using the definition of function space, $w_0 \equiv 0$ a.e. in U^c, implies that $w_0 \in L^\infty(\mathbb{R}^n)$. Suppose $\lambda \geq 0$ and recall that w_0 is a solution of (P_λ) (in the sense of Definition (4.1)), and since $w_0 \geq 0$ a.e. in U, for every $\varphi \in \mathcal{X}_+^{1,2}(U)$, we find

$$\int_O \nabla w \cdot \nabla \varphi \, dx + \iint_Q \frac{(w(x) - w(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \, dx \, dy = \int_O (w^{2^*-1} + \lambda w^p) \varphi \, dx \geq 0, \tag{4.1}$$

for each $\varphi \in \mathcal{X}_+^{1,2}(U)$. By using the Strong Maximum Principle, refer to [29], we have $w_0 > 0$ a.e. in U.

Thanks to Theorem 4.2, we are able to prove Theorem 1.3.

Next, for $\lambda > 0$, we can start our study separately for the linear case $p = 1$ and the superlinear case $1 < p < 2^* - 1$ of the solvability of (P_λ).

4.1. *Case: (if* $p = 1$*)*

Let us start studying to solve the following problem (P_λ)

$$\begin{cases} \mathcal{T}w = w^{2^*-1} + \lambda w, \quad w > 0 \quad \text{in } \mathcal{O}, \\ w = 0 \ \text{ in } \ U^c, \\ \mathcal{N}_s(w) = 0 \ \text{ in } \ \Pi_2, \\ \frac{\partial w}{\partial \nu} = 0 \ \text{ in } \ \partial\mathcal{O} \cap \overline{\Pi_2}. \end{cases} \tag{P_λ}$$

In this context, we study for any $\lambda > 0$, solutions to problem (P_λ) correlates the first eigenvalues $\lambda_{1,s}$ and λ' of $(-\Delta)^s$ and of $\mathcal{T}$ with mixed boundary conditions respectively.

Definition 4.3. *For $\mathcal{O}$ is the bounded open set, we define*

(1) [see [36]], the first eigenvalue of $(-\Delta)^s$ with mixed boundary conditions as

$$\lambda_{1,s} = \inf\{[w]_s^2 : \ u \in C_0^\infty(U) \text{ and } \|w\|_{L^2(\mathcal{O})} = 1\}; \tag{4.2}$$

(2) [see [29]] the first eigenvalue of $\mathcal{T}$ with mixed boundary conditions as

$$\lambda' = \inf\{\zeta(w)^2 : \ w \in C_0^\infty(U) \text{ and } \|w\|_{L^2(\mathcal{O})} = 1\}. \tag{4.3}$$

We subsequently provide a brief overview of the main properties of $\lambda_{1,s}$, and λ' in the following remark.

Remark 4.4. *(Some Properties of $\lambda_{1,s}$ and λ'). As regards $\lambda_{1,s}$ we observe that, since the map $u \mapsto [u]_s = N(u)$ is a norm for $C_0^\infty(U)$ that is equivalent to the full $\|.\|_{H^s(\mathbb{R}^n)}$, we refer to [32, Theorem 6.5] and recalling U is bounded since $\mathcal{O} \cup \Pi_2$ is bounded, one has*

$$\lambda_{1,s} = \inf\{[w]_s^2 : \ w \in \mathcal{D}_0^{s,2}(U) \text{ and } \|w\|_{L^2(\mathcal{O})} = 1\},$$

where $\mathcal{D}_0^{s,2}(U) \subseteq L^2(\mathcal{O})$ is the completion of $C_0^\infty(U)$ under the norm $N(u)$. Furthermore, we can easily see that $\lambda_{1,s}$ is truly achieved in this bigger space $\mathcal{D}_0^{s,2}(U)$ since the embedding $\mathcal{D}_0^{s,2}(U) \hookrightarrow L^2(\mathcal{O})$ is compact. So,

$$\exists\ \varphi_0 \in \mathcal{D}_0^{s,2}(U) : \ \|\varphi_0\|_{L^2(\mathcal{O})} = 1 \text{ and } [\varphi_0]^2 = \lambda_{1,s} > 0.$$

Indeed, it is possible to choose the function φ_0 to be strictly positive a.e. in U.

In addition, see in [36] or [Proposition 9 in [37]], since φ_0 is a constrained minimizer of the functional $w \mapsto [w]_s^2$ and using the Lagrange Multiplier method,

$$\iint_Q \frac{(\varphi_0(x) - \varphi_0(y))(v(x) - v(y))}{|x-y|^{n+2s}}\, dx\, dy = \lambda_{1,s} \int_{\mathcal{O}} \varphi_0 v\, dx \qquad \forall\ v \in \mathcal{D}_0^{s,2}(U),$$

and it shows that φ_0 is a weak solution to the following eigenvalue problem

$$\begin{cases} (-\Delta)^s w = \lambda_{1,s} w \ \ w > 0 \ \ \text{in } \mathcal{O} \\ w = 0 \ \ \text{in } \ U^c, \\ \mathcal{N}_s(w) = 0 \ \ \text{in } \ \Pi_2, \\ \frac{\partial w}{\partial \nu} = 0 \ \ \text{in } \ \partial\mathcal{O} \cap \overline{\Pi_2}. \end{cases}$$

Similarly, we can see, since $\zeta(\cdot)$ defines a norm on $C_0^\infty(U)$ that is equivalent to the full H^1-norm in $\mathbb{R}^n$, and

$$\lambda' = \inf\{\zeta(w)^2 : \ w \in \mathcal{X}^{1,2}_{\Pi_1}(\mathcal{O}) \text{ and } \|w\|_{L^2(\mathcal{O})} = 1\}.$$

Furthermore, it is easy to see that λ' is truly achieved in the larger space $\mathcal{X}^{1,2}_{\Pi_1}(U)$ and we know embedding $\mathcal{X}^{1,2}_{\Pi_1}(U) \hookrightarrow L^2(\mathcal{O})$ is compact, that is,

$$\exists\ \psi_0 \in \mathcal{X}^{1,2}_{\Pi_1}(U) : \ \|\psi_0\|_{L^2(\mathcal{O})} = 1 \text{ and } \zeta(\psi_0)^2 = \lambda' > 0.$$

Indeed, it is possible to choose the function $\psi_0 > 0$ a.e. in U.

Moreover, see in [29], since ψ_0 is a constrained minimizer of the functional $\zeta(\cdot)^2$ and using the Lagrange Multiplier method,

$$\int_{\mathcal{O}} \nabla\psi_0 \cdot \nabla v \, dx + \iint_{\mathcal{Q}} \frac{(\psi_0(x) - \psi_0(y))(v(x) - v(y))}{|x-y|^{n+2s}} \, dx \, dy$$
$$= \lambda' \int_{\mathcal{O}} \psi_0 v \, dx \qquad \forall\ v \in \mathcal{X}^{1,2}_{\Pi_1}(U),$$

and it shows that ψ_0 is a solution to the following eigenvalue problem

$$\begin{cases} \mathcal{T} w = \lambda' w, \ \ w > 0 \text{ in } \mathcal{O}, \\ w = 0 \ \text{ in } \ U^c, \\ \mathcal{N}_s(w) = 0 \ \text{ in } \ \Pi_2, \\ \frac{\partial w}{\partial \nu} = 0 \ \text{ in } \ \partial\mathcal{O} \cap \overline{\Pi_2}. \end{cases}$$

We are approaching the proof of Theorem 1.4. Several independent results will help us to get this. Initially, we establish a lemma that connects the existence of solutions for (P_λ) with the existence of constrained minimizers for an appropriate functional $\mathcal{R}_\lambda$. We define

$$\mathcal{R}_\lambda(w) = \zeta(w)^2 - \lambda\|w\|^2_{L^2(\mathcal{O})}, \qquad \text{for any } w \in \mathcal{X}^{1,2}_{\Pi_1}(U), \tag{4.4}$$

constrained to the manifold $\mathcal{V}(\mathcal{O}) = \mathcal{X}^{1,2}_{\Pi_1}(U) \cap \mathcal{M}(\mathcal{O})$.

We identify some useful properties of $\mathcal{S}(\Pi_2)(\lambda) = \inf_{u\in\mathcal{V}(\mathcal{O})} \mathcal{R}_\lambda(w)$ in the following remark.

Remark 4.5. *By using the definition of $\mathcal{S}(\Pi_2)(\lambda)$ and Hölder's inequality, for every $w \in \mathcal{V}(\mathcal{O})$, we have*

1. $\mathcal{S}(\Pi_2)(\lambda) \leq \mathcal{S}(\Pi_2), \ \forall\ \lambda > 0,$
2. $\mathcal{S}(\Pi_2)(\lambda) \leq \mathcal{S}(\Pi_2)(\lambda^*), \ \forall\ 0 < \lambda^* < \lambda.$

Additionally, keep in mind take note of Remark 4.4) and the definition of λ' in (4.3), λ' is attained in the space $\mathcal{X}^{1,2}_{\Pi_1}(U)$, easy to check

$$\mathcal{S}(\Pi_2)(\lambda) \geq 0 \iff 0 < \lambda \leq \lambda'.$$

Using properties of $\mathcal{S}(\Pi_2)(\lambda)$, we prove the next result.

Lemma 4.6. *We consider*

$$\mathcal{S}(\Pi_2)(\lambda) = \inf_{w \in \mathcal{V}(O)} \mathcal{R}_\lambda(w), \quad \forall\, \lambda > 0. \tag{4.5}$$

Suppose that $\mathcal{S}(\Pi_2)(\lambda) > 0$ and $\mathcal{S}(\Pi_2)(\lambda)$ is achieved; that is, there exists some function $u \in \mathcal{V}(O)$ such that $\mathcal{R}_\lambda(u) = \mathcal{S}(\Pi_2)(\lambda)$. Then, there exists a solution of (P_λ).

Proof. By assumption, we know that there exists $u \in \mathcal{V}(O))$ as a constrained minimizer for $\mathcal{R}_\lambda$, that is, $\mathcal{R}_\lambda(u) = \mathcal{S}(\Pi_2)(\lambda)$. We can easily see $\mathcal{R}_\lambda(|u|) \leq \mathcal{R}_\lambda(u)$, then we may suppose that $u \geq 0$ a.e. in O. Furthermore, by the Lagrange multiplier method, $\exists\, \mu \in \mathbb{R}$, and we have

$$\int_O \nabla u \cdot \nabla \varphi + \iint_Q \frac{(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} = \int_O (\mu u^{2^*-1} + \lambda u)\varphi\, dx, \qquad \forall\, \varphi \in \mathcal{X}^{1,2}_{\Pi_1}(U). \tag{4.6}$$

Now, take $\varphi = u$ as a test function in (4.6), we obtain

$$\mu = \mu \|u\|^{2^*}_{L^{2^*}(\mathbb{R}^n)} = \zeta(u)^2 - \lambda \|u\|^2_{L^2(\mathbb{R}^n)} = \mathcal{R}_\lambda(u) = \mathcal{S}(\Pi_2)(\lambda) > 0. \tag{4.7}$$

As a consequence, setting $w = \mathcal{S}(\Pi_2)(\lambda)^{\frac{n-2}{4}} u$, we see that $w \geq 0$ a.e. in O and for every $\varphi \in \mathcal{X}^{1,2}_{\Pi_1}(U)$ using (4.6) and (4.7), we have

$$\frac{1}{\mathcal{S}(\Pi_2)(\lambda)^{\frac{n-2}{4}}} \left(\int_O \nabla w \cdot \nabla \varphi + \iint_Q \frac{(w(x) - w(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} \right) = \int_O \mu \left(\frac{w}{\mathcal{S}(\Pi_2)(\lambda)^{\frac{n-2}{4}}} \right)^{2^*-1} \varphi\, dx$$
$$+ \int_O \frac{\lambda}{\mathcal{S}(\Pi_2)(\lambda)^{\frac{n-2}{4}}} w\varphi\, dx, \ \forall\, \varphi \in \mathcal{X}^{1,2}_{\Pi_1}(U),$$

then we obtain that

$$\int_O \nabla w \cdot \nabla \varphi + \iint_Q \frac{(w(x) - w(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+2s}} = \int_O (w^{2^*-1} + \lambda w)\varphi\, dx, \ \forall\, \varphi \in \mathcal{X}^{1,2}_{\Pi_1}(U).$$

Hence, we obtain w as a solution to (P_λ). In this manner, we have completed the proof of this lemma.

On account of Lemma 4.6, the sign of the real number $\mathcal{S}(\Pi_2)(\lambda)$ plays a crucial role to study the solvability of (P_λ). In this perspective, we have already identified by Remark 4.5 that

$$\mathcal{S}(\Pi_2)(\lambda) \geq 0 \iff 0 < \lambda \leq \lambda'.$$

We are provided with additional information as a result of the following outcome together with Remark 4.5.

Lemma 4.7. *We have*

$$\mathcal{S}(\Pi_2)(\lambda) = \mathcal{S}(\Pi_2) > 0 \ \ \forall\, \lambda \in (0, \lambda_{1,s}].$$

Proof. Suppose $\lambda \in (0, \lambda_{1,s}]$ and using Remark 4.5, we have $\mathcal{S}(\Pi_2)(\lambda) \leq \mathcal{S}(\Pi_2)$. On the other hand, to prove the reverse part of it, using the definition of $\lambda_{1,s}$ that is defined by (4.2). For any $w \in C_0^\infty(U) \cap \mathcal{M}(O)$, we have

$$\mathcal{R}_\lambda(w) = \|\nabla w\|^2_{L^2(O)} + \left([w]_s^2 - \lambda \|w\|^2_{L^2(O)}\right) \geq \|\nabla w\|^2_{L^2(O)} + (\lambda_{1,s} - \lambda)\, \|w\|^2_{L^2(O)} \geq \|\nabla w\|^2_{L^2(O)}.$$

But we know, $C_0^\infty(U)$ is dense in $\mathcal{X}^{1,2}_{\Pi_1}(U)$, we can see

$$\mathcal{S}(\Pi_2)(\lambda) = \inf\{\mathcal{R}_\lambda(w) : w \in C_0^\infty(U) \cap \mathcal{M}(O)\} \geq \inf\left\{\|\nabla w\|^2_{L^2(O)} : w \in C_0^\infty(U) \cap \mathcal{M}(O)\right\} = \mathcal{S}(\Pi_2).$$

Thus we get the desired result $\mathcal{S}(\Pi_2)(\lambda) = \mathcal{S}(\Pi_2)$.

Lemma 4.8. *The function* $\lambda \mapsto \mathcal{S}(\Pi_2)(\lambda)$ *is continuous on* $(0, \infty)$.

Proof. First, we shall show the left continuity. Suppose $\lambda_0 > 0$, $\epsilon > 0$. So, using the definition of $\mathcal{S}(\Pi_2)(\lambda_0)$ there exists $v = v_{\epsilon,\lambda_0} \in \mathcal{V}(O)$ such that

$$\mathcal{S}(\Pi_2)(\lambda_0) \leq \mathcal{R}_{\lambda_0}(v) < \mathcal{S}_n(\lambda_0) + \frac{\epsilon}{2}.$$

using the monotonicity of $\mathcal{S}(\Pi_2)(\cdot)$ and $\forall\ \lambda < \lambda_0$, deduce that

$$\begin{aligned} 0 < \mathcal{S}(\Pi_2)(\lambda) - \mathcal{S}(\Pi_2)(\lambda_0) \leq \mathcal{R}_\lambda(v) - \mathcal{S}(\Pi_2)(\lambda_0) &= (\mathcal{R}_{\lambda_0}(v) - \mathcal{S}(\Pi_2)(\lambda_0)) + (\lambda_0 - \lambda)\|v\|^2_{L^2(O)} \\ &< \frac{\epsilon}{2} + (\lambda_0 - \lambda)\|v\|^2_{L^2(O)}. \end{aligned}$$

As a consequence, setting $\delta_\epsilon = \epsilon/\left(2\|v\|^2_{L^2(O)}\right)$, we conclude that

$$0 < \mathcal{S}(\Pi_2)(\lambda) - \mathcal{S}(\Pi_2)(\lambda_0) < \epsilon \text{ for every } \lambda_0 - \delta_\epsilon < \lambda \leq \lambda_0.$$

Hence, $\mathcal{S}(\Pi_2)(\cdot)$ is left continuous at λ_0.

We find some other properties of $\mathcal{S}(\Pi_2)(\lambda)$ in the following remark.

Remark 4.9. *By combining Remark 4.5 and Lemma 4.7, we conclude the following estimates:*

(i) $\mathcal{S}(\Pi_2)(\lambda) = \mathcal{S}(\Pi_2),\ \forall\ 0 < \lambda \leq \lambda_{1,s}$,

(ii) $\mathcal{S}(\Pi_2)(\lambda) \geq 0,\ \forall\ 0 < \lambda \leq \lambda'$,

(iii) $\mathcal{S}(\Pi_2)(\lambda) < 0,\ \forall\ \lambda > \lambda'$.

Proof of Theorem 1.4: First, we define

$$\mu = \sup\{\lambda > 0 : \ \mathcal{S}(\Pi_2)(\lambda^*) = \mathcal{S}(\Pi_2) \ \text{ for all } 0 < \lambda^* \leq \lambda\}. \tag{4.8}$$

Based on Lemma 4.7, it is evident that $\lambda_{1,s} \leq \mu < \infty$. Furthermore, from Lemma 4.8 that $\mathcal{S}(\Pi_2)(\cdot)$ is continuous, we can establish $\mathcal{S}(\Pi_2)(\mu) = \mathcal{S}(\Pi_2)$.

An observation worth noting is that $\mathcal{S}(\Pi_2)(\cdot) \geq 0$ on $(0, \lambda']$ and $\mathcal{S}(\Pi_2)(\cdot) < 0$ on (λ', ∞), again by the continuity of $\mathcal{S}(\Pi_2)(\cdot)$ we deduce that $\mathcal{S}(\Pi_2)(\lambda') = 0$, consequently recalling that $\mathcal{S}(\Pi_2)(\cdot)$ is nonincreasing on $(0, \infty)$, we have $\mu \in [\lambda_{1,s}, \lambda')$. Now, we aim to show that the assertion of Theorem 1.4 satisfies with choice of λ^*. Specifically, we treat the following three cases separately:

Case I: (If $0 < \lambda \leq \lambda_{1,s}$), by the contrary statement, let us assume that $\exists\ w \in B$ that is the soluton to problem (P_λ). Then, setting

$$u = w/\|w\|_{L^{2^*}(O)}, \ \text{ and } \ B = \{w \in L^{2^*}(O) : \ \|w\|_{L^{2^*}(O)} \leq \mathcal{S}(\Pi_2)^{(n-2)/4}\}.$$

Based on the characteristics of w and using the definition of weak solution (2.3) (take $\varphi = w$ and $p = 1$), we have

$$\begin{aligned}\mathcal{R}_\lambda(u) &= \frac{1}{\|w\|^2_{L^{2^*}(O)}}\mathcal{R}_\lambda(w) = \frac{1}{\|w\|^2_{L^{2^*}(O)}}(\zeta(u)^2 - \lambda\|w\|^2_{L^2(O)}) \\ &= \frac{1}{\|w\|^2_{L^{2^*}(O)}}\int_O w^{2^*}\,dx \\ &= \|w\|^{2^*-2}_{L^{2^*}(O)} \le \mathcal{S}(\Pi_2) \qquad (\text{since } u \in B),\end{aligned}$$

Consequently, from Lemma 4.7, we obtain

$$\mathcal{R}_\lambda(u) = \mathcal{S}(\Pi_2) = \mathcal{S}(\Pi_2)(\lambda). \tag{4.9}$$

Hence, we conclude that $\mathcal{S}(\Pi_2)(\lambda)$ is achieved at u. Now, recalling (3.2), and since $w \in \mathcal{X}^{1,2}_{\Pi_1}(U) \subseteq \mathcal{D}^{1,2}_0(U)$, from (4.9), we obtain

$$\begin{aligned}\mathcal{S}(\Pi_2) &\le \|\nabla w\|^2_{L^2(O)} = \mathcal{R}_\lambda(w) - ([w]^2_s - \lambda\|w\|^2_{L^2(O)}) \\ &\le \mathcal{R}_\lambda(w) = \mathcal{S}(\Pi_2),\end{aligned}$$

which shows that $\mathcal{S}(\Pi_2)$ is achieved. using the fact that $\lambda \le \lambda_{1,s}$ and $w \in \mathcal{X}^{1,2}_{\Pi_1}(U) \subseteq \mathcal{D}^{s,2}_0(U)$.
Thus, we conclude that $v \in \mathcal{D}^{1,2}_0(U)$ achieves the optimal Sobolev constant $\mathcal{S}(\Pi_2)$. We get a contradiction since O is bounded and from Remark 3.1 that $\mathcal{S}(\Pi_2)$ is never achieved in $\mathcal{X}^{1,2}_{\Pi_1}(U)$.

Case II: (If $\mu < \lambda < \lambda'$), from Lemma 4.8, we know $\mathcal{S}(\Pi_2)(\cdot)$ is continuous and $0 \le \mathcal{S}(\Pi_2)(\lambda) < \mathcal{S}(\Pi_2)$. (Using the definition of μ), i.e., (4.8), we have from this, following the same ideas of [Lemma 1.2, [26]], the best constant $\mathcal{S}(\Pi_2)(\lambda)$ is achieved, i.e., $v \in \mathcal{V}(O)$ such that

$$\mathcal{R}_\lambda(v) = \mathcal{S}(\Pi_2)(\lambda).$$

Since, we have $\lambda < \lambda'$ and $v \not\equiv 0$, then we deduce that

$$\begin{aligned}\mathcal{S}(\Pi_2)(\lambda) &= \|v\|^2_{L^2(O)}\Big(\frac{\zeta(v)^2}{\|v\|^2_{L^2(O)}} - \lambda\Big) \\ &\ge \|v\|^2_{L^2(O)}(\lambda' - \lambda) > 0.\end{aligned}$$

Thus, $\mathcal{S}(\Pi_2)(\lambda) > 0$ and it is achieved. Using Lemma 4.7, a solution to (P_λ), does exist.

Case III: (If $\lambda \ge \lambda'$) arguing by contradiction, we may assume that there exists a solution to (P_λ) i.e., that $0 \ne \varphi_0 \in \mathcal{X}^{1,2}_{\Pi_1}(U)$ such that $\varphi_0 > 0$ a.e. in U. We can see that

$$\begin{aligned}&\int_O \nabla\varphi_0 \cdot \nabla u\,dx + \iint_Q \frac{(\varphi_0(x) - \varphi_0(y))(u(x) - u(y))}{|x-y|^{n+2s}}\,dx\,dy \\ &\qquad = \lambda' \int_O \varphi_0 u\,dx \qquad \forall\ u \in \mathcal{X}^{1,2}_{\Pi_1}(U),\end{aligned} \tag{4.10}$$

that is, φ_0 is an eigenfunction for mixed operator $\mathcal{T}$ corresponding to the eigenvalue λ'.

In particular, choosing $w = u \in \mathcal{X}^{1,2}_{+}(U)$, more specifically, using the above-mentioned identity (4.10) and recalling that u is a solution to problem (P_λ), using the definition (2.3) (for $p = 1$),
we obtain

$$\begin{aligned}\lambda' \int_{O} \varphi_0 w\,dx &= \int_{O} \nabla\varphi_0 \cdot \nabla w\,dx + \iint_{Q} \frac{(\varphi_0(x) - \varphi_0(y))(w(x) - w(y))}{|x-y|^{n+2s}}\,dx\,dy \\ &= \int_{O} (w^{2^*-1} + \lambda w)\varphi_0\,dx \\ &> \lambda \int_{O} w\varphi_0\,dx,\end{aligned}$$

since by using Theorem 4.2, we have $w, \varphi_0 > 0$ a.e. in O, which is contradiction with $\lambda \geq \lambda'$.

4.2. ***Superlinear case*** $(1 < p < 2^* - 1)$***:***

In this section, we examine the proof of Theorem 1.5. We use the Mountain Pass Theorem to establish the existence of a solution for (P_λ) if $1 < p < 2^* - 1$ (superlinear case). The key obstacle when utilizing the Mountain Pass Theorem is confirming the fulfillment of a $(PS)_c$ condition at a specified level c, much like the scenario observed in the purely local case, see [26]. In particular, it is essential to demonstrate that the Palais-Smale condition holds for every c that is strictly less than the first critical level as defined by the Mountain Pass Theorem. Keeping with the methodology in [26], we examine a slightly improved following energy functional about (2.4), namely

$$\mathcal{J}_\lambda(w) = \frac{1}{2}\zeta(w)^2 - \frac{1}{2^*}\int_{O} (w_+)^{2^*}\,dx - \frac{\lambda}{p+1}\int_{O} (w_+)^{p+1}\,dx \quad \forall\, w \in \mathcal{X}^{1,2}_{\Pi_1}(U), \tag{4.11}$$

where $w_+ = \max\{w, 0\}$ denotes the positive part of w. Now, it is simple to see that any nonzero critical point of $\mathcal{J}_\lambda$ is a solution to (P_λ). In particular, since we are assuming $\lambda > 0$, we have $\mathcal{T}w \geq 0$ in O (in a weak sense) since we can see it by using the definition of the weak solution 2.3 and in the proof of Theorem 4.2. Note that $w \equiv 0$ a.e. in U^c, allows us to apply the Weak Maximum Principle with mixed boundary conditions, which follow from [38, Theorem 1.2]. We have $w \geq 0$ a.e. in $\mathbb{R}^n$ then we deduce that $w_+ \equiv w$. Hence $w > 0$ a.e. in U is a solution to problem (P_λ). In the following lemma, our aim is to establish that the functional $\mathcal{J}_\lambda$ satisfies a local (PS) condition at level $c \in \mathbb{R}$, which is related to the best Sobolev constant $\mathcal{S}(\Pi_2)$.

Lemma 4.10. *The functional $\mathcal{J}_\lambda$ satisfies the $(PS)_c$ for every $c < \frac{1}{n}(\mathcal{S}(\Pi_2))^{n/2}$.*

Proof. It is a well-known result. So, we can easily prove this lemma by following Lemma 4.10 in [25].

The final step required to show Theorem 1.5 is demonstrating a path with energy below the $\mathcal{S}(\Pi_2)/n$ critical obstacle. To do this, we first provide an auxiliary function similar to the one employed in [26].

Now, we define a non-increasing cut-off function by

$$\psi_0(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq \frac{1}{2}, \\ 0, & \text{if } t \geq 1. \end{cases} \tag{4.12}$$

Then, we consider a function $\psi_\rho(x) : \mathbb{R}^n \to \mathbb{R}$ by

$$\psi_\rho(x) = \psi_0\left(\frac{|x|}{\rho}\right), \tag{4.13}$$

for given $\rho > 0$ such that $\overline{B_\rho(0)} \subset \mathcal{O}$. Finally, for all $\varepsilon > 0$, let

$$U_\varepsilon(x) = \frac{\varepsilon^{(n-2)/2}}{(|x|^2 + \varepsilon^2)^{(n-2)/2}} \qquad \text{and} \qquad \varrho_\varepsilon(x) = \frac{\psi_\rho(x)\, U_\varepsilon(x)}{\|\psi_\rho\, U_\varepsilon\|_{L^{2^*}(\mathcal{O})}} \in \mathcal{X}^{1,2}_{\Pi_1}(U).$$

Lemma 4.11. *Suppose $n \geq 3$ and $p \in (1, 2^* - 1)$. Furthermore, constants $\alpha_{s,n}$, $\tau_{p.n}$ are given in* (1.7). *Then, the following statements hold true.*

(1) If $\alpha_{s,n} > \tau_{p.n}$, then there exists $\varepsilon > 0$ such that

$$\sup_{t\geq 0} \mathcal{J}_\lambda(t\varrho_\varepsilon) < \frac{1}{n}(\mathcal{S}(\Pi_2))^{n/2}, \quad \forall\ \lambda > 0.$$

(2) If, instead, $\alpha_{s,n} \leq \tau_{p.n}$, then there exist $\varepsilon > 0$, $\lambda_0 > 0$ such that

$$\sup_{t\geq 0} \mathcal{J}_\lambda(t\varrho_\varepsilon) < \frac{1}{n}(\mathcal{S}(\Pi_2))^{n/2}, \quad \forall\ \lambda \geq \lambda_0.$$

Proof. We have

$$\begin{aligned} \mathcal{J}_\lambda(t\varrho_\varepsilon) = \frac{t^2}{2}\int_{\mathcal{O}} |\nabla \varrho_\varepsilon|^2\, dx + \frac{t^2}{2} \iint_{\mathcal{Q}} \frac{|\varrho_\varepsilon(x) - \varrho_\varepsilon(y)|^2}{|x-y|^{n+2s}}\, dxdy \\ - \frac{t^{2^*}}{2^*} - \lambda \frac{t^{p+1}}{p+1}\int_{\mathcal{O}} \varrho_\varepsilon(x)^{p+1}\, dx; \end{aligned} \tag{4.14}$$

we then turn to estimate the integrals of (4.14). We use the [Lemma 5.1 and Lemma 5.3 in [39]] and obtain the following integrals:

$$\int_{\mathcal{O}} |\nabla \varrho_\varepsilon|^2\, dx = \mathcal{S}(\Pi_2) + O(\varepsilon^{n-2}) \qquad \text{as } \varepsilon \to 0^+ \text{ and,}$$

$$\iint_{\mathcal{Q}} \frac{|\varrho_\varepsilon(x) - \varrho_\varepsilon(y)|^2}{|x-y|^{n+2s}}\, dxdy = O(\varepsilon^{k_{s,n}}), \text{ where } \alpha_{s,n} = \min\{2(1-s), n-2\}.$$

and also for more ideas, we refer to [Proposition 21 in [27]].

Next, it is easy to see that L^{p+1}-norm of ϱ_ε and

$$\begin{aligned} \int_{\mathbb{R}^n} \varrho_\varepsilon(x)^{p+1}\, dx &= C_1 \int_{B_\rho(0)} U_\varepsilon(x)^{p+1}\, dx \\ &= C_1\, \varepsilon^{-(p+1)\frac{n-2}{2}} \int_0^\rho \frac{\sigma^{n-1}}{\left(\frac{\sigma^2}{\varepsilon^2} + 1\right)^{(p+1)\frac{n-2}{2}}}\, d\sigma \\ &\geq C_1\, \varepsilon^{n-(p+1)\frac{n-2}{2}} \int_1^{\rho/\varepsilon} \frac{t^{n-1}}{(t^2+1)^{(p+1)\frac{n-2}{2}}}\, dt \\ &= \frac{C_1}{n} \varepsilon^{n-(p+1)\frac{n-2}{2}} \left(\left(\frac{\varepsilon}{\rho}\right)^n + 1\right), \end{aligned} \tag{4.15}$$

where the constant $C_1 > 0$ is adjusted line to line. Now, combining the above integrals and using (4.14) we deduce that

$$\begin{aligned}\mathcal{J}_\lambda(t\varrho_\varepsilon) &\leq \frac{t^2}{2}(\mathcal{S}(\Pi_2) + O(\varepsilon^{n-2}) + O(\varepsilon^{\alpha_{s,n}})) - \frac{t^{2^*}}{2^*} - C\lambda\frac{t^{p+1}}{p+1}\varepsilon^{n-(p+1)\frac{n-2}{2}} \\ &= \frac{t^2}{2}(\mathcal{S}(\Pi_2) + O(\varepsilon^{\alpha_{s,n}})) - \frac{t^{2^*}}{2^*} - C\lambda\frac{t^{p+1}}{p+1}\varepsilon^{\tau_{p.n}} \\ &\leq \frac{t^2}{2}(\mathcal{S}(\Pi_2) + C\varepsilon^{\alpha_{s,n}}) - \frac{t^{2^*}}{2^*} - C\lambda\frac{t^{p+1}}{p+1}\varepsilon^{\tau_{p.n}},\end{aligned} \tag{4.16}$$

where $\varepsilon > 0$ and $C > 0$, suitable constants. Let us set

$$f_{\varepsilon,\lambda}(t) = \frac{t^2}{2}(\mathcal{S}(\Pi_2) + C\varepsilon^{\alpha_{s,n}}) - \frac{t^{2^*}}{2^*} - C\lambda\frac{t^{p+1}}{p+1}\varepsilon^{\tau_{p.n}}, \tag{4.17}$$

such that $f_{\varepsilon,\lambda}(0) = 0$ and $f_{\varepsilon,\lambda}(t) \to -\infty$ as $t \to \infty$. So, there exists $t_{\varepsilon,\lambda} \geq 0$ such that

$$\sup_{t\geq 0} f_{\varepsilon,\lambda}(t) = f_{\varepsilon,\lambda}(t_{\varepsilon,\lambda}).$$

We can see that for $t_{\varepsilon,\lambda} = 0$, we have $f_{\varepsilon,\lambda}(t) \leq 0$ for all $t \geq 0$, and the lemma is trivially established as a consequence of (4.16). On the other hand, if $t_{\varepsilon,\lambda} > 0$, we obtain

$$0 = f'_{\varepsilon,\lambda}(t_{\varepsilon,\lambda}) = t_{\varepsilon,\lambda}(\mathcal{S}(\Pi_2) + C\varepsilon^{\alpha_{s,n}}) - t_{\varepsilon,\lambda}^{2^*-1} - C\lambda t_{\varepsilon,\lambda}^{p}\varepsilon^{\tau_{p.n}}. \tag{4.18}$$

from which we may easily conclude that

$$t_{\varepsilon,\lambda} < (\mathcal{S}(\Pi_2) + C\varepsilon^{\alpha_{s,n}})^{1/(2^*-2)}.$$

We now distinguish two cases, according to the assumptions.

Case (1): (If $\alpha_{s,n} > \tau_{p.n}$), for $\varepsilon > 0$ small enough, using equation (4.18), we have

$$t_{\varepsilon,\lambda} \geq \mu_\lambda > 0,$$

and using the fact that the following map

$$t \mapsto \frac{t^2}{2}(\mathcal{S}(\Pi_2) + C\varepsilon^{\alpha_{s,n}}) - \frac{t^{2^*}}{2^*}$$

is increasing in the closed interval $[0, \mathcal{S}(\Pi_2) + C\varepsilon^{\alpha_{s,n}})^{1/(2^*-2)}]$ containing $t_{\varepsilon,\lambda}$, we have

$$\begin{aligned}\sup_{t\geq 0} f_{\varepsilon,\lambda}(t) &= f_{\varepsilon,\lambda}(t_{\varepsilon,\lambda}) \\ &< \frac{(\mathcal{S}(\Pi_2) + C\varepsilon^{\alpha_{s,n}})^{1+2/(2^*-2)}}{2} - \frac{(\mathcal{S}(\Pi_2) + C\varepsilon^{\alpha_{s,n}})^{2^*/(2^*-2)}}{2^*} - C\varepsilon^{\tau_{p.n}} \\ &= \frac{1}{n}(\mathcal{S}(\Pi_2) + C\varepsilon^{\alpha_{s,n}})^{n/2} - C\varepsilon^{\tau_{p.n}} \\ &< \frac{1}{n}(\mathcal{S}(\Pi_2))^{n/2}, \quad \text{for } \varepsilon > 0 \text{ sufficiently small.}\end{aligned}$$

Case (2): (for $\alpha_{s,n} \leq \tau_{p.n}$), let us start by claiming that

$$\lim_{\lambda\to\infty} t_{\varepsilon,\lambda} = 0. \tag{4.19}$$

By contrary, we assume that $e = \limsup_{\lambda\to\infty} t_{\varepsilon,\lambda} > 0$, then choosing a sequence $\{\lambda_k\}_{k\geq 1}$ diverging to ∞, as $k \to \infty$ and from (4.18) we obtain

$$\mathcal{S}(\Pi_2) + C\varepsilon^{\alpha_{s,n}} = t_{\varepsilon,\lambda_k}^{2^*-2} + C\lambda_k t_{\varepsilon,\lambda_k}^{p-1}\varepsilon^{\tau_{p.n}} \to \infty$$

which is a contradiction. Hence, we have $\lim_{\lambda\to\infty} t_{\varepsilon,\lambda} = 0$.

So, using (4.16) and (4.19), we have

$$0 \leq \sup_{t\geq 0} \mathcal{J}_\lambda(t\varrho_\varepsilon) \leq f_{\varepsilon,\lambda}(t_{\varepsilon,\lambda}) \leq \frac{t_{\varepsilon,\lambda}^2}{2}(\mathcal{S}(\Pi_2) + C\varepsilon^{\alpha_{s,n}}) - \frac{t_{\varepsilon,\lambda}^{2^*}}{2^*} \to 0 \quad \text{as } \lambda \to \infty,$$

which implies the existence of $\lambda_0 = \lambda_0(p, s, n, \varepsilon) > 0$ such that

$$\sup_{t\geq 0} \mathcal{J}_\lambda(t\varrho_\varepsilon) < \frac{1}{n}(\mathcal{S}(\Pi_2))^{2/n} \quad \text{for all } \lambda \geq \lambda_0,$$

for fix $\varepsilon > 0$ is small enough. Thus, our work is done.

In the next lemma, we verify the functional $\mathcal{J}_\lambda$ satisfies Mountain Pass geometry. We follow the same arguments of [Proposition 3.1 in [31]].

Lemma 4.12. *There exist positive constants $a, c_1, c_2 > 0$ such that*

(1) For any $w \in \mathcal{X}^{1,2}_{\Pi_1}(U)$ with $\zeta(w) = a$, it holds that $\mathcal{J}_\lambda(w) \geq c_2$,
(2) There exists a $t_\varepsilon > 0$ large enough so that $\zeta(\varrho_\varepsilon t_\varepsilon) > c_1$ and $\mathcal{J}_\lambda(t_\varepsilon\varrho_\varepsilon) < c_2$.

Proof. By the Sobolev embedding and Remark (3.1), we have

$$\mathcal{J}_\lambda(w) \geq \frac{1}{2}\zeta(w)^2 - \frac{C}{2^*}\zeta(w)^{2^*} - \lambda(\mathcal{S}(\Pi_2))^{\frac{2^*}{p+1}}\zeta(w)^{p+1},$$

for any $w \in \mathrm{X}^{1,2}_{\Pi_1}(U)$. Since $1 < p < 2^* - 1$, we can easily obtain the part (1) assuming $\zeta(w)$ is small enough. On the other hand, we have

$$\lim_{t\to\infty} J_\lambda(tv_\varepsilon) = -\infty,$$

from which we easily complete our proof.

Proof of Theorem 1.5. Hence, by using the Mountain Pass theorem and thanks to Lemmas 4.12, 4.10, and 4.11, we complete the proof of Theorem 1.5.

5. Conclusions

In this paper, firstly we identify the optimal constant in the mixed Sobolev inequality under mixed boundary conditions. We prove optimal constant $\mathcal{S}_{n,s}(O, \Pi_1) = \mathcal{S}(\Pi_2)$, which is our Theorem 1.1, and Theorem (1.2) says that constant $\mathcal{S}(\Pi_2)$ is never achieved in the space $\mathcal{X}^{1,2}_{\Pi_1}(U)$, but if $U = \mathbb{R}^n$ then it is achieved in the limiting sense. In addition, the aim of this manuscript is to prove the existence and non-existence of a positive solution for (P_λ) using the variational methods. Moreover, we note that the case $p = 1$ in Theorem 1.4 is different compared to the case $1 < p < 2^* - 1$ in Theorem 1.5 in a structural aspect. However, Theorem 1.5 ensures the existence of solutions for all λ large enough, while Theorem 1.4 only recognizes solutions for λ in a certain interval, demonstrating that no solutions exist when λ is too large. Therefore, the case $p = 1$ cannot be considered as the limit case of the setting $1 < p < 2^* - 1$.

Lastly, we will come back to the study of $(P_\lambda)$for sublinear perturbations $(0 < p < 1)$ and with some singular type nonlinearity with critical exponent under the mixed boundary conditions in future work.

Author contributions

The author is responsible for all aspects of the research and manuscript.

Acknowledgments

The authors express their sincerely gratitude to the anonymous referees for their useful comments and suggestions. The author received assistance from the UGC Grant with reference no. 191620169606 funded by the Government of India.

Conflict of interest

The authors declare there is no conflict of interest.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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