Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 16(4): 858–871.
DOI: 10.3934/cam.2024037
Received: 02 January 2024
Revised: 17 October 2024
Accepted: 07 November 2024
Published: 22 November 2024

Research article

Hardy-Sobolev spaces of higher order associated to Hermite operator

Jizheng Huang*and Shuangshuang Ying

Key Laboratory of Mathematics and Information Networks(Ministry of Education) and School of Science, Beijing University of Posts and Telecommunications, Beijing 100876, P.R.China

* **Correspondence:** Email: huangjz@bupt.edu.cn; Tel: 010-62281800.

Abstract: Let $L = -\Delta + |x|^2$ be the Hermite operator on $\mathbb{R}^d$, where Δ is the Laplacian on $\mathbb{R}^d$. In this paper, we will consider the Hardy-Sobolev spaces of higher order associated with L. We also give some new characterizations of the Hardy spaces associated with L.

Keywords: Hardy spaces; Riesz transform; Hardy-Sobolev spaces; Hermite operator
Mathematics Subject Classification: 42B35, 47A60, 32U20

1. Introduction

The Hermite operator L on $\mathbb{R}^d$ is defined by

$$L = -\Delta + |x|^2, \quad x \in \mathbb{R}^d.$$

The operator L is positive and symmetric in $L^2(\mathbb{R}^d)$, it can be decomposed as

$$L = \frac{1}{2}\sum_{i=1}^{d} A_i A_{-i} + A_{-i} A_i,$$

where

$$A_i = \frac{\partial}{\partial x_i} + x_i, \quad A_{-i} = -\frac{\partial}{\partial x_i} + x_i, \quad 1 \le i \le d.$$

When we study the problems associated with L [1–4], the operators A_i play the role of the partial differential operators $\frac{\partial}{\partial x_i}$ in the classical Euclidean case. For example, we can define the Riesz transform associated with Hermite operator by

$$R_i^L = A_i L^{-1/2}, \quad R_{-i}^L = A_{-i} L^{-1/2}, \quad i = 1, 2, \cdots, d.$$

Thangavelu [5] proved that R_i^L and R_{-i}^L were bounded on $L^p(\mathbb{R}^d)$ and used them to study the wave equations associated with L, where $1 < p < \infty$. Their boundedness on the local Hardy spaces [6] can be found in [4]. Moreover, whether we can characterize the local Hardy spaces by the Riesz transform associated with L? This problem was pointed out by Thangavelu in [4] and given a negative answer in [7]. In fact, the Riesz transform associated with L can characterize a new space which is called Hardy space associated with L [8]. Therefore, when we want to prove some results for L similar to the classical case, we must introduce new function spaces for L. In [9], the authors defined the Sobolev spaces associated with L and used them to study the Schrödinger equation for L. In [10, 11], the authors defined the Besov spaces associated with L and proved the boundedness of Riesz transforms on these spaces. In order to prove the endpoint version of the div-curl theorem for the Hermite operator, the Hardy-Sobolev space was defined in [12]. When we consider the equation $L^m F = f$ with m is a positive integer and f in the Hardy spaces associated to L, we need to define the higher-order Hardy-Sobolev spaces associated with L. In this paper, we will define and give several characterizations of these spaces.

In order to state our main results, we first introduce some notations. Let $H_k(x)$ denote the Hermite polynomials on $\mathbb{R}$, which can be defined as

$$H_k(x) = (-1)^k \frac{d^k}{dx^k}(e^{-x^2})e^{x^2}, \quad k = 0, 1, 2, \cdots$$

The normalized Hermite functions are defined by

$$h_k(x) = (\pi^{1/2} 2^k k!)^{-1/2} H_k(x) \exp\big(-x^2/2\big), \;\; k = 0, 1, \cdots$$

The higher-dimensional Hermite functions on $\mathbb{R}^d$, can be defined in the following way: for $\alpha = (\alpha_1, \cdots, \alpha_d)$, $\alpha_i \in \{0, 1, \cdots\}$, $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$,

$$h_\alpha(x) = \prod_{j=1}^{d} h_{\alpha_j}(x_j).$$

The Hermite functions $\{h_\alpha\}$ form a complete orthonormal basis of $L^2(\mathbb{R}^d)$. Let $|\alpha| = \alpha_1 + \cdots + \alpha_d$. Then we have

$$L h_\alpha = (2|\alpha| + d) h_\alpha.$$

Let $\{T_t^L\}_{t \geq 0}$ be the heat semigroup defined by

$$T_t^L f = e^{-tL} f = \sum_{n=0}^{\infty} e^{-t(2n+d)} \mathcal{P}_n f,$$

for $f \in L^2(\mathbb{R}^d)$ and

$$\mathcal{P}_n f = \sum_{|\alpha|=n} < f, h_\alpha > h_\alpha.$$

Then the Poisson semigroup is defined as

$$P_t^L f = e^{-tL^{1/2}} f = \sum_{n=0}^{\infty} e^{-t(2n+d)^{1/2}} \mathcal{P}_n f, \quad f \in L^2(\mathbb{R}^d).$$

We define Hardy space $H_L^1(\mathbb{R}^d)$ for $d \geq 3$ as follows (cf. [8])

$$H_L^1(\mathbb{R}^d) = \{f \in L^1(\mathbb{R}^d) : \;\; \mathcal{M}_L f \in L^1(\mathbb{R}^d)\},$$

where $\mathcal{M}_L f(x) = \sup_{t>0} |(T_t^L f)(x)|$.

The Riesz transforms of higher order can be defined as follows:

Definition 1.1. Let m be a positive integer. The operator $L^{-\frac{m}{2}}$ is defined by

$$L^{-\frac{m}{2}}h_\alpha = (2|\alpha| + d)^{-\frac{m}{2}}h_\alpha,$$

and the Riesz transform of order m is defined by

$$R^L_{i_1 i_2 \cdots i_m} = A_{i_1}A_{i_2}\cdots A_{i_m}L^{-\frac{m}{2}},$$

where $1 \le |i_j| \le d$ for $1 \le j \le m$ and for any $\alpha \in \{0, 1, 2, \cdots\}^d$.

The first result of this paper is that we can characterize the Hardy space $H^1_L(\mathbb{R}^d)$ by $R^L_{i_1 i_2 \cdots i_m}$.

Theorem 1.2. *$f \in H^1_L(\mathbb{R}^d)$ if and only if $R^L_{i_1 i_2 \cdots i_m}f \in L^1(\mathbb{R}^d)$ for all $1 \le |i_j| \le d$ and $f \in L^1(\mathbb{R}^d)$, i.e., there exists $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_L} \le \sum_{-d \le i_1, \cdots, i_m \le d} \|R^L_{i_1 i_2 \cdots i_m}f\|_{L^1} + \|f\|_{L^1} \le C\|f\|_{H^1_L}.$$

Let $L_b = L + b$ with $b \in \mathbb{R}^+$ and P^b_t be the semigroup with the infinitesimal generator $\sqrt{L_b}$. Then we can define the following version of higher-order Littlewood-Paley g-functions.

Definition 1.3. Let m be a positive integer and $f \in L^p(\mathbb{R}^d)$. The Littlewood-Paley g-function of higher-order is defined by

$$g_{m,b}(f)(x) = \left(\int_0^\infty \sum_{-d \le i_1, \cdots, i_m \le d} \left| t^m A_{i_1}A_{i_2}\cdots A_{i_m}P^b_t f(x) \right|^2 \frac{dt}{t} \right)^{1/2}.$$

The next result of this paper is that the Hardy space $H^1_L(\mathbb{R}^d)$ can be characterized by the higher order Littlewood-Paley g-function $g_{m,b}$.

Theorem 1.4. *For $f \in L^1(\mathbb{R}^d)$, $f \in H^1_L(\mathbb{R}^d)$ if and only if $g_{m,b}(f) \in L^1(\mathbb{R}^d)$ and there exists $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_L} \le \|g_{m,b}(f)\|_{L^1} \le C\|f\|_{H^1_L}.$$

Now, we introduce the Hardy-Sobolev space of higher order associated to L.

Definition 1.5. We define the Hardy-Sobolev space $H^{m,1}_L(\mathbb{R}^d)$ of order m as the set of functions $f \in L^1(\mathbb{R}^d)$ such that

$$A_{i_1}A_{i_2}\cdots A_{i_m}f \in H^1_L(\mathbb{R}^d),\ 1 \le |i_j| \le d,$$

with the norm

$$\|f\|_{H^{m,1}_L} = \sum_{-d \le i_1, \cdots, i_m \le d} \|A_{i_1}A_{i_2}\cdots A_{i_m}f\|_{H^1_L} + \|f\|_{L^1}.$$

Definition 1.6. A locally integrable function b is called a $(1, q)$-atom of $H^{m,1}_L(\mathbb{R}^d)$ if it satisfies

$$supp\ b \subset B(x_0, r);$$
$$\|L^{\frac{m}{2}}b\|_q \le |B(x_0, r)|^{1/q-1}.$$

The atomic quasi-norm in $H_L^{m,1}(\mathbb{R}^d)$ is defined by

$$\|f\|_{H_L^{m,1}-atom} = \inf\left\{\sum |c_j|\right\},$$

where the infimum is taken over all decompositions $f = \sum c_j a_j$, where a_j are $H_L^{m,1}$-atoms. We can give the atomic decomposition of $H_L^{m,1}(\mathbb{R}^d)$.

Theorem 1.7. *The norms $\|\cdot\|_{H_L^{m,1}}$ and $\|\cdot\|_{H_L^{m,1}-atom}$ are equivalent, that is, there exists a constant $C > 0$ such that for $f \in H_L^{m,1}(\mathbb{R}^d)$,*

$$C^{-1}\|f\|_{H_L^{m,1}} \le \|f\|_{H_L^{m,1}-atom} \le C\|f\|_{H_L^{m,1}}.$$

If we define the following version of the maximal function

$$(M_{m,L}f)(x) = \sup_{t>0} |\partial_t^m P_t^L f(x)|,$$

then we have

Theorem 1.8. *A function f in $H_L^{m,1}(\mathbb{R}^d)$ if and only if $M_{m,L}f \in H_L^1(\mathbb{R}^d)$ and $f \in L^1(\mathbb{R}^d)$. Moreover, there exists a constant $C > 0$ such that*

$$C^{-1}\|f\|_{H_L^{m,1}} \le \|M_{m,L}f\|_{H_L^1} + \|f\|_{L^1} \le C\|f\|_{H_L^{m,1}}.$$

The paper is organized as follows: in section 1, we will give several characterizations of the Hardy space $H_L^1(\mathbb{R}^d)$. The Hardy-Sobolev spaces will be studied in section 2.

Throughout the article, we will use A and C to denote the positive constants, which are independent of main parameters and may be different at each occurrence. By $B_1 \sim B_2$, we mean that there exists a constant $C > 1$ such that $C^{-1} \le B_1/B_2 \le C$ and $A \lesssim B$ means that there exists a positive constant C such that $A \le CB$.

2. Square function characterizations of $H_L^1(\mathbb{R}^d)$

In this section, we will give several new characterizations of $H_L^1(\mathbb{R}^d)$.

Define (cf. [13, (1.5)])

$$\rho_L(x) = \frac{1}{1+|x|}. \tag{2.1}$$

The function $\rho_L(x)$ has the following propositions (cf. [13, Lemma 1.4]).

Proposition 2.1. *There exists $k_0 > 0$ such that*

$$\frac{1}{C}\left(1 + \frac{|x-y|}{\rho_L(x)}\right)^{-k_0} \le \frac{\rho_L(y)}{\rho_L(x)} \le C\left(1 + \frac{|x-y|}{\rho_L(x)}\right)^{\frac{k_0}{k_0+1}}.$$

In particular, $\rho_L(y) \sim \rho_L(x)$ if $|x-y| < C\rho_L(x)$.

We say $a(x)$ is an atom for the space $H_L^1(\mathbb{R}^d)$, if there exists a ball $B(x_0, r)$ such that

(1) $\quad supp\ a \subset B(x_0, r)$,

$$(2)\quad \|a\|_{L^\infty} \le |B(x_0, r)|^{-1},$$
$$(3)\quad if\;\; r < \rho_L(x_0),\;\; then \int a(x)dx = 0.$$

The atomic quasi-norm in $H^1_L(\mathbb{R}^d)$ can be defined as

$$\|f\|_{L-atom} = \inf\left\{\sum |c_j|\right\},$$

where the infimum is taken over all atomic decomposition of f.

In [8, Theorem 1.12], the author proved the following result.

Proposition 2.2. *There exists $C > 0$ satisfying*

$$C^{-1}\|f\|_{H^1_L} \le \|f\|_{L-atom} \le C\|f\|_{H^1_L}.$$

For $b > 0$, since $\rho_L(x) = \rho_{L+b}(x)$, then by the atomic decomposition of $H^1_L(\mathbb{R}^d)$, we can obtain(cf. [7, Lemma 9])

Lemma 2.3. *For $f \in L^1(\mathbb{R}^d)$, $f \in H^1_L(\mathbb{R}^d)$ is equivalent to $f \in H^1_{L+b}(\mathbb{R}^d)$ for $b > 0$.*

The proof of the following lemma can be found in [9, Lemma 4].

Lemma 2.4. *If $\beta \in \mathbb{R}$ and $f \in L^2(\mathbb{R}^d)$, then for $j = 1, 2, \cdots, d$*

$$A_j L^\beta f = (L+2)^\beta A_j f,$$

and for $j = -1, -2, \cdots, -d$

$$A_j L^\beta f = (L-2)^\beta A_j f.$$

The boundedness of $R^L_{i_1,\cdots,i_m}$ on $L^p(\mathbb{R}^d)$ can be found in [1, Theorem B].

Proposition 2.5. *The Riesz transforms associated to L of higher order $R^L_{i_1,\cdots,i_m}$ are bounded on $L^p(\mathbb{R}^d)$, where $1 < p < \infty$.*

The boundedness of Riesz transforms on Hardy spaces has been proved in [7, Theorem 2] and [14, Theorem 2].

Proposition 2.6. *(1) $f \in H^1_L(\mathbb{R}^d)$ if and only if $R^L_i f \in L^1(\mathbb{R}^d)$ for $1 \le |i| \le d$ and $f \in L^1(\mathbb{R}^d)$. Moreover, the operators R^L_i are bounded on $H^1_L(\mathbb{R}^d)$, that is, there exists $C > 0$ satisfying*

$$\left\|R^L_i f\right\|_{H^1_L} \le C\left\|f\right\|_{H^1_L}.$$

(2) The Riesz transforms of higher order $R^L_{i_1,i_2,\cdots,i_m}$ are also bounded on $H^1_L(\mathbb{R}^d)$, i.e., there exists $C > 0$ satisfied

$$\left\|R^L_{i_1,i_2,\cdots,i_m} f\right\|_{H^1_L} \le C\left\|f\right\|_{H^1_L},$$

where $1 \le |i_j| \le d$ and $j = 1, 2, \cdots, m$.

Now, we can prove Theorem 1.2. In the following, we let $\Sigma(i) = 2$ for $i > 0$ and $\Sigma(i) = -2$ for $i < 0$. We use $\Sigma(i_1, i_2, \cdots, i_m)$ to denote $\Sigma(i_1) + \Sigma(i_2) + \cdots \Sigma(i_m)$.

Proof of Theorem 1.2. First, we let $f \in H^1_L(\mathbb{R}^d)$. Then, by Proposition 2.6, we obtain

$$\left\|R^L_{i_1 i_2\cdots i_m} f\right\|_{L^1} \le \left\|R^L_{i_1 i_2\cdots i_m} f\right\|_{H^1_L} \le C\|f\|_{H^1_L}.$$

For the reverse, by Lemma 2.3, it is sufficient to prove $f \in H^1_{L+b}(\mathbb{R}^d)$ for some $b > 0$. We will prove this by induction. If $m = 1$, this can be given by Proposition 2.6. We assume $m = n - 1$ holds, then for $m = n$, by Lemma 2.4

$$\begin{aligned} R^b_{i_1 i_2\cdots i_n} f &= A_{i_1}(L + b + \Sigma(i_2, \cdots, i_n))^{-\frac{1}{2}} A_{i_2} \cdots A_{i_n}(L + b)^{-\frac{m-1}{2}} f \\ &= R^{b+\Sigma(i_2,\cdots,i_n)}_{i_1}(A_{i_2} \cdots A_{i_n}(L + b)^{-\frac{m-1}{2}} f) \in L^1(\mathbb{R}^d). \end{aligned}$$

Therefore, we can choose $b \in \mathbb{R}^+$ such that $b + \Sigma(i_2, \cdots, i_n) > 0$, then

$$A_{i_2} \cdots A_{i_n}(L + b)^{-\frac{m-1}{2}} f \in H^1_{L+b+\Sigma(i_2,\cdots,i_n)}(\mathbb{R}^d) = H^1_L(\mathbb{R}^d).$$

Therefore $f \in H^1_L(\mathbb{R}^d)$ follows from the inductive assumption, and Theorem 1.2 is proved.

Let

$$g^0_{m,b}(f)(x) = \left(\int_0^\infty \left|t^m \frac{\partial^m}{\partial t^m} P^b_t f(x)\right|^2 \frac{dt}{t}\right)^{1/2}.$$

Then, we can characterize $H^1_L(\mathbb{R}^d)$ by $g^0_{m,b}$ [15, Theorem 4] and $g_{1,b}$ (cf. [14, Theorem 1]).

Proposition 2.7. *(a) For $f \in L^1(\mathbb{R}^d)$, $f \in H^1_L(\mathbb{R}^d)$ if and only if $g^0_{m,b}(f) \in L^1(\mathbb{R}^d)$ and there exists $C > 0$ such that*

$$C^{-1}\|f\|_{H^1_L} \le \|g^0_{m,b}(f)\|_{L^1} \le C\|f\|_{H^1_L}.$$

(b) For $f \in L^1(\mathbb{R}^d)$, $f \in H^1_L(\mathbb{R}^d)$ if and only if $g_{1,b}(f) \in L^1(\mathbb{R}^d)$ and there exists $C > 0$ such that

$$C^{-1}\|f\|_{H^1_L} \le \|g_{1,b}(f)\|_{L^1} \le C\|f\|_{H^1_L}.$$

In the following, we will prove Theorem 1.4.

Proof of Theorem 1.4. If $f \in H^1_L(\mathbb{R}^d)$, then by Proposition 2.2, we have $f = \sum_{k=1}^\infty \lambda_k a_k$, where a_k are atoms.

By Lemma 2.4, we know

$$\begin{aligned} &(-1)^m A_{i_1} \cdots A_{i_m} P^b_t h_\alpha \\ =\ & A_{i_1} \cdots A_{i_m} \frac{\partial^m}{\partial t^m}(e^{-t(L+b)^{1/2}})(L + b)^{-\frac{m}{2}} h_\alpha \\ =\ & \frac{\partial^m}{\partial t^m}\left(e^{-t(L+b+\Sigma(i_1,\cdots,i_m))^{1/2}}\right) A_{i_1} \cdots A_{i_m}(L + b)^{-\frac{m}{2}} h_\alpha \\ =\ & \frac{\partial^m}{\partial t^m} P^{b+\Sigma(i_1,\cdots,i_m)}_t (R^b_{i_1,\cdots,i_m} h_\alpha), \end{aligned}$$

then

$$\frac{\partial^m}{\partial t^m} P^{b+\Sigma(i_1,\cdots,i_m)}_t (R^b_{i_1,\cdots,i_m} a_k) = (-1)^m A_{i_1} \cdots A_{i_m} P^b_t a_k,$$

where a_k are atoms for f.

Therefore

$$g^2_{m,b}(a_k) = \sum_{-d\le i_1,\cdots,i_m\le d} [g^0_{m,b+\Sigma(i_1,\cdots,i_m)}(R^b_{i_1,\cdots,i_m}a_k)]^2.$$

Then, by Proposition 2.7 and Proposition 2.6

$$\begin{aligned}\|g_{m,b}(a_k)\|_{L^1} &\le \sum_{-d\le i_1,\cdots,i_m\le d} \|g^0_{m,b+\Sigma(i_1,\cdots,i_m)}(R^b_{i_1,\cdots,i_m}a_k)\|_{L^1}\\ &\le C\sum_{-d\le i_1,\cdots,i_m\le d}\|R^b_{i_1,\cdots,i_m}a_k\|_{H^1_L}\\ &\le C\|a_k\|_{H^1_L}\le C.\end{aligned}$$

For the reverse, we assume $g_{m,b}(f)\in L^1(\mathbb{R}^d)$, then we will prove the theorem by induction of m. When $m=1$, it follows from Proposition 2.7. We assume the case of m holds, then we will prove $m+1$ holds. We first prove

$$\int_0^\infty |t^{m+1}A_{i_1}\cdots A_{i_{m+1}}(P^b_t f(x))|^2\frac{dt}{t} = 2m(2m+1)\int_0^\infty\int_0^\infty |st^m A_{i_1}\cdots A_{i_{m+1}}(P^b_{t+s}f(x))|^2\frac{ds}{s}\frac{dt}{t}. \tag{2.2}$$

This can be proved by changing variables as follows:

$$\begin{aligned}&\int_0^\infty\int_0^\infty |st^m A_{i_1}\cdots A_{i_{m+1}}(P^b_{t+s}f(x))|^2\frac{ds}{s}\frac{dt}{t}\\ =&\int_0^\infty\int_0^\infty |A_{i_1}\cdots A_{i_{m+1}}(P^b_{t+s}f(x))|^2 t^{2m-1}s\,dt\,ds\\ =&\int_0^\infty\int_s^\infty |A_{i_1}\cdots A_{i_{m+1}}(P^b_t f(x))|^2(t-s)^{2m-1}s\,dt\,ds\\ =&\int_0^\infty\int_0^t (t-s)^{2m-1}s\,ds|A_{i_1}\cdots A_{i_{m+1}}(P^b_t f(x))|^2dt\\ =&\frac{1}{2m(2m+1)}\int_0^\infty t^{2m+1}|A_{i_1}\cdots A_{i_{m+1}}(P^b_t f(x))|^2dt\\ =&\frac{1}{2m(2m+1)}\int_0^\infty |t^{m+1}A_{i_1}\cdots A_{i_{m+1}}(P^b_t f(x))|^2\frac{dt}{t}.\end{aligned}$$

Let $\mathbf{K}$ be the Hilbert space defined as $h\in\mathbf{K}$ if and only if $h=\{h_{i_1,\cdots,i_m}(t)\}$, where $-d\le i_1,\cdots,i_m\le d$ and $0<t<\infty$ with

$$\|h\|^2_{\mathbf{K}} = \int_0^\infty \sum_{-d\le i_1,\cdots,i_m\le d}|h_{i_1,\cdots,i_m}(t)|^2\frac{dt}{t}<\infty.$$

Let $h=\{t^m A_{i_1}\cdots A_{i_m}(P^b_t f(x))\}$. Then, by the inductive assumption, we know $h\in\mathbf{K}$, and (2.2) shows

$$\int_0^\infty\sum_{j=-d}^{d}\|sA_jP^b_s h\|^2_{\mathbf{K}}\frac{ds}{s} = g_{m+1,b}(f)\in L^1(\mathbb{R}^d).$$

If we use $H^1_{\mathbf{K}}(\mathbb{R}^d)$ to denote the $\mathbf{K}-$valued Hardy spaces associated to L, then $h\in H^1_{\mathbf{K}}(\mathbb{R}^d)\subset L^1_{\mathbf{K}}(\mathbb{R}^d)$, i.e.,

$$\|h\|^2_{\mathbf{K}} = \int_0^\infty\sum_{-d\le i_1,\cdots,i_m\le d}|t^m A_{i_1}\cdots A_{i_m}(P^b_t f(x))|^2\frac{dt}{t}\in L^1(\mathbb{R}^d).$$

Therefore, by the inductive assumption, we know $f\in H^1_L(\mathbb{R}^d)$ and Theorem 1.4 is proved.

3. Hardy-Sobolev spaces

We first prove that $H_L^{m,1}(\mathbb{R}^d)$ is a Banach space. In order to do that, we need the following lemma (cf. p.122 in [16]).

Lemma 3.1. *Let* $1 \leq p < \infty$, $f \in W^{k,p}(\mathbb{R}^d)$ *and* $\{f_n\}$ *be a sequence such that* $\|f_n - f\|_p \to 0$. *Then, for any* $|\alpha| \leq k$, *we have*

$$\left\| \frac{\partial^\alpha f_n}{\partial x^\alpha} - \frac{\partial^\alpha f}{\partial x^\alpha} \right\|_p \to 0,$$

where $W^{k,p}$ *is the classical Sobolev spaces.*

By Lemma 3.1, we can prove

Proposition 3.2. $H_L^{m,1}(\mathbb{R}^d)$ *is a Banach space.*

Proof. Let $\{f_n\}$ be a Cauchy sequence in $H_L^{m,1}(\mathbb{R}^d)$. Then $\{x_i^\mu \partial_j^\nu f_n\}_{\mu+\nu=m}$ is a Cauchy sequence in $H_L^1(\mathbb{R}^d)$. Since $H_L^1(\mathbb{R}^d)$ is a Banach space, there exists $g \in H_L^1(\mathbb{R}^d)$ such that

$$\left\| x_i^\mu \partial_j^\nu f_n - g \right\|_{H_L^1} \to 0. \tag{3.1}$$

Let f be the limit of $\{f_n\}$ in $L^1(\mathbb{R}^d)$. Then, by Lemma 2.3,

$$\left\| x_i^\mu \partial_j^\nu f_n - x_i^\mu \partial_j^\nu f \right\|_{L^1} \to 0. \tag{3.2}$$

By (3.1) and (3.2), we obtain $g = x_i^\mu \partial_j^\nu f$. This proves $\|A_{i_1}A_{i_2}\cdots A_{i_m} f_n - A_{i_1}A_{i_2}\cdots A_{i_m} f\|_{H_L^1} \to 0$ for $1 \leq |i_j| \leq d$, i.e., $\|f_n - f\|_{H_L^{m,1}} \to 0$, then we get $H_L^{m,1}(\mathbb{R}^d)$ is a Banach space.

Now, we give an equivalent characterization of $H_L^{m,1}(\mathbb{R}^d)$.

Definition 3.3. Let $\mathcal{H}_L^{m,1}(\mathbb{R}^d) = L^{-\frac{m}{2}}(H_L^1(\mathbb{R}^d))$ or

$$\mathcal{H}_L^{m,1}(\mathbb{R}^d) = \{f \in L^1(\mathbb{R}^d) : \ L^{\frac{m}{2}} f \in H_L^1(\mathbb{R}^d)\},$$

with the norm $\|f\|_{\mathcal{H}_L^{m,1}} = \|L^{\frac{m}{2}} f\|_{H_L^1} + \|f\|_{L^1}$.

Theorem 3.4. *The norms* $\|\cdot\|_{H_L^{m,1}}$ *and* $\|\cdot\|_{\mathcal{H}_L^{m,1}}$ *are equivalent, that is, there exists a constant* $C > 0$ *such that for* $f \in H_L^{m,1}(\mathbb{R}^d)$,

$$C^{-1}\|f\|_{H_L^{m,1}} \leq \|f\|_{\mathcal{H}_L^{m,1}} \leq C\|f\|_{H_L^{m,1}}.$$

Proof. Let $f \in H_L^{m,1}(\mathbb{R}^d)$. Then, by Theorem 1.2,

$$\begin{aligned}
\|f\|_{\mathcal{H}_L^{m,1}} &= \|L^{\frac{m}{2}} f\|_{H_L^1} + \|f\|_{L^1} \leq \sum_{-d \leq i_1, \cdots, i_m \leq d} \|R^L_{i_1 i_2 \cdots i_m} L^{\frac{m}{2}} f\|_{L^1} + \|f\|_{L^1} \\
&= \sum_{-d \leq i_1, \cdots, i_m \leq d} \|A_{i_1}A_{i_2}\cdots A_{i_m} f\|_{L^1} + \|f\|_{L^1} \\
&\leq \sum_{-d \leq i_1, \cdots, i_m \leq d} \|A_{i_1}A_{i_2}\cdots A_{i_m} f\|_{H_L^1} + \|f\|_{L^1}
\end{aligned}$$

$$\leq \quad C\|f\|_{H_L^{m,1}},$$

i.e., $f \in \mathcal{H}_L^{m,1}(\mathbb{R}^d)$.

If $f \in \mathcal{H}_L^{m,1}(\mathbb{R}^d)$, by Proposition 2.6,

$$\begin{aligned}
\|f\|_{H_L^{m,1}} &= \sum_{-d\leq i_1,\cdots,i_m\leq d} \|A_{i_1}A_{i_2}\cdots A_{i_m}f\|_{H_L^1} + \|f\|_{L^1} \\
&= \sum_{-d\leq i_1,\cdots,i_m\leq d} \|R^L_{i_1i_2\cdots i_m}L^{\frac{m}{2}}f\|_{H_L^1} + \|f\|_{L^1} \\
&\leq C\|L^{\frac{m}{2}}f\|_{H_L^1} + \|f\|_{L^1} \\
&\leq C\|f\|_{\mathcal{H}_L^{m,1}}.
\end{aligned}$$

This gives the proof of Theorem 3.4.

In the following, we consider the atomic decomposition of $H_L^{m,1}(\mathbb{R}^d)$. Given $a > 0$, we define the operator

$$L^{-a}f(x) = \frac{1}{\Gamma(a)}\int_0^\infty e^{-tL}f(x)t^a\frac{dt}{t}, \quad x \in \mathbb{R}^d, \tag{3.3}$$

where $f \in \mathcal{S}(\mathbb{R}^d)$. Then, we have (cf. [9, Proposition 2])

Lemma 3.5. *The operator L^{-a} has the integral representation*

$$L^{-a}f(x) = \int_{\mathbb{R}^d} K_a(x,y)f(y)dy, \quad x \in \mathbb{R}^d,$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. Moreover, there exists $\Phi_a \in L^1(\mathbb{R}^d)$ and a constant $C > 0$ such that

$$K_a(x,y) \leq C\Phi_a(x-y), \ \textit{for all} \ \ x,y \in \mathbb{R}^d.$$

Let $G_t(x,y)$ denote the heat kernel of L, i.e.,

$$e^{-tL}f(x) = \int_{\mathbb{R}^d} G_t(x,y)f(y)dy.$$

Fayman-Kac formula gives

$$G_t(x,y) \leq h_t(x-y),$$

where $h_t(x)$ is the Gauss kernel.

The heat kernel $G_t^b(x,y)$ of the semigroup $\{e^{-t(L+b)}\}$ is

$$G_t^b(x,y) = e^{-bt}G_t(x,y).$$

It is easy to know

$$G_t^b(x,y) \leq G_t(x,y).$$

Therefore, we have the following estimations for $G_t^b(x,y)$ (cf. [17, Proposition 2-3]).

Lemma 3.6. (a) *For $N \in \mathbb{N}$, there exists $C_N > 0$ such that*

$$0 \le G_t^b(x,y) \le C_N t^{-\frac{d}{2}} e^{-(5t)^{-1}|x-y|^2}\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-N}. \tag{3.4}$$

(b) *For every $N > 0$, there are $C_N > 0$ and $C > 0$ such that for all $|h| \le \frac{|x-y|}{2}$,*

$$|G_t^b(x+h,y) - G_t^b(x,y)| \le C_N \left(\frac{|h|}{\sqrt{t}}\right) t^{-\frac{d}{2}} e^{-Ct^{-1}|x-y|^2}\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-N}. \tag{3.5}$$

In order to prove the atomic decomposition of $H_L^{m,1}(\mathbb{R}^d)$, we need the following lemma.

Lemma 3.7. *Let $a(x)$ be an $(1,q)$-atom associated to ball $B(x_0,r)$ of $H_L^1(\mathbb{R}^d)$. Then*

$$|L^{-\frac{m}{2}} a(x)| \le C \frac{r}{|x-x_0|^{d+1}}$$

for $|x - x_0| \ge 2r$.

Proof. For $f \in \mathcal{S}(\mathbb{R}^d)$, we have

$$\begin{aligned} L^{-\frac{m}{2}} f(x) &= \frac{1}{\Gamma(\frac{m}{2})}\int_0^\infty e^{-tL} f(x) t^{\frac{m}{2}-1} dt \\ &= \frac{1}{\Gamma(\frac{m}{2})}\int_0^\infty \int_{\mathbb{R}^d} G_t^L(x,y) f(y) dy t^{\frac{m}{2}-1} dt. \end{aligned}$$

Therefore

$$K_a(x,y) = \frac{1}{\Gamma(\frac{m}{2})}\int_0^\infty G_t^L(x,y) t^{\frac{m}{2}-1} dt.$$

Then, by Lemma 3.6 and note that $\rho_L(x) \le 1$, when $|h| \le \frac{|x-y|}{2}$, we have

$$\begin{aligned} & |K_a(x,y+h) - K_a(x,y)| \\ \le\ & \frac{1}{\Gamma(\frac{m}{2})}\int_0^\infty |G_t^L(x,y+h) - G_t^L(x,y)| t^{\frac{m}{2}-1} dt \\ \le\ & C\int_0^\infty \left(\frac{|h|}{\sqrt{t}}\right) t^{-\frac{d}{2}} e^{-Ct^{-1}|x-y|^2}\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-N} t^{\frac{m}{2}-1} dt \\ =\ & C\int_0^{|x-y|^2} \left(\frac{|h|}{\sqrt{t}}\right) t^{-\frac{d}{2}} e^{-Ct^{-1}|x-y|^2}\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-N} t^{\frac{m}{2}-1} dt \\ & + C\int_{|x-y|^2}^\infty \left(\frac{|h|}{\sqrt{t}}\right) t^{-\frac{d}{2}} e^{-Ct^{-1}|x-y|^2}\Big(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\Big)^{-N} t^{\frac{m}{2}-1} dt \\ \le\ & C|h|\int_0^{|x-y|^2} t^{-\frac{d+3}{2}+\frac{m}{2}} e^{-Ct^{-1}|x-y|^2}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{-m} dt \\ & + C|h|\int_{|x-y|^2}^\infty t^{-\frac{d+3}{2}+\frac{m}{2}}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{-m} dt \end{aligned}$$

$$
\begin{aligned}
&\leq C\frac{|h|}{|x-y|^{d+3}}\int_0^{|x-y|^2}\left(\frac{|x-y|^2}{t}\right)^{\frac{d+3}{2}}e^{-Ct^{-1}|x-y|^2}dt+C|h|\int_{|x-y|^2}^{\infty}t^{-\frac{d+3}{2}}dt\\
&\leq C\frac{|h|}{|x-y|^{d+1}}.
\end{aligned}
$$

If $r<\rho_L(x_0)$, then a satisfies the vanishing condition, so

$$
\begin{aligned}
|L^{-\frac{m}{2}}a(x)| &\leq \int_{B(x_0,r)}|K_{\frac{m}{2}}(x,y)-K_{\frac{m}{2}}(x,x_0)||a(y)|dy\\
&\leq C\int_{B(x_0,r)}\frac{r}{|x-x_0|^{d+1}}|a(y)|dy\leq C\frac{r}{|x-x_0|^{d+1}}.
\end{aligned}
$$

If $r\geq\rho_L(x_0)$, by Proposition 2.1, we can obtain $\rho(x)\leq Cr$ for $x\in B(x_0,r)$. Then, following from Lemma 3.6, we have

$$
\begin{aligned}
|K_{\frac{m}{2}}(x,y)| &\leq C\int_0^{\infty}t^{-\frac{d}{2}}e^{-At^{-1}|x-y|^2}\left(1+\frac{\sqrt{t}}{\rho(x)}+\frac{\sqrt{t}}{\rho(y)}\right)^{-N}t^{\frac{m}{2}-1}dt\\
&= C\int_0^{|x-y|^2}t^{-\frac{d+2}{2}+\frac{m}{2}}e^{-At^{-1}|x-y|^2}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{-(m+1)}dt\\
&\quad+C\int_{|x-y|^2}^{\infty}t^{-\frac{d+2}{2}+\frac{m}{2}}\left(\frac{\sqrt{t}}{\rho(x)}\right)^{-(m+1)}dt\\
&\leq C\frac{\rho(x)}{|x-y|^{d+3}}\int_0^{|x-y|^2}\left(\frac{|x-y|^2}{t}\right)^{\frac{d+3}{2}}e^{-At^{-1}|x-y|^2}dt\\
&\quad+C\rho(x)\int_{|x-y|^2}^{\infty}t^{-\frac{d+3}{2}}dt\\
&\leq C\frac{r}{|x-y|^{d+1}}.
\end{aligned}
$$

When $y\in B(x_0,r)$ and $|x-x_0|>2r$, we obtain

$$
|x-y|\geq|x-x_0|-|x_0-y|\geq|x-x_0|-\frac{|x-x_0|}{2}=\frac{|x-x_0|}{2}.
$$

Therefore

$$
\begin{aligned}
|L^{-\frac{m}{2}}a(x)| &\leq \int_{B(x_0,r)}|K_{\frac{m}{2}}(x,y)||a(y)|dy\\
&\leq C\int_{B(x_0,r)}\frac{r}{|x-y|^{d+1}}|a(y)|dy\\
&\leq C\int_{B(x_0,r)}\frac{r}{|x-x_0|^{d+1}}|a(y)|dy\\
&\leq C\frac{r}{|x-x_0|^{d+1}}.
\end{aligned}
$$

This gives the proof of Lemma 3.7.

Now we can give the proof of Theorem 1.7.

Proof of Theorem 1.7. To show $f = \sum \lambda_i b_i \in H_L^{m,1}(\mathbb{R}^d)$, it suffices to prove that for any $(1, q)$-atom b, we have $\|b\|_{H_L^{m,1}} \leq C$ with C independent of b. By Theorem 3.4 and Proposition 2.5,

$$\begin{aligned}
\|b\|_{H_L^{m,1}} &= \|L^{m/2}b\|_{H_L^1} = \sum_{-d\leq i_1,\cdots,i_m\leq d} \|R^L_{i_1 i_2\cdots i_m} L^{m/2}b\|_{L^1} + \|L^{m/2}b\|_{L^1} \\
&= \sum_{-d\leq i_1,\cdots,i_m\leq d} \|A_{i_1}A_{i_2}\cdots A_{i_m}b\|_{L^1} + \|L^{m/2}b\|_{L^1} \\
&= \sum_{-d\leq i_1,\cdots,i_m\leq d} \int_{B(x_0,r)} |R^L_{i_1 i_2\cdots i_m} L^{m/2}b(x)|dx + \int_{B(x_0,r)} |L^{m/2}b(x)|dx \\
&\leq |B|^{\frac{1}{q'}} \sum_{-d\leq i_1,\cdots,i_m\leq d} \|R^L_{i_1 i_2\cdots i_m} L^{m/2}b\|_{L^q} + |B|^{\frac{1}{q'}}\|L^{m/2}b\|_{L^q} \\
&\leq C|B|^{\frac{1}{q'}}|B|^{\frac{1}{q}-1} \leq C.
\end{aligned}$$

For the reverse, if $f \in H_L^{m,1}(\mathbb{R}^d)$, there exists $g \in H_L^1(\mathbb{R}^d)$ such that $f = L^{-m/2}g$. Since $g = \sum \lambda_i a_i$, where a_i are $(1, q)$-atoms in $H_L^1(\mathbb{R}^d)$, we get $f = \sum \lambda_i L^{-m/2}a_i$ with $\sum |\lambda_j| < \infty$. Since $L^{-m/2}a_i$ does not have compact support, it is not an atom for $H_L^{m,1}(\mathbb{R}^d)$.

Let a be a $(1, q)$-atom of $H_L^1(\mathbb{R}^d)$ such that $supp\, a \subset B(x_0, r)$ and $b(x) = L^{-m/2}a$. We choose a smooth partition of unity $1 = \phi_0 + \sum_{j=1}^{\infty} \phi_j$, where $\phi_0 \equiv 1$ and $\phi_1 \equiv 0$ on $|x - x_0| < 2r$.

$$supp\, \phi_0 \subset \{x : |x - x_0| \leq 4r\},\ supp\, \phi_1 \subset \{x : 2r \leq |x - x_0| \leq 8r\}$$

and $\phi_j(x) = \phi_1(2^{1-j}x)$ for $j \geq 2$. Then $b(x) = \phi_0 b + \sum_{j=1}^{\infty} \phi_j b$. We will show $\phi_j b = \lambda_j b_j$ for appropriate scalars λ_j, where b_j are $(1, q)$-atoms in $H_L^{m,1}(\mathbb{R}^d)$ and $\sum |\lambda_j| < C$.

It is obvious, $supp\, b_j \subset B(x_0, 2^{4+j}r)$. Let

$$\lambda_j = [2^{(4+j)}r]^{d(1-\frac{1}{q})}\|L^{m/2}(\phi_j b)\|_{L^q}.$$

For $j = 0$, since $\|L^{m/2}b\|_{L^q} = 1$, we get $\|L^{m/2}\phi_0 b\|_{L^q} \leq C$. For $j \geq 1$, since L is self-adjoint and Lemma 3.7, we have

$$\begin{aligned}
\|L^{\frac{m}{2}}(\phi_j b)\|_{L^q} &= \sup_{\|g\|_{L^{q'}}=1} \int_{\mathbb{R}^d} L^{\frac{m}{2}}(\phi_j b)(x)g(x)dx \\
&= \sup_{\|g\|_{L^{q'}}=1} \int_{\mathbb{R}^d} (\phi_j b)(x)(L^{-\frac{m}{2}}g)(x)dx \\
&\leq \sup_{\|g\|_{L^{q'}}=1} \int_{2^{1+j}r\leq|x-x_0|\leq 2^{4+j}r} \phi_j(x)L^{-\frac{m}{2}}a(x)L^{-\frac{m}{2}}g(x)dx \\
&\leq C(2^j r)^{d/q}\frac{r}{(2^j r)^{d+1}}\|g\|_{L^{q'}} \\
&\leq C2^{-j}(2^j r)^{-\frac{d}{q'}}.
\end{aligned}$$

So $\lambda_j \leq C2^{-j}$, which gives $\sum |\lambda_j| \leq C$.

In order to give the proof of Theorem 1.8, we need the following Poisson maximal function characterization of $H_L^1(\mathbb{R}^d)$(cf. [18, Theorem 8.2]).

Lemma 3.8. *For $f \in L^1(\mathbb{R}^d)$, we have $f \in H^1_L(\mathbb{R}^d)$ if and only if $M_P(f) \in L^1(\mathbb{R}^d)$, where*

$$M_P(f)(x) = \sup_{t>0} |P^L_t(f)(x)|.$$

Moreover, there exists $C > 0$ such that

$$C^{-1}\|f\|_{H^1_L} \leq \|M_P(f)\|_{L^1} + \|f\|_{L^1} \leq C\|f\|_{H^1_L}.$$

Proof of Theorem 1.8. By Theorem 3.4 and Lemma 3.8, we obtain

$$\begin{aligned} & \|f\|_{H^{m,1}_L(\mathbb{R}^d)} \approx \|L^{\frac{m}{2}} f\|_{H^1_L} \\ \approx \ & \|M_P(L^{\frac{m}{2}} f)\|_{L^1} \\ = \ & \left\| \sup_{t>0} |P^L_t(L^{\frac{m}{2}} f)| \right\|_{L^1} \\ = \ & \left\| \sup_{t>0} |L^{\frac{m}{2}} P^L_t(f)| \right\|_{L^1} \\ = \ & \|M_{m,L}(f)\|_{L^1}. \end{aligned}$$

This completes the proof of Theorem 1.8.

Author contributions

All authors have the same contribution to the paper.

Acknowledgements

Jizheng Huang is supported by Fundamental Research Funds for the Central Universities (# 500423101) and Beijing Natural Science Foundation of China(#1232023).

Conflict of interest

The authors declare there is no conflict of interest.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

References

1. E. Harboure, L. de Rosa, C. Segovia, J. L. Torrea, L^p-dimension free boundedness for Riesz transforms associated to Hermite functions, *Math. Ann.*, **328** (2004), 653–682. https://doi.org/10.1007/s00208-003-0501-2
2. F. Lust-Piquard, Dimension free estimates for Riesz transforms associated to the harmonic oscillator on $\mathbb{R}^n$, *Potential Anal.*, **24** (2006), 47–62. https://doi.org/10.1007/s11118-005-4389-1

3. K. Stempak, J. L. Torrea, Poisson integrals and Riesz transforms for the Hermite function expansions with weights, *J. Funct. Anal.*, **202** (2003), 443–472. https://doi.org/10.1016/S0022-1236(03)00083-1
4. S. Thangavelu, *Lectures on Hermite and Laguerre expansions*, Mathematical Notes, Vol. 42, Princeton University Press, Princeton, NJ, 1993. https://doi.org/10.1515/9780691213927
5. S. Thangavelu, Riesz transforms and wave equation for the Hermite operator, *Comm. Partial Differential Equations*, **15** (1990), 1199–1215. https://doi.org/10.1080/03605309908820720
6. D. Goldberg, A local version of real Hardy spaces, *Duke Math. J.*, **46** (1979), 27–42. https://doi.org/10.1215/S0012-7094-79-04603-9
7. J. Z. Huang, The boundedness of Riesz transforms for Hermite expansions on the Hardy spaces, *J. Math. Anal. Appl.*, **385** (2012), 559–571. https://doi.org/10.1016/j.jmaa.2011.06.075
8. J. Dziubański, Atomic Decomposition of H^p Spaces Associated with some Schrödinger Operators, *Indiana University Math. J.*, **47** (1998), 75–98. https://doi.org/10.1512/iumj.1998.47.1479
9. B. Bongioanni, J. L. Torrea, Sobolev spaces associated to the harmonic oscillator, *Proc. Indian. Acad. Sci. Math. Sci.*, **116** (2003), 337–360. https://doi.org/10.1007/BF02829750
10. T. A. Bui, X. T. Duong, Higher-order Riesz transforms of Hermite operators on new Besov and Triebel-Lizorkin spaces, *Constr. Approx.*, **53** (2021), 85–120. https://doi.org/10.1007/s00365-019-09493-y
11. T. A. Bui, X. T. Duong, Besov and Triebel-Lizorkin spaces associated to Hermite operators, *J. Fourier Anal. Appl.*, **21** (2015), 405–448. https://doi.org/10.1007/s00041-014-9378-6
12. J. Z. Huang, Hardy-Sobolev spaces associated with Hermite expansions and interpolation, *Nonlinear Anal.*, **157** (2017), 104–122. https://doi.org/10.1016/j.na.2017.03.014
13. Z. W. Shen, L^p estimates for Schrödinger operators with certain potentials, *Ann. Inst. Fourier (Grenoble)*, **45** (1995), 513–546. https://doi.org/10.5802/aif.1463
14. J. Z. Huang, Higher order Riesz transforms for Hermite expansions, *J. Inequal. Appl.*, **2017** (2017), 1–17. https://doi.org/10.1186/s13660-017-1376-1
15. C. Lin, H. Liu, $BMO_L(\mathbb{H}^n)$ spaces and Carleson measures for Schrödinger operators, *Adv. Math.*, **228** (2011), 1631–1688. https://doi.org/10.1016/j.aim.2011.06.024
16. E. Stein, *Singular integral and differentiability properties of functions*, Princeton University Press, Princeton, NJ, 1970.
17. J. Dziubański, G. Garrigós, T. Martinez, J. L. Torrea, J. Zienkiewicz, BMO spaces related to Schrödinger operator with potential satisfying reverse Hölder inequality, *Math. Z.*, **249** (2005), 329–356. https://doi.org/10.1007/s00209-004-0701-9
18. S. Hofmann, G. Z. Lu, D. Mitres, M. Mitrea, L. X. Yan, Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates, *Mem. Am. Math. Soc.*, **214** (2011), 1007. https://doi.org/10.1090/S0065-9266-2011-00624-6

AIMS Press