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Research article

Discontinuous differential equation for modelling the Antarctic Circumpolar Current

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Abstract: In this paper, we were concerned with the existence of the solution related to the discontinuous differential equation, which corresponded to the stratification phenomenon in the Antarctic Circumpolar Current (ACC). By considering the piecewise vorticity function, we demonstrated the existence of solution corresponding to the discontinuous differential equation using Green's function, fixed point theory, and topological degree theory. This primarily included cases with piecewise constant vorticity, piecewise linear vorticity, and piecewise nonlinear vorticity. Additionally, we provided some examples to verify our results.

Keywords: Antarctic Circumpolar Current; stratification; Green's function; discontinuous differential equation; topological degree theory
Mathematics Subject Classification: 34B15, 34B27, 34B60

1. Introduction

In recent years, the mathematical study of geophysical flows has been of great interest and has attracted much attention, particularly the equatorial flow, Antarctic Circumpolar Current (ACC), and Arctic gyres. We recommend readers to refer to monographs and the latest mathematical achievements to understand these oceanic phenomena. In addition, fluid stratification is an inherent characteristic of physical oceanography and is closely related to large-scale oceanic movements. The baroclinic instability is the main cause of the lateral and vertical stratification in ACC [1, 2]. Moreover, the generation of stratification phenomenon also relies on eddies. This reason is that the interaction of imposed patterns of mechanical and buoyancy forcing by the eddies can establish stratification in both the horizontal and vertical directions [1, 3].

For the motion of ACC, the existence of stratification that accommodates the observed sharp changes in water density (due to variations in temperature and salinity [4–7]), Constantin and Johnson [8, 9] have constructed exact solutions to the governing equations of geophysical fluid dynamics in terms of spherical coordinates. These solutions represent purely azimuthal and depth-varying flows, which can be used to model equatorial undercurrent and ACC, respectively, and describe a purely homogeneous flow without stratification. For the investigation of stratified flow, Henry and Martin [10–12] constructed the exact solutions for geophysical fluid dynamics, which allows for the continuous stratification of equatorial flow that varies with the change of depth and latitude. Further, this method is extended to discontinuous density [13] and discontinuous varying density together with forcing terms in ACC [14, 15].

In this paper, we investigate the existence of solutions for the stratified fluid flow corresponding to the different nonlinear vorticity in ACC. In response to the effects of density variation and forcing terms, we propose a new approach to determine the existence of stratified flow solutions. This method utilizes the Green's function and different nonlinear vorticity terms to derive the specific expression of the solution on both sides of the stratification and further proves the existence of stratification caused by density in the ACC.

The arrangement of this paper is presented as follows. In Section 2, we will state the governing equations with boundary conditions for ACC in terms of spherical coordinates. In Sections 3 and 4, based on the boundary conditions, we establish the existence of solutions corresponding to different nonlinear vorticity terms including piecewise constant coefficient, piecewise linear, and piecewise nonlinear, and verify the results graphically to represent the stratified flow in ACC. Section 5 extends to a general discontinuous case.

2. Preliminary

Consider the spherical coordinates first, with $\theta \in [0, \pi)$ as the polar angle (with $\theta = 0$ corresponding to the North Pole) and $\varphi \in [0, 2\pi)$ as the angle of longitude (or azimuthal angle). In terms of the stream function $\psi(\varphi, \theta)$, the horizontal gyre flow on the spherical Earth has azimuthal and polar velocity components given by

$$v = \frac{1}{\sin\theta}\psi_{\varphi}, \quad w = -\psi_{\theta}.$$

Then, we introduce the stream function $\Psi(\theta, \varphi)$ associated with the vorticity of the ocean motion (the effects of the Earth's rotation are not taken into account), defined as

$$\Psi(\theta, \varphi) = \psi(\theta, \varphi) + \omega \cos\theta,$$

and the vorticity equation of the gyre flow is

$$\frac{1}{\sin^2\theta}\Psi_{\varphi\varphi} + \Psi_{\theta}\cot\theta + \Psi_{\theta\theta} = F(\Psi - \omega\cos\theta), \tag{2.1}$$

where $\omega > 0$ is the nondimensional form of Coriolis parameter. $F(\Psi - \omega\cos\theta)$ and $2\omega\cos\theta$ are the oceanic vorticity and the planetary vorticity. By applying the stereographic projection of the unit sphere centered at the origin from the North Pole to the equatorial plane, the model (2.1) in spherical coordinates can be transformed into an equivalent plane elliptic partial differential equation, defined by

$$\xi = re^{i\phi} \quad \text{with} \quad r = \cot\left(\frac{\theta}{2}\right) = \frac{\sin\theta}{1-\cos\theta}, \tag{2.2}$$

where (r, ϕ) are the polar coordinates in the equatorial plane. Thus (2.2) can be transformed into

$$\psi_{\xi\bar{\xi}} + 2\omega\frac{1-\xi\bar{\xi}}{(1+\xi\bar{\xi})^3} - \frac{F(\psi)}{(1+\xi\bar{\xi})^2} = 0.$$

Using the Cartesian coordinates (x, y) in the complex ξ-plane, the above equation is equivalent the semi-linear elliptic partial differential equation

$$\Delta\psi + 8\omega\frac{1-(x^2+y^2)}{(1+x^2+y^2)^3} - \frac{4F(\psi)}{(1+x^2+y^2)^2} = 0. \tag{2.3}$$

If the gyre flow has no variations in the azimuth, then the radially symmetric solutions of (2.3) have a form of $\psi = \psi(r)$. In terms of the change of variables $\psi = U(s)$, $s_1 < s < s_2$ with

$$r = e^{-s/2} \ \text{ for } \ 0 < s_1 = -2\ln(r_+) < s_2 = -2\ln(r_-),$$

for $0 < r_- < r_+ < 1$, the equation (2.3) can be transformed into the second-order differential equation

$$U''(s) - \frac{e^s}{(1+e^s)^2}F(U(s)) + \frac{2\omega e^s(1-e^s)}{(1+e^s)^3} = 0, \quad s_1 < s < s_2. \tag{2.4}$$

Considering the change of variables

$$u(t) = U(s), \quad t = \frac{s-s_1}{s_2-s_1},$$

we have

$$u'' = a(t)F(u) + b(t), \quad 0 \le t \le 1, \tag{2.5}$$

where $a(\cdot)$, $b(\cdot)$ are two positive continuous functions given by

$$\begin{cases} a(t) = \dfrac{(s_2-s_1)^2 e^{(s_2-s_1)t+s_1}}{(1+e^{(s_2-s_1)t+s_1})^2}, \\ b(t) = -\dfrac{2\omega(s_2-s_1)^2 e^{(s_2-s_1)t+s_1}(1-e^{(s_2-s_1)t+s_1})}{(1+e^{(s_2-s_1)t+s_1})^3}, \qquad 0 \le t \le 1. \end{cases}$$

and with the new boundary conditions

$$u(0) = m, \quad u(1) = n. \tag{2.6}$$

This means that the ACC is a streamline at the boundary. Moreover, we consider when $F(u)$ is a discontinuous function (including three cases), which means the ACC is stratified.

3. The case $m < \gamma < n$

We consider in this section that

$$m < \gamma < n. \tag{3.1}$$

3.1. Preliminary results

We first solve a simple discontinuous problem

$$\begin{aligned} u''(t) &= h(t), \\ u(0) = m, &\quad u(1) = n, \end{aligned} \tag{3.2}$$

where

$$h(u) = \begin{cases} h_+(t) & \text{for } u(t) > \gamma, \\ h_-(t) & \text{for } u(t) < \gamma \end{cases}$$

for $m, n, \gamma \in \mathbb{R}$, and $h_\pm \in C[0,1]$. We look for a C^1−smooth solution of (3.2) such that

p1) There is an $t_1 \in (0,1)$ with $u(t_1) = \gamma$.

p2) $u(t) < \gamma$ for $t \in (0, t_1)$ and $\gamma < u(t)$ for $t \in (t_1, 1)$.

We see that if such a $u(t)$ exists, then it must have by p1) a form

$$u(t) = \begin{cases} u_-(t) = \int_0^{t_1} G_-(t_1,t,s)h_-(s)ds + m + \frac{\gamma-m}{t_1}t & \text{for } t \in [0,t_1], \\ u_+(t) = \int_{t_1}^1 G_+(t_1,t,s)h_+(s)ds + n + \frac{\gamma-n}{1-t_1}(1-t) & \text{for } t \in [t_1,1], \end{cases} \tag{3.3}$$

where $G_\pm(t_1,t,s)$ are Green functions

$$G_-(t_1,t,s) = \begin{cases} \frac{(t-t_1)s}{t_1}, & 0 \le s \le t \le t_1, \\ \frac{(s-t_1)t}{t_1}, & 0 \le t \le s \le t_1, \end{cases}$$

and

$$G_+(t_1,t,s) = \begin{cases} \frac{(t-1)(s-t_1)}{1-t_1}, & t_1 \le s \le t \le 1, \\ \frac{(s-1)(t-t_1)}{1-t_1}, & t_1 \le t \le s \le 1. \end{cases}$$

Clearly,

$$G_\pm(t_1,t,s) < 0. \tag{3.4}$$

A condition $u_-'(t_1) = u_+'(t_1)$, $t_1 \in (0,1)$ is equivalent to

$$\begin{gathered} 0 = t_1(1-t_1)(u_-'(t_1) - u_+'(t_1)) = \\ t_1(1-t_1)\left(\int_0^{t_1} \partial_t G_-(t_1,t_1,s)h_-(s)ds + \frac{\gamma-m}{t_1} - \int_{t_1}^1 \partial_t G_+(t_1,t_1,s)h_+(s)ds + \frac{\gamma-n}{1-t_1}\right) = \\ t_1(1-t_1)\left(\frac{1}{t_1}\int_0^{t_1} sh_-(s)ds + \frac{\gamma-m}{t_1} + \frac{1}{1-t_1}\int_{t_1}^1 (1-s)h_+(s)ds + \frac{\gamma-n}{1-t_1}\right) = \\ (1-t_1)\int_0^{t_1} sh_-(s)ds + t_1\int_{t_1}^1 (1-s)h_+(s)ds + \gamma - m(1-t_1) - nt_1. \end{gathered}$$

Hence, we solve

$$0 = \psi(t_1) = (1 - t_1) \int_0^{t_1} s h_-(s) ds + t_1 \int_{t_1}^{1} (1 - s) h_+(s) ds + \gamma - m(1 - t_1) - n t_1. \tag{3.5}$$

Since

$$\psi(0+) = \lim_{t \to 0^+} \psi(t) = \gamma - m > 0, \quad \psi(1-) = \lim_{t \to 1^-} \psi(t) = \gamma - n < 0,$$

there is a $t_1 \in (0, 1)$ solving (3.5). We take such t_1. To satisfy p2), we suppose either

$$h_+(t) < 0,\ h_-(t) \neq 0, \quad t \in [0, 1] \tag{3.6}$$

or

$$h_+(t) \neq 0,\ h_-(t) > 0, \quad t \in [0, 1]. \tag{3.7}$$

If (3.6) holds, then $u_+(t)$ is strictly concave on $[t_1, 1]$, which implies $u_+(t) > n + \frac{\gamma - n}{1 - t_1}(1 - t) > \gamma$ for $t \in (t_1, 1]$, and $u_+(t)$ has at most one extreme point in $[t_1, 1]$. Hence, $u'_+(t_1) > 0$ and, thus, $u'_-(t_1) > 0$. Hence, $u_-(t) < \gamma$ for $t < t_1$ near t_1. If there is an $t_2 \in (0, t_1)$ with $u_-(t_2) = \gamma$, then there are $t_3 \in (0, t_2]$ and $t_4 \in (t_2, t_1)$ with $u'_-(t_3) = u'_-(t_4) = 0$, so there is $t_5 \in (t_3, t_4)$ with $u''(t_5) = 0$, which contradicts to (3.6). Thus, $u_-(t) < \gamma$ for $0 \leq t < t_1$. Similar arguments work under (3.7). Summarizing, we arrive at the following result.

Theorem 3.1. *Under assumption* (3.1)*, either* (3.6) *or* (3.7) *holds, then there is a solution* (3.3) *of* (3.2) *with properties of p1) and p2), where* t_1 *solves* (3.5).

3.2. Piecewise constant oceanic vorticity

We apply results from Subsection 3.1 by considering

$$F(u) = \begin{cases} b_1, & u > \gamma, \\ b_2, & u < \gamma, \end{cases} \tag{3.8}$$

where b_1, b_2, γ are constants. Then,

$$h_-(t) = b_2 a(t) + \omega c(t), \quad h_+(t) = b_1 a(t) + \omega c(t),$$

where

$$c(t) = -\frac{2(s_2 - s_1)^2 e^{(s_2 - s_1)t + s_1}(1 - e^{(s_2 - s_1)t + s_1})}{(1 + e^{(s_2 - s_1)t + s_1})^3}.$$

Since $-\frac{c(t)}{a(t)} = 2\frac{1 - e^{t(s_2 - s_1) + s_1}}{1 + e^{t(s_2 - s_1) + s_1}}$, set

$$p_1 = \min_{t \in [0,1]} -\frac{c(t)}{a(t)} = 2\frac{1 - e^{s_2}}{1 + e^{s_2}} < 0, \quad p_2 = \max_{t \in [0,1]} -\frac{c(t)}{a(t)} = 2\frac{1 - e^{s_1}}{1 + e^{s_1}} < 0. \tag{3.9}$$

Now, condition (3.6) means

$$\frac{b_1}{\omega} < p_1,\ \frac{b_2}{\omega} \notin [p_1, p_2] \tag{3.10}$$

and condition (3.7) means

$$\frac{b_1}{\omega} \notin [p_1, p_2],\ \frac{b_2}{\omega} > p_2. \tag{3.11}$$

By Theorem 3.1, we get the following result.

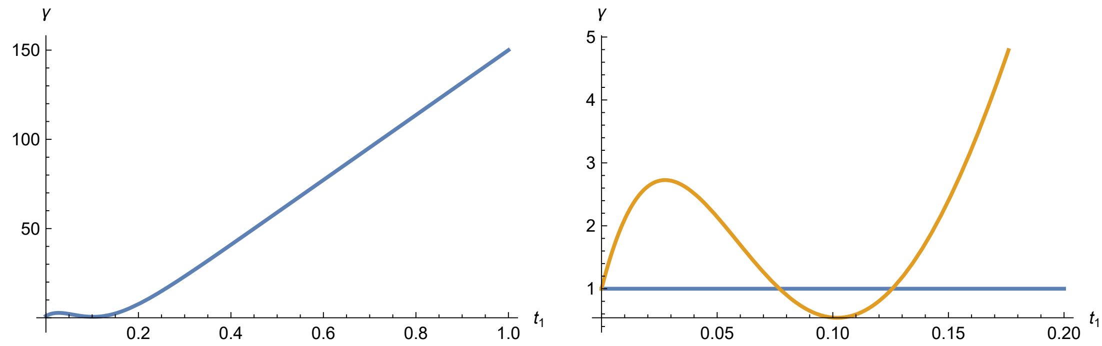

Figure 1. The graph of $\gamma(t_1)$ with (3.13) on $[0, 1]$ (left) and on $[0, 0.2]$ with the line $\gamma = 1$ (right).

Theorem 3.2. *Consider* (3.8) *and assumption* (3.1). *If either* (3.10) *or* (3.11) *holds, then there is a solution* (3.3) *of* (3.2) *with properties of p1) and p2), where t_1 solves* (3.5).

Note that now (3.5) is a linear function of b_1, b_2, ω, m, n, and γ when $t_1 \in (0, 1)$ is considered as a parameter of the form

$$\begin{aligned} & b_1 t_1 \int_{t_1}^{1} (1-s)a(s)ds + b_2(1-t_1) \int_0^{t_1} sa(s)ds \\ & +\omega\left(t_1 \int_{t_1}^{1} (1-s)c(s)ds + (1-t_1)\int_0^{t_1} sc(s)ds\right) - m(1-t_1) - nt_1 + \gamma = 0. \end{aligned} \tag{3.12}$$

So, we get a surface in the space $(t_1, b_1, b_2, \omega, m, n)$ when either (3.10) or (3.11) holds. The righthand side of (3.12) has an explicit formula, but it is rather awkward, so we do not go into more details, we just add a concrete example.

Example 3.3. *We take*

$$b_1 = 10, \quad b_2 = 100, \quad s_2 = 20, \quad s_1 = 1, \quad \omega = 3, \quad m = 1, \quad n = 150. \tag{3.13}$$

Clearly (3.11) *holds for these values* (3.13), *since* $p_1 = 2\frac{1-e^{20}}{1+e^{20}} < 0$ *and* $p_2 = \frac{1-e^{1}}{1+e^{1}} < 0$. *We solve* $\gamma = \gamma(t_1)$ *from* (3.12) *with* (3.13) *(see Figure 1), which is again a huge formula not suitable for writing it here. However, we numerically find that* $\gamma(t_1)$ *is increasing on* $[0, 0.0275]$ *from* 1 *to* 2.72678, *then decreasing on* $[0.0275, 0.101948]$ *from* 2.72678 *to* 0.538563, *and increasing on* $[0.101948, 1]$ *from* 0.538563 *to* 150. *This gives that for* $\gamma \in (0, 2.72678)$, *we have* 3 *solutions of* (3.12), *then* 2 *solutions for* $\gamma = 2.72678$ *and* 1 *solution for* $\gamma \in (2.72678, 150)$. *Thus,* γ *has a cusp type bifurcation.*

3.3. Piecewise linear oceanic vorticity

Now, we extend results of Subsection 3.2 by considering a piecewise linear vorticity case

$$F(u) = \begin{cases} k_1 u + b_1, & u > \gamma, \\ k_2 u + b_2, & u < \gamma, \end{cases} \tag{3.14}$$

where k_1, k_2, b_1, b_2 are constants. Then,

$$\begin{aligned} h_+(t) &= (k_1 \max\{u(t), \gamma\} + b_1)a(t) + \omega c(t), \\ h_-(t) &= (k_2 \min\{u(t), \gamma\} + b_2)a(t) + \omega c(t). \end{aligned} \tag{3.15}$$

We take $\mu < m < n < \nu$ and consider a subset

$$\Omega_1 = \{u \in C_0 = C([0,1]) \mid \mu \leq u(t) \leq \nu,\ \forall t \in [0,1]\}. \tag{3.16}$$

Set

$$\begin{aligned} \xi_- &= \min\{k_2\gamma + b_2, k_2\mu + b_2\}, \quad \chi_- = \max\{k_2\gamma + b_2, k_2\mu + b_2\}, \\ \xi_+ &= \min\{k_1\gamma + b_1, k_1\nu + b_1\}, \quad \chi_+ = \max\{k_1\gamma + b_1, k_1\nu + b_1\}. \end{aligned} \tag{3.17}$$

Then, we derive

$$\begin{aligned} \xi_- a(t) + \omega c(t) \leq h_-(t) \leq \chi_- a(t) + \omega c(t), \\ \xi_+ a(t) + \omega c(t) \leq h_+(t) \leq \chi_+ a(t) + \omega c(t) \end{aligned} \tag{3.18}$$

for any $u \in \Omega_1$. Now, condition (3.6) means

$$\frac{\chi_+}{\omega} < p_1,\ \left[\frac{\xi_-}{\omega}, \frac{\chi_-}{\omega}\right] \cap [p_1, p_2] = \emptyset \tag{3.19}$$

and condition (3.7) means

$$\left[\frac{\xi_+}{\omega}, \frac{\chi_+}{\omega}\right] \cap [p_1, p_2] = \emptyset,\ \frac{\xi_-}{\omega} > p_2. \tag{3.20}$$

For any $t_1 \in [0,1]$ and $u \in \Omega_1$, we consider (3.3) with (3.15). Then, (3.4) and (3.18) imply

$$\begin{aligned} &\int_0^{t_1} G_-(t_1,t,s)(\chi_- a(s) + \omega c(s))ds + m + \frac{\gamma - m}{t_1} t \\ &\leq u_-(t) \leq \\ &\int_0^{t_1} G_-(t_1,t,s)(\xi_- a(s) + \omega c(s))ds + m + \frac{\gamma - m}{t_1} t,\ t \in (0, t_1), \\ &\int_{t_1}^{1} G_+(t_1,t,s)(\chi_+ a(s) + \omega c(s))ds + n + \frac{\gamma - n}{1 - t_1}(1 - t) \\ &\leq u_+(t) \leq \\ &\int_{t_1}^{1} G_+(t_1,t,s)(\xi_+ a(s) + \omega c(s))ds + n + \frac{\gamma - n}{1 - t_1}(1 - t),\ t \in (t_1, 1). \end{aligned} \tag{3.21}$$

By setting

$$\begin{aligned} \zeta_- &= \inf_{t_1 \in (0,1], t \in [0,t_1]} \left(\int_0^{t_1} G_-(t_1,t,s)(\chi_- a(s) + \omega c(s))ds + m + \frac{\gamma - m}{t_1} t \right), \\ \eta_- &= \sup_{t_1 \in (0,1], t \in [0,t_1]} \left(\int_0^{t_1} G_-(t_1,t,s)(\xi_- a(s) + \omega c(s))ds + m + \frac{\gamma - m}{t_1} t \right), \\ \zeta_+ &= \inf_{t_1 \in [0,1), t \in [t_1,1]} \left(\int_{t_1}^{1} G_+(t_1,t,s)(\chi_+ a(s) + \omega c(s))ds + n + \frac{\gamma - n}{1 - t_1}(1 - t) \right), \\ \eta_+ &= \sup_{t_1 \in [0,1), t \in [t_1,1]} \left(\int_{t_1}^{1} G_+(t_1,t,s)(\xi_+ a(s) + \omega c(s))ds + n + \frac{\gamma - n}{1 - t_1}(1 - t) \right), \end{aligned} \tag{3.22}$$

(3.21) gives

$$\begin{aligned} \zeta_- \leq u_-(t) \leq \eta_-,\ t \in [0, t_1], \\ \zeta_+ \leq u_+(t) \leq \eta_+,\ t \in [t_1, 1]. \end{aligned} \tag{3.23}$$

By assuming

$$\mu < \min\{\zeta_-, \zeta_+\}, \quad \nu > \max\{\eta_-, \eta_+\}, \tag{3.24}$$

(3.23) implies

$$\mu < u_-(t) < \nu,\ t \in [0, t_1], \quad \mu < u_+(t) < \nu,\ t \in [t_1, 1]. \tag{3.25}$$

We denote (3.3) with (3.15) by $\Psi(u, t_1)(t)$, i.e,

$$\Psi(u, t_1)(t) = \begin{cases} \int_0^{t_1} G_-(t_1, t, s)\,((k_2 \min\{u(s), \gamma\} + b_2)a(s) + \omega c(s))\,ds + m + \frac{\gamma - m}{t_1} t & t \in [0, t_1], \\ \int_{t_1}^1 G_+(t_1, t, s)\,((k_1 \max\{u(s), \gamma\} + b_1)a(s) + \omega c(s))\,ds + n + \frac{\gamma - n}{1 - t_1}(1 - t) & t \in [t_1, 1]. \end{cases} \tag{3.26}$$

By following (3.5), we set $\psi : \Omega \to \mathbb{R}$, $\Omega = \Omega_1 \times (0, 1)$ as

$$\begin{aligned} \psi(u, t_1) = (1 - t_1) \int_0^{t_1} sh_-(s)ds + t_1 \int_{t_1}^1 (1 - s)h_+(s)ds + \gamma - m(1 - t_1) - nt_1 = \\ (1 - t_1) \int_0^{t_1} s\,((k_2 \min\{u(s), \gamma\} + b_2)a(s) + \omega c(s))\,ds + \\ t_1 \int_{t_1}^1 (1 - s)\,((k_1 \max\{u(s), \gamma\} + b_1)a(s) + \omega c(s))\,ds + \gamma - m(1 - t_1) - nt_1 \end{aligned} \tag{3.27}$$

see (3.15). Summarizing, we obtain the following result.

Theorem 3.4. *Assuming either* (3.19) *or* (3.20)*, and* (3.24) *with* (3.22)*, there is a solution* (3.3) *of* (3.2) *with properties of p1), p2), and* (3.16)*.*

Proof. It is standard to verify that

$$\Psi : \Omega \to C_0$$

is continuous and compact. We introduce a homotopy $H = (H_1, H_2) : [0, 1] \times \Omega \to C_0 \times \mathbb{R}$ as follows:

$$H_1(\lambda, u, t_1) = (1 - \lambda)\frac{\mu + \nu}{2} + \lambda \Psi(u, t_1),$$
$$H_2(\lambda, u, t_1) = (1 - \lambda)(\gamma - m(1 - t_1) - nt_1) + \lambda \psi(u, t_1).$$

Clearly, (3.25) gives

$$H_1([0, 1] \times \Omega) \subset \operatorname{int} \Omega_1,$$

and noting

$$\lim_{t_1 \to 0^+} H_2(\lambda, u, t_1) = H_2(\lambda, u, 0+) = \gamma - m > 0 > \lim_{t_1 \to 1^-} H_2(\lambda, u, t_1) = H_2(\lambda, u, 1-) = \gamma - n,$$

we see that

$$(I - H_1, H_2)([0, 1] \times \partial\Omega_\delta) \neq 0$$

for a small $\delta > 0$ and $\Omega_\delta = \Omega_1 \times (\delta, 1 - \delta)$. This means that we can compute a Leray-Schauder degree

$$\deg((I - H_1(1, \cdot), H_2(1, \cdot)), \Omega_\delta, 0) = \deg((I - H_1(0, \cdot), H_2(0, \cdot)), \Omega_\delta, 0) = \deg(I, H_2(0, \cdot)), \Omega_\delta, 0) =$$
$$\deg(\gamma - m(1 - t_1) - nt_1, (\delta, 1 - \delta), 0) = -1 \neq 0.$$

Thus, there is an $(u, t_1) \in \Omega_\delta$ such that

$$u = \Psi(u, t_1), \quad \psi(u, t_1) = 0.$$

By (3.19), (3.20), and Theorem 3.1, we get a desired solution. The proof is finished.

Next, it is clear that h_+ and h_- in (3.15) are Lipschitz continuous in u with constants $|k_1|$ and $|k_2|$, respectively. By introducing

$$\begin{aligned}\kappa_- &= \max_{t_1\in[0,1],t\in[0,t_1]} \int_0^{t_1} |G_-(t_1,t,s)a(s)|ds,\\ \kappa_+ &= \max_{t_1\in[0,1],t\in[t_1,1]} \int_{t_1}^{1} |G_+(t_1,t,s)a(s)|ds,\end{aligned} \tag{3.28}$$

we can easily check that $\Psi(u,t_1)$ is Lipschitz continuous in u on Ω_1 with a constant

$$L_\Psi = \max\{|k_1|\kappa_+, |k_2|\kappa_-\} \tag{3.29}$$

uniformly with respect to $t_1 \in (0,1)$. By using a Banach fixed point theorem, we can improve Theorem 3.4 as follows.

Theorem 3.5. *Assume either* (3.19) *or* (3.20)*, and* (3.24) *with* (3.22)*. If*

$$L_\Psi < 1, \tag{3.30}$$

then for any $t_1 \in (0,1)$*, there is a unique fixed point of*

$$u_{t_1} = \Psi(u_{t_1}, t_1) \tag{3.31}$$

with (3.16)*. Inserting it into* (3.27)*, we get a scalar equation,*

$$\psi(u_{t_1}, t_1) = 0. \tag{3.32}$$

It has a solution which can be approximately located by

$$\psi(u^n_{t_1}, t_1) = 0, \quad u^n_{t_1} = \Psi^n\left(\frac{\mu+\nu}{2}, t_1\right).$$

Thus, a solution (3.3) *of* (3.2) *with properties of p1) and p2) can be approximately located.*

Example 3.6. *We extend Example 3.3 for* (3.14)*, satisfying additional conditions. To begin, we suppose* (3.30) *with* (3.29)*. Then,* (3.26) *is a global contraction on* C_0 *and we split it to*

$$\Psi(u,t_1)(t) = \Psi_1(u,t_1)(t) + \Psi_2(t_1)(t) \tag{3.33}$$

for

$$\Psi_1(u,t_1)(t) = \begin{cases} k_2 \int_0^{t_1} G_-(t_1,t,s)\min\{u(s),\gamma\}a(s)ds & t\in[0,t_1]\\ k_1 \int_{t_1}^{1} G_+(t_1,t,s)\max\{u(s),\gamma\}a(s)ds & t\in[t_1,1]\end{cases} \tag{3.34}$$

and

$$\Psi_2(t_1)(t) = \begin{cases} \int_0^{t_1} G_-(t_1,t,s)\,(b_2a(s)+\omega c(s))\,ds + m + \frac{\gamma-m}{t_1}t & t\in[0,t_1]\\ \int_{t_1}^{1} G_+(t_1,t,s)\,(b_1a(s)+\omega c(s))\,ds + n + \frac{\gamma-n}{1-t_1}(1-t) & t\in[t_1,1]\end{cases} \tag{3.35}$$

We solve (3.31) *by the Banach fixed point theorem to get*

$$u_{t_1} = (I - \Psi_1(\cdot,t_1))^{-1}\Psi_2(t_1). \tag{3.36}$$

Finally, we get (3.32)*, which is solvable on* $(0,1)$ *according to the above arguments. Clearly,* (3.36) *gives*

$$\sup_{t_1\in(0,1)} \|u_{t_1}\|_0 \le (1-L_\Psi)^{-1} \sup_{t_1\in(0,1)} \|\Psi_2(t_1)\|_0. \tag{3.37}$$

Next, (3.13) *implies*

$$(1-L_\Psi)^{-1} \sup_{t_1\in(0,1)} \|\Psi_2(t_1)\|_0 > (1-L_\Psi)^{-1}\gamma > \gamma,$$

thus applying also (3.15) *and* (3.37)*, we obtain*

$$\begin{aligned} h_+(t) &\ge \min_{t\in[0,1]}(b_1 a(t) + \omega c(t)) - |k_1|(1-L_\Psi)^{-1} \sup_{t_1\in(0,1)} \|\Psi_2(t_1)\|_0\|a\|_0 > 0, \\ h_-(t) &\ge \min_{t\in[0,1]}(b_2 a(t) + \omega c(t)) - |k_2|(1-L_\Psi)^{-1} \sup_{t_1\in(0,1)} \|\Psi_2(t_1)\|_0\|a\|_0 > 0, \end{aligned}$$

if

$$\begin{aligned} |k_1|(1-L_\Psi)^{-1} &< \frac{\min_{t\in[0,1]}(b_1 a(t) + \omega c(t))}{\sup_{t_1\in(0,1)} \|\Psi_2(t_1)\|_0\|a\|_0}, \\ |k_2|(1-L_\Psi)^{-1} &< \frac{\min_{t\in[0,1]}(b_2 a(t) + \omega c(t))}{\sup_{t_1\in(0,1)} \|\Psi_2(t_1)\|_0\|a\|_0}. \end{aligned} \tag{3.38}$$

Summarizing, we arrive at the following conclusion.

Theorem 3.7. *Consider* (3.14) *with parameters from Example 3.3. Suppose* (3.30) *and* (3.38) *with* (3.29)*. Then,* (3.2) *has a solution with properties of p1) and p2).*

3.4. Piecewise nonlinear oceanic vorticity

Finally, we extend results of Subsection 3.3 by considering a nonlinear case given as

$$F(u) = \begin{cases} F_1(u), & u > \gamma, \\ F_2(u), & u < \gamma, \end{cases} \tag{3.39}$$

where F_1, F_2 are continuous. Then,

$$\begin{aligned} h_+(t) &= F_1(\max\{u(t),\gamma\})a(t) + \omega c(t), \\ h_-(t) &= F_2(\min\{u(t),\gamma\})a(t) + \omega c(t). \end{aligned}$$

We again consider a subset Ω_1 from (3.16). We set

$$\begin{aligned} \xi_- &= \min\{F_2(u) \mid \mu \le u \le \gamma\}, \quad \chi_- = \max\{F_2(u) \mid \mu \le u \le \gamma\}, \\ \xi_+ &= \min\{F_1(u) \mid \gamma \le u \le \nu\}, \quad \chi_- = \max\{F_1(u) \mid \gamma \le u \le \nu\}. \end{aligned} \tag{3.40}$$

Clearly, (3.26) now takes a form

$$\Psi(u,t_1)(t) = \begin{cases} \int_0^{t_1} G_-(t_1,t,s)\,(F_2(\min\{u(s),\gamma\})a(s) + \omega c(s))\,ds + m + \frac{\gamma-m}{t_1}t & t\in[0,t_1] \\ \int_{t_1}^1 G_+(t_1,t,s)\,(F_1(\max\{u(s),\gamma\})a(s) + \omega c(s))\,ds + n + \frac{\gamma-n}{1-t_1}(1-t) & t\in[t_1,1] \end{cases} \tag{3.41}$$

Now we can directly follow step by step arguments of Subsection 3.3 to extend the above results for a nonlinear case. So, we get the following results.

Theorem 3.8. *Theorem 3.4 is valid for* (3.39) *with* (3.40).

Theorem 3.9. *Assuming*

$$k_{1,2} = \sup_{\mu \le v < w \le \nu} \frac{|F_{1,2}(w) - F_{1,2}(v)|}{w - v} < \infty, \tag{3.42}$$

Theorem 3.5 is valid for (3.39) *with* (3.40).

Example 3.10. *We extend Example 3.6 for*

$$F_1(u) = k_1|\sin u| + b_1, \quad F_2(u) = k_2|\cos u| + b_2. \tag{3.43}$$

To begin, we suppose (3.30) *with* (3.42). *Then,* (3.41) *is a global contraction on* C_0 *and we can directly repeat the argument of Example 3.6 to get the following result.*

Theorem 3.11. *Consider* (3.39) *with parameters from Example 3.3. Suppose* (3.30) *with* (3.42) *and also (see* (3.38)*)*

$$\begin{aligned} |k_1|(1 - L_\Psi)^{-1} &< \frac{\min_{t\in[0,1]}(b_1 a(t) + \omega c(t))}{\|a\|_0}, \\ |k_2|(1 - L_\Psi)^{-1} &< \frac{\min_{t\in[0,1]}(b_2 a(t) + \omega c(t))}{\|a\|_0}. \end{aligned} \tag{3.44}$$

Then, (3.2) *has a solution with properties of p1) and p2).*

By using (3.13), *we compute from* (3.28)

$$\kappa_- = \max_{t_1\in[0,1], t\in[0,t_1]} \frac{\ln(1+e)(t_1 - t) - t_1 \ln\left(e^{19t+1} + 1\right) + t \ln\left(e^{19t_1+1} + 1\right)}{t_1} \sim 0.245715,$$

$$\kappa_+ = \max_{t_1\in[0,1], t\in[t_1,1]} \frac{\ln\left(1 + e^{20}\right)(t_1 - t) - (t_1 - 1)\ln\left(e^{19t+1} + 1\right) + (t-1)\ln\left(e^{19t_1+1} + 1\right)}{t_1 - 1} \sim 0.245711,$$

and (3.29) *has a value*

$$L_\Psi \sim \max\{|k_1| 0.245711, |k_2| 0.245715\} \sim 0.245715 \max\{|k_1|, |k_2|\},$$

so (3.30) *reads*

$$\max\{|k_1|, |k_2|\} < 4.06976.$$

Furthermore, we calculate

$$\|a\|_0 \sim 70.9769,$$

$$\min_{t\in[0,1]}(b_1 a(t) + \omega c(t)) = \min_{t\in[0,1]} \frac{1444 e^{19z+1}\left(4e^{19z+1} + 1\right)}{\left(e^{19z+1} + 1\right)^3} \sim 0.0000119052,$$

$$\min_{t\in[0,1]}(b_2 a(t) + \omega c(t)) = \min_{t\in[0,1]} \frac{722 e^{19z+1}\left(53e^{19z+1} + 47\right)}{\left(e^{19z+1} + 1\right)^3} \sim 0.0000788721.$$

Hence, (3.44) *becomes*

$$|k_1|(1 - 0.245715\max\{|k_1|, |k_2|\})^{-1} < \frac{\min_{t\in[0,1]}(b_1 a(t) + \omega c(t))}{\|a\|_0} \sim 1.67734.10^{-7},$$

$$|k_2|(1 - 0.245715\max\{|k_1|, |k_2|\})^{-1} < \frac{\min_{t\in[0,1]}(b_2 a(t) + \omega c(t))}{\|a\|_0} \sim 1.11124.10^{-6}.$$

The smallness of the above values is caused by the exponential terms in $a(t)$, $c(t)$, *and* $s_2 = 20$.

Remark 3.12. *A reverse inequality of* (3.1) *given by*

$$n < \gamma < m \tag{3.45}$$

is symmetric to (3.1) *by taking a symmetric transformation* $t \leftrightarrow 1 - t$, $t \in [0,1]$. *So considering* $u(t) \leftrightarrow u(1-t)$, *results for the case* (3.1) *in this section are extended straightforwardly to the case* (3.45).

4. The case $m < \gamma$ and $n < \gamma$

We consider in this section that

$$m < \gamma, \quad n < \gamma. \tag{4.1}$$

This situation (4.1) is different from both (3.1) and (3.45), but we can extend a method of Section 3.

4.1. Preliminary results

We look for a C^1−smooth solution of (3.2) such that

p3) There are $0 < t_1 < t_2 < 1$ with $u(t_1) = u(t_2) = \gamma$.

p4) $u(t) < \gamma$ for $t \in (0, t_1) \bigcup (t_2, 1)$ and $\gamma < u(t)$ for $t \in (t_1, t_2)$.

We see that if such a $u(t)$ exists, then it must have by p3) a form

$$u(t) = \begin{cases} u_-(t) = \int_0^{t_1} G_-(t_1, t, s)h_-(s)ds + m + \frac{\gamma-m}{t_1}t & \text{for } t \in [0, t_1], \\ u_+(t) = \int_{t_1}^{t_2} G_+(t_2, t_1, t, s)h_+(s)ds + \gamma & \text{for } t \in [t_1, t_2], \\ u_-(t) = \int_{t_2}^{1} G_-(t_2, t, s)h_-(s)ds + n + \frac{\gamma-n}{1-t_2}(1-t) & \text{for } t \in [t_2, 1], \end{cases} \tag{4.2}$$

where $G_\pm$ are Green functions

$$G_-(t_1, t, s) = \begin{cases} \frac{(t-t_1)s}{t_1}, & 0 \le s \le t \le t_1, \\ \frac{(s-t_1)t}{t_1}, & 0 \le t \le s \le t_1, \end{cases}$$

$$G_+(t_2, t_1, t, s) = \begin{cases} \frac{(t-t_2)(s-t_1)}{t_2-t_1}, & t_1 \le s \le t \le t_2, \\ \frac{(s-t_2)(t-t_1)}{t_2-t_1}, & t_1 \le t \le s \le t_2, \end{cases}$$

and

$$G_-(t_2, t, s) = \begin{cases} \frac{(t-1)(s-t_2)}{1-t_2}, & t_2 \le s \le t \le 1, \\ \frac{(s-1)(t-t_2)}{1-t_2}, & t_2 \le t \le s \le 1. \end{cases}$$

Clearly, $G_\pm < 0$. A condition $u_-'(t_1) = u_+'(t_1)$, $0 < t_1 < t_2 < 1$ is equivalent to

$$\begin{aligned} 0 = \psi_1(t_1, t_2) = t_1(u_-'(t_1) - u_+'(t_1)) = \\ t_1\left(\int_0^{t_1} \partial_t G_-(t_1, t_1, s)h_-(s)ds + \frac{\gamma-m}{t_1} - \int_{t_1}^{t_2} \partial_t G_+(t_2, t_1, t_1, s)h_+(s)ds\right) = \\ t_1\left(\frac{1}{t_1}\int_0^{t_1} sh_-(s)ds + \frac{\gamma-m}{t_1} + \frac{1}{t_2-t_1}\int_{t_1}^{t_2}(t_2-s)h_+(s)ds\right) = \\ \int_0^{t_1} sh_-(s)ds + \frac{t_1}{t_2-t_1}\int_{t_1}^{t_2}(t_2-s)h_+(s)ds + \gamma - m. \end{aligned} \tag{4.3}$$

Next, a condition $u'_-(t_2) = u'_+(t_2)$, $0 < t_1 < t_2 < 1$ is equivalent to

$$
\begin{gathered}
0 = \psi_2(t_1, t_2) = (1 - t_2)(u'_-(t_2) - u'_+(t_2)) = \\
(1 - t_2)\left(\int_{t_2}^{1} \partial_t G_-(t_2, t_2, s)h_-(s)ds - \frac{\gamma - n}{1 - t_2} - \int_{t_1}^{t_2} \partial_t G_+(t_2, t_1, t_2, s)h_+(s)ds\right) = \\
(1 - t_2)\left(\frac{1}{1 - t_2}\int_{t_2}^{1}(s - 1)h_-(s)ds - \frac{\gamma - n}{1 - t_2} + \frac{1}{t_2 - t_1}\int_{t_1}^{t_2}(t_1 - s)h_+(s)ds\right) = \\
\int_{t_2}^{1}(s - 1)h_-(s)ds + \frac{1 - t_2}{t_2 - t_1}\int_{t_1}^{t_2}(t_1 - s)h_+(s)ds + n - \gamma
\end{gathered}
\tag{4.4}
$$

We consider

$$
\psi = (\psi_1, \psi_2) : \triangle = \{(t_1, t_2) \in [0, 1]^2 \mid t_1 \leq t_2\} \to \mathbb{R}^2.
$$

Since

$$
\begin{aligned}
\left|\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}(t_2 - s)h_+(s)ds\right| &\leq \frac{\|h_+\|_0}{2}\int_{t_1}^{t_2}(t_2 - s)ds = \frac{\|h_+\|_0}{2}(t_2 - t_1), \\
\left|\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}(t_1 - s)h_+(s)ds\right| &\leq \frac{\|h_+\|_0}{2}\int_{t_1}^{t_2}(s - t_1)ds = \frac{\|h_+\|_0}{2}(t_2 - t_1),
\end{aligned}
$$

we derive

$$
\begin{aligned}
\psi_1(t, t) &= \int_0^t sh_-(s)ds + \gamma - m, \quad \psi_1(0, t) = \gamma - m > 0, \\
\psi_2(t, t) &= \int_t^1 (s - 1)h_-(s)ds + n - \gamma \quad \psi_2(t, 1) = n - \gamma < 0.
\end{aligned}
\tag{4.5}
$$

By supposing (3.6) and following arguments in Subsection 3.1, we can see that p4) is satisfied.

Of course, solving (4.3) and (4.4) is not easy. If there is an $t_1 \in (0, 1)$ such that

$$
\psi_1(t_1, t_1) = 0 \quad \psi_2(t_1, t_1) = 0,
\tag{4.6}
$$

then using (4.3), (4.4), (4.5), and (3.6), we see that $h_-(t) < 0$ on $[0, 1]$ and $u'_-(t_1-) = u'_-(t_1+)$. This gives a solution (4.2) for $t_1 = t_2$, which is grazing the line $u = \gamma$ at $t = t_1$.

On the other hand, if

$$
\psi_1(t, t)^2 + \psi_2(t, t)^2 > 0, \ \forall t \in (0, 1),
\tag{4.7}
$$

which holds if

$$
h_-(t) > 0, \ \forall t \in [0, 1],
\tag{4.8}
$$

then we can consider a Brouwer degree or a winding number of a planar vector field (ψ_1, ψ_2)

$$
\deg((\psi_1, \psi_2), \triangle, 0),
$$

which can be computed by using (4.5). Summarizing, we have the following.

Theorem 4.1. *Under assumption* (4.1)*,* (3.6) *holds, then there is a solution* (3.3) *of* (3.2) *with properties of p3) and p4), where t_1 and t_2 solves* (4.3) *and* (4.4)*. Moreover, such t_1 and t_2 exist if supposing* (4.7) *and*

$$
\deg((\psi_1, \psi_2), \triangle, 0) \neq 0
\tag{4.9}
$$

holds.

Next, we compute (see (4.5))

$$(\psi_1(0,t),\psi_2(0,t)) = \left(\gamma - m, \int_t^1 (s-1)h_-(s)ds - \frac{1-t}{t}\int_0^t sh_+(s)ds + n - \gamma\right)$$

$$(\psi_1(t,1),\psi_2(t,1)) = \left(\int_0^t sh_-(s)ds + \frac{t}{1-t}\int_t^1 (1-s)h_+(s)ds + \gamma - m, n - \gamma\right) \tag{4.10}$$

$$(\psi_1(t,t),\psi_2(t,t)) = \left(\int_0^t sh_-(s)ds + \gamma - m, \int_t^1 (s-1)h_-(s)ds + n - \gamma\right),$$

along a counterclockwise oriented border

$$\partial\triangle = \{[0,t],[t,1],[t,t] \mid t \in [0,1]\}.$$

We see that the vector field $\{(\psi_1(0,t),\psi_2(0,t))\}_{t\in[0,1]}$ is located in the right half plane $x_1 > 0$, while the vector field $\{(\psi_1(t,1),\psi_2(t,1))\}_{t\in[0,1]}$ is located in the lower half plane $x_2 < 0$. Thus, there is no whole rotation from a vector

$$(\psi_1(0,0),\psi_2(0,0)) = \left(\gamma - m, \int_0^1 (s-1)h_-(s)ds + n - \gamma\right)$$

to a vector

$$(\psi_1(1,1),\psi_2(1,1)) = \left(\int_0^1 sh_-(s)ds + \gamma - m, n - \gamma\right)$$

along the borders $\{[0,t],[t,1] \mid t \in [0,1]\}$. As a matter of fact, we can use along the borders $\{[0,t],[t,1] \mid t \in [0,1]\}$ in the following homotopy:

$$h_l(t,\lambda) = (1-\lambda)\psi(0,t) + \lambda((1-t)\psi(0,0) + t(\gamma - m, n-\gamma)),\ t,\lambda \in [0,1],$$
$$h_r(t,\lambda) = (1-\lambda)\psi(t,1) + \lambda(t\psi(1,1) + (1-t)(\gamma - m, n-\gamma)),\ t,\lambda \in [0,1],$$

which is suitable, since $h_{l1}(t,\lambda) = (\gamma - m) \neq 0$ and $h_{r2}(t,\lambda) = (n-\gamma) \neq 0$. Note that $h_l(t,1)$, $h_r(t,1)$ gives linear lines connecting $\psi(0,0)$ with $\psi(1,1)$ via $(\gamma - m, n-\gamma)$.

Next, under (4.8), the vector field $\{(\psi_1(t,t),\psi_2(t,t))\}_{t\in[0,1]}$ is located into a quadrant $x_1 > 0$, $x_2 < 0$, which gives that there is no whole rotation along a counterclockwise oriented $\partial\triangle$. This means

$$\deg((\psi_1,\psi_2),\triangle,0) = 0. \tag{4.11}$$

Consequently, under assumptions

$$h_+(t) < 0,\ h_-(t) > 0, \quad t \in [0,1],$$

we cannot use a topological degree argument of Theorem 4.1, but Theorem 4.1 is still applicable. We introduce a mapping

$$\chi(t_1,t_2) = (\chi_1(t_1,t_2),\chi_2(t_1,t_2)) =$$
$$\left(\int_0^{t_1} sh_-(s)ds + \frac{t_1}{t_2-t_1}\int_{t_1}^{t_2}(t_2-s)h_+(s)ds, \int_{t_2}^1 (s-1)h_-(s)ds + \frac{1-t_2}{t_2-t_1}\int_{t_1}^{t_2}(t_1-s)h_+(s)ds\right), \tag{4.12}$$

and consider the set

$$\Upsilon = \{(t_1, t_2) \in \triangle \mid \chi(t_1, t_2) \in \{x_1 < 0, x_2 > 0\}\}. \tag{4.13}$$

Note $\chi(t_1, t_2) = \psi(t_1, t_2) - (\gamma - m, n - \gamma)$. So for $(t_1, t_2) \in \Upsilon$, we take

$$(\gamma - m, n - \gamma) = -\chi(t_1, t_2). \tag{4.14}$$

To complete our discussion, we assume

$$h_\pm(t) < 0, \quad t \in [0, 1]. \tag{4.15}$$

Then, (4.10) shows that both $\psi_{1,2}(t, t)$ are decreasing on $[0, t]$. So, if (4.7) holds along with

$$\exists\, \hat{t} \in (0, 1) : \psi_1(\hat{t}, \hat{t}) < 0,\ \psi_2(\hat{t}, \hat{t}) > 0, \tag{4.16}$$

then the vector field ψ along a counterclockwise oriented border $\partial\triangle$ takes one counterclockwise turn, so (4.9) holds with = 1. Otherwise, (4.11) holds. Furthermore, (4.16) requires $\psi(1, 1) < 0$, i.e.,

$$\int_0^1 sh_-(s)ds + \gamma - m < 0. \tag{4.17}$$

Since $\psi_1(0, 0) = \gamma - m > 0$, under (4.17), there is a unique $t_0 \in (0, 1)$ such that $\psi_1(t_0, t_0) = 0$, i.e.,

$$\int_0^{t_0} sh_-(s)ds + \gamma - m = 0. \tag{4.18}$$

Thus, $\psi_1(t, t) < 0$ for $t \in (t_0, 1]$ and $\psi_1(t, t) > 0$ for $t \in [0, t_0)$. If $\psi_2(t_0, t_0) \leq 0$, then $\psi_2(t, t) \leq 0$ for $t \in (t_0, 1]$. Hence, (4.16) is not satisfied. On the other hand, if $0 < \psi_2(t_0, t_0)$, i.e.,

$$0 < \int_{t_0}^1 (s - 1)h_-(s)ds + n - \gamma, \tag{4.19}$$

then (4.16) holds. Clearly, (4.7) also holds. Summarizing, we arrive at the following result.

Lemma 4.2. *Under assumptions* (4.15)*,* (4.17)*, and* (4.19)*, where t_0 uniquely solves* (4.18)*, we have*

$$\deg((\psi_1, \psi_2), \triangle, 0) = 1.$$

4.2. Piecewise constant oceanic vorticity

We apply results from Subsection 4.1 by considering (3.8). We can follow the method of Subsection 3.2.

Theorem 4.3. *Consider* (3.8) *and assumption* (4.1)*. If* (3.10) *holds, there is a solution* (3.3) *of* (3.2) *with properties of p3) and p4), when t_1 and t_2 solves* (4.3) *and* (4.4)*.*

Now, (4.3) and (4.4) have forms

$$\begin{gathered} b_1 \frac{t_1}{t_2 - t_1} \int_{t_1}^{t_2} (t_2 - s)a(s)ds + b_2 \int_0^{t_1} sa(s)ds + \\ \omega\left(\int_0^{t_1} sc(s)ds + \frac{t_1}{t_2 - t_1} \int_{t_1}^{t_2} (t_2 - s)c(s)ds\right) + \gamma - m = 0, \end{gathered} \tag{4.20}$$

and

$$b_1 \frac{1-t_2}{t_2-t_1} \int_{t_1}^{t_2} (t_1 - s)a(s)ds + b_2 \int_{t_2}^{1} (s-1)a(s)ds + \\ \omega\left(\int_{t_2}^{1} (s-1)c(s)ds + \frac{1-t_2}{t_2-t_1} \int_{t_1}^{t_2} (t_1 - s)c(s)ds \right) + n - \gamma = 0. \tag{4.21}$$

Again, (4.20) and (4.21) are linear functions of b_1, b_2, ω, m, n, and γ when $(t_1, t_2) \in \triangle$ are considered as parameters. So, we get surfaces in the space $(t_1, t_2, b_1, b_2, \omega, m, n)$ when (3.10) holds. The righthand sides of (4.20) and (4.21) have explicit formulae, but they are rather awkward, so we do not go into more details, but just add a concrete example.

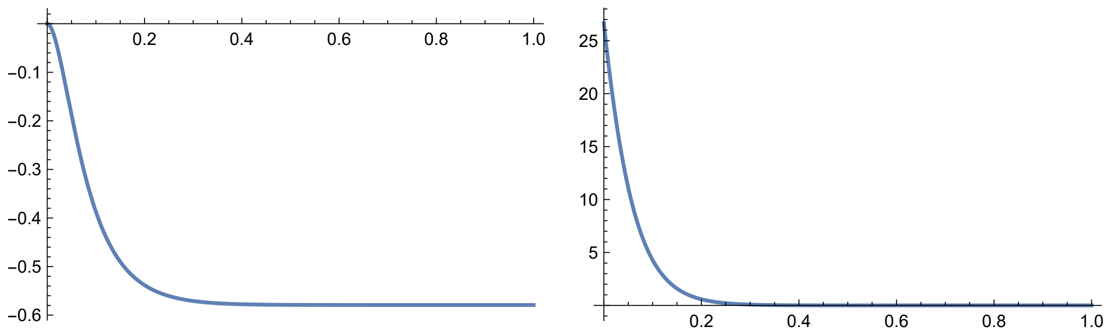

Figure 2. The graphs of $\chi_1(t, t)$ (left) and $\chi_2(t, t)$ (right) on $t \in [0, 1]$ with (4.22).

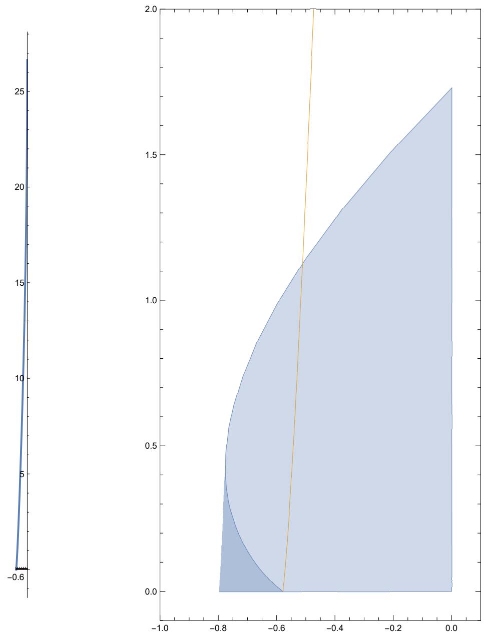

Figure 3. The graph of $\chi(t, t)$, $t \in [0, 1]$ with (4.22) (left), and the graphs of $\chi(t_1, t_1)$ and $\chi(t_1, t_2)$ for $t_1 \in [0, 0.5]$ and $t_2 \in [0.6, 1]$ with (4.22).

Example 4.4. *We take*

$$b_1 = -10, \quad b_2 = -7, \quad s_2 = 20, \quad s_1 = 1, \quad \omega = 3. \tag{4.22}$$

Then

$$p_1\omega = \frac{6\left(1-e^{20}\right)}{1+e^{20}} \sim -6, \quad p_2\omega = \frac{6(1-e)}{1+e} \sim -2.7727,$$

and (3.10) *gives* (4.15)*. We recall* (4.14)*. We numerically check that* $\chi_1(t,t)$ *is decreasing on* $[0,1]$ *from* 0 *to* -0.579183*, while* $\chi_2(t,t)$ *is decreasing on* $[0,1]$ *from* 26.643 *to* 0 *(see Figures 2 and 3). This means* $(t,t) \in \Upsilon$*,* $t \in [0,1]$ *and that for a suitable* $(\gamma - m, n - \gamma)$*, we have* (4.9) *with* $= 1$*:* $-(\gamma - m, n - \gamma)$ *from a region between the graph of* $\chi(t,t)$ *and left axis* x_1 *and upper axis* x_2*, so* (4.16) *holds, for instance, we can take* $(\gamma - m, n - \gamma) = (0.2, -0.5)$*. On the other hand, we see from Figure 3 that there are* $(\gamma - m, n - \gamma)$ *when* (4.11) *holds, but Theorem 4.1 is still applicable, i.e,* (4.14) *holds for some* $(t_1,t_2) \in \Upsilon$*:* $-(\gamma - m, n - \gamma)$ *from a region* $\chi(\Upsilon)$ *on the left side of the graph of* $\chi(t,t)$*, for instance, we can take* $(\gamma - m, n - \gamma) = (0.7, -0.5)$*. These results are consistent with our discussion at the end of Subsection 4.1.*

4.3. Piecewise nonlinear oceanic vorticity

Finally, we extend results of Subsection 3.4 by considering a nonlinear case given as (3.39). From (4.2), (3.41), now possesses a form

$$\Psi(u,t_2,t_1)(t) = \begin{cases} \int_0^{t_1} G_-(t_1,t,s)\,(F_2(\min\{u(s),\gamma\})a(s) + \omega c(s))\,ds + m + \frac{\gamma-m}{t_1}t & t \in [0,t_1] \\ \int_{t_1}^{t_2} G_+(t_2,t_1,t,s)\,(F_1(\max\{u(s),\gamma\})a(s) + \omega c(s))\,ds + \gamma & t \in [t_1,t_2] \\ \int_{t_2}^{1} G_-(t_2,t,s)\,(F_2(\max\{u(s),\gamma\})a(s) + \omega c(s))\,ds + n + \frac{\gamma-n}{1-t_2}(1-t) & t \in [t_2,1] \end{cases} \tag{4.23}$$

We take

$$\mu < \min\{m,n\} < \gamma < \nu.$$

We again consider a subset Ω_1 from (3.16) and (3.40). By setting

$$\begin{aligned} \zeta_- &= \min\Bigg\{ \inf_{t_1\in(0,1],t\in[0,t_1]} \left(\int_0^{t_1} G_-(t_1,t,s)(\chi_- a(s) + \omega c(s))ds + m + \frac{\gamma-m}{t_1}t \right), \\ &\qquad \inf_{t_2\in[0,1),t\in[t_2,1]} \left(\int_{t_2}^{1} G_-(t_2,t,s)(\chi_- a(s) + \omega c(s))ds + n + \frac{\gamma-n}{1-t_2}(1-t) \right) \Bigg\}, \\ \eta_- &= \max\Bigg\{ \sup_{t_1\in(0,1],t\in[0,t_1]} \left(\int_0^{t_1} G_-(t_1,t,s)(\xi_- a(s) + \omega c(s))ds + m + \frac{\gamma-m}{t_1}t \right), \\ &\qquad \sup_{t_2\in[0,1),t\in[t_2,1]} \left(\int_{t_2}^{1} G_-(t_2,t,s)(\xi_- a(s) + \omega c(s))ds + n + \frac{\gamma-n}{1-t_2}(1-t) \right) \Bigg\}, \\ \zeta_+ &= \min_{t_1\in[0,1],t_2\in[t_1,1],t\in[t_1,t_2]} \left(\int_{t_1}^{t_2} G_+(t_2,t_1,t,s)(\chi_+ a(s) + \omega c(s))ds + \gamma \right), \\ \eta_+ &= \max_{t_1\in[0,1],t_2\in[t_1,1],t\in[t_1,t_2]} \left(\int_{t_1}^{t_2} G_+(t_2,t_1,t,s)(\xi_+ a(s) + \omega c(s))ds + \gamma \right), \end{aligned} \tag{4.24}$$

similar estimates like (3.21) give

$$\begin{aligned} \zeta_- \le u_-(t) \le \eta_-,\ t \in [0, t_1] \cup [t_2, 1], \\ \zeta_+ \le u_+(t) \le \eta_+,\ t \in [t_2, 1]. \end{aligned} \tag{4.25}$$

By assuming

$$\mu < \min\{\zeta_-, \zeta_+\}, \quad \nu > \max\{\eta_-, \eta_+\}, \tag{4.26}$$

(4.25) implies

$$\mu < u_-(t) < \nu,\ t \in [0, t_1] \cup [t_2, 1], \quad \mu < u_+(t) < \nu,\ t \in [t_1, t_2]. \tag{4.27}$$

Note, (4.23) can be continuously extended to $0 < t_1 = t_2 < 1$. Summarizing, we obtain the following result.

Theorem 4.5. *Assuming* (4.26) *with* (4.24)*, there is a fixed point* $u(t_1, t_2) = \Psi(u(t_1, t_2), t_2, t_1)$ *in* (3.16) *for any* $(t_1, t_2) \in \triangle_D = \{0 < t_1 \le t_2 < 1\}$.

Proof. It is standard to verify that

$$\Psi : \Omega_D = \Omega_1 \times \triangle_D \to C_0$$

is continuous and compact. Clearly, (4.27) gives

$$\Psi(\Omega_D) \subset \operatorname{int} \Omega_1.$$

The proof follows from the Schauder fixed point theorem.

Next, by introducing

$$\begin{aligned} \kappa_- &= \max\left\{ \max_{t_1\in[0,1], t\in[0,t_1]} \int_0^{t_1} |G_-(t_1, t, s)a(s)| ds,\ \max_{t_2\in[0,1], t\in[t_2,1]} \int_{t_2}^{1} |G_-(t_2, t, s)a(s)| ds \right\} \\ \kappa_+ &= \max_{t_1\in[0,1], t_2\in[t_1,1], t\in[t_1,t_2]} \int_{t_1}^{t_2} |G_+(t_2, t_1, t, s)a(s)| ds, \end{aligned} \tag{4.28}$$

and assuming (3.42), we can easily check that $\Psi(u, t_2, t_1)$ is Lipschitz continuous in u on Ω_1 with a constant (3.29) uniformly with respect to $t_1, t_2 \in \triangle_D$. By using a Banach fixed point theorem, Theorem 3.5 states as follows.

Theorem 4.6. *If* (3.30) *holds from* (3.29)*, with* (4.28) *in addition in Theorem 4.5, then* $u(t_1, t_2)$ *is unique.*

Inserting $u(t_1, t_2)$ into (4.3) and (4.4), we get equations

$$\begin{aligned} 0 = \phi_1(t_1, t_2) = \int_0^{t_1} s\,(F_2(\max\{u(t_1, t_2)(s), \gamma\})a(s) + \omega c(s))\,ds + \\ \frac{t_1}{t_2 - t_1} \int_{t_1}^{t_2} (t_2 - s)\,(F_1(\max\{u(t_1, t_2)(s), \gamma\})a(s) + \omega c(s))\,ds + \gamma - m, \end{aligned} \tag{4.29}$$

and

$$\begin{aligned} 0 = \phi_2(t_1, t_2) = \int_{t_2}^{1} (s-1)\,(F_2(\max\{u(t_1, t_2)(s), \gamma\})a(s) + \omega c(s))\,ds + \\ \frac{1 - t_2}{t_2 - t_1} \int_{t_1}^{t_2} (t_1 - s)\,(F_1(\max\{u(t_1, t_2)(s), \gamma\})a(s) + \omega c(s))\,ds + n - \gamma \end{aligned} \tag{4.30}$$

We arrive at the following result.

Theorem 4.7. *Assume* (3.19) *in addition in Theorems 4.5 and 4.6. If* $0 < t_1 < t_2 < 1$ *solves* (4.29) *and* (4.30)*, then* $u(t_1, t_2)$ *solves* (3.2) *with* (4.1) *for* (3.15)*, and it has properties of p3) and p4).*

Solving (4.29) and (4.30) is difficult. We can use an approximation method of Theorem 3.5 for Theorem 4.6 as follows:

$$\begin{aligned} u_n(t_1, t_2)(t) &= \Psi^n\left(\frac{\mu+\nu}{2}, t_2, t_1\right)(t), \\ \phi_{n,1}(t_1, t_2) &= \int_0^{t_1} s\,(F_2(\max\{u_n(t_1,t_2)(s),\gamma\})a(s) + \omega c(s))\,ds + \\ \frac{t_1}{t_2 - t_1}&\int_{t_1}^{t_2} (t_2 - s)\,(F_1(\max\{u_n(t_1,t_2)(s),\gamma\})a(s) + \omega c(s))\,ds + \gamma - m = 0, \\ \phi_{n,2}(t_1, t_2) &= \int_{t_2}^{1} (s-1)\,(F_2(\max\{u_n(t_1,t_2)(s),\gamma\})a(s) + \omega c(s))\,ds + \\ \frac{1-t_2}{t_2 - t_1}&\int_{t_1}^{t_2} (t_1 - s)\,(F_1(\max\{u_n(t_1,t_2)(s),\gamma\})a(s) + \omega c(s))\,ds + n - \gamma = 0. \end{aligned} \tag{4.31}$$

Furthermore, following arguments of Subsection 4.1, we get

$$\begin{aligned} \phi_1(0^+, t) &= \gamma - m > 0, \quad \phi_2(t, 1^-) = \gamma - n < 0, \\ \phi_1(t, t) &= \int_0^t s\,(F_2(\max\{u(t,t)(s),\gamma\})a(s) + \omega c(s))\,ds + \gamma - m, \\ \phi_2(t, t) &= \int_t^1 (s-1)\,(F_2(\max\{u(t,t)(s),\gamma\})a(s) + \omega c(s))\,ds + n - \gamma \end{aligned}$$

for $t \in (0, 1)$. To get (4.15), we assume (see (3.19))

$$\chi_{\pm} < \omega p_1. \tag{4.32}$$

At the end, we have to compute

$$\deg((\phi_{n,1}, \phi_{n,2}), \triangle_{\frac{1}{n}}, 0)$$

for large n by using a method from Subsection 4.1.

Remark 4.8. *We can take a concrete example by extending Example 4.4 for* $F_1(u) = k_1|\sin u| + b_1$ *and* $F_2(u) = k_2|\cos u| + b_2$ *with sufficiently small* $k_1 > 0$ *and* $k_2 > 0$*. However, we do not present details since they are similar.*

Remark 4.9. *A reverse inequality of* (4.1) *given by*

$$m > \gamma, \quad n > \gamma$$

leads to (4.1) *by taking a transformation* $u \leftrightarrow -u$*, and the results of this section can be directly reformulated.*

5. Final remarks

In the future work, we can consider general discontinuities, then we must pass to differential inclusions via Filippov convexification of (3.39)

$$u''(t) \in a(t)F(u) + c(t) \quad \text{for almost all (aa) } t \in [0,1], \tag{5.1}$$

where

$$F(u) = \begin{cases} F_1(u), & u > \gamma, \\ [F_1(\gamma), F_2(\gamma)], & u = \gamma, \\ F_2(u), & u > \gamma. \end{cases}$$

We rewrite (5.1) as a fixed point problem

$$u(t) = \int_0^1 G(t,s)h(s)ds + m + (n-m)t \text{ for aa } t \in [0,1] \tag{5.2}$$

where

$$G(t,s) = \begin{cases} (t-1)s, & 0 \le s \le t \le 1, \\ (s-1)t, & 0 \le t \le s \le 1, \end{cases}$$

and

$$h \in L_2(0,1), \quad h(t) \in F(u(t)) \text{ for aa } t \in [0,1].$$

So, we introduce a multifunction

$$M(u)(t) = \left\{ \int_0^1 G(t,s)h(s)ds + m + (n-m)t \mid h \in L_2(0,1), \quad h(t) \in F(u(t)) \text{ for aa } t \in [0,1] \right\},$$

and (5.2) is equivalent to

$$u \in M(u). \tag{5.3}$$

Now, if $F_1(u)$ and $F_2(u)$ are globally bounded and continuous, then the Kakutani fixed point theorem gives the existence of the solution of (5.3) [16–18]. The advantage of the above Sections 3 and 4 is finding detailed solutions, which can be constructed or approximated, while (5.3) gives a vague form of solutions. For a numerical analysis of (5.1), we use an approximation of (5.2) given by

$$u(t) = \int_0^1 G(t,s)\left(a(s)\left(\frac{F_2(u) - F_1(u)}{2}\tanh\frac{u-\gamma}{\varepsilon} + \frac{F_2(u) + F_1(u)}{2}\right) + c(s)\right)ds$$
$$+m + (n-m)t, \quad t \in [0,1]$$

for $\varepsilon > 0$ sufficiently small.

Author contributions

Michal Fečkan: Writing-Original draft preparation, Writing-Reviewing and Editing; Shan Li: Writing-Reviewing and Editing; JinRong Wang: Conceptualization, Writing-Reviewing and Editing.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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