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Research article

Analysis of global dynamics in an attraction-repulsion model with nonlinear indirect signal and logistic source

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Abstract: The following chemotaxis system has been considered:

$$\begin{cases} v_t = \Delta v - \xi\nabla\cdot(v\nabla w_1) + \chi\nabla\cdot(v\nabla w_2) + \lambda v - \mu v^{\kappa}, & x\in\Omega,\ t>0,\\ w_{1t} = \Delta w_1 - w_1 + w^{\kappa_1},\ 0 = \Delta w - w + v^{\kappa_2}, & x\in\Omega,\ t>0,\\ 0 = \Delta w_2 - w_2 + v^{\kappa_3}, & x\in\Omega,\ t>0, \end{cases}$$

under the boundary conditions of $\frac{\partial v}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial w_2}{\partial \nu}$ on $\partial\Omega$, where Ω was a bounded smooth domain of $\mathbb{R}^n (n \geq 1)$, ν was the normal vector of $\partial\Omega$, and the parameters were $\lambda, \mu, \xi, \chi, \kappa_1, \kappa_2, \kappa_3 > 0$, and $\kappa > 1$. In this paper, we showed that if either $\kappa_1\kappa_2 < \max\{\frac{2}{n}, \kappa_3, \kappa - 1\}$ or $\kappa_1\kappa_2 = \max\{\frac{2}{n}, \kappa_3, \kappa - 1\}$ with the coefficients and initial data satisfying appropriate conditions, then the system possessed a global classical solution. Furthermore, we also have studied the convergence of solutions to a special case of the above system with $\kappa = \delta + 1, \kappa_1 = 1, \kappa_2 = \kappa_3 = \delta$ for $\delta > 0$. It has been proven that if $\mu > 0$ is large enough, then the corresponding classical solutions exponentially converged to $((\frac{\lambda}{\mu})^{\frac{1}{\delta}}, \frac{\lambda}{\mu}, \frac{\lambda}{\mu}, \frac{\lambda}{\mu})$, where the convergence rate could be formally expressed by the parameters of the system.

Keywords: attraction-repulsion model; indirect signal mechanism; global existence; convergence
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1. Introduction

To investigate the movement of microglia in Alzheimer's disease, the authors in [1] proposed a chemotaxis system, the so-called attraction-repulsion system, which can be formulated as

$$\begin{cases} v_t = \Delta v - \xi \nabla \cdot (v \nabla w_1) + \chi \nabla \cdot (v \nabla w_2) + f(v), & x \in \Omega,\ t > 0, \\ \tau w_{1t} = \Delta w_1 + \alpha v - \beta w_1, & x \in \Omega,\ t > 0, \\ \tau w_{2t} = \Delta w_2 + \gamma v - \delta w_2, & x \in \Omega,\ t > 0, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a bounded smooth domain and the parameters satisfy $\alpha, \beta, \gamma, \delta, \xi, \chi > 0$. Here, the unknown function v represents the microglia density, and w_1, w_2 represent the concentration of two different chemical signals secreted by microglia. $f(v)$ is the logistic term to characterize the proliferation and death of microglia. So far, the numerous studies have been done on dynamical behavior with regard to model (1.1), such as the global classical solvability, the long-time behavior, and the blow-up analysis of classical solutions. Let us briefly summarize some of these achievements in this aspect.

On the one hand, assume that model (1.1) does not contain a logistic source term. For $\tau = 0$, Tao and Wang [2] gained the global well-posedness of the solution in high-dimensional space provided that the repulsion mechanism plays a dominant role in the sense that $\chi\gamma > \xi\alpha$; meanwhile, for $\tau = 0$, the blow-up analysis of solutions was also explored therein in two dimensions under the conditions that $\chi\gamma < \xi\alpha$ and $\beta = \delta$. Espejo and Suzuki [3] removed the restriction $\beta = \delta$ in [2] for $\Omega \subset \mathbb{R}^2$ and also showed that the blow-up result still holds provided that $\chi\gamma < \xi\alpha$. If repulsion mechanism dominates over attraction mechanism with $\chi\gamma > \xi\alpha$, Jin [4] proved the existence of global classical solutions in two-dimensional space and the existence of global weak solutions in three-dimensional space, respectively. For $\tau = 1$, Lin and Mu [5] obtained the global classical solvability in the two-dimensional setting provided that initial data satisfy $\|v_0\|_{L^1(\Omega)} < \frac{1}{k\xi\alpha}$ with $k > 0$ depending only on Ω. It has been proven by Li and Li [6] that the non-radial solutions of the parabolic-elliptic-elliptic version of system (1.1) will be unbounded in finite-time for $\Omega \subset \mathbb{R}^2$ in the sense that $\chi\gamma < \xi\alpha$ and $\beta - \delta \neq 0$. Later on, Yu-Guo-Zheng made an extension [7] and further showed that the blow-up can be guaranteed by the condition that $\int_\Omega v_0 > 8\pi/(\xi\alpha - \chi\gamma)$ with $\chi\gamma < \xi\alpha$.

On the other hand, assume that the system has a logistic damping. As for $f(s) \leq \varrho s(1 - s)$ for all $s \geq 0$ with $\varrho > 0$, Zhang and Li [8] showed that the system with $\tau = 0$ is globally well-posed if one of the following assumptions holds: (a) $\xi\alpha - \chi\beta \leq \varrho$; (b) $n \leq 2$; (c) $\frac{n-2}{n}\xi\alpha - \chi\beta \leq \varrho, n \geq 3$. Meanwhile, the global convergence of the solutions was established for the logistic term $f(s) = \varrho s(1-s)$. Moreover, they also investigated the global weak solvability of the system provided that logistic damping is rather mild. As for a generalized logistic damping $f(s) = \lambda - \mu s^\theta$ with $\lambda, \mu > 0$ and $\theta > 1$, Wang-Zhuang-Zheng [9] demonstrated that if $\xi\alpha = \chi\gamma$, then the global classical solvability with $\tau = 1$ can be guaranteed by the space dimension n and parameter θ. In three-dimensional setting, for $f(s) = s - \mu s^{1+\theta}$ with $\mu > 0, \theta \geq 1$, then the conclusions in [10] imply that the fully-parabolic system of (1.1) is globally well-posed provided that $\beta, \delta \geq \frac{1}{2}$ and $\mu \geq \max\left\{ \left(\frac{41}{2}\xi\alpha + 9\chi\gamma\right)^\theta, \left(\frac{41}{2}\chi\gamma + 9\xi\alpha\right)^\theta \right\}$. Moreover, whenever $v_0 \not\equiv 0$ and for any $\theta \in \mathbb{N}$, the global convergence of solutions was also established. However, it should be mentioned that the convergence rate was still unknown therein. For the case $f(u) = \lambda s - \mu s^2$ with $\lambda \geq 0$ and $\mu > 0$, the higher-dimensional boundedness problem with $n \geq 3$ has been shown in [11] in the sense that if $\beta = \delta$, there exists $\theta_0 > 0$ such that $\xi\alpha + \chi\gamma < \mu\theta_0$.

The signal production in the above literature discussed is usually linear. Recently, the system involving nonlinear signal secretion mechanism has been widely studied. For instance, when $f(s) \leq s(\lambda - \mu s^{\theta})$ and the last two equations of (1.1) were replaced by $0 = \Delta w_1 - \alpha w_1 + \beta v^k$ and $0 = \Delta w_2 - \gamma w_2 + \delta v^l$, respectively, with $\lambda, \mu, \alpha, \beta, \theta, \gamma, \delta, k, l > 0$, Hong-Tian-Zheng [12] obtained the global well-posedness of the system in the sense that $k < \max\{l, \theta, \frac{2}{n}\}$. Moreover, when $k = \max\{l, \theta\} \geq \frac{2}{n}$, the same statement still holds provided that if one of the following assumptions is true: (a) $k = l = \theta, \frac{kn-2}{kn}(\alpha\xi - \gamma\chi) < \mu$; (b) $k = l > \theta, \alpha\xi - \gamma\chi < 0$; (c) $k = \theta > l, \frac{kn-2}{kn}\alpha\xi < \mu$. Zhou-Li-Zhao [13] further obtained the global boundedness under the corresponding critical cases: (a) $k = l = \theta, \frac{kn-2}{kn}(\alpha\xi - \gamma\chi) = \mu$; (b) $k = l > \theta, \alpha\xi - \gamma\chi = 0, nk(nk-2) < 4, 0 < k = l \leq 1$ with $n \geq 2$; (c) $k = \theta > l, \frac{kn-2}{kn}\alpha\xi = \mu$. Moreover, the long-time behavior of solutions was also developed therein. As a further exploration of these, some more generalized models, such as the attraction-repulsion chemotaxis model involving both production and consumption (see [14, 15]) and the attraction-repulsion chemotaxis model with nonlinear diffusions (see [16, 17]), have been considered and many colorful dynamical behaviors can be found therein.

When removing the repulsion mechanism in system (1.1), we get the Keller-Segel [18], which reads

$$\begin{cases} v_t = \Delta v - \xi \nabla \cdot (v \nabla w_1) + f(v), & x \in \Omega,\ t > 0, \\ \tau w_{1t} = \Delta w_1 + g(v, w_1), & x \in \Omega,\ t > 0. \end{cases} \tag{1.2}$$

If $f(v) = 0$ and $g(v, w_1)$ satisfies $-w_1 \leq g(v, w_1) \leq K v^{\alpha} - w_1$ with $K, \alpha > 0$, then the global solvability with $\tau = 1$ established by Liu and Tao [19] can be ensured by the condition $0 < \alpha < \frac{2}{n}$. Moreover, assuming that $f(v) = 0$ and the second equation has taken the form of $0 = \Delta w_1 - \frac{1}{|\Omega|}\int_\Omega v^{\kappa} + v^{\kappa}$ with $\kappa > 0$, Winkler [20] proved that if the number $\kappa > \frac{2}{n}$, then the classical solutions would be unbounded in finite time in radial setting; otherwise, if $\kappa < \frac{2}{n}$, the solutions remain bounded in $\Omega \times (0, \infty)$. For more studies on (1.2) and its variants, we refer the reader to [21–24] for more details.

In the attraction-repulsion model and Keller-Segel model mentioned above, the attraction and repulsion signals are produced by cell itself, directly. In the realistic environment, the secretion of signal substance may undergo some complicated processes. Attraction or repulsion signals may not come directly from the cell, but rather be secreted by another signal substance. The phenomenon may be formulated by the following system:

$$\begin{cases} v_t = \Delta v - \nabla \cdot (v \nabla w_1) + f(v), & x \in \Omega,\ t > 0, \\ \tau w_{1t} = \Delta w_1 - w_1 + w_2,\ \tau w_{2t} = \Delta w_2 - w_2 + v, & x \in \Omega,\ t > 0, \end{cases} \tag{1.3}$$

where w_2 stands for the indirect signal concentration. Suppose that $f(s) = \varrho(s - s^{\gamma})$ for all $s \geq 0$ with $\varrho, \gamma > 0$. Zhang-Niu-Liu [25] obtained global well-posedness of the system with $\tau = 1$ provided that $\gamma > \frac{n}{4} + \frac{1}{2}$ with $n \geq 2$. Later on, the similar statement was also discussed on some more generalized systems involving nonlinear diffusions in [26]. Ren [27] discussed the global generalized solvability of system (1.3) when the initial data satisfied some appropriate regularity conditions. In addition, the convergence of generalized solutions was also discussed therein. For system (1.3) with $\tau = 0$, Li and Li [28] explored the global well-posedness for a quasi-linear system and also explored the limit behavior of the homogeneous steady state.

The existing studies implied that the indirect signal is typically represented by a linear function of cell density. It is rare to see that attraction or repulsion mechanisms are both indirect and nonlinear in a chemotaxis model. Based on the complexity of signal production, including nonlinear indirect mechanisms of signals in a chemotaxis model may be more realistic. In [29] we have studied the global well-posedness for a nonlinear indirect parabolic-parabolic-elliptic system. In addition, more recently, the authors in [30] further extended this work to study the parabolic-parabolic-elliptic-elliptic system

$$\begin{cases} v_t = \Delta v - \xi\nabla\cdot(v\nabla w_1) + \chi\nabla\cdot(v\nabla w_2), & x\in\Omega,\ t>0,\\ w_{1t} = \Delta w_1 - w_1 + w^{\kappa_1},\ 0 = \Delta w - w + v^{\kappa_2}, & x\in\Omega,\ t>0,\\ 0 = \Delta w_2 - w_2 + v^{\kappa_3}, & x\in\Omega,\ t>0, \end{cases} \tag{1.4}$$

where the parameters satisfy $\xi,\chi,\kappa_1,\kappa_2,\kappa_3>0$. The authors explored the global existence of classical solutions in the sense that $\kappa_1\kappa_2<\max\{\frac{2}{n},\kappa_3\}$ and $\kappa_1\kappa_2=\max\{\frac{2}{n},\kappa_3\}$ therein. From the point of reality, considering the proliferation and death of the cell population is natural. Moreover, the existing results also imply that a chemotaxis model involving logistic damping may have more diverse dynamic properties. Thus, in this paper, we continue to consider the system (1.4) with logistic term as follows:

$$\begin{cases} v_t = \Delta v - \xi\nabla\cdot(v\nabla w_1) + \chi\nabla\cdot(v\nabla w_2) + \lambda v - \mu v^{\kappa}, & x\in\Omega,\ t>0,\\ w_{1t} = \Delta w_1 - w_1 + w^{\kappa_1},\ 0 = \Delta w - w + v^{\kappa_2}, & x\in\Omega,\ t>0,\\ 0 = \Delta w_2 - w_2 + v^{\kappa_3}, & x\in\Omega,\ t>0, \end{cases} \tag{1.5}$$

with $\frac{\partial v}{\partial \nu}=\frac{\partial w_1}{\partial \nu}=\frac{\partial w}{\partial \nu}=\frac{\partial w_2}{\partial \nu}=0$ on $\partial\Omega$ and initial data $v(x,0)=v_0(x), w_1(x,0)=w_{10}(x)$, where Ω is a bounded smooth domain of $\mathbb{R}^n(n\geq 1)$, ν is the normal vector of $\partial\Omega$, and the parameters are $\lambda,\mu,\xi,\chi,\kappa_1,\kappa_2,\kappa_3>0$, and $\kappa>1$. Our purpose is to obtain the effects of random diffusion, attraction, and repulsion mechanisms as well as logistic damping on the dynamical behavior of solutions.

The first conclusion of the paper is given below.

Theorem 1.1. *Assume that $\lambda,\mu,\xi,\chi,\kappa_1,\kappa_2,\kappa_3>0$, and $\kappa>1$. Let $v_0\in C^{\vartheta}(\overline{\Omega})$ with $0<\vartheta<1$ and $w_{10}\in W^{1,\infty}(\Omega)$ be nonnegative.*

(i) *If $\kappa_1\kappa_2<\max\{\frac{2}{n},\kappa_3,\kappa-1\}$, then the system (1.5) admits a classical solution $(v,w_1,w,w_2)\in(C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)))^2\times(C^{2,0}(\overline{\Omega}\times(0,\infty)))^2$ fulfilling $\|v(\cdot,t)\|_{L^\infty(\Omega)}+\|w_1(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|w_2(\cdot,t)\|_{W^{1,\infty}(\Omega)}\leq C$, where $C>0$ is a constant independent of t.*

(ii) *Let $M_0=\max\{\int_\Omega v_0,(\frac{\lambda}{\mu})^{\frac{1}{\kappa-1}}|\Omega|\}$. When $\kappa_1\kappa_2=\max\{\frac{2}{n},\kappa_3,\kappa-1\}$, then there exist $m_1,m_2,m_3>0$ such that if one of the following assumptions is satisfied:*

- *(a) $\kappa_1\kappa_2=\frac{2}{n}=\kappa_3=\kappa-1$ with M_0 or ξ small, or χ or μ large satisfying $m_1\xi<m_2M_0^{-\frac{2}{n}}+m_3\chi+\mu$;*
- *(b) $\kappa_1\kappa_2=\frac{2}{n}=\kappa-1>\kappa_3$ with ξ or M_0 small, or μ large satisfying $m_1\xi\leq m_2M_0^{-\frac{2}{n}}+\mu$;*
- *(c) $\kappa_1\kappa_2=\kappa_3=\kappa-1>\frac{2}{n}$ with χ or μ large, or ξ small satisfying $m_1\xi<m_3\chi+\mu$;*
- *(d) $\kappa_1\kappa_2=\frac{2}{n}=\kappa_3>\kappa-1$ with ξ or M_0 small, or χ large satisfying $m_1\xi<m_2M_0^{-\frac{2}{n}}+m_3\chi$;*
- *(e) $\kappa_1\kappa_2=\kappa_3>\max\{\frac{2}{n},\kappa-1\}$ with ξ small, or χ large satisfying $m_1\xi<m_3\chi$;*
- *(f) $\kappa_1\kappa_2=\kappa-1>\max\{\frac{2}{n},\kappa_3\}$ with ξ small, or μ large satisfying $m_1\xi\leq\mu$;*
- *(g) $\kappa_1\kappa_2=\frac{2}{n}>\max\{\kappa_3,\kappa-1\}$ with ξ or M_0 small satisfying $m_1\xi\leq m_2M_0^{-\frac{2}{n}}$,*

then the system (1.5) possesses a classical solution $(v,w_1,w,w_2)\in(C^0(\overline{\Omega}\times[0,\infty))\cap C^{2,1}(\overline{\Omega}\times(0,\infty)))^2\times(C^{2,0}(\overline{\Omega}\times(0,\infty)))^2$ satisfying $\|v(\cdot,t)\|_{L^\infty(\Omega)}+\|w_1(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}+\|w_2(\cdot,t)\|_{W^{1,\infty}(\Omega)}\leq C$, where $C>0$ is a constant independent of t.

The results in Theorem 1.1 **(i)** show that apart from attraction mechanism, the remaining terms including random diffusion and repulsion mechanism as well as logistic source are beneficial to the global boundedness. Theorem 1.1 **(ii)** tells us that under the balance case with $\kappa_1\kappa_2 = \max\{\frac{2}{n}, \kappa_3, \kappa - 1\}$, the global boundedness can be controlled by the sizes of the initial data v_0 and the coefficients ξ, χ, μ. Compared to the boundedness results established in [12,31], since we consider the nonlinear and indirect attraction-signal mechanism in this paper, the boundedness results here seem to be more generalized. Compared to [30], due to considering the logistic source, the boundedness results achieved here are more complicated. It should be pointed out that since nonlinear indirect mechanisms involve parabolic equations in system (1.5), we cannot get the explicit coefficient relationships as in [12], but it also indirectly reflects the importance of coefficients in system (1.5). Moreover, compared to the previous studies in [2,4], our boundedness results remove the restriction on spatial dimension.

In the following, we study the long-time behavior to a specific form of chemotaxis system (1.5) (namely, $\kappa = \delta + 1, \kappa_1 = 1$, and $\kappa_2 = \kappa_3 = \delta$ with $\delta > 0$)

$$\begin{cases} v_t = \Delta v - \xi\nabla\cdot(v\nabla w_1) + \chi\nabla\cdot(v\nabla w_2) + v(\lambda - \mu v^{\delta}), & x\in\Omega,\ t>0,\\ w_{1t} = \Delta w_1 - w_1 + w,\ 0 = \Delta w - w + v^{\delta}, & x\in\Omega,\ t>0,\\ 0 = \Delta w_2 - w_2 + v^{\delta}, & x\in\Omega,\ t>0. \end{cases} \tag{1.6}$$

It is not difficult to check that if $\delta \geq \frac{2}{n}$ and the coefficients and initial data v_0 satisfy conditions as in Theorem 1.1 **(ii)** (a) or (c), then the system (1.6) is globally well-posed. In order to better state the convergence results to system (1.6), we make the following assumption:

$$0 < v(x,t) \leq R,\quad (x,t)\in\overline{\Omega}\times[0,\infty), \tag{1.7}$$

where $R > 0$ is independent of parameters of (1.6).

Thus, the second result is stated as follows.

Theorem 1.2. *Let Ω be a bounded smooth domain of $\mathbb{R}^n (n \geq 1)$ and the parameters satisfy $\lambda, \mu, \xi, \chi, \delta > 0$. Assume that $v_0 \in C^{\vartheta}(\overline{\Omega})$ with $0 < \vartheta < 1$ and $w_{10} \in W^{1,\infty}(\Omega)$ are nonnegative. If $\delta \geq \frac{2}{n}$ and $\mu > 0$ with*

$$\begin{cases} \mu > \sqrt{(\frac{\chi^2}{8} + \frac{\xi^2}{2})\lambda}, & \delta\in(0,1],\\ \mu > \dfrac{\frac{(\delta-1)R^{\delta}}{2}(\xi^2+\frac{\chi^2}{4}) + \sqrt{\frac{(\delta-1)^2R^{2\delta}}{4}(\xi^2+\frac{\chi^2}{4})^2 + 2\lambda(\xi^2+\frac{\chi^2}{4})}}{2}, & \delta\in(1,\infty), \end{cases}$$

such that

$$\delta_1 = \min\left\{\mu - (\frac{\chi^2}{8} + \frac{\xi^2}{2})\frac{\lambda}{\mu}, \frac{3N_1}{4}\right\} > 0$$

and

$$\delta_2 = \min\left\{\mu - N_2 - (\frac{\lambda\chi^2}{8\mu} + \frac{\chi^2(\delta-1)R^{\delta}}{8}), \frac{3N_2}{4}\right\} > 0,$$

with $\eta = (\frac{\lambda}{\mu})^{\frac{1}{\delta}}, N_1 = \frac{\eta\xi^2}{2}, N_2 = \frac{\xi^2}{2}\left[\frac{\lambda}{\mu} + (\delta-1)R^{\delta}\right]$, and $R > 0$ defined in (1.7), then there exist $C > 0$ sufficiently large and $T > 0$ such that

$$\begin{aligned} \|v(\cdot,t) - \eta\|_{L^\infty(\Omega)} &+ \|w_1(\cdot,t) - \eta^{\kappa}\|_{L^\infty(\Omega)} + \|w(\cdot,t) - \eta^{\kappa}\|_{L^\infty(\Omega)}\\ &+ \|w_2(\cdot,t) - \eta^{\kappa}\|_{L^\infty(\Omega)} \leq \begin{cases} Ce^{-\mu_1 t}, & \delta\in(0,1],\\ Ce^{-\mu_2 t}, & \delta\in(1,\infty), \end{cases} \end{aligned} \tag{1.8}$$

for all $t \geq T$, *where*

$$\mu_1 = \frac{\delta_1}{(n+2)\max\left\{\frac{1}{\delta\eta^\delta}, \frac{N_1}{2}\right\}} \tag{1.9}$$

and

$$\mu_2 = \frac{\delta_2}{(n+2)\max\{\frac{1}{\eta^\delta}, \frac{N_2}{2}\}}. \tag{1.10}$$

In Theorem 1.2, we have extended the convergence results established in [32, Theorem 3.3] and [13, Theorem 1.2]. In fact, compared to [32], our model is more generalized and we have to modify the methods developed in [32] to overcome the difficulties generated by dealing with the nonlinear indirect mechanism (please see the proof in Lemma 4.2). Compared to [13, Theorem 1.2], we have obtained a relatively accurate convergence rate, which can be formally expressed by the parameters of the system.

The remaining parts of this paper are carried out as follows. In Sect. 2, we first establish a result involving the local solvability and then give some basic properties. In Sect. 3, we first prove L^p-boundedness for v and then obtain $L^\infty-$boundedness of v by the method of Moser iteration. In Sect. 4, we study a special case of the system (1.5) and analyze the convergence of the corresponding classical solutions.

2. Preliminaries

This section is dedicated to a series of preparatory work. To this end, some basic properties on solutions are necessary and the related proofs can be referred to corresponding references.

Lemma 2.1. *Assume that the conditions in Theorem 1.1 hold. Then, for any nonnegative* $v_0 \in C^\vartheta(\overline{\Omega})$ *with* $0 < \vartheta < 1$ *and* $w_{10} \in W^{1,\infty}(\Omega)$, *there exist* $T_{\max} \in (0, \infty]$ *and nonnegative functions* (v, w_1, w, w_2) *satisfying*

$$(v, w_1, w, w_2) \in (C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max})))^2 \times (C^{2,0}(\overline{\Omega} \times (0, T_{\max})))^2,$$

which solve the system (1.5) *classically in* $\Omega \times (0, T_{\max})$. *Furthermore, if* $T_{\max} < \infty$, *we have*

$$\limsup_{t \nearrow T_{\max}} \|v(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.1}$$

Proof. The proof is quite standard in the framework of the fixed point argument. The reader can refer to [33, 34] for more details.

Lemma 2.2. *(cf. [12, Lemma 2.2]) For any* $\eta_i > 0, \tau_i > 1$ *with* $i = 1, 2$, *the following properties of solutions hold:*

$$\int_\Omega w_2^{\tau_1} \leq \eta_1 \int_\Omega v^{\kappa_3 \tau_1} + c_0 \text{ and } \int_\Omega w^{\tau_2} \leq \eta_2 \int_\Omega v^{\kappa_2 \tau_2} + c_1 \text{ for all } t \in (0, T_{\max}), \tag{2.2}$$

where $\kappa_2, \kappa_3 > 0$ *is given in system* (1.5), *and* $c_0, c_1 > 0$ *depend only on* κ_3, η_1, τ_1 *and* κ_2, η_2, τ_2, *respectively. Additionally, there holds*

$$\int_\Omega v \leq \max\left\{\int_\Omega v_0, \left(\frac{\lambda}{\mu}\right)^{\frac{1}{\kappa-1}} |\Omega|\right\} =: M_0 \;\; t \in (0, T_{\max}). \tag{2.3}$$

Lemma 2.3. *(cf. [35, Lemma 2.5]) For $\kappa_1 > 0$, assume that w_1 fulfills*

$$\begin{cases} w_{1t} = \Delta w_1 + w^{\kappa_1} - w_1, & x \in \Omega,\ t > 0, \\ \frac{\partial w_1}{\partial \nu} = 0, & x \in \partial\Omega,\ t > 0, \\ w_1(x,0) = w_{10}(x), & x \in \Omega. \end{cases} \tag{2.4}$$

Then, for all $w_{10} \in W^{2,r}(\Omega)$ with $\frac{\partial w_{10}}{\partial \nu}|_{\partial\Omega} = 0$ and $w \in L^r((0,T); L^r(\Omega))$ with $r \in (1,\infty)$, the equation (2.4) *admits a unique solution satisfying*

$$w_1 \in W^{1,r}((0,T); L^r(\Omega)) \cap L^r((0,T); W^{2,r}(\Omega)). \tag{2.5}$$

Furthermore, we can find $C_r > 0$ and $s_0 \in [0,T)$ with $T \in (0,\infty]$ such that if $w_1(\cdot, s_0) \in W^{2,r}(\Omega)$ with $\frac{\partial w_1(\cdot,s_0)}{\partial \nu}|_{\partial\Omega} = 0$, there holds

$$\int_{s_0}^T \int_\Omega e^{rs}|\Delta w_1|^r \le C_r \int_{s_0}^T \int_\Omega e^{rs} w^{r\kappa_1} + C_r e^{rs_0}(\|w_1(\cdot, s_0)\|^r_{L^r(\Omega)} + \|\Delta w_1(\cdot, s_0)\|^r_{L^r(\Omega)}). \tag{2.6}$$

3. Global existence and uniform boundedness

In this section, we shall illustrate the L^∞−boundedness of v by employing the maximum Sobolev regularity argument and the Moser iteration. For this purpose, we give some local properties of solutions. Let $s_0 \in (0, T_{\max})$ with $s_0 < 1$. Due to Lemma 2.1, we know that $v(\cdot, s_0), w_1(\cdot, s_0) \in C^2(\overline{\Omega})$ with $\frac{\partial w_1(\cdot,s_0)}{\partial \nu}\Big|_{\partial\Omega} = \frac{\partial v(\cdot,s_0)}{\partial \nu}\Big|_{\partial\Omega} = 0$. Hence, we can find $M_1 > 0$ such that

$$\begin{cases} \sup_{0\le s\le s_0} \|v(\cdot, s)\|_{L^\infty(\Omega)} \le M_1, \quad \sup_{0\le s\le s_0} \|w_1(\cdot, s)\|_{L^\infty(\Omega)} \le M_1, \\ \|\Delta w_1(\cdot, s_0)\|_{L^\infty(\Omega)} \le M_1. \end{cases} \tag{3.1}$$

Lemma 3.1. *Let Ω be a bounded smooth domain of $\mathbb{R}^n (n \ge 1)$ and the parameters fulfill $\lambda, \mu, \xi, \chi, \kappa_1, \kappa_2, \kappa_3 > 0, \kappa > 1$. Assume that the conditions in Theorem 1.1 hold. Then, for any $p > \max\{1, \kappa_2 - \kappa_1\kappa_2\}$, there exists $C > 0$ such that*

$$\int_\Omega v^p \le C, \quad t \in (0, T_{\max}). \tag{3.2}$$

Proof. Multiplying v^{p-1} on both sides of (1.5) for any $p > 1$, we infer that

$$\begin{aligned} \frac{1}{p}\frac{d}{dt}\int_\Omega v^p =& -\frac{4(p-1)}{p^2}\int_\Omega |\nabla v^{\frac{p}{2}}|^2 + \xi(p-1)\int_\Omega v^{p-1}\nabla v \cdot \nabla w_1 \\ & -\chi(p-1)\int_\Omega v^{p-1}\nabla v \cdot \nabla w_2 + \lambda \int_\Omega v^p - \mu \int_\Omega v^{p+\kappa-1} \end{aligned} \tag{3.3}$$

for all $t \in (0, T_{\max})$. For some $c_2 > 0$, in light of the Gagliardo-Nirenberg inequality, we conclude

$$\int_\Omega v^{p+\frac{2}{n}} = \|v^{\frac{p}{2}}\|^{\frac{2(p+\frac{2}{n})}{p}}_{L^{\frac{2(p+\frac{2}{n})}{p}}(\Omega)} \le c_2 \|\nabla v^{\frac{p}{2}}\|^{\frac{2(p+\frac{2}{n})}{p}\cdot\theta}_{L^2(\Omega)} \|v^{\frac{p}{2}}\|^{\frac{2(p+\frac{2}{n})}{p}\cdot(1-\theta)}_{L^{\frac{2}{p}}(\Omega)} + c_2\|v^{\frac{p}{2}}\|^{\frac{2(p+\frac{2}{n})}{p}}_{L^{\frac{2}{p}}(\Omega)} \tag{3.4}$$

for all $t \in (0, T_{\max})$, where $\theta = \frac{\frac{p}{2} - \frac{p}{2(p+\frac{2}{n})}}{\frac{p}{2} + \frac{1}{n} - \frac{1}{2}} \in (0, 1)$. Clearly, we see that $\frac{p+\frac{2}{n}}{p} \cdot \theta = 1$. Thus, we infer from (2.3) that

$$\int_\Omega v^{p+\frac{2}{n}} \leq c_2 M_0^{\frac{2}{n}} \|\nabla v^{\frac{p}{2}}\|_{L^2(\Omega)}^2 + c_2 M_0^{p+\frac{2}{n}}, \quad t \in (0, T_{\max}). \tag{3.5}$$

A simple calculation can yield

$$\xi(p-1) \int_\Omega v^{p-1} \nabla v \cdot \nabla w_1 \leq \frac{\xi(p-1)}{p} \int_\Omega v^p |\Delta w_1|, \quad t \in (0, T_{\max}). \tag{3.6}$$

Similarly, for the equation of w_2 in system (1.5), we know

$$-\chi(p-1) \int_\Omega v^{p-1} \nabla v \cdot \nabla w_2 = \frac{\chi(p-1)}{p} \int_\Omega v^p w_2 - \frac{\chi(p-1)}{p} \int_\Omega v^{p+\kappa_3}, \quad t \in (0, T_{\max}). \tag{3.7}$$

For any $\varepsilon_1 > 0$, it is not difficult to deduce from Young's inequality that

$$\frac{\xi(p-1)}{p} \int_\Omega v^p |\Delta w_1| \leq \varepsilon_1 \int_\Omega v^{p+\kappa_1\kappa_2} + \varepsilon_1^{-\frac{p}{\kappa_1\kappa_2}} \cdot \left(\frac{\xi(p-1)}{p}\right)^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}} \int_\Omega |\Delta w_1|^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}}, \quad t \in (0, T_{\max}). \tag{3.8}$$

For any $\varepsilon_2 > 0$, we conclude that by invoking Young's inequality again,

$$\frac{\chi(p-1)}{p} \int_\Omega v^p w_2 \leq \frac{\varepsilon_2}{2} \int_\Omega v^{p+\kappa_3} + \left(\frac{\varepsilon_2}{2}\right)^{-\frac{p}{\kappa_3}} \left(\frac{\chi(p-1)}{p}\right)^{\frac{p+\kappa_3}{\kappa_3}} \int_\Omega w_2^{\frac{p+\kappa_3}{\kappa_3}}, \quad t \in (0, T_{\max}). \tag{3.9}$$

Recalling Lemma 2.2, there holds

$$\left(\frac{\varepsilon_2}{2}\right)^{-\frac{p}{\kappa_3}} \left(\frac{\chi(p-1)}{p}\right)^{\frac{p+\kappa_3}{\kappa_3}} \int_\Omega w_2^{\frac{p+\kappa_3}{\kappa_3}} \leq \frac{\varepsilon_2}{2} \int_\Omega v^{p+\kappa_3} + c_0, \quad t \in (0, T_{\max}), \tag{3.10}$$

where $\eta_1 = \left(\frac{\varepsilon_2}{2}\right)^{1+\frac{p}{\kappa_3}} \left(\frac{\chi(p-1)}{p}\right)^{-\frac{p+\kappa_3}{\kappa_3}}$. Collecting (3.3) and (3.5)–(3.10), one may get

$$\begin{aligned} \frac{1}{p}\frac{d}{dt} \int_\Omega v^p \leq & -\frac{4(p-1)}{p^2 c_2 M_0^{\frac{2}{n}}} \int_\Omega v^{p+\frac{2}{n}} + \varepsilon_1 \int_\Omega v^{p+\kappa_1\kappa_2} + \varepsilon_1^{-\frac{p}{\kappa_1\kappa_2}} \cdot \left(\frac{\xi(p-1)}{p}\right)^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}} \int_\Omega |\Delta w_1|^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}} \\ & + \left(\varepsilon_2 - \frac{\chi(p-1)}{p}\right) \int_\Omega v^{p+\kappa_3} + \lambda \int_\Omega v^p - \mu \int_\Omega v^{p+\kappa-1} + c_3, \quad t \in (0, T_{\max}), \end{aligned} \tag{3.11}$$

where $c_3 = c_0 \left(\frac{\varepsilon_2}{2}\right)^{-\frac{p}{\kappa_3}} \left(\frac{\chi(p-1)}{p}\right)^{\frac{p+\kappa_3}{\kappa_3}} + \frac{4(p-1)}{p^2} M_0^p$. We first add $\frac{p+\kappa_1\kappa_2}{p\kappa_1\kappa_2} \int_\Omega v^p$ and then multiply $e^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2} t}$ on the both sides of (3.11) to derive

$$\begin{aligned} \frac{d}{dt}\left[e^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2} t} \frac{1}{p} \int_\Omega v^p\right] \leq & -\frac{4(p-1)}{p^2 c_2 M_0^{\frac{2}{n}}} \cdot e^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2} t} \int_\Omega v^{p+\frac{2}{n}} + \varepsilon_1 \cdot e^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2} t} \int_\Omega v^{p+\kappa_1\kappa_2} \\ & + \varepsilon_1^{-\frac{p}{\kappa_1\kappa_2}} \cdot \left(\frac{\xi(p-1)}{p}\right)^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}} \cdot e^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2} t} \int_\Omega |\Delta w_1|^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}} \end{aligned}$$

$$+\left(\varepsilon_2-\frac{\chi(p-1)}{p}\right)\cdot e^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}t}\int_\Omega v^{p+\kappa_3}-\mu\cdot e^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}t}\int_\Omega v^{p+\kappa-1}$$
$$+\left(\lambda+\frac{p+\kappa_1\kappa_2}{p\kappa_1\kappa_2}\right)\cdot e^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}t}\int_\Omega v^p+c_3\cdot e^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}t},\quad t\in(0,T_{\max}). \tag{3.12}$$

Integrating (3.12) from s_0 to t, we infer that

$$\begin{aligned}\frac{1}{p}\int_\Omega v^p\le&-\frac{4(p-1)}{p^2c_2M_0^{\frac{2}{n}}}\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\frac{2}{n}}+\varepsilon_1\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_1\kappa_2}\\&+\varepsilon_1^{-\frac{p}{\kappa_1\kappa_2}}\cdot\left(\frac{\xi(p-1)}{p}\right)^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}}\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega|\Delta w_1|^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}}\\&+\left(\varepsilon_2-\frac{\chi(p-1)}{p}\right)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_3}-\mu\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa-1}\\&+\left(\lambda+\frac{p+\kappa_1\kappa_2}{p\kappa_1\kappa_2}\right)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^p+c_4,\quad t\in(s_0,T_{\max}),\end{aligned} \tag{3.13}$$

where $c_4=\frac{c_3\kappa_1\kappa_2}{p+\kappa_1\kappa_2}+\frac{1}{p}\int_\Omega v^p(\cdot,s_0)$. Setting $r=\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}>1$ in Lemma 2.3, one may get from (2.6) that

$$\begin{aligned}\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega|\Delta w_1|^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}}\le&C_r\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega w^{\frac{p+\kappa_1\kappa_2}{\kappa_2}}\\&+C_re^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s_0)}\|w_1(\cdot,s_0)\|^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}}_{W^{2,\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}}},\quad t\in(s_0,T_{\max}).\end{aligned} \tag{3.14}$$

Based on Lemma 2.2 with $\tau_2=\frac{p+\kappa_1\kappa_2}{\kappa_2}>1$, we derive that

$$\int_\Omega w^{\frac{p+\kappa_1\kappa_2}{\kappa_2}}\le\eta_2\int_\Omega v^{p+\kappa_1\kappa_2}+c_1,\quad t\in(s_0,T_{\max}). \tag{3.15}$$

Substituting (3.14) and (3.15) into (3.13), we see

$$\begin{aligned}\frac{1}{p}\int_\Omega v^p\le&-\frac{4(p-1)}{p^2c_2M_0^{\frac{2}{n}}}\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\frac{2}{n}}+f(\varepsilon_1)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_1\kappa_2}\\&+\left(\lambda+\frac{p+\kappa_1\kappa_2}{p\kappa_1\kappa_2}\right)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^p+\left(\varepsilon_2-\frac{\chi(p-1)}{p}\right)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_3}\\&-\mu\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa-1}+c_5,\quad t\in(s_0,T_{\max}),\end{aligned} \tag{3.16}$$

where

$$f(\varepsilon_1)=\varepsilon_1+C_r\eta_2\left(\frac{\xi(p-1)}{p}\right)^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}}\varepsilon_1^{-\frac{p}{\kappa_1\kappa_2}}$$

and

$$c_5=c_1C_r\left(\frac{\xi(p-1)}{p}\right)^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}}\cdot\varepsilon_1^{-\frac{p}{\kappa_1\kappa_2}}\cdot\frac{\kappa_1\kappa_2}{p+\kappa_1\kappa_2}+C_r\left(\frac{\xi(p-1)}{p}\right)^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}}\cdot\varepsilon_1^{-\frac{p}{\kappa_1\kappa_2}}\|w_1(\cdot,s_0)\|^{\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}}_{W^{2,\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}}}+c_4.$$

It is not hard to see that $f(\varepsilon_1)$ gets the minimum value at $\varepsilon_1 = \left(\frac{C_r\eta_2 p}{\kappa_1\kappa_2}\right)^{\frac{\kappa_1\kappa_2}{p+\kappa_1\kappa_2}} \cdot \frac{\xi(p-1)}{p}$, namely,

$$\inf_{\varepsilon_1>0} f(\varepsilon_1) = \left(\frac{C_r\eta_2 p}{\kappa_1\kappa_2}\right)^{\frac{\kappa_1\kappa_2}{p+\kappa_1\kappa_2}} \cdot \frac{\xi(p+\kappa_1\kappa_2)(p-1)}{p^2} = m_1\xi, \tag{3.17}$$

where $m_1 = (\frac{C_r\eta_2 p}{\kappa_1\kappa_2})^{\frac{\kappa_1\kappa_2}{p+\kappa_1\kappa_2}} \cdot \frac{(p+\kappa_1\kappa_2)(p-1)}{p^2}$. Hence, letting $\varepsilon_1 = (\frac{C_r\eta_2 p}{\kappa_1\kappa_2})^{\frac{\kappa_1\kappa_2}{p+\kappa_1\kappa_2}} \cdot \frac{\xi(p-1)}{p}$ in (3.16), one can arrive at

$$\begin{aligned}\frac{1}{p}\int_\Omega v^p \le &-\frac{4(p-1)}{p^2 c_2 M_0^{\frac{2}{n}}}\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\frac{2}{n}} + m_1\xi\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_1\kappa_2}\\ &+\left(\lambda+\frac{p+\kappa_1\kappa_2}{p\kappa_1\kappa_2}\right)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^p + \left(\varepsilon_2-\frac{\chi(p-1)}{p}\right)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_3}\\ &-\mu\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa-1} + c_5, \quad t\in(s_0,T_{\max}).\end{aligned} \tag{3.18}$$

In the following, the proof of inequality (3.2) is divided into two different cases.

Case (i) $\kappa_1\kappa_2 < \max\{\frac{2}{n}, \kappa_3, \kappa-1\}$.

Let $\kappa_1\kappa_2 < \kappa_3$. From Young's inequality, there holds

$$m_1\xi\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_1\kappa_2} \le \frac{\chi(p-1)}{2p}\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_3} + c_6, \quad t\in(s_0,T_{\max}), \tag{3.19}$$

where $c_6 > 0$. Setting $\varepsilon_2 = \frac{\chi(p-1)}{2p}$ in (3.18), one may obtain

$$\frac{1}{p}\int_\Omega v^p \le \left(\lambda+\frac{p+\kappa_1\kappa_2}{p\kappa_1\kappa_2}\right)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^p - \mu\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa-1} + c_7 \tag{3.20}$$

for all $t\in(s_0,T_{\max})$ and $c_7 = c_5 + c_6 > 0$. Combining Young's inequality and (3.1), for $\kappa > 1$, (3.2) can be inferred.

Let $\kappa_1\kappa_2 < \frac{2}{n}$. Invoking Young's inequality, we can find $c_8 > 0$ such that

$$m_1\xi\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_1\kappa_2} \le \frac{4(p-1)}{p^2 c_2 M_0^{\frac{2}{n}}}\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\frac{2}{n}} + c_8, \quad t\in(s_0,T_{\max}). \tag{3.21}$$

Combining (3.21) and (3.18), and setting $\varepsilon_2 = \frac{\chi(p-1)}{p}$ in (3.18), we see

$$\frac{1}{p}\int_\Omega v^p \le \left(\lambda+\frac{p+\kappa_1\kappa_2}{p\kappa_1\kappa_2}\right)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^p - \mu\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa-1} + c_9, \tag{3.22}$$

where $c_9 = c_5 + c_8 > 0$. Due to (3.1), we can infer (3.2) from Young's inequality.

Let $\kappa_1\kappa_2 < \kappa - 1$. Invoking Young's inequality, one may deduce

$$m_1\xi\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_1\kappa_2} \le \frac{\mu}{2}\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa-1} + c_{10} \tag{3.23}$$

for all $t \in (s_0, T_{\max})$ and $c_{10} > 0$. Collecting (3.18) and (3.23), and setting $\varepsilon_2 = \frac{\chi(p-1)}{p}$ in (3.18), we get

$$\frac{1}{p}\int_\Omega v^p \le \left(\lambda + \frac{p+\kappa_1\kappa_2}{p\kappa_1\kappa_2}\right)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^p - \frac{b}{2}\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa-1} + c_{11} \tag{3.24}$$

for all $t \in (s_0, T_{\max})$, $c_{11} = c_5 + c_{10} > 0, \kappa > 1$. Thus, by applying Young's inequality and (3.1), it is not hard to obtain (3.2).

Case (ii) $\kappa_1\kappa_2 = \max\{\frac{2}{n}, \kappa_3, \kappa - 1\}$.

Set $m_2 = \frac{4(p-1)}{p^2 c_2}$ and $m_3 = \frac{p-1}{p}$. Thus, we rewrite the inequality (3.18) as

$$\begin{aligned}\frac{1}{p}\int_\Omega v^p \le & - m_2 M_0^{-\frac{2}{n}}\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\frac{2}{n}} + m_1\xi\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_1\kappa_2}\\ & + \left(\lambda + \frac{p+\kappa_1\kappa_2}{p\kappa_1\kappa_2}\right)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^p + (\varepsilon_2 - m_3\chi)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_3}\\ & - \mu\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa-1} + c_5, \quad t \in (s_0, T_{\max}).\end{aligned} \tag{3.25}$$

(a) Let $\kappa_1\kappa_2 = \frac{2}{n} = \kappa_3 = \kappa - 1$. Then, the inequality (3.25) turns into

$$\begin{aligned}\frac{1}{p}\int_\Omega v^p \le & (- m_2 M_0^{-\frac{2}{n}} - m_3\chi + m_1\xi + \varepsilon_2 - \mu)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_3}\\ & + \left(\lambda + \frac{p+\kappa_1\kappa_2}{p\kappa_1\kappa_2}\right)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^p + c_5, \quad t \in (s_0, T_{\max}).\end{aligned} \tag{3.26}$$

If $m_1\xi < m_2 M_0^{-\frac{2}{n}} + m_3\chi + \mu$, we may take $\varepsilon_2 > 0$ sufficiently small such that $-m_2 M_0^{-\frac{2}{n}} - m_3\chi + m_1\xi + \varepsilon_2 - \mu < 0$. Thus, applying Young's inequality to (3.26), one can obtain the desired result (3.2).

(b) Let $\kappa_1\kappa_2 = \frac{2}{n} = \kappa - 1 > \kappa_3$. Setting $\varepsilon_2 = \frac{m_3\chi}{2}$ in (3.25), we can get

$$\frac{1}{p}\int_\Omega v^p \le (-m_2 M_0^{-\frac{2}{n}} + m_1\xi - \mu)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\frac{2}{n}} + c_{12}, \quad t \in (s_0, T_{\max}), \tag{3.27}$$

with $c_{12} > 0$. Thus, if $m_1\xi \le m_2 M_0^{-\frac{2}{n}} + \mu$, we can obtain (3.2), directly.

(c) Let $\kappa_1\kappa_2 = \kappa_3 = \kappa - 1 > \frac{2}{n}$. Thanks to Young's inequality, we conclude from (3.25) that

$$\frac{1}{p}\int_\Omega v^p \le (\varepsilon_2 - m_3\chi + m_1\xi - \mu)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_3} + c_{13}, \quad t \in (s_0, T_{\max}), \tag{3.28}$$

with $c_{13} > 0$. If $m_1\xi < m_3\chi + \mu$, we may choose ε_2 small enough such that $\varepsilon_2 - m_3\chi + m_1\xi - \mu < 0$. Thus, the result (3.2) is concluded.

(d) Let $\kappa_1\kappa_2 = \kappa_3 = \frac{2}{n} > \kappa - 1$. The Young inequality enables us to deduce from (3.25) that

$$\frac{1}{p}\int_\Omega v^p \le (-m_2 M_0^{-\frac{2}{n}} + m_1\xi + \varepsilon_2 - m_3\chi)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_3} + c_{14}, \quad t \in (s_0, T_{\max}), \tag{3.29}$$

with $c_{14} > 0$. If $m_1\xi < m_2M_0^{-\frac{2}{n}} + m_3\chi$, we may take ε_2 small enough such that $-m_2M_0^{-\frac{2}{n}} + m_1\xi + \varepsilon_2 - m_3\chi < 0$. Hence, we can deduce the desired result (3.2).

(e) Let $\kappa_1\kappa_2 = \kappa_3 > \max\{\frac{2}{n}, \kappa - 1\}$. Using the same method, it can be deduced from (3.25) that

$$\frac{1}{p}\int_\Omega v^p \le (m_1\xi + \varepsilon_2 - m_3\chi)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa_3} + c_{15}, \;\; t \in (s_0, T_{\max}), \tag{3.30}$$

where $c_{15} > 0$. If $m_1\xi < m_3\chi$, we may choose ε_2 small enough such that $m_1\xi + \varepsilon_2 - m_3\chi < 0$. Thus, it is not difficult to obtain (3.2) from (3.30).

(f) Let $\kappa_1\kappa_2 = \kappa - 1 > \max\{\frac{2}{n}, \kappa_3\}$. Taking $\varepsilon_2 = m_3\chi$ in (3.25), we see that

$$\frac{1}{p}\int_\Omega v^p \le (m_1\xi - \mu)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\kappa-1} + c_{16}, \;\; t \in (s_0, T_{\max}), \tag{3.31}$$

with some $c_{16} > 0$. If $m_1\xi \le \mu$, we can obtain (3.2).

(g) Let $\kappa_1\kappa_2 = \frac{2}{n} > \max\{\kappa_3, \kappa - 1\}$. Setting $\varepsilon_2 = m_3\chi$ in (3.25), we can derive that

$$\frac{1}{p}\int_\Omega v^p \le (-m_2M_0^{-\frac{2}{n}} + m_1\xi)\int_{s_0}^t e^{-\frac{p+\kappa_1\kappa_2}{\kappa_1\kappa_2}(t-s)}\int_\Omega v^{p+\frac{2}{n}} + c_{17}, \;\; t \in (s_0, T_{\max}), \tag{3.32}$$

with some $c_{17} > 0$. If $m_1\xi \le m_2M_0^{-\frac{2}{n}}$, we can conclude that $\frac{1}{p}\int_\Omega v^p \le c_{17}$. Thus, combining with (3.1), we can get the desired result (3.2).

Now, we are in a position to prove Theorem 1.1.

The proof of Theorem 1.1 Let the parameters fulfill $\lambda, \mu, \xi, \chi, \kappa_1, \kappa_2, \kappa_3 > 0$, and $\kappa > 1$. For any $p > \max\{1, n\kappa_2, n\kappa_3, \kappa_2 - \kappa_1\kappa_2\}$, using Lemma 3.1 and the elliptic L^p–estimate, it can be concluded form the equations of w and w_2 in system (1.5) that $\|w(\cdot,t)\|_{W^{2,\frac{p}{\kappa_2}}(\Omega)} \le c_{18}$ and $\|w_2(\cdot,t)\|_{W^{2,\frac{p}{\kappa_3}}(\Omega)} \le c_{19}$ for all $t \in (0, T_{\max})$, with some $c_{18}, c_{19} > 0$. Invoking the Sobolev imbedding argument, it is sufficient to find $c_{20} > 0$ such that $\|w(\cdot,t)\|_{W^{1,\infty}(\Omega)}, \|w_2(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le c_{20}$ for all $t \in (0, T_{\max})$. With an application of the parabolic regularity, it is not difficult to deduce from the second equation in system (1.5) that $\|w_1(\cdot,t)\|_{W^{1,\infty}(\Omega)} \le c_{21}$ for all $t \in (0, T_{\max})$, with $c_{21} > 0$. By applying Moser iteration [22] and recalling Lemma 3.1, we can find $c_{22} > 0$ such that

$$\|v(\cdot,t)\|_{L^\infty} \le c_{22}, \;\; t \in (0, T_{\max}).$$

Hence, we can obtain $T_{\max} = \infty$ from Lemma 2.1. Thus, the proof of the Theorem 1.1 is finished.

4. Global asymptotic stability for a special model of system (1.5)

For this part, we are going to study the global convergence of solutions to the following system:

$$\begin{cases} v_t = \Delta v - \xi\nabla\cdot(v\nabla w_1) + \chi\nabla\cdot(v\nabla w_2) + v(\lambda - \mu v^\delta), & x \in \Omega, \; t > 0, \\ w_{1t} = \Delta w_1 - w_1 + w, \; 0 = \Delta w - w + v^\delta, & x \in \Omega, \; t > 0, \\ 0 = \Delta w_2 - w_2 + v^\delta, & x \in \Omega, \; t > 0, \\ \frac{\partial v}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w}{\partial \nu} = \frac{\partial w_2}{\partial \nu} = 0, & x \in \partial\Omega, \; t > 0, \\ v(x,0) = v_0(x), w_1(x,0) = w_{10}(x), & x \in \Omega, \; t > 0. \end{cases} \tag{4.1}$$

The system (4.1) can be seen as a special case of system (1.5) with $\kappa = \delta + 1, \kappa_1 = 1, \kappa_2 = \kappa_3 = \delta$ for $\delta > 0$. If $\delta \geq \frac{2}{n}$ and the coefficients and initial data v_0 satisfy the conditions in Theorem 1.1 **(ii)** (a) or (c), then the system (4.1) possesses a global classical solution. In addition, the solution is bounded, namely, we can find a constant $R > 0$ satisfying

$$0 < v(x,t) \leq R, \quad (x,t) \in \overline{\Omega} \times [0,\infty), \tag{4.2}$$

where R is independent of the parameters of the system (4.1).

We recall an important lemma established in [36].

Lemma 4.1. *(cf. [36, Lemma 3.1]) Let $f(t) \geq 0$ be a uniformly continuous function satisfying $\int_{t_0}^{\infty} f(t)dt < \infty$ with some $t_0 > 0$. Thus, we infer that*

$$f(t) \to 0 \quad as\ t \to \infty. \tag{4.3}$$

To develop the long-time behavior of solutions, the following L^2-convergence of solutions seems to be necessary.

Lemma 4.2. *Let $\lambda, \mu, \xi, \chi, \delta > 0$. If $\delta \geq \frac{2}{n}$ and the coefficients and initial value v_0 satisfy conditions as in Theorem 1.1* ***(ii)*** *(a) or (c), then there holds*

$$\int_\Omega (v-\eta)^2 + \int_\Omega (w_1 - \eta^\delta)^2 + \int_\Omega (w - c^\delta)^2 + \int_\Omega (w_2 - \eta^\delta)^2 \to 0 \ \ as\ t \to \infty, \ with\ \eta = (\frac{\lambda}{\mu})^{\frac{1}{\delta}}. \tag{4.4}$$

Proof. Letting $\delta \in (0,1]$, we establish the energy functional as below

$$A(t) = \int_\Omega v - \eta - \eta \ln(\frac{v}{\eta}) + \frac{N_1}{2}(w_1 - \eta^\delta)^2, \ \ \eta = (\frac{\lambda}{\mu})^{\frac{1}{\delta}}, \ \ N_1 = \frac{\xi^2 \eta}{2}, \ \ t \geq 0. \tag{4.5}$$

In fact, $(v, w_1) = (\eta, \eta^\kappa)$ is a global minimum value point of $A(t)$. We thus infer that $A(t) \geq 0$ for all $t \geq 0$. We take derivative to deduce

$$\begin{aligned}
\frac{d}{dt}A(t) =& \int_\Omega \frac{v-\eta}{v} v_t + N_1 \int_\Omega (w_1 - \eta^\kappa) w_{1t} \\
=& \int_\Omega \frac{v-\eta}{v}\Big[\Delta v - \xi \nabla \cdot (v \nabla w_1) + \chi \nabla \cdot (v \nabla w_2) + v(\lambda - \mu v^\delta)\Big] + N_1 \int_\Omega (w_1 - \eta^\delta)(\Delta w_1 - w_1 + w) \\
=& -\eta \int_\Omega \frac{|\nabla v|^2}{v^2} + \eta\xi \int_\Omega \frac{\nabla v \cdot \nabla w_1}{v} - \eta\chi \int_\Omega \frac{\nabla v \cdot \nabla w_2}{v} - \mu \int_\Omega (v-\eta)(v^\delta - \eta^\delta) \\
& - N_1 \int_\Omega |\nabla w_1|^2 - N_1 \int_\Omega (w_1 - \eta^\delta)^2 + N_1 \int_\Omega (w_1 - \eta^\delta)(w - \eta^\delta) \\
\leq& \frac{\eta\chi^2}{2} \int_\Omega |\nabla w_2|^2 - \mu \int_\Omega (v-\eta)(v^\delta - \eta^\delta) - \frac{3N_1}{4} \int_\Omega (w_1 - \eta^\delta)^2 + N_1 \int_\Omega (w - \eta^\delta)^2.
\end{aligned} \tag{4.6}$$

Testing the equation of w in (4.1) with $w - \eta^\delta$, we get

$$\int_\Omega |\nabla w|^2 = -\int_\Omega (w - \eta^\delta)^2 + \int_\Omega (w - \eta^\delta)(v^\delta - \eta^\delta). \tag{4.7}$$

By direct calculation, it can be concluded that

$$2N_1\int_\Omega|\nabla w|^2\le -N_1\int_\Omega(w-\eta^\delta)^2+N_1\int_\Omega(v^\delta-\eta^\delta)^2. \tag{4.8}$$

Combining (4.8) and (4.6), there holds

$$\frac{d}{dt}A(t)\le\frac{\eta\chi^2}{2}\int_\Omega|\nabla w_2|^2-\mu\int_\Omega(v-\eta)(v^\delta-\eta^\delta)-\frac{3N_1}{4}\int_\Omega(w_1-\eta^\delta)^2+N_1\int_\Omega(v^\delta-\eta^\delta)^2. \tag{4.9}$$

Testing the equation of w_2 in (4.1) with $w_2-\eta^\delta$, one may deduce

$$\int_\Omega|\nabla w_2|^2=-\int_\Omega(w_2-\eta^\delta)^2+\int_\Omega(w_2-\eta^\delta)(v^\delta-\eta^\delta). \tag{4.10}$$

Substituting (4.10) into (4.9), we can infer that

$$\begin{aligned}\frac{d}{dt}A(t)\le&-\frac{\eta\chi^2}{2}\int_\Omega(w_2-\eta^\delta)^2+\frac{\eta\chi^2}{2}\int_\Omega(w_2-\eta^\delta)(v^\delta-\eta^\delta)-\mu\int_\Omega(v-\eta)(v^\delta-\eta^\delta)\\&-\frac{3N_1}{4}\int_\Omega(w_1-\eta^\delta)^2+N_1\int_\Omega(v^\delta-\eta^\delta)^2\\\le&(\frac{\eta\chi^2}{8}+N_1)\int_\Omega(v^\delta-\eta^\delta)^2-\mu\int_\Omega(v-\eta)(v^\delta-\eta^\delta)-\frac{3N_1}{4}\int_\Omega(w_1-\eta^\delta)^2.\end{aligned} \tag{4.11}$$

Since $\delta\in(0,1]$, we thus deduce that

$$(v^\delta-\eta^\delta)^2\le\eta^{\delta-1}(v-\eta)(v^\delta-\eta^\delta). \tag{4.12}$$

Thus, from (4.12), we can rewrite (4.11) as

$$\begin{aligned}\frac{d}{dt}A(t)\le&-\Big[\mu-(\frac{\eta\chi^2}{8}+N_1)\eta^{\delta-1}\Big]\int_\Omega(v-\eta)(v^\delta-\eta^\delta)-\frac{3N_1}{4}\int_\Omega(w_1-\eta^\delta)^2\\=&-\Big[\mu-(\frac{\chi^2}{8}+\frac{\xi^2}{2})\frac{\lambda}{\mu}\Big]\int_\Omega(v-\eta)(v^\delta-\eta^\delta)-\frac{3N_1}{4}\int_\Omega(w_1-\eta^\delta)^2\\=&-\delta_1\Big[\int_\Omega(v-\eta)(v^\delta-\eta^\delta)+\int_\Omega(w_1-\eta^\delta)^2\Big],\end{aligned} \tag{4.13}$$

where $\delta_1=\min\Big\{\mu-(\frac{\chi^2}{8}+\frac{\xi^2}{2})\frac{\lambda}{\mu},\frac{3N_1}{4}\Big\}$ and $N_1=\frac{\eta\xi^2}{2}$ is defined in (4.5). For any $t_0\ge0$, we can get by integrating (4.13) from t_0 to t that

$$A(t)-A(t_0)\le-\delta_1\Big[\int_{t_0}^t\int_\Omega(v-\eta)(v^\delta-\eta^\delta)+\int_{t_0}^t\int_\Omega(w_1-\eta^\delta)^2\Big]. \tag{4.14}$$

The nonnegativity of δ_1 can be guaranteed by $\mu>\sqrt{(\frac{\chi^2}{8}+\frac{\xi^2}{2})\lambda}$. Since $A(t)\ge0$, we thus have

$$\int_{t_0}^t\int_\Omega(v-\eta)(v^\delta-\eta^\delta)+\int_{t_0}^t\int_\Omega(w_1-\eta^\delta)^2\le\frac{A(t_0)}{\delta_1}<\infty. \tag{4.15}$$

Owing to Theorem 1.1, it is not difficult to obtain the boundedness of the solution (v, w_1, w, w_2). Due to the parabolic regularity argument [34], we can find $\epsilon \in (0, 1)$ and $C > 0$ such that

$$\|(v, w_1, w, w_2)\|_{C^{2+\epsilon,1+\frac{\epsilon}{2}}(\overline{\Omega}\times[t,t+1])} \le C, \quad t \ge 0. \tag{4.16}$$

Thanks to (4.16), the uniform continuity and global boundedness of $\int_\Omega (v-\eta)(v^\delta - \eta^\delta) + \int_\Omega (w_1 - \eta^\delta)^2$ are obvious. Recalling Lemma 4.1, thus there holds

$$\int_\Omega (v-\eta)(v^\delta - \eta^\delta) + \int_\Omega (w_1 - \eta^\delta)^2 \to 0 \ \text{ as } t \to \infty. \tag{4.17}$$

Using (4.12) again, we have

$$\frac{1}{\eta^{\delta-1}} \int_\Omega (v^\delta - \eta^\delta)^2 \le \int_\Omega (v-\eta)(v^\delta - \eta^\delta) \to 0 \ \text{ as } t \to \infty. \tag{4.18}$$

From (4.7) and (4.10), we invoke Young's inequality to deduce that

$$\int_\Omega |\nabla w|^2 \le -\frac{1}{2}\int_\Omega (w-\eta^\delta)^2 + \frac{1}{2}\int_\Omega (v^\delta - \eta^\delta)^2 \tag{4.19}$$

and

$$\int_\Omega |\nabla w_2|^2 \le -\frac{1}{2}\int_\Omega (w_2-\eta^\delta)^2 + \frac{1}{2}\int_\Omega (v^\delta - \eta^\delta)^2. \tag{4.20}$$

So, in light of (4.18), we conclude from (4.19) and (4.20) that

$$\int_\Omega (w-\eta^\delta)^2 \le \int_\Omega (v^\delta - \eta^\delta)^2 \to 0 \ \text{ as } t \to \infty \tag{4.21}$$

and

$$\int_\Omega (w_2-\eta^\delta)^2 \le \int_\Omega (v^\delta - \eta^\delta)^2 \to 0 \ \text{ as } t \to \infty. \tag{4.22}$$

Define $h(s) = s^{\frac{1}{\delta}}$. By means of the mean value theorem and (4.2), we find

$$v - \eta = h(v^\delta) - h(\eta^\delta) = \frac{1}{\delta}\zeta^{\frac{1-\delta}{\delta}}(v^\delta - \eta^\delta), \tag{4.23}$$

with ζ between R^δ and η^δ. Therefore,

$$\int_\Omega (v-\eta)^2 \le \frac{1}{\delta^2}R^{2(1-\delta)} \int_\Omega (v^\delta - \eta^\delta)^2 \to 0 \ \text{ as } t \to \infty. \tag{4.24}$$

Thus, we can get (4.4) for $\delta \in (0, 1]$ by collecting (4.17), (4.21), (4.22), and (4.24).

For the case $\delta \in (1, \infty)$, we redefine the energy functional $W(t)$ as follows:

$$W(t) = \frac{1}{\delta}\int_\Omega \left[v^\delta - \frac{\lambda}{\mu} - \frac{\lambda}{\mu}\ln\left(\frac{\mu v^\delta}{\lambda}\right)\right] + \frac{N_2}{2}\int_\Omega \left(w_1 - \frac{\lambda}{\mu}\right)^2, \quad t \ge 0, \tag{4.25}$$

where $N_2 = \frac{\xi^2}{2}\left[\frac{\lambda}{\mu} + (\delta-1)R^\delta\right]$. From the integrand of $W(t)$, it is not difficult to see that $W(t) \geq 0$ for all $t \geq 0$. In light of (4.2), we conclude from Young's inequality that

$$\begin{aligned}
\frac{d}{dt}W(t) =& \int_\Omega \frac{v^\delta - \frac{\lambda}{\mu}}{v} v_t + N_2 \int_\Omega (w_1 - \frac{\lambda}{\mu}) w_{1t} \\
=& -(\delta-1)\int_\Omega v^{\delta-2}|\nabla v|^2 - \frac{\lambda}{\mu}\int_\Omega \frac{|\nabla v|^2}{v^2} + \xi(\delta-1)\int_\Omega v^{\delta-1}\nabla v\cdot\nabla w_1 \\
&+ \frac{\lambda}{\mu}\xi \int_\Omega \frac{\nabla v\cdot\nabla w_1}{v} - \chi(\delta-1)\int_\Omega v^{\delta-1}\nabla v\cdot\nabla w_2 - \frac{\lambda}{\mu}\chi\int_\Omega \frac{\nabla v\cdot\nabla w_2}{v} \\
&- \mu\int_\Omega (v^\delta - \frac{\lambda}{\mu})^2 - N_2\int_\Omega |\nabla w_1|^2 - N_2\int_\Omega (w_1 - \frac{\lambda}{\mu})^2 + N_2\int_\Omega (w_1 - \frac{\lambda}{\mu})(w - \frac{\lambda}{\mu}) \\
\leq& \frac{\lambda\xi^2}{2\mu}\int_\Omega |\nabla w_1|^2 + \frac{\lambda\chi^2}{2\mu}\int_\Omega |\nabla w_2|^2 - (\delta-1)\int_\Omega v^{\delta-2}|\nabla v|^2 + \xi(\delta-1)\int_\Omega v^{\delta-1}\nabla v\cdot\nabla w_1 \\
&- \chi(\delta-1)\int_\Omega v^{\delta-1}\nabla v\cdot\nabla w_2 - \mu\int_\Omega (v^\kappa - \frac{\lambda}{\mu})^2 - N_2\int_\Omega |\nabla w_1|^2 - \frac{3N_2}{4}\int_\Omega (w_1 - \frac{\lambda}{\mu})^2 \\
&+ N_2\int_\Omega (w - \frac{\lambda}{\mu})^2, \quad t \geq 0,
\end{aligned} \tag{4.26}$$

where $N_2 = \frac{\xi^2}{2}\left[\frac{\lambda}{\mu} + (\delta-1)R^\delta\right]$. Thus, there holds

$$\begin{aligned}
\frac{d}{dt}W(t) \leq& \frac{\lambda\chi^2}{2\mu}\int_\Omega |\nabla w_2|^2 - \frac{\delta-1}{2}\int_\Omega \left[v^{\frac{\delta}{2}-1}\nabla v - \xi v^{\frac{\delta}{2}}\nabla w_1\right]^2 - \frac{\delta-1}{2}\int_\Omega \left[v^{\frac{\delta}{2}-1}\nabla v + \chi v^{\frac{\delta}{2}}\nabla w_2\right]^2 \\
&+ \frac{\chi^2(\delta-1)R^\delta}{2}\int_\Omega |\nabla w_2|^2 - \mu\int_\Omega (v^\delta - \frac{\lambda}{\mu})^2 - \frac{3N_2}{4}\int_\Omega (w_1 - \frac{\lambda}{\mu})^2 + N_2\int_\Omega (w - \frac{\lambda}{\mu})^2 \\
\leq& \Big(\frac{\lambda\chi^2}{2\mu} + \frac{\chi^2(\delta-1)R^\delta}{2}\Big)\int_\Omega |\nabla w_2|^2 - \mu\int_\Omega (v^\delta - \frac{\lambda}{\mu})^2 - \frac{3N_2}{4}\int_\Omega (w_1 - \frac{\lambda}{\mu})^2 \\
&+ N_2\int_\Omega (w - \frac{\lambda}{\mu})^2, \quad t \geq 0,
\end{aligned} \tag{4.27}$$

where $N_2 = \frac{\xi^2}{2}\left[\frac{\lambda}{\mu} + (\delta-1)R^\delta\right]$. Testing the equation of w in (4.1) with $w - \frac{\lambda}{\mu}$, one may obtain

$$\int_\Omega |\nabla w|^2 = -\int_\Omega (w - \frac{\lambda}{\mu})^2 + \int_\Omega (w - \frac{\lambda}{\mu})(v^\delta - \frac{\lambda}{\delta}). \tag{4.28}$$

Hence, we have

$$2N_2\int_\Omega |\nabla w|^2 \leq -N_2\int_\Omega (w - \frac{\lambda}{\mu})^2 + N_2\int_\Omega (v^\delta - \frac{\lambda}{\mu})^2. \tag{4.29}$$

Combining (4.29) and (4.27), there holds

$$\begin{aligned}
\frac{d}{dt}W(t) \leq& \Big(\frac{\lambda\chi^2}{2\mu} + \frac{\chi^2(\delta-1)R^\delta}{2}\Big)\int_\Omega |\nabla w_2|^2 - \mu\int_\Omega (v^\delta - \frac{\lambda}{\mu})^2 \\
&- \frac{3N_2}{4}\int_\Omega (w_1 - \frac{\lambda}{\mu})^2 + N_2\int_\Omega (v^\delta - \frac{\lambda}{\mu})^2
\end{aligned}$$

$$=\Big(\frac{\lambda\chi^2}{2\mu}+\frac{\chi^2(\delta-1)R^\delta}{2}\Big)\int_\Omega|\nabla w_2|^2-(\mu-N_2)\int_\Omega(v^\delta-\frac{\lambda}{\mu})^2-\frac{3N_2}{4}\int_\Omega(w_2-\frac{\lambda}{\mu})^2. \tag{4.30}$$

Similarly, one may deduce

$$\int_\Omega|\nabla w_2|^2=-\int_\Omega(w_2-\frac{\lambda}{\mu})^2+\int_\Omega(w_2-\frac{\lambda}{\mu})(v^\delta-\frac{\lambda}{\mu}). \tag{4.31}$$

Substituting (4.31) into (4.30), we can obtain from Young's inequality

$$\begin{aligned}\frac{d}{dt}W(t)\le&-\Big(\frac{\lambda\chi^2}{2\mu}+\frac{\chi^2(\delta-1)R^\delta}{2}\Big)\int_\Omega(w_2-\frac{\lambda}{\mu})^2+\Big(\frac{\lambda\chi^2}{2\mu}+\frac{\chi^2(\delta-1)R^\delta}{2}\Big)\int_\Omega(w_2-\frac{\lambda}{\mu})(v^\delta-\frac{\lambda}{\mu})\\&-(\mu-N_2)\int_\Omega(v^\delta-\frac{\lambda}{\mu})^2-\frac{3N_2}{4}\int_\Omega(w_1-\frac{\lambda}{\mu})^2\\\le&\Big(\frac{\lambda\chi^2}{8\mu}+\frac{\chi^2(\delta-1)R^\delta}{8}\Big)\int_\Omega(v^\delta-\frac{\lambda}{\mu})^2-(\mu-N_2)\int_\Omega(v^\delta-\frac{\lambda}{\mu})^2-\frac{3N_2}{4}\int_\Omega(w_1-\frac{\lambda}{\mu})^2\\=&-\Big[\mu-N_2-\Big(\frac{\lambda\chi^2}{8\mu}+\frac{\chi^2(\delta-1)R^\delta}{8}\Big)\Big]\int_\Omega(v^\delta-\frac{\lambda}{\mu})^2-\frac{3N_2}{4}\int_\Omega(w_1-\frac{\lambda}{\mu})^2\\=&-\delta_2\Big[\int_\Omega(v^\delta-\frac{\lambda}{\mu})^2-\int_\Omega(w_1-\frac{\lambda}{\mu})^2\Big],\end{aligned} \tag{4.32}$$

where $\delta_2=\min\{\mu-N_2-(\frac{\lambda\chi^2}{8\mu}+\frac{\chi^2(\delta-1)R^\delta}{8}),\frac{3N_2}{4}\}$. Since

$$\mu>\frac{\frac{(\delta-1)R^\delta}{2}(\xi^2+\frac{\chi^2}{4})+\sqrt{\frac{(\delta-1)^2R^{2\delta}}{4}(\xi^2+\frac{\chi^2}{4})^2+2\lambda(\xi^2+\frac{\chi^2}{4})}}{2},$$

we have $\delta_2>0$. We integrate (4.32) to get

$$W(t)-W(t_0)\le-\delta_2\Big[\int_{t_0}^t\int_\Omega(v^\delta-\frac{\lambda}{\mu})^2+\int_{t_0}^t\int_\Omega(w_1-\frac{\lambda}{\mu})^2\Big]. \tag{4.33}$$

Due to $W(t)\ge0$ and $\delta_2>0$, we thus deduce from (4.33) that

$$\int_{t_0}^\infty\int_\Omega(v^\delta-\frac{\lambda}{\mu})^2+\int_{t_0}^\infty\int_\Omega(w_1-\frac{\lambda}{\mu})^2\le\frac{W(t_0)}{\delta_2}<\infty. \tag{4.34}$$

Using (4.16), we gain the uniform continuity and global boundedness of $\int_\Omega(v^\delta-\frac{\lambda}{\mu})^2+\int_\Omega(w_1-\frac{\lambda}{\mu})^2$ with respect to t. Thus, it may be concluded from Lemma 4.1 that

$$\int_\Omega(v^\delta-\frac{\lambda}{\mu})^2+\int_\Omega(w_1-\frac{\lambda}{\mu})^2\to0\ \text{ as } t\to\infty. \tag{4.35}$$

Invoking Young's inequality, we gain from (4.28) and (4.31) that

$$\int_\Omega|\nabla w|^2\le-\frac{1}{2}\int_\Omega(w-\frac{\lambda}{\mu})^2+\frac{1}{2}(v^\delta-\frac{\lambda}{\mu})^2 \tag{4.36}$$

and

$$\int_\Omega |\nabla w_2|^2 \le -\frac{1}{2}\int_\Omega (w_2 - \frac{\lambda}{\mu})^2 + \frac{1}{2}(v^\delta - \frac{\lambda}{\mu})^2. \tag{4.37}$$

Thus, we get from (4.35) that

$$\int_\Omega (w - \frac{\lambda}{\mu})^2 \le \int_\Omega (w_1 - \frac{\lambda}{\mu})^2 + (v^\delta - \frac{\lambda}{\mu})^2 \to 0 \ \text{ as } t \to \infty \tag{4.38}$$

and

$$\int_\Omega (w_2 - \frac{\lambda}{\mu})^2 \le \int_\Omega (w_1 - \frac{\lambda}{\mu})^2 + (v^\delta - \frac{\lambda}{\mu})^2 \to 0 \ \text{ as } t \to \infty. \tag{4.39}$$

Due to $\delta > 1$, we can find $N_3 > 0$ such that

$$N_3 = \sup_{s\in(0,\infty)} \frac{\left(s - (\frac{\lambda}{\mu})^{\frac{1}{\delta}}\right)^2}{\left(s^\delta - \frac{\lambda}{\mu}\right)^2} < \infty. \tag{4.40}$$

Hence, we can derive from (4.35) and (4.40) that

$$\int_\Omega (v - (\frac{\lambda}{\mu})^{\frac{1}{\delta}})^2 \le N_3 \int_\Omega (v^\delta - \frac{\lambda}{\mu})^2 \le N_3 \int_\Omega (v^\delta - \frac{\lambda}{\mu})^2 + \int_\Omega (w_1 - \frac{\lambda}{\mu})^2 \to 0 \tag{4.41}$$

as $t \to \infty$. This clearly gets the desired result of Lemma 4.2.

Proof of Theorem 1.2 The Gagliardo-Nirenberg inequality [37] enables us to infer from (4.4), (4.16), (4.24), and (4.41) that

$$\begin{aligned} \|v(\cdot,t) - (\frac{\lambda}{\mu})^{\frac{1}{\delta}}\|_{L^\infty(\Omega)} &\le C_{GN}\|v(\cdot,t) - (\frac{\lambda}{\mu})^{\frac{1}{\delta}}\|^{\frac{n}{n+2}}_{W^{1,\infty}(\Omega)}\|v(\cdot,t) - (\frac{\lambda}{\mu})^{\frac{1}{\delta}}\|^{\frac{2}{n+2}}_{L^2(\Omega)} \\ &\le C_1\|v(\cdot,t) - (\frac{\lambda}{\mu})^{\frac{1}{\delta}}\|^{\frac{2}{n+2}}_{L^2(\Omega)} \\ &\le C_1\|v^\delta(\cdot,t) - \frac{\lambda}{\mu}\|^{\frac{2}{n+2}}_{L^2(\Omega)} \to 0 \ \text{ as } t \to \infty, \end{aligned} \tag{4.42}$$

where $C_1 > 0$. For $\delta \in (0, 1]$, with an aid of the L'Hospital rule, there holds

$$\lim_{v\to\eta} \frac{v - \eta - \eta\ln(\frac{v}{\eta})}{(v-\eta)(v^\delta - \eta^\delta)} = \frac{1}{2\delta\eta^\delta}, \quad \eta = (\frac{\lambda}{\mu})^{\frac{1}{\delta}}. \tag{4.43}$$

We thus can find $t_1 > 0$ such that

$$\frac{1}{4\delta\eta^\delta}(v-\eta)(v^\delta - \eta^\delta) \le a(v) \le \frac{1}{\delta\eta^\delta}(v-\eta)(v^\delta - \eta^\delta), \quad t \ge t_1, \tag{4.44}$$

where $a(v) = v - \eta - \eta\ln(\frac{v}{\eta})$. We get from the definition of $A(t)$ that

$$\min\left\{\frac{1}{4\delta\eta^\delta}, \frac{N_1}{2}\right\}\left[\int_\Omega (v-\eta)(v^\delta - \eta^\delta) + \int_\Omega (w_1 - \eta^\delta)^2\right] \le A(t), \quad t \ge t_1, \tag{4.45}$$

and

$$A(t) \le \max\left\{\frac{1}{\delta\eta^{\delta}}, \frac{N_1}{2}\right\}\left[\int_\Omega (v-\eta)(v^{\delta}-\eta^{\delta}) + \int_\Omega (w_1-\eta^{\delta})^2\right], \quad t \ge t_1. \tag{4.46}$$

According to (4.13) and (4.46), there holds

$$\frac{d}{dt}A(t) \le -\frac{\delta_1}{\max\left\{\frac{1}{\delta\eta^{\delta}}, \frac{N_1}{2}\right\}}A(t), \quad t \ge t_1, \tag{4.47}$$

thus

$$A(t) \le A(t_1)\mathrm{e}^{-\frac{\delta_1}{\max\left\{\frac{1}{\delta\eta^{\delta}}, \frac{N_1}{2}\right\}}(t-t_1)}, \quad t \ge t_1, \tag{4.48}$$

where $\delta_1 = \min\left\{\mu - (\frac{\chi^2}{8} + \frac{\xi^2}{2})\frac{\lambda}{\mu}, \frac{3N_1}{4}\right\}$. Due to (4.12), (4.42), (4.45), and (4.48), we get

$$\begin{aligned}
\|v(\cdot,t)-\eta\|_{L^\infty(\Omega)} &\le C_1\left[\int_\Omega (v^{\delta}-\eta^{\delta})^2\right]^{\frac{1}{n+2}} \\
&\le C_1\left[\int_\Omega \eta^{\delta-1}(v-\eta)(v^{\delta}-\eta^{\delta})\right]^{\frac{1}{n+2}} \\
&\le C_1\left[\frac{\eta^{\delta-1}}{\min\left\{\frac{1}{4\delta\eta^{\delta}}, \frac{N_1}{2}\right\}}A(t)\right]^{\frac{1}{n+2}} \\
&\le C_1\left[\frac{\eta^{\delta-1}}{\min\left\{\frac{1}{4\delta\eta^{\delta}}, \frac{N_1}{2}\right\}}A(t_1)\right]^{\frac{1}{n+2}} e^{-\frac{\delta_1}{(n+2)\max\left\{\frac{1}{\delta\eta^{\delta}}, \frac{N_1}{2}\right\}}(t-t_1)}, \quad t \ge t_1.
\end{aligned} \tag{4.49}$$

Similarly, for v, we can choose $C_2 > 0$ such that

$$\|w_1(\cdot,t)-\eta^{\delta}\|_{L^\infty(\Omega)} \le C_2\left[\frac{\eta^{\delta-1}}{\min\left\{\frac{1}{4\delta\eta^{\delta}}, \frac{N_1}{2}\right\}}A(t_1)\right]^{\frac{1}{n+2}} e^{-\frac{\delta_1}{(n+2)\max\left\{\frac{1}{\delta\eta^{\delta}}, \frac{N_1}{2}\right\}}(t-t_1)}, \quad t \ge t_1. \tag{4.50}$$

According to (4.38), (4.42), and (4.49), there exists $C_3 > 0$ such that

$$\|w(\cdot,t)-\eta^{\delta}\|_{L^\infty(\Omega)} \le C_3\left[\frac{\eta^{\delta-1}}{\min\left\{\frac{1}{4\delta\eta^{\delta}}, \frac{N_1}{2}\right\}}A(t_1)\right]^{\frac{1}{n+2}} e^{-\frac{\delta_1}{(n+2)\max\left\{\frac{1}{\delta\eta^{\delta}}, \frac{N_1}{2}\right\}}(t-t_1)}, \quad t \ge t_1. \tag{4.51}$$

Similarly to the discussion of w, for w_2, it can be deduced from (4.39), (4.42), and (4.49) that

$$\|w_2(\cdot,t)-\eta^{\delta}\|_{L^\infty(\Omega)} \le C_4\left[\frac{\eta^{\delta-1}}{\min\left\{\frac{1}{4\delta\eta^{\delta}}, \frac{N_1}{2}\right\}}A(t_1)\right]^{\frac{1}{n+2}} e^{-\frac{\delta_1}{(n+2)\max\left\{\frac{1}{\delta\eta^{\delta}}, \frac{N_1}{2}\right\}}(t-t_1)}, \quad t \ge t_1, \tag{4.52}$$

where $C_4 > 0$. If $\delta \in (1, \infty)$, it can be seen from the L'Hospital rule that

$$\lim_{w_2 \to \eta^\delta} \frac{w_2 - \eta^\delta - \eta^\delta \ln(\frac{w_2}{\eta^\delta})}{(w_2 - \eta^\delta)^2} = \frac{1}{2\eta^\delta}. \tag{4.53}$$

We can find $t_2 > 0$ such that

$$\min\{\frac{1}{4\eta^\delta}, \frac{N_2}{2}\}\left[\int_\Omega (v^\delta - \eta^\delta)^2 + \int_\Omega (w_1 - \eta^\delta)^2\right] \le W(t), \quad t \ge t_2, \tag{4.54}$$

and

$$W(t) \le \max\{\frac{1}{\eta^\delta}, \frac{N_2}{2}\}\left[\int_\Omega (v^\delta - \eta^\delta)^2 + \int_\Omega (w_1 - \eta^\delta)^2\right], \quad t \ge t_2. \tag{4.55}$$

Combining (4.32) and (4.55), there holds

$$\frac{d}{dt}W(t) \le -\frac{\delta_2}{\max\{\frac{1}{\eta^\delta}, \frac{N_2}{2}\}}W(t), \quad t \ge t_2, \tag{4.56}$$

thus

$$W(t) \le W(t_2)e^{-\frac{\delta_2}{\max\{\frac{1}{\eta^\delta}, \frac{N_2}{2}\}}(t-t_2)}, \quad t \ge t_2, \tag{4.57}$$

where $\delta_2 = \min\{\mu - N_2 - (\frac{\lambda\chi^2}{8\mu} + \frac{\chi^2(\delta-1)R^\delta}{8}), \frac{3N_2}{4}\}$. We can infer from (4.40), (4.54), and (4.57) that

$$\begin{aligned}\|v(\cdot,t) - \eta\|_{L^\infty(\Omega)} &\le C_5\left[\int_\Omega (v^\delta - \eta^\delta)^2\right]^{\frac{1}{n+2}} \\ &\le C_5\left[\frac{1}{\min\{\frac{1}{4\eta^\delta}, \frac{N_2}{2}\}}W(t_2)\right]^{\frac{1}{n+2}} e^{-\frac{\delta_2}{(n+2)\max\{\frac{1}{\eta^\delta}, \frac{N_2}{2}\}}(t-t_2)}, \quad t \ge t_2,\end{aligned} \tag{4.58}$$

where $C_5 > 0$. We can also get the same result for v :

$$\|w_1(\cdot,t) - \eta^\delta\|_{L^\infty(\Omega)} \le C_6\left[\frac{1}{\min\{\frac{1}{4\eta^\delta}, \frac{N_2}{2}\}}W(t_2)\right]^{\frac{1}{n+2}} e^{-\frac{\delta_2}{(n+2)\max\{\frac{1}{\eta^\delta}, \frac{N_2}{2}\}}(t-t_2)}, \quad t \ge t_2, \tag{4.59}$$

where $C_6 > 0$. Furthermore, for w and w_2, we can deduce from (4.38), (4.39), (4.42), and (4.58) that

$$\|w(\cdot,t) - \eta^\delta\|_{L^\infty(\Omega)} \le C_7\left[\frac{1}{\min\{\frac{1}{4\eta^\delta}, \frac{N_2}{2}\}}W(t_2)\right]^{\frac{1}{n+2}} e^{-\frac{\delta_2}{(n+2)\max\{\frac{1}{\eta^\delta}, \frac{N_2}{2}\}}(t-t_2)}, \quad t \ge t_2 \tag{4.60}$$

and

$$\|w_2(\cdot,t) - \eta^\delta\|_{L^\infty(\Omega)} \le C_8\left[\frac{1}{\min\{\frac{1}{4\eta^\delta}, \frac{N_2}{2}\}}W(t_2)\right]^{\frac{1}{n+2}} e^{-\frac{\delta_2}{(n+2)\max\{\frac{1}{\eta^\delta}, \frac{N_2}{2}\}}(t-t_2)}, \quad t \ge t_2, \tag{4.61}$$

where $C_7, C_8 > 0$. Finally, plugging δ_1 and δ_2 into (4.49)–(4.52) and (4.58)–(4.61), respectively, then we can conclude the desired results in Theorem 1.2 by choosing $C > 0$ sufficiently large.

Author contributions

Chang-Jian Wang: Methodology, Writing-original draft, Writing-review editing, formal analysis; Jia-Yue Zhu: Writing-original draft, Writing-review editing.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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