Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 16(4): 786–812.
DOI: 10.3934/cam.2024034
Received: 29 February 2024
Revised: 30 July 2024
Accepted: 03 September 2024
Published: 28 October 2024

Research article

Hybrid quantum-classical control problems

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Abstract: The notion of hybrid quantum-classical control system was introduced as a control dynamical system which combined classical and quantum degrees of freedom. Classical and quantum objects were combined within a geometrical description of both types of systems. We also considered the notion of hybrid quantum-classical controllability by means of the usual definitions of geometric control theory, and we discussed how the different concepts associated to quantum controllability are lost in the hybrid context because of the nonlinearity of the dynamics. We also considered several examples of physically relevant problems, such as the spin-boson model or the notion of hybrid spline.

Keywords: hybrid control system; classical controllability; quantum controllability; hybrid controllability; splines
Mathematics Subject Classification: 93B05, 93B29

1. Introduction: why hybrid control?

Control theory is one of the fields of mathematics with a broader range of applications. Since the seminal work by Maxwell [1], it has been made clear that a solid mathematical framework is fundamental for an appropriate description of the interaction between mechanical systems and their environments or operators. Control theory provides such a framework [2, 3], and has shown its usefulness in many fields of physics [4, 5], chemistry [6, 7], quantum information [8] and many others.

In particular, the applications of control theory to quantum systems are fundamental in order to develop new quantum technologies with a wide range of applications [9–14]. Some of the focuses of research on quantum control are the controllability of quantum systems, i.e., the determination of which states of the system can be reached by suitable control functions [15–18], and optimal control problems, with the goal of designing efficient algorithms that may reach specific states in minimal time or with a minimal energy investment [5, 19–21].

The aim of this paper is to extend the controllability concepts to hybrid quantum-classical systems. Hybrid models are useful to describe quantum systems that present two different energy or mass scales; in those cases, slow or heavy degrees of freedom can be approximated as classical variables. These hybrid models are useful in describing both molecular and condensed matter systems [22–25], which present two distinctive energy scales (nuclei and electrons). An interesting framework for the analysis of hybrid quantum-classical systems is based on differential geometry, which allows combining the geometrical descriptions of both classical mechanics [26–28] and quantum mechanics [28–34]. In previous works, [33, 35–38], we introduced a geometric formulation for hybrid quantum-classical models, aiming to describe molecular systems [39–41], statistics [42–44] and quantum fields [45]. Our goal in this paper is to analyze the same geometrical description of hybrid quantum-classical systems from the perspective of control theory, providing sufficient conditions for the controllability of such systems.

The paper is organized as follows. Section 2 presents a summary of the main elements in the geometrical description of classical and quantum systems, as well as their combination to describe hybrid systems. Section 3 shows how controllability concepts can be applied to hybrid systems. Applications of the controllability results to some examples are provided in Section 4. Finally, conclusions and an outlook of future research in the field are presented in Section 5.

2. A geometrical description of hybrid quantum-classical dynamics

The geometric formalism of mechanics is a common framework for a description of dynamics based on the intrinsic elements of the relevant spaces. Here, we summarize the main aspects of this formalism for both classical and quantum systems, as well as its combination for the analysis of hybrid quantum-classical systems.

2.1. Geometric formalism of classical mechanics

For most applications, geometric formulation of classical mechanics is based on the description of the phase space as a cotangent bundle on a differentiable manifold and the geometric objects and structures that emerge from it. See [26–28] and references therein for a detailed description of the formalism. Thus, for most of the cases, we will consider a classical system with an n-dimensional configuration space N and coordinates $\boldsymbol{R} = (R_1, \dots, R_n) \in N$. The Hamiltonian formulation of mechanics focuses on the structures and properties of its cotangent bundle $M_C = T^*N$ with elements $\xi = (\boldsymbol{R}, \boldsymbol{P}) \in M_C$, where coordinates $\boldsymbol{P} = (P_1, \dots, P_n)$ are identified as the momenta associated to $\boldsymbol{R}$. Notice, though, that other relevant examples are defined directly on a general symplectic manifold, where these variables $\xi = (\boldsymbol{R}, \boldsymbol{P})$ will just be used as local coordinates on a Darboux chart. Remember that a manifold is said to be symplectic if it is endowed with a closed and nondegenerate differential two form which in Darboux coordinates is written as $\omega_C = dR_j \wedge dP_j$ (again, see the references above for details). As in the following we will just need a symplectic structure, we will just consider that (M_C, ω_C) is a general symplectic manifold, with ω_C representing the symplectic form. In some of the examples, we may also need a Riemannian structure defined on M_C, but this will be clarified below.

Let $\mathcal{F}(M_C)$ denote the set of smooth functions on M_C. On the symplectic manifold M_C, there exists a canonical Poisson bracket acting on smooth functions whose coordinate expression in the Darboux chart is

$$\{f, g\}_C = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial R_j} \frac{\partial g}{\partial P_j} - \frac{\partial f}{\partial P_j} \frac{\partial g}{\partial R_j} \right), \quad \forall f, g \in \mathcal{F}(M_C). \tag{2.1}$$

This Poisson bracket thus provides the set of smooth functions $\mathcal{F}(M_C)$ with a Poisson algebra structure.

For every smooth function $f \in \mathcal{F}(M_C)$, there exists a Hamiltonian vector field X_f^C defined by its action on smooth functions:

$$X_f^C = \{\cdot, f\}_C \;\Leftrightarrow\; X_f^C(g) = \{g, f\}_C = \omega_C(X_g^C, X_f^C),\; \forall f, g \in \mathcal{F}(M_C). \tag{2.2}$$

For instance, consider the following function H and its associated Hamiltonian vector field:

$$H = \frac{1}{2} \sum_{j=1}^{n} (P_j^2 + R_j^2),\; n \in \mathbb{N} \Rightarrow X_H = \sum_{j=1}^{n} \left(P_j \frac{\partial}{\partial R_j} - R_j \frac{\partial}{\partial P_j} \right), \tag{2.3}$$

Integral curves of X_H correspond to the trajectories of the harmonic oscillator in n-dimensions, whose Hamiltonian function is H. In the case $n = 1$, the trajectories correspond to the curves in Figure (1).

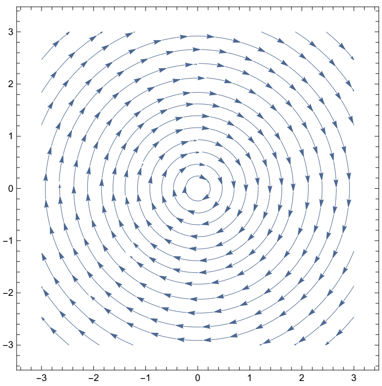

Figure 1. Integral curves of the Harmonic oscillator in 1 dimension.

The set of classical Hamiltonian vector fields on M_C is a Lie algebra whose Lie bracket (the commutator of vector fields) satisfies

$$[X_f^C, X_g^C] = X_{\{g,f\}_C}^C, \quad \forall f, g \in \mathcal{F}(M_C). \tag{2.4}$$

Observables on the classical system are represented by functions on M_C. In particular, the Hamiltonian function of the system $h \in \mathcal{F}(M_C)$ represents the energy of the system. This observable, together with the canonical symplectic structure on M_C, determines the dynamics of classical systems, as the integral curves of its Hamiltonian vector field X_h^C are the orbits of the systems on the phase space M_C. Dynamics thus obtained preserve the described symplectic structure.

In summary, dynamics of classical systems are governed by Hamiltonian vector fields defined on a symplectic manifold via a Poisson bracket. These key elements are the basis for the geometric formalism. Any geometrical analysis of other types of systems aims to find similar elements on the corresponding manifolds.

2.2. Geometric formalism of quantum mechanics

A geometrical description for quantum system analogous to that of classical systems can be achieved by identifying the underlying geometrical structure that governs dynamics. The geometric formulation of quantum mechanics was originally analyzed by prominent authors such as T. W. B. Kibble [34] and A. Heslot [32], and is a relevant field of study in modern physics both from a theoretical perspective and for its applications; see [28–31, 33] and references therein.

Postulates of quantum mechanics [46] establish that states of a quantum system are described by elements $|\psi\rangle$ in a complex Hilbert space $\mathcal{H}$. Let us assume, for the sake of simplicity, that $\mathcal{H}$ is finite-dimensional, with complex dimension n. The geometrical formulation of quantum mechanics is based on the description of this linear space of quantum states of complex dimension $n \in \mathbb{N}$ (isomorphic to $\mathbb{C}^n$), as a real manifold M_Q of real dimension $2n$ (isomorphic to $\mathbb{R}^{2n}$). In order to identify them, let us consider a basis $\{|e_j\rangle\}_{j=1}^n$ of $\mathcal{H}$ defines a set of complex coordinates on $\mathcal{H}$:

$$|\psi\rangle = \sum_{j=1}^{n} z_j |e_j\rangle, \quad z_j = \frac{1}{\sqrt{2}}\left(q_j + i p_j\right), \quad q_j, p_j \in \mathbb{R},\ j = 1, 2, \ldots, n. \tag{2.5}$$

Real numbers $\boldsymbol{q} = (q_1, q_2, \cdots)$ and $\boldsymbol{p} = (p_1, p_2, \cdots)$ can be understood as coordinates of points on a $2n$-dimensional real manifold M_Q, thus defining a one-to-one correspondence $\psi = (\boldsymbol{q}, \boldsymbol{p}) \in M_Q \mapsto |\psi\rangle \in \mathcal{H}$. The choice of different bases on $\mathcal{H}$ leads to different sets of real coordinates for M_Q. To encode the complex nature of $\mathcal{H}$, we need a complex structure J defined on M_Q, which identifies the q–coordinates as the real parts and to the p–coordinates as the imaginary ones. Futhermore, we know that $\mathcal{H}$ is endowed with a canonical Hermitian tensor h which defines a scalar product on the complex vector space. This Hermitian structure can also be encoded on the real manifold M_Q in the form of a pair of tensors: a Riemannian structure g (which represents the real part of h) and a symplectic form ω (representing the imaginary part) Notice that the different choices of complex coordinates preserving the Hermitian tensor h on $\mathcal{H}$ lead to different charts on M_Q defining different coordinate representations of the tensors J, g and ω, which are intrinsic objects on M_Q. Hermiticity of the tensor h on $\mathcal{H}$ is equivalent to the property of the triad (g, ω, J) defining a Kähler structure on the real manifold M_Q. This Kähler structure distinguishes the classical case, where no complex structure is considered and where the Riemannian structure may or may not be considered, from the quantum case where all the tensors are canonical and therefore they are always present (see [28, 30, 31, 33] for details).

Notice that despite the formal similarities with a classical manifold, the equivalence is purely mathematical, since no physical meaning as positions and momenta can be associated with variables $(\boldsymbol{q}, \boldsymbol{p})$, which simply represent the real and imaginary parts of the quantum states. Nonetheless, from the mathematical point of view, M_Q will be described as a symplectic manifold (specifically, a Kähler manifold, as stated), with analogous properties to the classical symplectic manifold M_C. This leads to the definition of Schrödinger equation as a Hamiltonian flow, which is one of the most remarkable consequences of geometric quantum mechanics. Besides, this formal similarity of both M_C and M_Q as symplectic manifolds is crucial to build the geometrical description of hybrid quantum-classical dynamics as a Hamiltonian flow, as we will see below.

Tensors ω_Q and $\mathfrak{g}$ endow the set $\mathcal{F}(M_Q)$ of smooth functions on M_Q with both a Poisson bracket $\{\cdot, \cdot\}_Q$ and a symmetric product of functions $(\cdot, \cdot)_Q$ determined by the properties of the Hermitian product in $\mathcal{H}$ [28]. Their coordinate expressions are

$$\{f, g\}_Q = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right), \quad (f, g)_Q = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial q_j} + \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial p_j} \right), \quad \forall f, g \in \mathcal{F}(M_Q). \tag{2.6}$$

On the other hand, observables of the quantum system are represented by Hermitian linear operators on $\mathcal{H}$, denoted as Herm(n), which has a Lie-Jordan algebra structure with respect to the following Lie bracket and symmetric product:

$$[\![A, B]\!] = -i(AB - BA), \quad A \circ B = AB + BA, \quad A, B \in \text{Herm}(n). \tag{2.7}$$

Notice that the Lie-Jordan algebra Herm(n) is isomorphic to the unitary algebra $\mathfrak{u}(n)$, while the subset of traceless Hermitian observables, which will be denoted as $\text{Herm}_0(n)$, is a Lie algebra with the same Lie bracket defined in (2.7) and isomorphic to the special unitary algebra $\mathfrak{su}(n)$.

The geometric formalism represents observables by their expectation values. Thus, any $A \in \text{Herm}(n)$ defines the following expectation value function f_A on M_Q:

$$f_A(\psi) = \langle \psi | A | \psi \rangle. \tag{2.8}$$

There is a one-to-one correspondence between expectation value functions of the form (2.8) and observables in Herm(n). The Kähler structure reproduces on these functions the Lie-Jordan algebraic structure on Herm(n) defined by the two brackets in (2.7) as follows

$$\{f_A, f_B\}_Q = f_{[\![A,B]\!]}, \quad (f_A, f_B)_Q = f_{A \circ B}, \quad \forall A, B \in \text{Herm}(n). \tag{2.9}$$

Compared to the classical case, the quantum formalism includes an analogous Poisson structure which can be used to describe the Schrödinger equation as a Hamiltonian system, and an additional structure, the symmetric product $(\cdot, \cdot)_Q$, related to the indeterministic nature of quantum mechanics [33,47]. Notice that not any smooth function on M_Q can represent a physical observable of the quantum system, but only those of the form in (2.8).

Smooth functions on M_Q determine Hamiltonian vector fields on the manifold through the Poisson bracket. In particular, for any observable $A \in \text{Herm}(n)$, we will denote by X_A^Q the vector field defined as

$$X_A^Q = \{\cdot, f_A\}_Q \Leftrightarrow X_A^Q(f_B) = \{f_B, f_A\}_Q = \omega_Q(X_B^Q, X_A^Q) = f_{[\![B,A]\!]}, \quad \forall g \in \mathcal{F}(M_Q) \tag{2.10}$$

Analogously to the property of classical Hamiltonian vector fields encoded in Equation (2.4), commutators of these quantum Hamiltonian vector fields satisfy

$$[X_A^Q, X_B^Q] = X_{[\![B,A]\!]}^Q. \tag{2.11}$$

Hence, the set of quantum Hamiltonian vector fields of observables is a Lie sub-algebra of the whole set of quantum Hamiltonian vector fields on M_Q.

In the particular case of the Hamiltonian observable H of the quantum system, integral curves of its associated vector field X_H correspond to the solutions to Schrödinger's equation [28]. Dynamics thus obtained preserve the canonical Kähler structure described above. Notice again that, despite the similarities with classical dynamics, these are purely formal, since there is no physical meaning of the quantum Darboux coordinates.

It should be noticed that, as it is clear from the postulates of quantum mechanics, states of quantum systems are not in one-to-one correspondence with elements in a Hilbert space $\mathcal{H}$. Instead, states can be identified as elements of the projective space $P\mathcal{H}$, defined as the equivalence classes on $\mathcal{H} - \{0\}$ of vectors which belong to the same complex line through the origin of the linear space. The complex projective space $P\mathcal{H}$ can also be descried as a Kähler manifold, whose tensor fields are related to those of the Kähler structure on $\mathcal{H}$ [33, 48]. The existence of this double description of the quantum state at the level of the linear space or the complex projective space has deep implications in the definition of the different notions of quantum controllability [49].

An efficient representation of the complex projective space is provided by the set of projectors onto one-dimensional subspaces of $\mathcal{H} - \{0\}$. Indeed, there is a one-to-one correspondence between the equivalence classes of $P\mathcal{H}$ and the set of projectors:

$$P\mathcal{H} \ni [\psi] \longleftrightarrow \rho_\psi = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} . \tag{2.12}$$

These projectors belong to the space of Hermitian operators Herm(n). Nonetheless, it is well-known that Herm(n) is isomorphic to the Lie algebra of the unitary group $\mathfrak{u}(n)$ and that this algebra, which can be endowed with a scalar product $\langle A, B\rangle = \mathrm{Tr}(A^\dagger B)$, is also isomorphic to its dual $\mathfrak{u}^*(n)$. As a result, Herm(n) can also be identified with $\mathfrak{u}^*(n)$, and therefore the subset of projectors of the form (2.12) is diffeomorphic (with respect to the natural differentiable linear structures) to a $2n - 2$ sub-manifold $\mathcal{D}^1(\mathcal{H})$ of the n^2-dimensional space $\mathfrak{u}^*(n)$. Furthermore, being the dual of a Lie algebra, $\mathfrak{u}^*(n)$ is endowed with a canonical Poisson bracket (the Lie-Poisson structure corresponding to the Lie bracket of $\mathfrak{u}(n)$ or, equivalently, to the bracket $[\![\cdot, \cdot]\!]$ on Herm(n)). $\mathcal{D}^1(\mathcal{H})$ defines one of the leaves of its symplectic foliation, with the symplectic form being equivalent to the one of the Kähler structure of the projective space $P\mathcal{H}$ (see [33, 48] for details). As a result, using the identification with a subset of $\mathfrak{u}^*(n)$, we have been able to endow the set of projectors of the form (2.12) with a canonical symplectic structure. This is the ingredient we need to define hybrid quantum-classical dynamics while representing the quantum state as the projector ρ_ψ. In the following, to simplify the notation, we will use $\mathcal{D}^1(\mathcal{H})$ to represent the set of pure states in any of the two representations, as a subset of $\mathfrak{u}^*(n)$ or as a subset of Herm(n), just clarifying when necessary.

The simplest example corresponds to the case of a qubit, whose Hilbert space is $\mathbb{C}^2$, the corresponding projective space being $\mathbb{CP}^1$. This projective space can be determined as a sub-manifold of $\mathfrak{u}^*(2)$ corresponding to the self-adjoint rank-one projectors on $\mathbb{C}^2$. Using a basis composed of the identity matrix $\mathbb{I}$ and the three Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.13}$$

Projectors ρ_ψ can be written as

$$\rho_\psi = \frac{1}{2}\left(\rho_0\mathbb{I} + \rho_1\sigma_1 + \rho_2\sigma_2 + \rho_3\sigma_3\right), \quad \rho_0 = 1, \quad \rho_1^2 + \rho_2^2 + \rho_3^2 = 1 \tag{2.14}$$

where the constraints on the coordinates ensure that ρ_ψ is a rank-one projector onto $\mathbb{C}^2$. Therefore, $\mathcal{D}^1(\mathbb{C}^2)$ is diffeomorphic to the two-dimensional sphere S^2. This set is known as the Bloch sphere.

Unitary dynamics correspond to the different actions of the unitary group $U(n)$ on these spaces of states:

- The action on the Hilbert space $\mathcal{H}$ where the unitary group defines the canonical isometries, determines orbits $\psi(t)$ whose tangent vectors are defined by the Schrödinger equation,

$$i\hbar\frac{d\psi}{dt} = H\psi(t). \tag{2.15}$$

- The action on the projective space $\mathcal{PH}$ is diffeomorphic to the action on the space of projectors on one-dimensional subspaces, which is determined by the co-adjoint action of the $U(n)$ group on the dual of its Lie algebra $\mathfrak{u}^*(n)$. From our discussion above, we know that we can consider the states ρ_ψ either as Hermitian operators corresponding to the projectors $\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$ or as points in $\mathfrak{u}^*(n)$. Considered as projectors, tangent vectors to trajectories $\rho_\psi(t)$ are defined now by the von Neumann equation:

$$i\hbar\frac{d\rho_\psi}{dt} = [H, \rho_\psi(t)]. \tag{2.16}$$

  The isomorphisms between the set of Hermitian operators and the Lie algebra $\mathfrak{u}(n)$ given by the multiplication by the imaginary unit, and of $\mathfrak{u}(n)$ and its dual, given by the scalar product $\langle A, B\rangle = \mathrm{Tr}(A^\dagger B)$, allows us to write the solutions of this equation as the image under the isomorphisms above of the flow on $\mathfrak{u}^*(n)$ of the canonical Hamiltonian vector field associated to the linear function $f_H(\rho) = \rho(iH) = \mathrm{Tr}(\rho_\psi H)$ for $\rho \in \mathfrak{u}^*(n), \rho_\psi, H \in \mathrm{Herm}(n)$ and the canonical Lie-Poisson structure defined on the dual of any Lie algebra (see [26]). Thus the space of rank-one projectors becomes diffeomorphic to a symplectic leaf of the canonical foliation of this Lie-Poisson structure, which is also symplectomorphic to the complex projective space. There is a single orbit of the action of $U(n)$ on $\mathcal{PH}$, equal to the whole space. Geometrically, the full tangent space to each point of $\mathcal{PH}$ is generated by Hamiltonian vector fields; hence, from now on, if necessary we can also consider the quantum manifold as $M_Q \simeq \mathcal{PH}$, with the tangent space at each point generated by observables in $\mathrm{Herm}_0(n)$ via (2.16).

We can also consider a simple example of these dynamical systems. Consider a two-level quantum system, whose Hilbert space is thus $\mathcal{H} = \mathbb{C}^2$, with a Hamiltonian written in the energy eigenbasis as

$$H = \begin{pmatrix} E_0 & 0 \\ 0 & E_1 \end{pmatrix}, \quad E_0, E_1 \in \mathbb{R}, \quad E_0 < E_1. \tag{2.17}$$

In the description in terms of Hilbert spaces, if we consider the eigenbasis of the Hamiltonian operator H as the basis for $\mathcal{H}$ and (q_1, p_1, q_2, p_2) the corresponding real coordinates, the function $f_H(\psi)$ becomes

$$f(q_1, p_1, q_2, p_2) = \langle\psi|H\psi\rangle = E_0(q_1^2 + p_1^2) + E_1(q_2^2 + p_2^2),$$

and the Hamiltonian vector field X_{f_H} with respect to the symplectic form $\omega_Q = dq_1 \wedge dp_1 + dq_2 \wedge dp_2$ becomes

$$X_{f_H} = E_0\left(p_1\frac{\partial}{\partial q_1} - q_1\frac{\partial}{\partial p_1}\right) + E_1\left(p_2\frac{\partial}{\partial q_2} - q_2\frac{\partial}{\partial p_2}\right). \tag{2.18}$$

If we consider the representation (2.14) for the projector ρ_ψ with coordinates $(\rho_1, \rho_2, \rho_3) \in \mathcal{D}^1(\mathbb{C}^2)$, the linear function for the energy becomes

$$f_H(\rho_\psi) = \mathrm{Tr}(H\rho_\psi) = \frac{E_1 + E_0}{2} - \frac{E_1 - E_0}{2}\rho_3. \tag{2.19}$$

With this function, the corresponding Hamiltonian vector field reads:

$$X_{f_H} = (E_1 - E_0)\left(\rho_2 \frac{\partial}{\partial \rho_1} - \rho_1 \frac{\partial}{\partial \rho_2}\right) \tag{2.20}$$

Integral curves of this vector field describe the evolution of pure states. Figure (2) shows the representation of (2.20) not only for pure states (satisfying $\rho_1^2 + \rho_2^2 + \rho_3^2 = 1$), but also for mixed states, with $\rho_1^2 + \rho_2^2 + \rho_3^2 < 1$, for which the Hamiltonian vector field is identical to (2.20) but defined on the whole interior of the Bloch sphere.

Integral curves of vector field (2.18) define the integral curves of vector field (2.20) when taken to the Bloch sphere by considering the corresponding projectors $\frac{|\psi(t)\rangle\langle\psi(t)|}{\langle\psi(t)|\psi(t)\rangle}$.

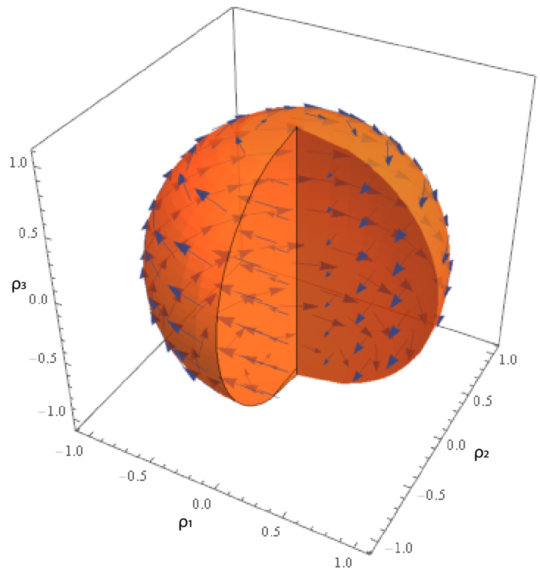

Figure 2. Integral curves of the Hamiltonian vector field on and inside the Bloch sphere, valid, respectively, for pure and mixed quantum states.

Notice that both descriptions, on the Hilbert space or on the projective space, are physically equivalent. The solutions of the Schrödinger equation $\psi(t)$ and the solutions of the von Neumann equation on the space of projectors $\rho_\psi(t)$ correspond to orbits of the solution of the same equation on the unitary group:

$$i\hbar \frac{d}{dt} U(t) = HU(t); \qquad H \in \mathrm{Herm}(n); \quad U(t) \in U(n),$$

as

$$\psi(t) = U(t)\psi(0); \qquad \rho_\psi(t) = U(t)\rho_\psi(0)U(t)^\dagger,$$

for $\psi(0) \in \mathcal{H}$ and $\rho_\psi(0) \in \mathcal{D}^1(\mathcal{H})$ being the initial conditions. Therefore, both dynamics contain the same physical information. From the mathematical point of view, both solutions can also be obtained as flows of Hamiltonian vector fields with respect to the canonical symplectic forms (on the vector space description or on the description in terms of projectors) presented above. As the existence of a symplectic form is the only technical requirement we need to build hybrid quantum-classical dynamics, in the following we will use both formulations indistinctly, just indicating with the representation of the state ψ or ρ_ψ which one are we considering. Nonetheless, these alternative formulations of quantum dynamics determine different approaches to the problem of quantum controllability [49], as it can be considered as the problem of building the suitable trajectory on the unitary group $U(n)$ or on the corresponding orbit. We will see later that this situation is very different when we consider hybrid quantum-classical dynamics, which are no longer linear.

2.3. Hybrid systems and their geometric description

The above summaries show how a common formalism can be used to describe two different types of dynamics. The existence of this common framework allows combining both descriptions in a natural way, giving as a result a geometric formalism for hybrid quantum-classical systems [33, 35–38].

Consider the product manifold $M_{QC} = M_C \times M_Q$ as the space of states of hybrid systems, where (M_C, ω_C) and (M_Q, ω_Q) are, respectively, the classical and the quantum symplectic manifolds considered above. Each manifold contains the states of different types of systems: the points of M_C represent the states of the classical degrees of freedom, while the points of M_Q represent the states of the quantum degrees of freedom. States of a hybrid system are represented by elements $(\xi, \psi) \in M_{QC}$, with $\xi \in M_C$, $\psi \in M_Q$. The composition of manifolds is immediate from the mathematical perspective: the phase space of the hybrid system is the Cartesian product of the phase spaces of the subsystems. The canonical projections onto the two subsystems will be denoted as

$$\begin{array}{llll} \pi_C: & M_{QC} & \to M_C & \\ & (\xi,\psi) & \mapsto \pi_C(\xi,\psi) = \xi \end{array} \qquad \begin{array}{llll} \pi_Q: & M_{QC} & \to M_Q \\ & (\xi,\psi) & \mapsto \pi_Q(\xi,\psi) = \psi \end{array} \tag{2.21}$$

The symplectic forms on both M_C and M_Q, described above, define the following canonical symplectic form on the product manifold M_{QC}:

$$\omega_{QC} = \pi_C^*(\omega_C) + \hbar^{-1}\pi_Q^*(\omega_Q), \tag{2.22}$$

with π_C^* and π_Q^* being the pullbacks of the projections π_C and π_Q, respectively. In the following, we will use natural units and assume $\hbar = 1$.

The hybrid manifold is thus a symplectic manifold with a Poisson bracket $\{\cdot,\cdot\}_{QC}$ acting on the set $\mathcal{F}(M_{QC})$ of smooth functions, and the corresponding hybrid Hamiltonian vector fields:

$$X_f^{QC} = \{\cdot, f\}_{QC} \Leftrightarrow X_f^{QC}(g) = \{g, f\}_{QC} = \omega_{QC}(X_g^{QC}, X_f^{QC}), \quad \forall f, g \in \mathcal{F}(M_{QC}). \tag{2.23}$$

As usual, hybrid Hamiltonian vector fields conform a Lie algebra with respect to their commutator:

$$[X_f^{QC}, X_g^{QC}] = X_{\{g,f\}_{QC}}^{QC}, \quad \forall f, g \in \mathcal{F}(M_{QC}). \tag{2.24}$$

Due to (2.22), the hybrid Poisson bracket can also be decomposed as the sum of its classical and quantum parts:

$$\{f, g\}_{QC} = \{f, g\}_C + \{f, g\}_Q, \quad \forall f, g \in \mathcal{F}(M_{QC}), \tag{2.25}$$

where in this context $\{\cdot,\cdot\}_C$ and $\{\cdot,\cdot\}_Q$ denote, respectively, the Poisson bracket defined by $\pi_C^*(\omega_C)$ and $\pi_Q^*(\omega_Q)$ on M_{QC}. These are directly defined, but strictly different due to their space of definition, to the Poisson brackets on the classical and quantum sub-manifolds. They are however similar enough to justify the abuse of notation. Hamiltonian vector fields with respect to both brackets will also reuse the notation of the subsystems, (2.2) and (2.10).

Analogously to the previous description, physical observables of the hybrid system are functions on M_{QC}. As quantum observables are described as Hermitian operators on $\mathcal{H}$, every hybrid observable is represented by a field of Hermitian operators $A(\xi)$, which in turn defines a smooth function on M_{QC} as its expectation value function:

$$f_A(\xi,\psi) = \langle\psi|A(\xi)|\psi\rangle. \tag{2.26}$$

This is the most general element of the hybrid algebra of observables defined by the tensor product of the classical and the quantum ones [40]. Hamiltonian vector fields on M_{QC} associated to a hybrid observable $A(\xi)$ will be denoted as

$$X_A^C = \{\cdot, f_A\}_C, \quad X_A^Q = \{\cdot, f_A\}_Q, \quad X_A^{QC} = \{\cdot, f_A\}_{QC}. \tag{2.27}$$

These vector fields satisfy the relation

$$X_A^{QC} = X_A^C + X_A^Q, \tag{2.28}$$

which has the following property as a consequence.

Lemma 2.1. *For any hybrid observable $A(\xi)$ and any point $(\xi,\psi) \in M_{QC}$:*

$$\pi_{C*}^{(\xi,\psi)}\left(X_A^{QC}\mid_{(\xi,\psi)}\right) = \pi_{C*}^{(\xi,\psi)}\left(X_A^{C}\mid_{(\xi,\psi)}\right), \quad \pi_{Q*}^{(\xi,\psi)}\left(X_A^{QC}\mid_{(\xi,\psi)}\right) = \pi_{Q*}^{(\xi,\psi)}\left(X_A^{Q}\mid_{(\xi,\psi)}\right). \tag{2.29}$$

with $\pi_{C*}^{(\xi,\psi)}$ and $\pi_{Q*}^{(\xi,\psi)}$ as the push-forwards or differentials of the canonical projections π_C and π_Q, respectively, at (ξ,ψ).

Proof. By construction, $X_A^C\mid_{(\xi,\psi)} \in \ker\left(\pi_{Q*}^{(\xi,\psi)}\right)$ and $X_A^Q\mid_{(\xi,\psi)} \in \ker(\pi_{C*}^{(\xi,\psi)})$. Hence, writing X_A^{QC} as (2.28) gives its push-forwards at every point by (2.29).

Notice that, unlike the quantum case, hybrid Hamiltonian vector fields X_A^{QC} of physical observables do not conform a Lie algebra. For any two hybrid observables, $A(\xi)$, $B(\xi)$, consider their Hamiltonian vector fields X_A^{QC}, X_B^{QC}. Their commutator, given by (2.24), involves the hybrid Poisson bracket $\{f_A, f_B\}_{QC}$. Due to the specific form (2.26) of expectation value functions (quadratic on the quantum variables), the result of the hybrid Poisson bracket is not, in general, an expectation value function. This motivates the analysis of hybrid systems, and, in particular, of control problems, from a perspective close to the classical one.

As in the previous cases, the Hamiltonian observable $H(\xi)$ that describes the energy of a hybrid system defines the vector field governing the symplectic-preserving dynamics of the system. The vector field governing hybrid dynamics is:

$$X_H^{QC} = \{\cdot, f_H\}_{QC} = \{\cdot, f_H\}_C + \{\cdot, f_H\}_Q = X_H^C + X_H^Q, \tag{2.30}$$

whose integral curves are the trajectories of the hybrid system on M_{QC}. Particular expressions of $H(\xi)$ allow studying different quantum-classical models, such as the Ehrenfest equations of molecular dynamics [35].

It is important to remark that, despite the formal similarities, there is a very important difference of these hybrid dynamics with respect to the purely quantum ones described in the previous section. In this case, the classical degrees of freedom make the dynamics defined by Equation (2.30) non-linear. This is the reason why the problem of hybrid controllability exhibits important differences with respect to the quantum controllability problem since, being a non-linear system in general, it resembles much more closely to the problem of classical controllability. Even in the case of considering a partial controllability of the quantum degrees of freedom, unitary dynamics have no meaning now.

3. Controllability of hybrid systems

The geometric formalism provides us with a framework to study control problems in hybrid systems. Our goal is to adapt the formalism for the analysis of hybrid control systems and to describe controllability conditions, as it has already been done for classical [2] and quantum control systems [15–18, 49]. Combining both types of control systems, which are not completely equivalent, introduces new types of problems which are interesting to consider.

Consider a hybrid system with Hamiltonian $H(\xi)$. Its natural evolution is governed by a drift vector field $X_0 = X_H^{QC}$, described in (2.30). We will consider linear controls over this system, introduced by means of a set of time-dependent control functions $u(t) = (u_1(t), \dots, u_k(t))$, taking values in a set $U \subset \mathbb{R}^n$ of admissible controls. There are no strong restrictions on the nature of these functions: In particular, admissible control functions can be non-smooth and even noncontinuous, as long as they take values in U. Some simple examples of typical control functions include the Heaviside function, square waves and so on, while practical implementations of control problems make use, among others, of Fourier series [50] and the chopped random basis ansatz (CRAB) [51]. The controllability properties developed here are valid for all kinds of control functions.

Control functions act as coefficients for control vector fields $X_1, \dots, X_k$, which are added to the drift; for simplicity, control vector fields will be assumed to be hybrid Hamiltonian vector fields associated to hybrid observables $A_1(\xi), \dots, A_k(\xi)$ defined by (2.26) and (2.27). Thus, the total vector field of the controlled system is a t-dependent family X_u of vector fields on M_{QC} defined as

$$X_u(t) = X_0 + \sum_{j=1}^{k} u_j(t) X_j, \quad X_j = X_{A_j}^{QC} = \{\cdot, f_{A_j}\}_{QC}, \quad j = 1, 2, \dots, k. \tag{3.1}$$

Without loss of generality, the vector fields $X_0, X_1, \dots, X_k$ can be assumed to be linearly independent. For simplicity, we will also assume them to be analytic and complete. Notice that, for every t, the vector field $X_u(t)$ is the hybrid Hamiltonian vector field of the hybrid function

$$F_u(\xi, \psi, t) = f_H(\xi, \psi) + \sum_{j=1}^{k} u_j(t) f_{A_j}(\xi, \psi) \tag{3.2}$$

which itself is the expectation value function of the hybrid observable $H(\xi) + \sum_{j=1}^{k} u_j(t) A_j(\xi)$, the controlled Hamiltonian of the system. Due to the definitions of previous sections, there exist one-to-one correspondences between vector fields, functions and hybrid observables.

As a first step in the study of hybrid control systems, we shall adapt the notion of controllability to our problems. We will not only focus on total controllability on the hybrid manifold M_{QC} (what we denote as hybrid controllability), but also to the analogous concepts on each subsystem (classical and quantum controllability).

Definition 3.1. *Let X_u be a controlled vector field on the hybrid manifold M_{QC}, defined by* (3.1)*. We say that the associated control system is* ***hybrid controllable*** *if, for any two arbitrary points on M_{QC}, there exists a set of control functions u' with values in the set of admissible controls such that an integral curve of $X_{u'}$ joins both points in finite time.*

Two weaker notions of controllability, referred to the analogous property restricted to the classical or the quantum sub-manifolds, can be also considered.

Definition 3.2. *Let X_u be a controlled vector field on the hybrid manifold M_{QC}, defined by* (3.1)*. We say that the associated control system is* ***classical controllable*** *if, for any two arbitrary points on M_C, there exists a set of control functions u' with values in the set of admissible controls and an integral curve $\gamma(t)$ of $X_{u'}$ such that the projection $\pi_C(\gamma(t))$ joins both points in finite time.*

Analogously, we say that the associated control system is ***quantum controllable*** *if, for any two arbitrary points on M_Q, there exists a set of control functions u' with values in the set of admissible controls and an integral curve $\gamma(t)$ of $X_{u'}$ such that the projection $\pi_Q(\gamma(t))$ joins both points in finite time.*

Both notions introduce relevant physical properties of the system, although, as we discussed above, the notion of classical-controllability is more natural because of the slower characteristic timescale of the classical degrees of freedom.

It is also important to remark that the different notions of quantum controllability we mentioned above cannot be transferred to the hybrid framework in a simple way. The reason is the nonlinearity of the hybrid dynamics, and therefore, the non-unitarity of the evolution on the quantum sub-manifold. Hence, it does not make sense considering the control problem on the unitary group or relating the problem defined on the space of quantum vectors and the problem defined on the set of rank-one projectors.

A sufficient condition for classical controllability on differentiable manifolds is that, at every point of the manifold, the control vector fields generate the whole tangent space to the manifold. Equivalently, we can ask the drift and the control vector fields to do it, but in this case some extra conditions on the topology of the orbits of the drift must be imposed [2]. The relevant results for our study are summarized in the following statements.

Definition 3.3. *Given some vector fields $Z_1, Z_2, \ldots$ on a manifold M, we shall denote as* $\mathrm{Lie}(Z_1, Z_2, \ldots)$ *the Lie algebra of vector fields on M generated by them.*

Theorem 3.4. *Let $X_0 + u_1(t)X_1 + \cdots + u_k(t)X_k$ be a control vector field on a differentiable and connected manifold M. If* $\mathrm{Lie}\,(X_1, \ldots, X_k)\,|_m = T_mM$*, for every $m \in M$ and there are no restrictions over the control functions $u_1, \ldots, u_k$, then the system is controllable.*

This theorem states the fact that, if controls allow for movement along any direction from any starting point, then with strong enough control functions one can overcome the drift X_0 and reach any desired destination. Indeed, without limitations on the control functions, this can be performed in any desired time; optimal control problems deal with these situations, with the usual goal of optimizing both the time required to reach the chosen points and the total strength of the controls needed for this operation.

Notice that movement parallel to vector fields $[X_i, X_j]$ can be achieved by consecutive movement by controls X_i, X_j, $-X_i$ and $-X_j$ (this is actually obtained as the time-zero limit of the corresponding evolutions). This can be performed because control functions can be either positive or negative, which allows to consider not only evolution by any control field X_i, but also by its inverse $-X_i$. This is not true for the drift X_0, which in general cannot be inverted. In those cases for which control fields $X_1, \dots, X_k$ are not enough to generate movement along all directions, additional requirements are needed for controllability.

Definition 3.5. *Let Z be a complete vector field on a manifold M, and let Φ_t^X denote the transformations on M such that $\Phi_t^Z(m)$ is the integral curve of X starting from $m \in M$. A point m is called recurrent for Z if there exists a sequence (t_n) with $\lim_{n\to\infty} t_n = +\infty$ such that $m = \lim_{n\to\infty} \Phi_{t_n}^Z(m)$. A complete vector field on M is called recurrent if it has a dense set on M of recurrent points.*

Fixed points of the dynamics are trivially recurrent. More interesting examples of recurrent points are the starting points of either periodic or quasi-periodic orbits. Consider a periodic orbit γ with period T; then, its starting point $m = \gamma(0)$ (as well as any other point in the orbit) is recurrent, as it satisfies the definition for a sequence $t_n = nT$ with $n \in \mathbb{N}$. Additionally, if a system starting in m evolves in such a way that the asymptotic limit of the orbit is again m (i.e., it returns to m in infinite time), then m is also recurrent.

Recurrent vector fields can thus be effectively inverted by simply allowing the system to evolve freely, possibly for a long time. For this reason, Theorem 3.4 can be generalized to control systems whose drift is recurrent as follows.

Theorem 3.6. *Let $X_0 + u_1(t)X_1 + \cdots + u_k(t)X_k$ be a control vector field on a differentiable and connected manifold M. If X_0 is recurrent, $\mathrm{Lie}\,(X_0, X_1, \dots, X_k)\mid_m = T_m M$, for every $m \in M$, and there are no restrictions over the control functions $u_1, \dots, u_k$, then the system is controllable.*

See Theorem 5 in Chapter 4 of [2] for a detailed proof. As indicated above, directions along Lie brackets of vector fields can be controlled if the involved vector fields can be inverted. Thus, if X_0 is recurrent, it can be effectively inverted and directions along $[X_0, X_i]$ can be reached. Notice that, in this case, it is not possible in general to reach the chosen destination in arbitrary time. Also, as the drift is now part of the Lie algebra, the system is controllable even if there exist restriction on the control functions, as long as $u_1 = \cdots = u_k = 0$ is an admissible control.

In order to combine these two types of systems, the following property will be used to simplify the computations.

Corollary 3.7. *Let $X_0 + u_1(t)X_1 + \cdots + u_k(t)X_k$ be a control vector field on a differentiable and connected manifold M. If X_0 is recurrent, then the system is controllable if the system with the controlled vector field $u_0(t)X_0 + u_1(t)X_1 + \cdots + u_k(t)X_k$ is controllable and satisfies the conditions of Theorem 3.4.*

Proof. The statement is proven by applying Theorems 3.4 and 3.6 to the proposed controlled vector fields.

The stated sufficient conditions for controllability need to be adapted for hybrid systems. We will achieve this by considering the projections onto M_C and M_Q of the whole system.

Theorem 3.8. *Let X_u be a controlled vector field on the hybrid manifold M_{QC}, defined by* (3.1)*, and consider a dense sub-manifold $S \subset M_{QC}$. If at least one of the following conditions is satisfied*

1. *for any $(\xi, \psi) \in S$:*
$$\pi_{C*}^{(\xi,\psi)}\left(\mathrm{Lie}\,(X_1, \dots, X_k)\mid_{(\xi,\psi)}\right) = T_\xi M_C; \tag{3.3}$$
2. *X_0 is recurrent and, for any $(\xi, \psi) \in S$:*
$$\pi_{C*}^{(\xi,\psi)}\left(\mathrm{Lie}\,(X_0, X_1, \dots, X_k)\mid_{(\xi,\psi)}\right) = T_\xi M_C; \tag{3.4}$$

and there are no restrictions on the admissible controls, then the system is classical controllable.

Proof. Let us assume hypothesis (3.3). Proofs of Theorem 3.4 are based on the fact that the controlled system can move along any direction parallel to vector fields in $\mathrm{Lie}\,(X_1, \ldots, X_k)$ [2]. That is, at every $(\xi, \psi) \in S$, there exist sets of control functions u such that the integral curve $\gamma_u(t)$ of X_u, with $\gamma(0) = (\xi, \psi)$ is parallel to each vector in $\mathrm{Lie}\,(X_1, \ldots, X_k)\,|_{(\xi,\psi)}$. In our case, the projection onto M_C shows that the control system can move in any direction tangent to $T_\xi M_C$. Hence, if there are no restrictions on the admissible controls, the system can reach any destination point in M_C and the system is classical controllable.

If X_0 is recurrent and hypothesis (3.4) is satisfied, then by Corollary 3.7, the above proof shows that the system is again classical controllable.

An analogous theorem describes quantum controllability. In the following, we will consider $M_Q = \mathcal{PH}$, so that its tangent space at each point is generated by the action of the unitary group. The theorems and results of the paper, however, can be immediately translated to the situations in which $M_Q = \mathcal{H}$. As only normalized elements of $\mathcal{H}$ with constant global phase are physically relevant, conditions involving $T_\psi M_Q$ can be relaxed as to not involve directions that change the norm or the global phase of vectors.

Theorem 3.9. *Let X_u be a controlled vector field on a connected hybrid manifold M_{QC}, defined by* (3.1). *If at least one of the following conditions is satisfied*

1. *for a dense set of points $(\xi, \psi) \in M_{QC}$:*

$$\pi_{Q*}^{(\xi,\psi)}\left(\mathrm{Lie}\,(X_1, \ldots, X_k)\,|_{(\xi,\psi)}\right) = T_\psi M_Q; \tag{3.5}$$

2. *X_0 is recurrent and, for a dense set of points $(\xi, \psi) \in M_{QC}$:*

$$\pi_{Q*}^{(\xi,\psi)}\left(\mathrm{Lie}\,(X_0, X_1, \ldots, X_k)\,|_{(\xi,\psi)}\right) = T_\psi M_Q; \tag{3.6}$$

and there are no restrictions on the admissible controls, then the system is quantum controllable.

Proof. This theorem complements Theorem 3.8, and it is proven in an analogous way, replacing classical elements by quantum ones.

Notice that, by definition, if a system is hybrid controllable, it is also classical controllable and quantum controllable. The converse, however, is not true, as it has to be guaranteed that the whole tangent space is generated at each point of M_{QC}, which cannot be deduced simply by its projections.

Controllability conditions in Theorems 3.4 and 3.6 involve Lie brackets of control fields. Intuitively, similar results should exist regarding classical and quantum controllability, replacing Lie bracket by Poisson bracket or commutators, as they are related by (2.4) and (2.11). These results are proven in the following theorems.

Theorem 3.10. *Let X_u be a controlled vector field on a connected hybrid manifold M_{QC}, defined by* (3.1). *If there exists a dense subset $S \subset M_{QC}$ such that, for every $(\xi, \psi) \in S$, hybrid control observables satisfy* $\mathrm{Lie}(A_1(\xi), \ldots, A_k(\xi), \mathbb{I}) = \mathrm{Herm}(n)$*, the system is quantum controllable.*

Proof. Control vector fields are assumed to always be analytic, hence control hybrid observables vary smoothly. Thus, consider an orthogonal basis $\sigma_1, \ldots, \sigma_{n^2}$ for $\mathrm{Herm}(n)$, with expectation value functions $s_j(\psi) = \langle\psi|\sigma_j|\psi\rangle$ and quantum Hamiltonian vector fields $Y_j^Q = X_{\sigma_j}^Q$, for $j = 1, 2, \ldots, n^2$. Structure constants $c_{ij}^r \in \mathbb{R}$ determine the commutators of elements in the basis as $[\![\sigma_i, \sigma_j]\!] = \sum_{r=1}^{n^2} c_{ij}^r \sigma_r$. Let $a_i^j(\xi)$

be the smooth functions on M_C representing the coefficients of each $A_i(\xi)$ hybrid control observable at each point $\xi \in M_C$:

$$A_i(\xi) = \sum_{j=1}^{n^2} a_i^j(\xi)\sigma_j, \quad f_i(\xi,\psi) = f_{A_i}(\xi,\psi) = \sum_{j=1}^{n^2} a_i^j(\xi)s_j(\psi), \quad X_i = X_{A_i}^{QC} = \sum_{j=1}^{n^2}\left(s_j X_{a_i^j}^C + a_i^j Y_j^Q\right), \quad (3.7)$$

for $i = 1, 2, \ldots, k$. By direct computation, commutators of control vector fields are

$$[X_i, X_j] = X_{\{f_i,f_j\}_C}^C + \sum_{l,m=1}^{n^2} s_{[l,m]}\left(a_i^l X_{a_j^m}^C + a_j^m X_{a_i^l}^C\right) + X_{[\![A_j,A_i]\!]}^Q - \sum_{l,m=1}^{n^2}\{a_i^l, a_j^m\}_C\left(s_m Y_l^Q + s_l Y_m^Q\right), \quad (3.8)$$

with $s_{[l,m]}(\psi) = \langle\psi|[\![\sigma_l,\sigma_m]\!]|\psi\rangle$. Consider now the projection $\pi_Q : M_{QC} \to M_Q$. Due to Lemma 2.1, only the quantum Hamiltonian vector fields project onto M_Q.

Without loss of generality, the hybrid control observables can be assumed to be linearly independent and orthogonal, and the basis of Herm(n) can be chosen so that, at a certain point $(\xi,\psi) \in M_{QC}$, the first elements of the basis satisfy $\sigma_1 = A_1(\xi), \ldots, \sigma_k = A_k(\xi)$. Thus, the push-forward by π_Q at (ξ,ψ) of the commutators are

$$\pi_{Q*}^{(\xi,\psi)}\left([X_i, X_j]\,|_{(\xi,\psi)}\right) = \sum_{l=1}^{n^2}\left(c_{ij}^l - \sum_{m=1}^{n^2}\left(\{a_i^l, a_j^m\}_C(\xi) + \{a_i^m, a_j^l\}_C(\xi)\right)s_m(\psi)\right)Y_l^Q\,|_\psi \quad (3.9)$$

Coefficients of $Y_l^Q|_\psi$ can only be zero if $c_{ij}^l = 0$ and either $a_i^l = a_j^l = 0$, $s_m(\psi) = 0$ or all Poisson brackets are zero, or if both terms cancel each other. The first case is impossible for all coefficients under the assumption of the theorem, while the second one is only possible on a no-where dense set of M_{QC}. Thus, projection onto M_Q determines all tangent directions parallel to quantum Hamiltonian vector fields of $\sigma_1, \ldots, \sigma_k$, as well as those of their commutators. The remaining directions, if any, can be obtained by higher orders commutators of vector fields, thus satisfying the conditions of Theorem 3.9 and proving the quantum controllability of the system.

Notice that the condition in Theorem 3.10 is sufficient but not necessary for quantum controllability. As one can notice in (3.9), the required direction could be obtained even if all the relevant structure constants are zero, as long as the classical Poisson brackets do not vanish. This may only occur if the coefficient functions are not zero at every point, so that the control hybrid observables $A_1(\xi), \ldots, A_k(\xi)$ involve all the relevant direction in a neighborhood of ξ. This means that, even if a purely quantum system is not controllable, it can be controllable as a part of a hybrid system as long as movement along the classical variables causes the controls to span all the directions tangent to M_Q, especially those not available originally.

Notice also that Theorem 3.10 involves the identity $\mathbb{I}$ in its hypothesis. This is due to the fact that control hybrid observables need not be traceless, although its trace is lost when projected onto M_Q, as $X_{\mathbb{I}}^Q = 0$. An alternative hypothesis would be $\mathrm{Herm}_0(n) \subset \mathrm{Lie}(A_1(\xi), \ldots, A_k(\xi))$, with the same validity but a slightly lengthier proof.

A similar theorem can be proven for the classical subsystem. We will denote by $\mathrm{Poi}_C(f_{A_1}, \ldots, f_{A_k})$ the Poisson algebra of functions in $C^\infty(M_{QC})$ generated by $f_{A_1}, \ldots, f_{A_k}$ with respect to the classical Poisson bracket $\{\cdot,\cdot\}_C$.

Theorem 3.11. *Let X_u be a controlled vector field on a connected hybrid manifold M_{QC}, defined by* (3.1). *If there exists a dense subset $S \subset M_{QC}$ such that, for every $(\xi, \psi) \in S$, there exists a set of smooth hybrid functions $\{g_1, \ldots, g_l\} \subset \mathrm{Poi_C}(f_{A_1}, \ldots, f_{A_k})$ such that $\pi_{C*}^{(\xi,\psi)}\left(\mathrm{Lie}\left(X_{g_1}^C, \ldots, X_{g_l}^C\right)|_{(\xi,\psi)}\right) = T_\xi M_C$, the system is classical controllable.*

Proof. As in the previous theorem, consider an orthogonal basis and compute the commutators of control vector fields as in (3.8). Its projection onto M_C at a point $(\xi, \psi) \in M_{QC}$ is

$$\pi_{C*}^{(\xi,\psi)}\left([X_i, X_j]\,|_{(\xi,\psi)}\right) = \pi_{C*}^{(\xi,\psi)}\left(X_{\{f_i,f_j\}_C}^C\,|_{(\xi,\psi)}\right) + \sum_{l,m=1}^{n^2} s_{[l,m]}(\psi)\left(a_i^l(\xi)X_{a_j^m}^C\,|_\xi + a_j^m(\xi)X_{a_i^l}^C\,|_\xi\right). \tag{3.10}$$

If the condition in the theorem is satisfied, then the first term produces all the tangent directions at ξ; the second term could cancel it on a nowhere-dense set of M_{QC}, as s_i functions are quadratic on ψ, while $\{f_i, f_j\}_C$ is a degree-4 function on the quantum variables. Conditions of Theorem 3.8 are thus satisfied, hence the system is classical controllable.

Notice that, in particular, if $R_1, R_2, \ldots, P_1, P_2, \ldots \in \mathrm{Poi_C}(f_{A_1}, \ldots, f_{A_k})$, then the system is classical controllable.

4. Applications

In order to illustrate the applications of our results, in this section we present some examples in which the developed criteria are useful in order to analyze the controllability of the systems.

4.1. Classical subsystem on the plane and a qubit

As a first example, let us consider a hybrid system whose classical manifold is the cotangent space to the plane, $M_C = T^*\mathbb{R}^2$, with coordinates $\xi = (R_1, R_2, P_1, P_2)$, and whose quantum subsystem is a qubit, hence $\mathcal{H} = \mathbb{C}^2$. Without controls, the evolution of the hybrid system is governed by the hybrid Hamiltonian $H(\xi)$. We will analyze the situation in which two linearly-independent controls are added to this drift. Thus, let us consider a simple control system with two control functions $u_1(t)$ and $u_2(t)$, and with the following particular expression for the expectation value function of the controlled Hamiltonian:

$$F_u(\xi, \psi, t) = f_H(\xi, \psi) + u_1(t)R_1 f_A(\psi) + u_2(t)g(\xi)f_B(\psi), \tag{4.1}$$

for some (constant) quantum observables $A, B \in \mathrm{Herm}(2)$ and some smooth function $g(\xi)$.

Let us apply the results shown in the previous section to analyze the controllability of the system:

- If A and B are linearly independent, then $\mathrm{Lie}(R_1 A, g(\xi)B, I) = \mathrm{Herm}(2)$ as long as $R_1 \neq 0$ and $g(\xi) \neq 0$. Thus, assuming g is a nonzero analytic function, Theorem 3.9 indicates that the system is quantum-controllable.
- Classical controllability requires $\mathrm{Poi_C}(R_1 f_A, g f_B)$ to generate the whole classical Poisson algebra of polynomials. As $\{R_1 f_A, g f_B(\psi)\}_C = f_A f_B \{R_1, g\}_C$, it is enough to consider a smooth function g whose successive Poisson brackets with R_1 generates enough functions to cover all the classical tangent space. Such is the case of the following function:

$$g(\xi) = \frac{1}{2}\left(P_1^2 + P_2^2 - 2R_1R_2\right), \tag{4.2}$$

The first elements in $\mathrm{Poi}_{\mathrm{C}}(R_1 f_A, g(\xi) f_B)$ are thus

$$\begin{aligned} \{R_1 f_A, g f_B\}_C(\xi,\psi) &= f_A(\psi) f_B(\psi) P_1, \\ \{\{R_1 f_A, g f_B\}_C, g f_B\}_C(\xi,\psi) &= f_A(\psi) f_B^2(\psi) R_2, \\ \{\{\{R_1 f_A, g f_B\}_C, g f_B\}_C, g f_B\}_C(\xi,\psi) &= f_A(\psi) f_B^3(\psi) P_2. \end{aligned}$$

Assume that $f_A(\psi) \neq 0$ and $f_B(\psi) \neq 0$, which occurs in a dense subset of M_Q if A and B are not the null operator, and consider the hybrid Hamiltonian vector fields of these three functions, together with $f_A R_1$. Their projections at each $(\xi,\psi) \in M_{QC}$ generate the whole tangent space $T_\xi M_C$, satisfying Theorem 3.8, so the system is classical controllable.

Alternatively, one could compute explicitly the control vector fields and their commutators, in order to prove controllability by Theorems 3.8 and 3.9. Consider the expression for $g(\xi)$ given in (4.2); control vector fields are thus

$$\begin{aligned} X_1 &= X^{QC}_{R_1 f_A} = -f_A \frac{\partial}{\partial P_1} + R_1 X^Q_A, \\ X_2 &= X^{QC}_{g f_B} = P_1 f_B \frac{\partial}{\partial R_1} + R_2 f_B \frac{\partial}{\partial P_1} + P_2 f_B \frac{\partial}{\partial R_2} + R_1 f_B \frac{\partial}{\partial P_2} + g X^Q_B. \end{aligned} \tag{4.3}$$

The commutator of both control vector fields can be directly computed to be:

$$\begin{aligned} [X_1, X_2] = &- f_A f_B \frac{\partial}{\partial R_1} + f_{[\![A,B]\!]} \left(-R_1 P_1 \frac{\partial}{\partial R_1} + (g - R_1 R_2) \frac{\partial}{\partial P_1} - R_1 P_2 \frac{\partial}{\partial R_2} - R_1^2 \frac{\partial}{\partial P_2} \right) \\ &- R_1 g X^Q_{[\![A,B]\!]} - P_1 \left(f_B X^Q_A + f_A X^Q_B \right) \end{aligned} \tag{4.4}$$

Notice the terms, both in classical and quantum directions, obtained from the commutation between the quantum part of X_1 and the classical part of X_2, and vice-versa, corresponding to the last terms in (3.9) and (3.10).

As long as $[\![A,B]\!] \neq 0$, projections onto M_Q of X_1, X_2 and $[X_1, X_2]$ already generate the whole tangent space to the quantum sub-manifold, so the system is quantum controllable. Additional commutators are needed in order to prove the same for the classical part. Lastly, with enough commutators one can prove that the whole tangent space $T_{(\xi,\psi)} M_{QC}$ can be generated at each point, so the system is classical, quantum and hybrid controllable.

4.2. *Hybrid controllability of the spin-boson model*

A qubit is a quantum system with only two possible energy levels, hence its Hilbert space is $\mathcal{H} = \mathbb{C}^2$. Although simple, this system has a great relevance due to the complex behaviors that show and its applications in many fields. A basis for the algebra of observables Herm(2) is given by the identity and the Pauli matrices $\sigma_1, \sigma_2, \sigma_3$, defined in (2.13).

The interaction between a qubit and a thermal reservoir can be described by the following Hamiltonian function [25]:

$$f_H(\xi,\psi) = h_r(\xi) + f_{H_Q}(\psi) - u E(\boldsymbol{R}) f_{\sigma_3}(\psi), \quad \xi \in M_C = \mathbb{R}^{2N},\ \psi \in \mathcal{H}, \tag{4.5}$$

where h_r denotes the Hamiltonian function for the thermal reservoir, modeled by a non-interacting group of N unidimensional bosons with masses $m_1, \ldots, m_N$ and frequencies $\omega_1, \ldots, \omega_n$; H_Q is the purely quantum Hamiltonian of the qubit; u is a constant coefficient; and E denotes the polarization energy of the reservoir:

$$h_r(\xi) = \sum_{j=1}^{N} \left(\frac{P_j^2}{2m_j} + \frac{m_j \omega_j^2 R_j^2}{2} \right), \quad H_Q = -\frac{1}{2} \begin{pmatrix} \epsilon & \Delta \\ \Delta & -\epsilon \end{pmatrix} = -\Delta\sigma_1 - \epsilon\sigma_3, \quad E(\boldsymbol{R}) = \sum_{j=1}^{N} c_j R_j, \tag{4.6}$$

for some positive numbers $\epsilon, \Delta, c_1, \dots, c_N \in \mathbb{R}$.

Let us consider a bath with a single boson with mass $m_1 = m$, frequency $\omega_1 = \omega$ and energy coefficient $c_1 = c$. We will replace the coefficient u by a control function $u(t)$, which turns (4.5) into a controlled Hamiltonian function:

$$F_u(\xi, \psi, t) = f_0(\xi, \psi) + u(t) f_1(\xi, \psi), \quad f_0(\xi, \psi) = \frac{P^2}{2m} + \frac{m\omega^2 R^2}{2} + f_{H_Q}(\psi), \quad f_1(\xi, \psi) = -cR f_{\sigma_3}(\psi). \tag{4.7}$$

For each $t \in \mathbb{R}$, this is the expectation value function of the following hybrid observable, acting as the controlled Hamiltonian of the hybrid system:

$$H_u(\xi, t) = H_0(\xi) + u(t) A_1(\xi), \quad H_0(\xi) = \left(\frac{P^2}{2m} + \frac{m\omega^2 R^2}{2} \right) \mathbb{I} + H_Q, \quad A_1(\xi) = -cR\sigma_3. \tag{4.8}$$

The controlled dynamics of the system are governed by the family of hybrid Hamiltonian vector fields of $F_u(t)$:

$$X_u(t) = X_{F_u}^{QC}(t) = X_0 + u(t) X_1, \quad X_0 = \frac{P}{m} \frac{\partial}{\partial R} - m\omega^2 R \frac{\partial}{\partial P} + X_{H_Q}^Q, \quad X_1 = c f_{\sigma_3} \frac{\partial}{\partial P} - cR X_{\sigma_3}^Q. \tag{4.9}$$

As the system has a single control, Theorem 3.4 cannot be directly applied to prove its controllability. However, in this case, notice that the drift X_0 describes a rotation of the system simultaneously on M_C and on M_Q. Therefore, X_0 is recurrent and Corollary 3.7 allows us to treat the drift as an additional control.

First, let us analyze the classical controllability of the system. By direct computation, the classical Poisson bracket of f_0 and f_1 is:

$$\{f_0, f_1\}_C = \frac{c}{m} P f_{\sigma_3}. \tag{4.10}$$

Therefore, at every point $(\xi, \psi) \in M_{QC}$ such that $f_{\sigma_3}(\psi) \neq 0$, push-forwards by π_C of the Hamiltonian vector fields of f_1 and $\{f_0, f_1\}_C$ generate the whole tangent space $T_\xi M_C$. By Theorem 3.8, the system is classical controllable (as long as $c \neq 0$, i.e. there exists interaction between the classical and the quantum subsystems).

On the other hand, the successive commutators of $H_0(\xi)$ and $A_1(\xi)$ at any $\xi \in M_C$ are

$$[\![H_0(\xi), A_1(\xi)]\!] = -cR[\![H_Q, \sigma_3]\!] = -c\Delta R\sigma_2, \quad [\![[\![H_0(\xi), A_1(\xi)]\!], A_1(\xi)]\!] = c^2 \Delta R^2 \sigma_1. \tag{4.11}$$

Hence, $\mathrm{Lie}(H_0(\xi), A_1(\xi)) = \mathrm{Herm}(2)$ except for the set $R = 0$, and as long as $c \neq 0$ and $\Delta \neq 0$, so by Theorem 3.9 the system is quantum controllable.

Consider now the case of 2 bosons, replacing functions f_0 and f_1 in (4.7) by

$$f_0(\xi, \psi) = \frac{P_1^2}{2m_1} + \frac{m_1 \omega_1^2 R_1^2}{2} + \frac{P_2^2}{2m_2} + \frac{m_2 \omega_2^2 R_2^2}{2} + f_{H_Q}(\psi), \quad f_1(\xi, \psi) = -(c_1 R_1 + c_2 R_2) f_{\sigma_3}(\psi). \tag{4.12}$$

We will assume $c_1 \neq 0$, $c_2 \neq 0$ and $\Delta \neq 0$.

As before, the drift is recurrent. As there are no changes in the quantum subsystem, the system is again quantum controllable Regarding the classical subsystem, let us compute some of the first Poisson brackets:

$$\{f_0, f_1\}_C = \left(\frac{c_1}{m_1}P_1 + \frac{c_2}{m_2}P_2\right) f_{\sigma_3}, \quad \{f_0, \{f_0, f_1\}_C\}_C = \left(c_1\omega_1^2 R_1 + c_2\omega_2^2 R_2\right) f_{\sigma_3}. \tag{4.13}$$

As long as $|\omega_1| \neq |\omega_2|$, functions f_1 and $\{f_0, \{f_0, f_1\}_C\}_C$ are linearly independent. Therefore, we have $R_1 f_{\sigma_3}, R_2 f_{\sigma_3} \in \mathrm{Poi}_\mathrm{C}(f_0, f_1)$; using $R_1 f_{\sigma_3}$ and $R_2 f_{\sigma_3}$ to compute more Poisson brackets, we find

$$\{f_0, R_1 f_{\sigma_3}\}_C = -\frac{P_1}{m_1} f_{\sigma_3}, \quad \{f_0, R_2 f_{\sigma_3}\}_C = -\frac{P_2}{m_2} f_{\sigma_3}. \tag{4.14}$$

Having $R_1 f_{\sigma_3}, R_2 f_{\sigma_3}, P_1 f_{\sigma_3}, P_2 f_{\sigma_3} \in \mathrm{Poi}_\mathrm{C}(f_0, f_1)$ is enough to generate, at every $(\xi, \psi) \in M_{QC}$ with $f_{\sigma_3}(\psi) \neq 0$, the whole tangent space $T_\xi M_C$ by push-forward of their hybrid Hamiltonian vector fields. The system is thus classical-controllable as long as both bosons have different frequencies.

Notice that the same construction can be extrapolated to any number of bosons: the spin-boson system is classical and quantum controllable for any number of bosons with different frequencies.

4.3. Hybrid splines

Splines have been studied for several decades due to their interesting applications in different fields, such as design and computer graphics. Originally introduced for linear spaces, and later generalized to arbitrary Riemannian manifolds [52–54], the idea behind them is to define curves joining a given set of points of the state space of a system at the corresponding values of the curve parameter (time, for instance). Curves are chosen as geodesics of the corresponding Riemannian metric. In 2012, the concept was generalized to the quantum state case, for quantum pure states first [20] and later to the case of mixed states [19].

The combination of classical and quantum splines leads naturally to considering the case of the interpolation problem for a hybrid state space, and from a threefold perspective. The first is the direct problem of interpolation of a set of points in the hybrid phase space $(\xi_0, \psi_0), (\xi_1, \psi_1), \cdots, (\xi_k, \psi_k) \in M_{QC}$ by means of a certain controlled hybrid dynamics. However, with respect to those same dynamics, we may consider the problem of interpolation restricted to just one of the two manifolds:

- interpolating the points $\{\xi_0, \xi_1, \cdots, \xi_k\}$ in M_C, or
- interpolating the points $\{\psi_0, \psi_1, \cdots, \psi_k\}$ in M_Q.

From the physical point of view, the first case makes more sense, since the classical degrees of freedom are slower than the quantum ones. Nonetheless, from the mathematical point of view, it is the most complicated one due to the nonlinearity and unboundedness of the dynamics.

As we saw in the references above, the problem of cubic splines can be considered an optimal control problem, which can be solved either by a variational approach as above (we can consider it a Lagrangian version of the problem) or by the Pontryagin Maximum principle [55, 56], which represents the corresponding Hamiltonian version. In this paper, we will consider the second case, which generalizes to a hybrid framework the (quantum) construction considered in [19].

We will recover the example analyzed in Section 4.1: a classical system with two degrees of freedom, parametrized by classical coordinates $\xi = (R_1, R_2, P_1, P_2) \in M_C$, and one qubit as the quantum subsystem, parametrized by a quantum state $\psi \in M_Q = \mathbb{C}^2$. The system will present two control functions $u_1(t)$, $u_2(t)$, so its controlled Hamiltonian has the following expression:

$$H_u(\xi, t) = H_0(\xi) + u_1(t)A_1(\xi) + u_2(t)A_2(\xi) \tag{4.15}$$

The drift Hamiltonian H_0 is taken to describe a purely classical harmonic oscillator and a static qubit,

$$H_0(\xi) = \left(\frac{R_1^2}{2} + \frac{R_2^2}{2} + \frac{P_1^2}{2} + \frac{P_2^2}{2}\right)\mathbb{I}, \tag{4.16}$$

while the control hybrid observables are taken as particular examples of (4.1) with Pauli matrices (2.13):

$$A_1(\xi) = R_1\sigma_x, \quad A_2(\xi) = -\frac{1}{2}\left(P_1^2 - P_2^2 + 2R_1R_2\right)\sigma_y. \tag{4.17}$$

The system thus described is classical, quantum and hybrid controllable, as described in Section 4.1. Its dynamics are governed by the family of Hamiltonian vector fields of (4.15):

$$X_u(t) = X_0 + u_1(t)X_1 + u_2(t)X_2, \tag{4.18}$$

where the drift term is a purely classical vector field,

$$X_0 = X_{H_0}^{QC} = P_1\frac{\partial}{\partial R_1} + P_2\frac{\partial}{\partial R_2} - R_1\frac{\partial}{\partial P_1} - R_2\frac{\partial}{\partial P_2}, \tag{4.19}$$

and the control vector fields are the same as in (4.3).

The effect of the control vector fields can be visualized in Figure (3) where the trajectory on the Bloch sphere and on the two classical planes is visualized together with the expression of the two control functions $u_1(t)$ and $u_2(t)$.

Given these controlled dynamics, we now proceed to studying the following optimal control problem. For a certain set of target points $(\xi_0, \psi_0), (\xi_1, \psi_1), \cdots, (\xi_k, \psi_k) \in M_{QC}$ and instants of time $t_0 < t_1 < \ldots < t_k$, our goal is to determine the control functions $\{u_1(t), u_2(t)\}$ such that the resulting trajectory $(\xi(t), \psi(t))$ of the system satisfying $\xi(t_0) = \xi_0$, $\psi(t_0) = \psi_0$ minimizes the functional:

$$\mathcal{J}(u_1, u_2) = \int_{t_0}^{t_k} dt\ (u_1^2(t) + u_2^2(t)) + \epsilon\sum_{j=0}^{k} d((\xi(t_j)\psi(t_j)), (\xi_j, \psi_j)), \quad \epsilon > 0, \tag{4.20}$$

with d a distance function in M_{QC}. The parameter ϵ is included to balance the strength of the two conditions in the functional: the power introduced in the system by the control functions, and the closeness of the trajectory to the target points.

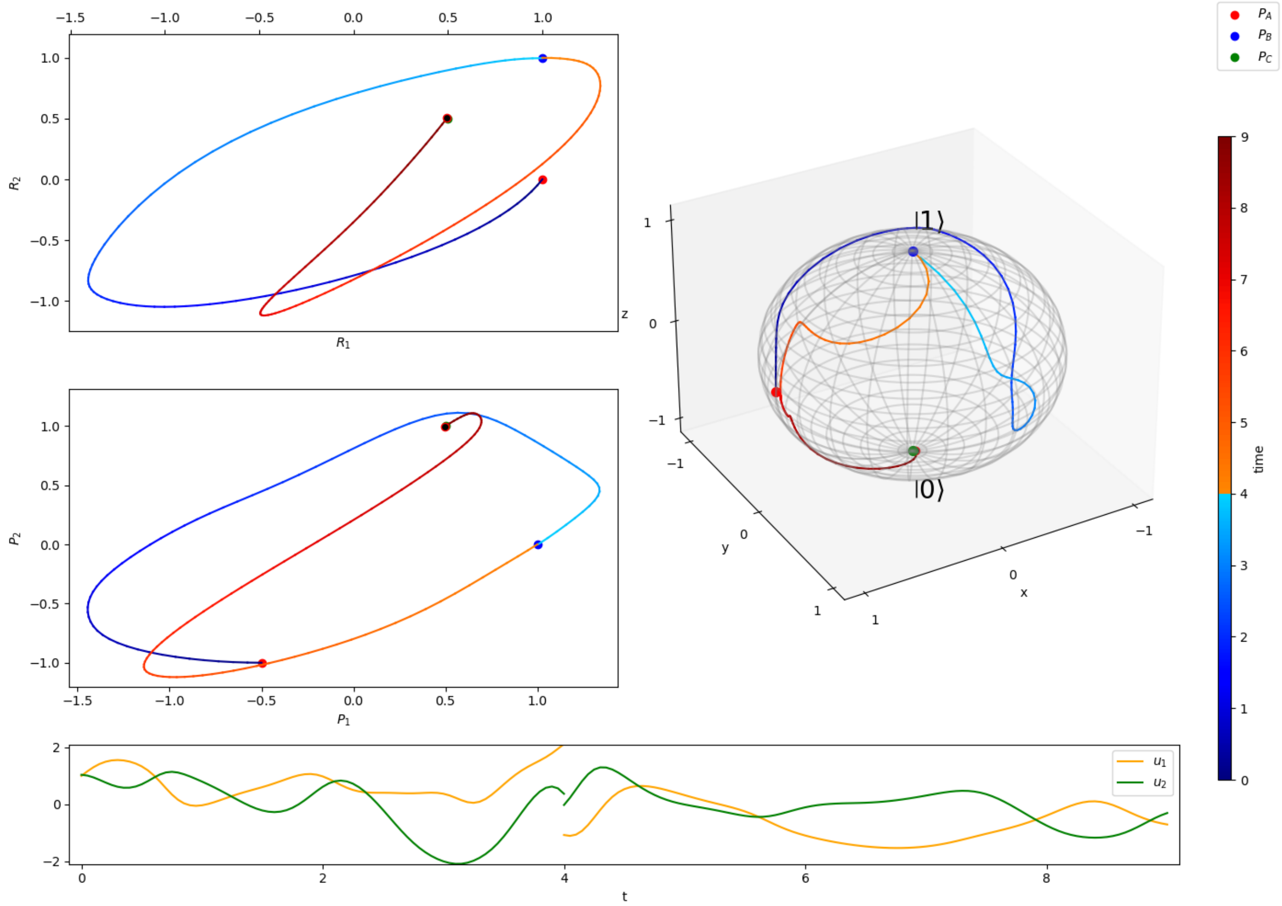

Figure 3. Solution of the optimal control problem. On the upper part left, the two projections on the R_1, R_2 and P_1, P_2 planes can be seen. On the righthand side, the trajectories on the Bloch sphere are depicted. At the bottom, the two control functions $u_1(t)$ and $u_2(t)$ are presented.

Notice that, as the hybrid manifold M_{QC} is a cartesian product, the problem can be easily adapted to interpolate between either hybrid points, or just classical and quantum ones, depending on the distance function we consider. Thus, if the function d above represents only the classical distance of points on M_C, the interpolation problem refers to classical points only. Notice that, in this case, a Riemannian structure for the classical manifold M_C is required. Analogously, considering a purely quantum distance function on M_Q, the interpolation problem is purely quantum. If the distance function is the sum of the two distances, the problem is a hybrid interpolation.

For the sake of simplicity, let us consider a problem with just three target points, defined as follows:

- Target point $\xi_0 = (R_{10}, R_{20}, P_{10}, P_{20}) = \left(1, 0, -\frac{1}{2}, -1\right)$, $|\psi_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ at time $t_0 = 0$.
- Target point $\xi_1 = (R_{11}, R_{21}, P_{11}, P_{21}) = (1, 1, 1, 0)$, $|\psi_1\rangle = |1\rangle$ at time $t_1 = 4$.
- Target point $\xi_2 = (R_{12}, R_{22}, P_{12}, P_{22}) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 1\right)$, $|\psi_2\rangle = |0\rangle$ at time $t_2 = 9$.

In order to do that, we define the corresponding Pontryagin Hamiltonian, by introducing the co-state variables $\Pi = (\Pi_{R_1}, \Pi_{R_2}, \Pi_{P_1}, \Pi_{P_2}, \Pi_\psi)$ and the Hamiltonian

$$\mathcal{H}_P = \Pi(X_u) - \mathcal{J}. \tag{4.21}$$

Hamilton equations are obtained straightforwardly from Equation (4.21) using the first term in Equation (4.20), for each of the two time intervals $[0, 4]$ and $[4, 9]$. The discrete part of the functional in (4.20) provides the corresponding boundary conditions for states and co-states at points $t = 0$, $t = 4$ and $t = 9$. In this optimal control problem, analogously to what happened in the discussion on the controllability notion in the previous section, we can also consider different levels while attending to just classical, just quantum, or the complete set of degrees of freedom in the functional $\mathcal{J}$. If the distance function d in (4.20) is purely classical, there will be no boundary constraints for the quantum degrees of freedom. Analogously, for a purely quantum distance function, there will be no boundary constraints for the classical ones. Having a hybrid distance function will introduce boundary conditions for both types of degrees of freedom.

The resulting set of fourteen equations (seven for the states and seven for the co-states) must be solved numerically with the boundary conditions of each interval. The problem is numerically complex because of the large number of equations and the two sets of conditions (one for each time interval). Our numerical solver, implemented in Mathematica, had difficulties with the balance between the large kicks and the required oscillatory regimes to balance the optimization functional. In any case, an exact solution can be seen in Figure 3.

An alternative approach to the resolution of this problem could be obtained by extending to the hybrid case the algorithm we introduced in [19] to find approximate solutions for the quantum spline problem. We will consider this extension in a future paper.

5. Conclusions and outlook

In this paper we have introduced the notion of hybrid control system, as a control dynamical system which combines classical and quantum degrees of freedom. The interest of this type of system comes mostly from physics, where some full quantum systems admit an approximate description where the slowest degrees of freedom can be modeled as classical objects. If we use a geometrical description for quantum systems, which allows us a formally analogous description to the classical ones, a simple dynamical description as Hamiltonian systems can be achieved for hybrid control dynamical systems.

Once defined, we have analyzed the notion of controllability adapted to hybrid systems. We generalized the theory of controllability of classical systems incorporating the quantum degrees of freedom described in geometrical terms. It is important to remark that the usual definitions of quantum controllability which involve the unitary group have lost their meaning, because the hybrid dynamics are no longer linear due to the coupling with the classical subsystem. Furthermore, the definition in terms of product manifolds opens the possibility of considering partial controllability on just the classical or the quantum degrees of freedom, the first case being more realistic from the physical point of view (remember that the classical degrees of freedom are slower than the quantum ones). Lie algebras of control observables and classical Poisson algebras of their expectation value function play a key role in the description of sufficient conditions for controllability.

Finally, we have considered some practical applications. In particular, we have proved the controllability of the spin-boson model, which has remarkable applications in different fields, the best known being transition of electronic states coupled to nuclear vibrations in molecules or the coupling of protein motion to electron transfer in a photosynthetic reaction center (references). We have also introduced the notion of hybrid spline, as a generalization of their classical and quantum analogues and have presented a simple case of interpolation between three hybrid points. A more detailed geometric analysis of this problem and further examples will be the goal of future papers. In particular, notice that the nature of the control functions is not subject to strong restrictions. Thus, other applications based on recent quantum control protocols such as the *choped random basis ansatz* [51], as well as control functions described in terms of Fourier series [50], are also being considered. Additionally, future works will deal with the practical applications of controllability and optimal control in numerical simulations and physical experiments in order to verify the validity of the theory and to comprehend its applicability to actual physical control systems.

Author contribution

Emanuel-Cristian Boghiu: Numerical and symbolic computations, reviewing and editing; Jesús Clemente-Gallardo: Conceptualization, analysis, co-writing original draft, reviewing and editing; Jorge A. Jover-Galtier: Analysis, co-writing original draft, reviewing and editing; David Martínez-Crespo: Initial version of the proof of Theorems of Section 3, numerical and symbolic computations, reviewing and editing.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors acknowledge partial financial support of Grants PID2021- 123251NB-I00 and PID2021-122961NB-I00 (JAJ-G) funded by MCIN/AEI/10.13039/501100011033 and by the European Union, and of Grants E48-23R and E24-23R(JAJ-G) funded by Government of Aragón. D.M-C acknowledges financial support by Gobierno de Aragón through the grant defined in ORDEN CUS/581/2020. E.-C.B. acknowledges financial support from MCIN/AEI/10.13039/501100011033 (CEX2019-000910-S, PRE2019-088482), FSE "El FSE invierte en tu futuro", Fundació Cellex, Fundació Mir-Puig, and Generalitat de Catalunya through the CERCA program.

Conflict of interest

The authors declare there is no conflict of interest.

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