Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 16(4): 717–737.
DOI: 10.3934/cam.2024032
Received: 22 November 2023
Revised: 27 February 2024
Accepted: 07 August 2024
Published: 14 October 2024

Research article

Existence and blow up for viscoelastic hyperbolic equations with variable exponents

Ying Chu*, **Bo Wen and Libo Cheng**

School of Mathematics and Statistics, Changchun University of Science and Technology, Changchun, 130022, China

* **Correspondence:** Email: chuying–12345@sina.com; Tel: 1-594-879-0402.

Abstract: In this article, we consider a nonlinear viscoelastic hyperbolic problem with variable exponents. By using the Faedo−Galerkin method and the contraction mapping principle, we obtain the existence of weak solutions under suitable assumptions on the variable exponents $m(x)$ and $p(x)$. Then we prove that a solution blows up in finite time with positive initial energy as well as nonpositive initial energy.

Keywords: variable exponents; weak solutions; viscoelastic; existence; blow up
Mathematics Subject Classification: 35L35, 35B40, 35B44

1. Introduction

In this paper, we study the initial boundary value problem of the nonlinear viscoelastic hyperbolic problem with variable exponents:

$$\begin{cases} u_{tt} + \triangle^2 u + \triangle^2 u_{tt} - \displaystyle\int_0^t g(t-\tau)\triangle^2 u(\tau)d\tau + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset R^n (n \geq 1)$ is a bounded domain in R^n with a smooth boundary $\partial\Omega$, ν is the unit outer normal to $\partial\Omega$, the exponents $m(x)$ and $p(x)$ are continuous functions on $\overline{\Omega}$ with the logarithmic module of continuity:

$$\forall x, y \in \Omega,\ |x-y| < 1,\ |m(x) - m(y)| + |p(x) - p(y)| \leq \omega(|x-y|), \tag{1.2}$$

where

$$\lim_{\tau \to 0^+} sup\, \omega(\tau) \ln \frac{1}{\tau} = C < \infty. \tag{1.3}$$

In addition to this condition, the exponents satisfy the following:

$$2 \le m^- := ess \inf_{x\in\Omega} m(x) \le m(x) \le m^+ := ess \sup_{x\in\Omega} m(x) < \frac{2(n-2)}{n-4}, \tag{1.4}$$

$$2 \le p^- := ess \inf_{x\in\Omega} p(x) \le p(x) \le p^+ := ess \sup_{x\in\Omega} p(x) < \frac{2(n-2)}{n-4}, \tag{1.5}$$

$g : R^+ \to R^+$ is a C^1 function satisfying

$$g(0) > 0, \ g'(\tau) \le 0, \ 1 - \int_0^\infty g(\tau)d\tau = l > 0. \tag{1.6}$$

The equation of Problem (1.1) arises from the modeling of various physical phenomena such as the viscoelasticity and the system governing the longitudinal motion of a viscoelastic configuration obeying a nonlinear Boltzmann's model, or electro-rheological fluids, viscoelastic fluids, processes of filtration through a porous medium, and fluids with temperature-dependent viscosity and image processing which give rise to equations with nonstandard growth conditions, that is, equations with variable exponents of nonlinearities. More details on these problems can be found in previous studies [1–6].

When $m(x)$ and $p(x)$ are constants, Messaoudi [7] discussed the nonlinear viscoelastic wave equation

$$u_{tt} - \triangle u + \int_0^t g(t-\tau)\triangle u(\tau)d\tau + |u_t|^{m-2}u_t = |u|^{p-2}u,$$

he proved that any weak solution with negative initial energy blows up in finite time if $p > m$, and a global existence result for $p \le m$. The results were improved later by Messaoudi [8], where the blow-up result in finite time with positive initial energy was obtained. Moreover, Song [9] showed the finite-time blow-up of some solutions whose initial data had arbitrarily high initial energy. In the same year, Song [10] studied the initial-boundary value problem

$$|u_t|^\rho u_{tt} - \triangle u + \int_0^t g(t-\tau)\triangle u(\tau)d\tau + |u_t|^{m-2}u_t = |u|^{p-2}u,$$

and proved the nonexistence of global solutions with positive initial energy. Cavalcanti, Domingos, and Ferreira [11] were concerned with the non-linear viscoelastic equation

$$|u_t|^\rho u_{tt} - \triangle u - \triangle u_{tt} + \int_0^t g(t-\tau)\triangle u(\tau)d\tau - \gamma\triangle u_t = 0,$$

and proved the global existence of weak solutions. Moreover, they obtained the uniform decay rates of the energy by assuming a strong damping $\triangle u_t$ acting in the domain and providing the relaxation function which decays exponentially.

In 2017, Messaoudi [12] considered the following nonlinear wave equation with variable exponents:

$$u_{tt} - \triangle u + a|u_t|^{m(x)-2}u_t = b|u|^{p(x)-2}u,$$

where a, b are positive constants. By using the Faedo−Galerkin method, the existence of a unique weak solution is established under suitable assumptions on the variable exponents $m(x)$ and $p(x)$. Then this paper also proved the finite-time blow-up of solutions and gave a two-dimensional numerical example to illustrate the blow up result. Park [13] showed the blow up of solutions for a viscoelastic wave equation with variable exponents

$$u_{tt} - \triangle u + \int_0^t g(t-s)\triangle u(s)ds + a|u_t|^{m(x)-2}u_t = b|u|^{p(x)-2}u,$$

where the exponents of nonlinearity $p(x)$ and $m(x)$ are given functions and $a, b > 0$ are constants. For nonincreasing positive function g, they prove the blow-up result for the solutions with positive initial energy as well as nonpositive initial energy. Alahyane [14] discussed the nonlinear viscoelastic wave equation with variable exponents

$$u_{tt} - \triangle u + \int_0^t g(t-\tau)\triangle u(\tau)d\tau + \mu u_t = |u|^{p(x)-2}u,$$

where μ is a nonnegative constant and the exponent of nonlinearity $p(x)$ and g are given functions. Under arbitrary positive initial energy and specific conditions on the relaxation function g, they prove a finite-time blow-up result and give some numerical applications to illustrate their theoretical results. Ouaoua and Boughamsa [15] considered the following boundary value problem:

$$u_{tt} + \triangle^2 u - \triangle u + |u_t|^{m(x)-2}u_t = |u|^{p(x)-2}u,$$

the authors established the local existence by using the Faedo−Galerkin method with positive initial energy and suitable conditions on the variable exponents $m(x)$ and $r(x)$. In addition, they also proved that the local solution is global and obtained the stability estimate of the solution. Ding and Zhou [16] considered a Timoshenko-type equation

$$u_{tt} + \triangle^2 u - M(\|\nabla u\|_2^2)\triangle u + |u_t|^{p(x)-2}u_t = |u|^{q(x)-2}u,$$

they prove that the solutions blow up in finite time with positive initial energy. Therefore, the existence of finite-time blow-up solutions with arbitrarily high initial energy is established, and the upper and lower bounds of the blow-up time are derived. More related references can be found in [17–22].

Motivated by [7,13,14], we considered the existence of the solutions and their blow-up for the nonlinear damping and viscoelastic hyperbolic problem with variable exponents. Our aim in this work is to prove the existence of the weak solutions and to find sufficient conditions on $m(x)$ and $p(x)$ for which the blow-up takes place.

This article consists of three sections in addition to the introduction. In Section 2, we recall the definitions and properties of $L^{p(x)}(\Omega)$ and the Sobolev spaces $W^{1,p(x)}(\Omega)$. In Section 3, we prove the existence of weak solutions for Problem (1.1). In Section 4, we state and prove the blow-up result for solutions with positive initial energy as well as nonpositive initial energy.

2. Preliminaries

In this section, we review some results regarding Lebesgue and Sobolev spaces with variable exponents first. All of these results and a comprehensive study of these spaces can be found in [23]. Here $(\cdot,\cdot)$ and

$\langle\cdot,\cdot\rangle$ denote the inner product in space $L^2(\Omega)$ and the duality pairing between $H^{-2}(\Omega)$ and $H_0^2(\Omega)$.

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined by

$$L^{p(x)}(\Omega)=\left\{u(x): u \text{ is measurable in } \Omega,\ \rho_{p(x)}(u)=\int_\Omega |u|^{p(x)}dx<\infty\right\},$$

this space is endowed with the norm

$$\|u\|_{p(x)}=\inf\left\{\lambda>0:\int_\Omega\left|\frac{u(x)}{\lambda}\right|^{p(x)}\mathrm{d}x\le 1\right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega)=\left\{u\in L^{p(x)}(\Omega) \text{ such that } \nabla u \text{ exists and } |\nabla u|\in L^{p(x)}(\Omega)\right\},$$

the corresponding norm for this space is

$$\|u\|_{1,p(x)}=\|u\|_{p(x)}+\|\nabla u\|_{p(x)},$$

define $W_0^{1,p(x)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ with respect to the $W^{1,p(x)}(\Omega)$ norm. The spaces $L^{p(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces when $1<p^-\le p^+<\infty$, where $p^-:=ess\inf_\Omega p(x)$ and $p^+:=ess\sup_\Omega p(x)$.

As usual, we denote the conjugate exponent of $p(x)$ by $p'(x)=p(x)/(p(x)-1)$ and the Sobolev exponent by

$$p^*(x)=\begin{cases}\frac{np(x)}{n-kp(x)}, & \text{if } p(x)<n,\\ \infty, & \text{if } p(x)\ge n.\end{cases}$$

Lemma 2.1. *If* $p_1(x),\ p_2(x)\in C_+(\overline{\Omega})=\{h\in C(\overline{\Omega}):\min_{x\in\overline{\Omega}}h(x)>1\}$, $p_1(x)\le p_2(x)$ *for any* $x\in\Omega$, *then there exists the continuous embedding* $L^{p_2(x)}(\Omega)\hookrightarrow L^{p_1(x)}(\Omega)$, *whose norm does not exceed* $|\Omega|+1$.

Lemma 2.2. *Let* $p(x),\ q(x)\in C_+(\overline{\Omega})$. *Assuming that* $q(x)<p^*(x)$, *there is a compact and continuous embedding* $W^{k,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$.

Lemma 2.3. *(Hölder's inequality)* [24] *For any* $u\in L^{p(x)}(\Omega)$ *and* $v\in L^{q(x)}(\Omega)$, *then the following inequality holds:*

$$\left|\int_\Omega uvdx\right|\le\left(\frac{1}{p^-}+\frac{1}{q^-}\right)\|u\|_{p(x)}\|v\|_{q(x)}\le 2\|u\|_{p(x)}\|v\|_{q(x)}.$$

Lemma 2.4. *For* $u\in L^{p(x)}(\Omega)$, *the following relations hold:*

$$u\ne 0\Rightarrow\left(\|u\|_{p(x)}=\lambda\Leftrightarrow\rho_{p(x)}\left(\frac{u}{\lambda}\right)=1\right),$$

$$\|u\|_{p(x)}<1(=1;>1)\Leftrightarrow\rho_{p(x)}(u)<1(=1;>1),$$

$$\|u\|_{p(x)}>1\Rightarrow\|u\|_{p(x)}^{p^-}\le\rho_{p(x)}(u)\le\|u\|_{p(x)}^{p^+},$$

$$\|u\|_{p(x)}<1\Rightarrow\|u\|_{p(x)}^{p^+}\le\rho_{p(x)}(u)\le\|u\|_{p(x)}^{p^-}.$$

Next, we give the definition of the weak solution to Problem (1.1).

Definition 2.1. *A function u(x,t) is called a weak solution for Problem* (1.1) *, if* $u \in C(0,T;H_0^2(\Omega)) \cap C^1(0,T;H_0^2(\Omega)) \cap C^2(0,T;H^{-2}(\Omega))$ *with* $u_{tt} \in L^2(0,T;H_0^2(\Omega))$ *and u satisfies the following conditions:*
(1) For every $\omega \in H_0^2(\Omega)$ *and for a.e.* $t \in (0,T)$

$$\langle u_{tt},\omega\rangle + (\triangle u,\triangle\omega) + (\triangle u_{tt},\triangle\omega) - \int_0^t g(t-\tau)(\triangle u(\tau),\triangle\omega)d\tau + (|u_t|^{m(x)-2}u_t,\omega) = (|u|^{p(x)-2}u,\omega),$$

(2) $u(x,0) = u_0(x) \in H_0^2(\Omega),\ u_t(x,0) = u_1(x) \in H_0^2(\Omega).$

3. The local existence of weak solution

In this section, we prove the existence of a weak solution for Problem (1.1) by making use of the Faedo–Galerkin method and the contraction mapping principle. For a fixed $T > 0$, we consider the space $\mathscr{H} = C(0,T;H_0^2(\Omega)) \cap C^1(0,T;H_0^2(\Omega))$ with the norm $\|v\|_{\mathscr{H}}^2 = \max\limits_{0\le t\le T}(\|\triangle v_t\|_2^2 + l\|\triangle v\|_2^2)$.

Lemma 3.1. *Assume that* (1.4)*,* (1.5)*, and* (1.6) *hold, let* $(u_0,u_1) \in H_0^2(\Omega)\times H_0^2(\Omega)$*, for any* $T > 0$*,* $v \in \mathscr{H}$*, then there exists* $u \in C(0,T;H_0^2(\Omega)) \cap C^1(0,T;H_0^2(\Omega)) \cap C^2(0,T;H^{-2}(\Omega))$ *with* $u_{tt} \in L^2(0,T;H_0^2(\Omega))$ *satisfying*

$$\begin{cases} u_{tt} + \triangle^2 u + \triangle^2 u_{tt} - \displaystyle\int_0^t g(t-\tau)\triangle^2 u(\tau)d\tau + |u_t|^{m(x)-2}u_t = |v|^{p(x)-2}v, & (x,t)\in\Omega\times(0,T),\\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, & (x,t)\in\partial\Omega\times(0,T),\\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x\in\Omega. \end{cases} \tag{3.1}$$

Proof. Let $\{\omega_j\}_{j=1}^\infty$ be the orthogonal basis of $H_0^2(\Omega)$, which is the standard orthogonal basis in $L^2(\Omega)$ such that

$$-\triangle\omega_j = \lambda_j\omega_j \text{ in } \Omega,\ \omega_\mathrm{j} = 0 \text{ on } \partial\Omega,$$

we denote by $V_k = \mathrm{span}\{\omega_1,\omega_2,\cdot\cdot\cdot,\omega_k\}$ the subspace generated by the first k vectors of the basis $\{\omega_j\}_{j=1}^\infty$. By normalization, we have $\|\omega_j\|_2 = 1$. For all $k \ge 1$, we seek k functions $c_1^k(t), c_2^k(t), \ldots, c_k^k(t) \in C^2[0,T]$ such that

$$u^k(x,t) = \sum_{j=1}^k c_j^k(t)\omega_j(x),$$

satisfying the following approximate problem

$$\begin{cases} (u_{tt}^k,\omega_i) + (\triangle u^k,\triangle\omega_i) + (\triangle u_{tt}^k,\triangle\omega_i) - \displaystyle\int_0^t g(t-\tau)(\triangle u^k,\triangle\omega_i)d\tau\\ +(|u_t^k|^{m(x)-2}u_t^k,\omega_i) = \displaystyle\int_\Omega |v|^{p(x)-2}v\omega_i dx,\\ u^k(0) = u_0^k,\quad u_t^k(0) = u_1^k,\quad i = 1,2,\ldots k, \end{cases} \tag{3.2}$$

where

$$u_0^k = \sum_{i=1}^k (u_0,\omega_i)\omega_i \to u_0 \text{ in } \mathrm{H}_0^2(\Omega),$$

$$u_1^k=\sum_{i=1}^{k}(u_1,\omega_i)\omega_i\to u_1\quad\text{in }\ \mathrm{H}_0^2(\Omega),$$

thus, (3.2) generates the initial value problem for the system of second-order differential equations with respect to $c_i^k(t)$:

$$\begin{cases}(1+\lambda_i^2)c_{itt}^k(t)+\lambda_i^2c_i^k(t)=G_i(c_{1t}^k(t),\dots,c_{kt}^k(t))+g_i(c_i^k(t)), & i=1,2,\dots,k,\\ c_i^k(0)=\int_\Omega u_0\omega_idx,\qquad c_{it}^k(0)=\int_\Omega u_1\omega_idx, & i=1,2,\dots,k.\end{cases}\tag{3.3}$$

where

$$G_i(c_{1t}^k(t),\dots,c_{kt}^k(t))=-\int_\Omega|\sum_{j=1}^{k}c_{jt}^k(t)\omega_j(x)|^{m(x)-2}\sum_{j=1}^{k}c_{jt}^k(t)\omega_j(x)\omega_i(x)dx,$$

and

$$g_i(c_i^k(t))=\lambda_i^2\int_0^t g(t-\tau)c_i^k(\tau)d\tau+\int_\Omega|v|^{p(x)-2}v\omega_idx,$$

by Peano's Theorem, we infer that the Problem (3.3) admits a local solution $c_i^k(t)\in C^2[0,T]$.

The first estimate. Multiplying (3.2) by $c_{it}^k(t)$ and summing with respect to i, we arrive at the relation

$$\begin{aligned}&\frac{d}{dt}(\frac{1}{2}\|u_t^k\|_2^2+\frac{1}{2}\|\triangle u^k\|_2^2+\frac{1}{2}\|\triangle u_t^k\|_2^2)+\int_\Omega|u_t^k|^{m(x)}dx-\int_0^t g(t-\tau)\int_\Omega\triangle u^k(\tau)\triangle u_t^kdxd\tau\\&=\int_\Omega|v|^{p(x)-2}vu_t^kdx.\end{aligned}\tag{3.4}$$

By simple calculation, we have

$$\begin{aligned}&-\int_0^t g(t-\tau)\int_\Omega\Delta u^k(\tau)\Delta u_t^kdxd\tau\\&=\frac{1}{2}\frac{d}{dt}(g\diamond\triangle u^k)-\frac{1}{2}(g'\diamond\triangle u^k)-\frac{1}{2}\frac{d}{dt}\int_0^t g(\tau)d\tau\|\Delta u^k\|_2^2+\frac{1}{2}g(t)\|\Delta u^k\|_2^2,\end{aligned}\tag{3.5}$$

where

$$(\varphi\diamond\triangle\psi)=\int_0^t\varphi(t-\tau)\|\triangle\psi(t)-\triangle\psi(\tau)\|_2^2d\tau,$$

inserting (3.5) into (3.4), using Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|u_t^k\|_2^2+\frac{1}{2}\|\triangle u_t^k\|_2^2+\frac{1}{2}(g\diamond\triangle u^k)+\frac{1}{2}(1-\int_0^t g(\tau)d\tau)\|\Delta u^k\|_2^2\right]\\&=\frac{1}{2}(g'\diamond\triangle u^k)-\frac{1}{2}g(t)\|\triangle u^k\|_2^2+\int_\Omega|v|^{p(x)-2}vu_t^kdx-\int_\Omega|u_t^k|^{m(x)}dx\\&\le\int_\Omega|v|^{p(x)-2}vu_t^kdx\le\left\||v|^{p(x)-2}v\right\|_2\|u_t^k\|_2\\&\le\frac{\eta}{2}\int_\Omega|v|^{2(p(x)-1)}dx+\frac{1}{2\eta}\|u_t^k\|_2^2,\end{aligned}\tag{3.6}$$

using the embedding $H_0^2(\Omega) \hookrightarrow L^{2(p(x)-1)}(\Omega)$ and Lemma 2.4, we easily obtain

$$\begin{aligned}\int_\Omega |v|^{2(p(x)-1)}dx &\le \max\left\{\|v\|_{2(p(x)-1)}^{2(p^--1)}, \|v\|_{2(p(x)-1)}^{2(p^+-1)}\right\}\\ &\le C\max\left\{\|\triangle v\|_2^{2(p^--1)}, \|\triangle v\|_2^{2(p^+-1)}\right\}\\ &\le C,\end{aligned} \tag{3.7}$$

where C is a positive constant. We denote by C various positive constants that may be different at different occurrences.

Combining (3.6) and (3.7), we obtain

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|u_t^k\|_2^2 + \frac{1}{2}\|\triangle u_t^k\|_2^2 + \frac{1}{2}(g \diamond \triangle u^k) + \frac{1}{2}(1 - \int_0^t g(\tau)d\tau)\|\Delta u^k\|_2^2\right]\\ \le&\frac{\eta}{2}C + \frac{1}{2\eta}\|u_t^k\|_2^2,\end{aligned}$$

by Gronwall's inequality, there exists a positive constant C_T such that

$$\|u_t^k\|_2^2 + \|\triangle u_t^k\|_2^2 + (g \diamond \triangle u^k) + l\|\Delta u^k\|_2^2 \le C_T, \tag{3.8}$$

therefore, there exists a subsequence of $\{u^k\}_{k=1}^\infty$, which we still denote by $\{u^k\}_{k=1}^\infty$, such that

$$\begin{aligned}&u^k \overset{*}{\rightharpoonup} u \text{ weakly star in } L^\infty(0,T; H_0^2(\Omega)),\\ &u_t^k \overset{*}{\rightharpoonup} u_t \text{ weakly star in } L^\infty(0,T; H_0^2(\Omega)),\\ &u^k \rightharpoonup u \text{ weakly in } L^2(0,T; H_0^2(\Omega)),\\ &u_t^k \rightharpoonup u_t \text{ weakly in } L^2(0,T; H_0^2(\Omega)).\end{aligned} \tag{3.9}$$

The second estimate. Multiplying (3.2) by $c_{itt}^k(t)$ and summing with respect to i, we obtain

$$\begin{aligned}&\|u_{tt}^k\|_2^2 + \|\Delta u_{tt}^k\|_2^2 + \frac{d}{dt}(\int_\Omega \frac{1}{m(x)}|u_t^k|^{m(x)}dx)\\ =&-\int_\Omega \triangle u^k \triangle u_{tt}^k dx + \int_0^t g(t-\tau)\int_\Omega \Delta u^k(\tau)\Delta u_{tt}^k dx d\tau + \int_\Omega |v|^{p(x)-2}v u_{tt}^k dx.\end{aligned} \tag{3.10}$$

Note that we have the estimates for $\varepsilon > 0$

$$\left|\int_\Omega \triangle u^k \triangle u_{tt}^k dx\right| \le \varepsilon\|\triangle u_{tt}^k\|_2^2 + \frac{1}{4\varepsilon}\|\triangle u^k\|_2^2, \tag{3.11}$$

$$\int_\Omega |v|^{p(x)-2}v u_{tt}^k dx \le \left\||v|^{p(x)-2}v\right\|_2 \left\|u_{tt}^k\right\|_2 \le \varepsilon\|u_{tt}^k\|_2^2 + \frac{1}{4\varepsilon}\int_\Omega |v|^{2(p(x)-1)}dx, \tag{3.12}$$

and

$$\begin{aligned}&\left|\int_0^t g(t-\tau)\int_\Omega \triangle u^k(\tau)\triangle u_{tt}^k dxd\tau\right|\\ \le&\frac{1}{4\varepsilon}\int_\Omega(\int_0^t g(t-\tau)\triangle u^k(\tau)d\tau)^2dx+\varepsilon\|\triangle u_{tt}^k\|_2^2\\ \le&\varepsilon\|\triangle u_{tt}^k\|_2^2+\frac{1}{4\varepsilon}\int_0^t g(s)ds\int_0^t g(t-\tau)\int_\Omega|\triangle u^k(\tau)|^2dxd\tau\\ \le&\varepsilon\|\triangle u_{tt}^k\|_2^2+\frac{(1-l)g(0)}{4\varepsilon}\int_0^t\|\triangle u^k(\tau)\|_2^2d\tau,\end{aligned}\tag{3.13}$$

similar to (3.6) and (3.7), from $H_0^2(\Omega)\hookrightarrow L^2(\Omega)$, we have

$$\int_\Omega |v|^{p(x)-2}vu_{tt}^k dx\le\varepsilon C\|\triangle u_{tt}^k\|_2^2+\frac{C}{4\varepsilon}.\tag{3.14}$$

Taking into account (3.10) – (3.14), we obtain

$$\begin{aligned}&\|u_{tt}^k\|_2^2+(1-2\varepsilon-C\varepsilon)\|\triangle u_{tt}^k\|_2^2+\frac{d}{dt}(\int_\Omega\frac{1}{m(x)}|u_t^k|^{m(x)}dx)\\ \le&\frac{1}{4\varepsilon}\|\triangle u^k\|_2^2+\frac{(1-l)g(0)}{4\varepsilon}\int_0^t\|\triangle u^k(\tau)\|_2^2d\tau+\frac{C}{4\varepsilon},\end{aligned}\tag{3.15}$$

integrating (3.15) over $(0,t)$, we obtain

$$\begin{aligned}&\int_0^t\|u_{tt}^k\|_2^2d\tau+(1-2\varepsilon-C\varepsilon)\int_0^t\|\triangle u_{tt}^k\|_2^2d\tau+\int_\Omega\frac{1}{m(x)}|u_t^k|^{m(x)}dx\\ \le&\frac{C}{4\varepsilon}\int_0^t(\|\triangle u^k\|_2^2+\int_0^\tau\|\triangle u^k(s)\|_2^2ds)d\tau+C_T,\end{aligned}\tag{3.16}$$

taking ε small enough in (3.16), for some positive constant C_T, we obtain

$$\int_0^t\|u_{tt}^k\|_2^2d\tau+\int_0^t\|\triangle u_{tt}^k\|_2^2d\tau\le C_T,\tag{3.17}$$

we observe that estimate (3.17) implies that there exists a subsequence of $\{u^k\}_{k=1}^\infty$, which we still denote by $\{u^k\}_{k=1}^\infty$, such that

$$u_{tt}^k\rightharpoonup u_{tt}\ \textit{weakly in}\ L^2(0,T;H_0^2(\Omega)).\tag{3.18}$$

In addition, from (3.9), we have

$$(u_{tt}^k,\omega_i)=\frac{d}{dt}(u_t^k,\omega_i)\overset{*}{\rightharpoonup}\frac{d}{dt}(u_t,\omega_i)=(u_{tt},\omega_i)\ \ \textit{weakly star in}\ \ L^\infty(0,T;H^{-2}(\Omega)).\tag{3.19}$$

Next, we will deal with the nonlinear term. Combining (3.9), (3.18), and Aubin–Lions theorem [25], we deduce that there exists a subsequence of $\{u^k\}_{k=1}^\infty$ such that

$$u_t^k\to u_t\ \textit{strongly in}\ C(0,T;L^2(\Omega)),\tag{3.20}$$

then

$$|u_t^k|^{m(x)-2}u_t^k \to |u_t|^{m(x)-2}u_t \;\; a.e. \; (x,t) \in \Omega \times (0,T), \tag{3.21}$$

using the embedding $H_0^2(\Omega) \hookrightarrow L^{2(m(x)-1)}(\Omega)$ and Lemma 2.4, we have

$$\left\| |u_t^k|^{m(x)-2}u_t^k \right\|_2^2 = \int_\Omega |u_t^k|^{2(m(x)-1)}dx \le \max\left\{ \|\triangle u_t^k\|_2^{2(m^- -1)}, \|\triangle u_t^k\|_2^{2(m^+ -1)} \right\} \le C, \tag{3.22}$$

hence, using (3.21) and (3.22), we obtain

$$|u_t^k|^{m(x)-2}u_t^k \overset{*}{\rightharpoonup} |u_t|^{m(x)-2}u_t \;\; weakly \;\; star \; in \; L^\infty(0,T;L^2(\Omega)). \tag{3.23}$$

Setting up $k \to \infty$ in (3.2), combining with (3.9), (3.18), (3.19), and (3.23), we obtain

$$\langle u_{tt}, \omega\rangle + (\triangle u, \triangle\omega) + (\triangle u_{tt}, \triangle\omega) - \int_0^t g(t-\tau)(\triangle u(\tau), \triangle\omega)d\tau + (|u_t|^{m(x)-2}u_t, \omega) = (|v|^{p(x)-2}v, \omega).$$

To handle the initial conditions. From (3.9) and Aubin–Lions theorem, we can easily get $u^k \to u$ in $C(0,T;L^2(\Omega))$, thus $u^k(0) \to u(0)$ in $L^2(\Omega)$, and we also have that $u^k(0) = u_0^k \to u_0$ in $H_0^2(\Omega)$, hence $u(0) = u_0$ in $H_0^2(\Omega)$. Similarly, we get that $u_t(0) = u_1$.

Uniqueness. Suppose that (3.1) has solutions u and z, then $\omega = u - z$ satisfies

$$\begin{cases} \omega_{tt} + \triangle^2\omega + \triangle^2\omega_{tt} - \displaystyle\int_0^t g(t-\tau)\triangle^2\omega(\tau)d\tau + |u_t|^{m(x)-2}u_t - |z_t|^{m(x)-2}z_t = 0, & (x,t) \in \Omega \times (0,T), \\ \omega(x,t) = \frac{\partial\omega}{\partial\nu}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ \omega(x,0) = 0, \; \omega_t(x,0) = 0, & x \in \Omega. \end{cases}$$

Multiplying the first equation of Problem (3.1) by ω_t and integrating over Ω, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left[\|\omega_t\|_2^2 + (1 - \int_0^t g(\tau)d\tau)\|\triangle\omega\|_2^2 + \|\triangle\omega_t\|_2^2 + (g \diamond \triangle\omega)\right] + \frac{1}{2}g(t)\|\triangle\omega\|_2^2 \\ &= -\int_\Omega \left(|u_t|^{m(x)-2}u_t - |z_t|^{m(x)-2}z_t\right)(u_t - z_t)dx + \frac{1}{2}(g' \diamond \triangle\omega), \end{aligned}$$

from the inequality

$$(|a|^{m(x)-2}a - |b|^{m(x)-2}b)(a-b) \ge 0, \tag{3.24}$$

for all $a, b \in R^n$ and a.e. $x \in \Omega$, we obtain

$$\|\omega_t\|_2^2 + l\|\triangle\omega\|_2^2 + \|\triangle\omega_t\|_2^2 = 0,$$

which implies that $\omega = 0$. This completes the proof.

Theorem 3.1. *Assume that* (1.4) *and* (1.6) *hold, let the initial date* $(u_0, u_1) \in H_0^2(\Omega) \times H_0^2(\Omega)$, *and*

$$2 \le p^- \le p(x) \le p^+ \le \frac{2(n-3)}{n-4},$$

then there exists a unique local solution of Problem (1.1).

Proof. For any $T > 0$, consider $M_T = \{u \in \mathscr{H} : u(0) = u_0, u_t(0) = u_1, \|u\|_{\mathscr{H}} \le M\}$. Lemma 3.1 implies that for $\forall v \in M_T$, there exists $u = S(v)$ such that u is the unique solution to Problem 3.1. Next, we prove that for a suitable $T > 0$, S is a contractive map satisfying $S(M_T) \subset M_T$.

Multiplying the first equation of the Problem (3.1) by u_t and integrating it over $(0, t)$, we obtain

$$\begin{aligned}&\|u_t\|_2^2 + \|\triangle u_t\|_2^2 + (g \diamond \triangle u) + l\|\Delta u\|_2^2\\ \le&\|u_1\|_2^2 + \|\triangle u_1\|_2^2 + \|\Delta u_0\|_2^2 + 2\int_0^t \int_\Omega |v|^{p(x)-2} v u_t dx d\tau,\end{aligned} \tag{3.25}$$

using Hölder's inequality and Young's inequality, we have

$$\begin{aligned}\left|\int_\Omega |v|^{p(x)-2} v u_t dx\right| &\le \gamma\|u_t\|_2^2 + \frac{1}{4\gamma}\int_\Omega |v|^{2p(x)-2}dx\\ &\le \gamma\|u_t\|_2^2 + \frac{1}{4\gamma}\left[\int_\Omega |v|^{2p^--2}dx + \int_\Omega |v|^{2p^+-2}dx\right]\\ &\le \gamma\|u_t\|_2^2 + \frac{C}{4\gamma}\left[\|\triangle v\|_2^{2p^--2} + \|\triangle v\|_2^{2p^+-2}\right],\end{aligned}$$

thus, (3.25) becomes

$$\begin{aligned}&\|u_t\|_2^2 + \|\triangle u_t\|_2^2 + l\|\Delta u\|_2^2\\ \le&\lambda_0 + 2\int_0^t \int_\Omega |v|^{p(x)-2} v u_t dx d\tau\\ \le&\lambda_0 + 2\gamma T \sup_{(0,T)}\|u_t\|_2^2 + \frac{TC}{2\gamma}\sup_{(0,T)}\left[\|\triangle v\|_2^{2p^--2} + \|\triangle v\|_2^{2p^+-2}\right],\end{aligned}$$

hence, we have

$$\begin{aligned}&\sup_{(0,T)}\|u_t\|_2^2 + \sup_{(0,T)}\|\triangle u_t\|_2^2 + l\sup_{(0,T)}\|\Delta u\|_2^2\\ \le&\lambda_0 + 2\gamma T \sup_{(0,T)}\|u_t\|_2^2 + \frac{TC}{2\gamma}\sup_{(0,T)}\left[\|v\|_{\mathscr{H}}^{2p^--2} + \|v\|_{\mathscr{H}}^{2p^+-2}\right],\end{aligned}$$

where $\lambda_0 = \|u_1\|_2^2 + \|\triangle u_1\|_2^2 + \|\Delta u_0\|_2^2$, choosing $\gamma = \frac{1}{2T}$ such that

$$\|u\|_{\mathscr{H}}^2 \le \lambda_0 + T^2 C \sup_{(0,T)}\left[\|v\|_{\mathscr{H}}^{2p^--2} + \|v\|_{\mathscr{H}}^{2p^+-2}\right].$$

For any $v \in M_T$, by choosing M large enough so that

$$\|u\|_{\mathscr{H}}^2 \le \lambda_0 + 2T^2 C M^{2(p^+-1)} \le M^2,$$

and $T > 0$, sufficiently small so that

$$T \le \sqrt{\frac{M^2 - \lambda_0}{2CM^{2(p^+-1)}}},$$

we obtain $\|u\|_{\mathscr{H}} \le M$, which shows that $S(M_T) \subset M_T$.

Let $v_1, v_2 \in M_T, u_1 = S(v_1), u_2 = S(v_2), u = u_1 - u_2$, then u satisfies

$$\begin{cases} u_{tt} + \triangle^2 u + \triangle^2 u_{tt} - \int_0^t g(t-\tau)\triangle^2 u(\tau)d\tau \\ +|u_{1t}|^{m(x)-2}u_{1t} - |u_{2t}|^{m(x)-2}u_{2t} = |v_1|^{p(x)-2}v_1 - |v_2|^{p(x)-2}v_2, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = 0,\ u_t(x,0) = 0, & x \in \Omega. \end{cases}$$

Multiplying by u_t and integrating over $\Omega \times (0,t)$, we obtain

$$\begin{aligned} &\frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(1 - \int_0^t g(\tau)d\tau)\|\triangle u\|_2^2 + \frac{1}{2}\|\triangle u_t\|_2^2 + \frac{1}{2}(g \diamond \triangle u) \\ &+ \int_0^t \int_\Omega \left[|u_{1t}|^{m(x)-2}u_{1t} - |u_{2t}|^{m(x)-2}u_{2t}\right](u_{1t} - u_{2t})dxd\tau \le \int_0^t \int_\Omega (f(v_1) - f(v_2))u_t dxd\tau, \end{aligned} \tag{3.26}$$

where $f(v) = |v|^{p(x)-2}v$. From (1.6) and (3.24), we obtain

$$\frac{1}{2}\|u_t\|_2^2 + \frac{l}{2}\|\triangle u\|_2^2 + \frac{1}{2}\|\triangle u_t\|_2^2 + \frac{1}{2}(g \diamond \triangle u) \le \int_0^t \int_\Omega (f(v_1) - f(v_2))u_t dxd\tau. \tag{3.27}$$

Now, we evaluate

$$I = \int_\Omega |(f(v_1) - f(v_2))||u_t|dx = \int_\Omega |f'(\xi)||v||u_t|dx,$$

where $v = v_1 - v_2$ and $\xi = \alpha v_1 + (1-\alpha)v_2$, $0 \le \alpha \le 1$. Thanks to Young's inequality and Hölder's inequality, we have

$$\begin{aligned} I &\le \frac{\delta}{2}\|u_t\|_2^2 + \frac{1}{2\delta}\int_\Omega |f'(\xi)|^2|v|^2dx \\ &\le \frac{\delta}{2}\|u_t\|_2^2 + \frac{(p^+-1)^2}{2\delta}\int_\Omega |\xi|^{2(p(x)-2)}|v|^2dx \\ &\le \frac{\delta}{2}\|u_t\|_2^2 + \frac{(p^+-1)^2}{2\delta}\left(\int_\Omega |v|^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}}\left[\int_\Omega |\xi|^{n(p(x)-2)}dx\right]^{\frac{2}{n}} \\ &\le \frac{\delta}{2}\|u_t\|_2^2 + \frac{(p^+-1)^2}{2\delta}\left(\int_\Omega |v|^{\frac{2n}{n-2}}dx\right)^{\frac{n-2}{n}}\left[\left(\int_\Omega |\xi|^{n(p^+-2)}dx\right)^{\frac{2}{n}} + \left(\int_\Omega |\xi|^{n(p^--2)}dx\right)^{\frac{2}{n}}\right] \\ &\le \frac{\delta}{2}\|u_t\|_2^2 + \frac{(p^+-1)^2C}{2\delta}\|\Delta v\|_2^2\left[\|\triangle\xi\|_2^{2(p^+-2)} + \|\triangle\xi\|_2^{2(p^--2)}\right] \\ &\le \frac{\delta}{2}\|u_t\|_2^2 + \frac{(p^+-1)^2C}{2\delta}\|\Delta v\|_2^2\left(M^{2(p^+-2)} + M^{2(p^--2)}\right). \end{aligned} \tag{3.28}$$

Inserting (3.28) into (3.27), choosing δ small enough, we obtain

$$\|u\|_{\mathscr{H}}^2 \le \frac{(p^+-1)^2CT}{\delta}(M^{2(p^--2)} + M^{2(p^+-2)})\|v\|_{\mathscr{H}}^2,$$

taking T small enough so that $\frac{(p^+-1)^2CT}{\delta}(M^{2(p^--2)} + M^{2(p^+-2)}) < 1$, we conclude

$$\|u\|_{\mathscr{H}}^2 = \|S(v_1) - S(v_2)\|_{\mathscr{H}}^2 \le \|v_1 - v_2\|_{\mathscr{H}}^2,$$

thus, the contraction mapping principle ensures the existence of a weak solution to Problem (1.1). This completes the proof.

4. The blow-up the solution

In this section, we show that the solution to Problem (1.1) blows up in finite time when the initial energy lies in positive as well as nonpositive. For this task, we define

$$E(t) = \frac{1}{2}\|u_t\|_2^2 + \frac{1}{2}(1 - \int_0^t g(\tau)d\tau)\|\triangle u\|_2^2 + \frac{1}{2}\|\triangle u_t\|_2^2 + \frac{1}{2}(g \diamond \triangle u) - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx, \tag{4.1}$$

by the definition of $E(t)$, we also have

$$E'(t) = -\int_\Omega |u_t|^{m(x)}dx + \frac{1}{2}(g' \diamond \triangle u) - \frac{1}{2}g(t)\|\triangle u\|_2^2 \leq 0. \tag{4.2}$$

Now, we set

$$B_1 = \max\left\{1, \frac{B}{l^{\frac{1}{2}}}\right\}, \quad \lambda_1 = \left(B_1^2\right)^{\frac{-2}{p^- - 2}}, \quad E_1 = (\frac{1}{2} - \frac{1}{p^-})(B_1^2)^{\frac{-p^-}{p^- - 2}},$$

and

$$H(t) = E_2 - E(t), \tag{4.3}$$

where the constant $E_2 \in (E(0), E_1)$ will be discussed later, and B is the best constant of the Sobolev embedding $H_0^2(\Omega) \hookrightarrow L^{p(x)}(\Omega)$. It follows from (4.2) that

$$H'(t) = -E'(t) \geq 0, \tag{4.4}$$

and $H(t)$ is a non−decreasing function.

To prove Theorem 4.1, we need the following two lemmas:

Lemma 4.1. *Suppose that* (1.6) *holds and the exponents* $m(x)$ *and* $p(x)$ *satisfy condition* (1.4) *and* (1.5). *Assume further that*

$$E(0) < E_1 \ \text{ and } \ \lambda_1 < \lambda(0) = B_1^2 l\|\triangle u_0\|_2^2,$$

then there exists a constant $\lambda_2 > \lambda_1$ *such that*

$$B_1^2 l\|\triangle u\|_2^2 \geq \lambda_2, \ \ t \geq 0. \tag{4.5}$$

Proof. Using (1.6), (4.1), Lemma 2.4, and the embedding $H_0^2(\Omega) \hookrightarrow L^{p(x)}(\Omega)$, we find that

$$\begin{aligned}
E(t) &\geq \frac{1}{2}(1 - \int_0^t g(\tau)d\tau)\|\triangle u\|_2^2 - \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx \geq \frac{l}{2}\|\triangle u\|_2^2 - \frac{1}{p^-}\int_\Omega |u|^{p(x)}dx \\
&\geq \frac{l}{2}\|\triangle u\|_2^2 - \frac{1}{p^-}\max\left\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\right\} \\
&\geq \frac{l}{2}\|\triangle u\|_2^2 - \frac{1}{p^-}\max\left\{B^{p^-}\|\triangle u\|_2^{p^-}, B^{p^+}\|\triangle u\|_2^{p^+}\right\} \\
&\geq \frac{l}{2}\|\triangle u\|_2^2 - \frac{1}{p^-}\max\left\{B_1^{p^-} l^{\frac{p^-}{2}}\|\triangle u\|_2^{p^-}, B_1^{p^+} l^{\frac{p^+}{2}}\|\triangle u\|_2^{p^+}\right\} \\
&\geq \frac{1}{2B_1^2}\lambda - \frac{1}{p^-}\max\left\{\lambda^{\frac{p^-}{2}}, \lambda^{\frac{p^+}{2}}\right\} := G(\lambda),
\end{aligned} \tag{4.6}$$

where $\lambda := \lambda(t) = B_1^2 l\|\triangle u\|_2^2$. Analyzing directly the properties of $G(\lambda)$, we deduce that $G(\lambda)$ satisfies the following properties:

$$G'(\lambda) = \begin{cases} \frac{1}{2B_1^2} - \frac{p^+}{2p^-}\lambda^{\frac{p^+-2}{2}} < 0, & \lambda > 1, \\ \frac{1}{2B_1^2} - \frac{1}{2}\lambda^{\frac{p^--2}{2}}, & 0 < \lambda < 1, \end{cases}$$

$$G'_+(1) = \frac{1}{2B_1^2} - \frac{p^+}{2p^-} < 0, \quad G'_-(1) = \frac{1}{2B_1^2} - \frac{1}{2} < 0,$$

$$G'(\lambda_1) = 0, \quad 0 < \lambda_1 < 1.$$

It is easily verified that $G(\lambda)$ is strictly increasing for $0 < \lambda < \lambda_1$, strictly decreasing for $\lambda_1 < \lambda$, $G(\lambda) \to -\infty$ as $\lambda \to +\infty$, and $G(\lambda_1) = E_1$. Since $E(0) < E_1$, there exists a $\lambda_2 > \lambda_1$ such that $G(\lambda_2) = E(0)$. By (4.6), we see that $G(\lambda(0)) \leq E(0) = G(\lambda_2)$, which implies $\lambda(0) \geq \lambda_2$ since the condition $\lambda(0) > \lambda_1$. To prove (4.5), we suppose by contradiction that for some $t_0 > 0$, $\lambda_{t_0} = B_1^2 l\|\triangle u(t_0)\|_2^2 < \lambda_2$. The continuity of $B_1^2 l\|\triangle u\|_2^2$ illustrates that we could choose t_0 such that $\lambda_1 < \lambda_{t_0} < \lambda_2$, then we have $E(0) = G(\lambda_2) < G(\lambda_{t_0}) \leq E(t_0)$. This is a contradiction. The proof is completed.

Lemma 4.2. *Let the assumption in Lemma* 4.1 *be satisfied. For $t \in [0, T)$, we have*

$$0 < H(0) \leq H(t) \leq \frac{1}{p^-}\rho_{p(x)}(u).$$

Proof. (4.4) indicates that $H(t)$ is nondecreasing with respect to t, thus

$$H(t) \geq H(0) = E_2 - E(0) > 0, \quad \forall t \in [0, T).$$

It follows from (1.6), (4.1), and Lemma 4.1 that

$$\begin{aligned} H(t) &= E_2 - E(t) \\ &= E_2 - \frac{1}{2}\|u_t\|_2^2 - \frac{1}{2}(1 - \int_0^t g(\tau)d\tau)\|\triangle u\|_2^2 - \frac{1}{2}(g \diamond \triangle u) - \frac{1}{2}\|\triangle u_t\|_2^2 + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx \\ &\leq E_1 - \frac{l}{2}\|\triangle u\|_2^2 + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx \leq E_1 - \frac{1}{2B_1^2}\lambda_2 + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx \\ &\leq E_1 - \frac{1}{2B_1^2}\lambda_1 + \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx \leq \int_\Omega \frac{1}{p(x)}|u|^{p(x)}dx \leq \frac{1}{p^-}\rho_{p(x)}(u). \end{aligned}$$

The proof is completed.

Our blow-up result reads as follows:

Theorem 4.1. *Suppose that*

$$2 \leq m^- \leq m(x) \leq m^+ < p^- \leq p(x) \leq p^+ \leq \frac{2(n-3)}{n-4},$$

and

$$1 - l = \int_0^\infty g(\tau)d\tau < \frac{\frac{p^-}{2} - 1}{\frac{p^-}{2} - 1 + \frac{1}{2p^-}}, \tag{4.7}$$

hold, if the following conditions

$$E(0) < \frac{1}{2}(\frac{1}{2} - \frac{1}{p^-})\left(1 - \frac{1}{p^-(p^- - 2)}\frac{1 - l}{l}\right)(B_1^2)^{\frac{-p^-}{p^- - 2}} \quad and \quad \lambda_1 < \lambda(0) = B_1^2 l \|\triangle u_0\|_2^2,$$

are satisfied, then there exists $T^* < +\infty$ *such that*

$$\lim_{t \to T^{*-}} \left(\|u_t\|_2^2 + \|\triangle u_t\|_2^2 + \|\triangle u\|_2^2 + \|u\|_{p^+}^{p^+}\right) = +\infty. \tag{4.8}$$

Proof. Assume by contradiction that (4.8) does not hold true, then for $\forall T^* < +\infty$ and all $t \in [0, T^*]$, we get

$$\|u_t\|_2^2 + \|\triangle u_t\|_2^2 + \|\triangle u\|_2^2 + \|u\|_{p^+}^{p^+} \leq C_*, \tag{4.9}$$

where C_* is a positive constant.

Now, we define $L(t)$ as follows:

$$L(t) = H^{1-\alpha}(t) + \epsilon \int_\Omega u_t u dx + \epsilon \int_\Omega \triangle u_t \triangle u dx, \tag{4.10}$$

where $\varepsilon > 0$, small enough to be chosen later, and

$$0 \leq \alpha \leq \min\left\{\frac{p^- - m^+}{p^-(m^+ - 1)}, \frac{p^- - 2}{2p^-}\right\}.$$

The remaining proof will be divided into two steps.

Step 1: Estimate for L'(t). By taking the derivative of (4.10) and using (1.1), we obtain

$$\begin{aligned}
L'(t) =&(1-\alpha)H^{-\alpha}(t)\left[\int_\Omega |u_t|^{m(x)} dx - \frac{1}{2}(g' \diamond \triangle u) + \frac{1}{2}g(t)\|\triangle u\|_2^2\right] \\
&+ \epsilon\|u_t\|_2^2 + \epsilon \int_\Omega \triangle u_{tt} \triangle u dx + \epsilon\|\triangle u_t\|_2^2 \\
&- \epsilon\|\triangle u\|_2^2 + \epsilon \int_\Omega \int_0^t g(t-\tau)\triangle u(\tau) d\tau \triangle u dx \\
&- \epsilon \int_\Omega |u_t|^{m(x)-2} u_t u dx + \epsilon \int_\Omega |u|^{p(x)} dx - \epsilon \int_\Omega \triangle u_{tt} \triangle u dx \\
&\geq (1-\alpha)H^{-\alpha}(t) \int_\Omega |u_t|^{m(x)} dx + \epsilon\|u_t\|_2^2 - \epsilon\|\triangle u\|_2^2 \\
&+ \epsilon \int_\Omega \int_0^t g(t-\tau)\triangle u(\tau) d\tau \triangle u dx - \epsilon \int_\Omega |u_t|^{m(x)-2} u_t u dx + \epsilon \int_\Omega |u|^{p(x)} dx + \epsilon\|\Delta u_t\|_2^2,
\end{aligned}$$

applying Hölder's inequality and Young's inequality, we have

$$\begin{aligned}
&\epsilon \int_\Omega \int_0^t g(t-\tau)\Delta u(\tau)\Delta u(t) d\tau dx \\
=&\epsilon \int_\Omega \int_0^t g(t-\tau)\Delta u(t)(\Delta u(\tau) - \Delta u(t)) d\tau dx + \epsilon \int_0^t g(t-\tau) d\tau \|\Delta u\|_2^2 \\
\geq& -\epsilon \int_0^t g(t-\tau)\|\Delta u(\tau) - \Delta u(t)\|_2 \|\Delta u(t)\|_2 d\tau + \epsilon \int_0^t g(t-\tau) d\tau \|\Delta u\|_2^2 \\
\geq& -\epsilon \frac{p^-(1-\varepsilon_1)}{2}(g \diamond \Delta u) + \epsilon\left(1 - \frac{1}{2p^-(1-\varepsilon_1)}\right) \int_0^t g(\tau) d\tau \|\Delta u\|_2^2,
\end{aligned}$$

where $0<\varepsilon_1<\frac{p^--2}{p^-}$, then

$$
\begin{aligned}
L'(t) \geq& (1-\alpha)H^{-\alpha}(t)\int_\Omega |u_t|^{m(x)}dx + \epsilon\|u_t\|_2^2 - \epsilon\|\Delta u\|_2^2 \\
&+ \epsilon\|\Delta u_t\|_2^2 - \epsilon\int_\Omega |u_t|^{m(x)-2}u_t u dx + \epsilon\int_\Omega |u|^{p(x)}dx \\
&- \epsilon\frac{p^-(1-\varepsilon_1)}{2}(g\diamond\triangle u) + \epsilon\left(1-\frac{1}{2p^-(1-\varepsilon_1)}\right)\int_0^t g(\tau)d\tau\|\Delta u\|_2^2,
\end{aligned}
$$

rewriting (4.7) to $(\frac{p^-}{2}-1)l-\frac{1}{2p^-}(1-l)>0$, using (4.1) and (4.3) to substitute for $(g\diamond\triangle u)$, choosing $\varepsilon_1>0$ sufficiently small, we obtain

$$
\begin{aligned}
L'(t) \geq& (1-\alpha)H^{-\alpha}(t)\int_\Omega |u_t|^{m(x)}dx + \epsilon p^-(1-\varepsilon_1)H(t) + \left(\epsilon+\frac{\epsilon p^-(1-\varepsilon_1)}{2}\right)(\|u_t\|_2^2+\|\triangle u_t\|_2^2) \\
&+ \epsilon\left\{\left(\frac{p^-(1-\varepsilon_1)}{2}-1\right)(1-\int_0^t g(\tau)d\tau) - \frac{1}{2p^-(1-\varepsilon_1)}\int_0^t g(\tau)d\tau\right\}\|\triangle u\|_2^2 \\
&- \epsilon p^-(1-\varepsilon_1)E_2 - \epsilon\int_\Omega |u_t|^{m(x)-2}u_t u dx + \epsilon\varepsilon_1\int_\Omega |u|^{p(x)}dx \\
\geq& (1-\alpha)H^{-\alpha}(t)\int_\Omega |u_t|^{m(x)}dx + \epsilon p^-(1-\varepsilon_1)H(t) + \left(\epsilon+\frac{\epsilon p^-(1-\varepsilon_1)}{2}\right)(\|u_t\|_2^2+\|\triangle u_t\|_2^2) \\
&+ \epsilon\frac{\left\{\left(\frac{p^-(1-\varepsilon_1)}{2}-1\right)\frac{l}{2}-\frac{1}{2p^-(1-\varepsilon_1)}\frac{1-l}{2}\right\}}{l}\frac{\lambda_2}{B_1^2} - \epsilon p^-(1-\varepsilon_1)E_2 - \epsilon\int_\Omega |u_t|^{m(x)-2}u_t u dx \qquad (4.11) \\
&+ \epsilon\left\{\left(\frac{p^-(1-\varepsilon_1)}{2}-1\right)\frac{l}{2}-\frac{1}{2p^-(1-\varepsilon_1)}\frac{1-l}{2}\right\}\|\triangle u\|_2^2 + \epsilon\varepsilon_1\int_\Omega |u|^{p(x)}dx. \\
\geq& (1-\alpha)H^{-\alpha}(t)\int_\Omega |u_t|^{m(x)}dx + \epsilon p^-(1-\varepsilon_1)H(t) + \left(\epsilon+\frac{\epsilon p^-(1-\varepsilon_1)}{2}\right)(\|u_t\|_2^2+\|\triangle u_t\|_2^2) \\
&+ \epsilon\frac{\left\{\left(\frac{p^-(1-\varepsilon_1)}{2}-1\right)\frac{l}{2}-\frac{1}{2p^-(1-\varepsilon_1)}\frac{1-l}{2}\right\}}{l}(B_1^2)^{\frac{-p^-}{p^--2}} - \epsilon p^-(1-\varepsilon_1)E_2 - \epsilon\int_\Omega |u_t|^{m(x)-2}u_t u dx \\
&+ \epsilon\left\{\left(\frac{p^-(1-\varepsilon_1)}{2}-1\right)\frac{l}{2}-\frac{1}{2p^-(1-\varepsilon_1)}\frac{1-l}{2}\right\}\|\triangle u\|_2^2 + \epsilon\varepsilon_1\int_\Omega |u|^{p(x)}dx.
\end{aligned}
$$

Step 1.1: Estimate for $\epsilon\frac{\left\{\left(\frac{p^-(1-\varepsilon_1)}{2}-1\right)\frac{l}{2}-\frac{1}{2p^-(1-\varepsilon_1)}\frac{1-l}{2}\right\}}{l}(B_1^2)^{\frac{-p^-}{p^--2}}-\epsilon p^-(1-\varepsilon_1)E_2$. It follows from the condition in Theorem 3.1 that

$$
E(0) < \frac{1}{2}(\frac{1}{2}-\frac{1}{p^-})\left(1-\frac{1-l}{p^-(p^--2)l}\right)(B_1^2)^{\frac{-p^-}{p^--2}} = \frac{(\frac{p^-}{2}-1)\frac{l}{2}-\frac{1}{2p^-}\frac{(1-l)}{2}}{lp^-}(B_1^2)^{\frac{-p^-}{p^--2}} < E_1,
$$

here, we can take $\varepsilon_1>0$ sufficiently small and choose $E_2\in(E(0),E_1)$ sufficiently close to $E(0)$ such that

$$
\begin{aligned}
&\epsilon\frac{\left(\frac{p^-(1-\varepsilon_1)}{2}-1\right)\frac{l}{2}-\frac{1}{2p^-(1-\varepsilon_1)}\frac{(1-l)}{2}}{l}(B_1^2)^{\frac{-p^-}{p^--2}}-\epsilon(1-\varepsilon_1)p^-E_2\\
\geq&\epsilon\frac{\left(\frac{p^-(1-\varepsilon_1)}{2}-1\right)\frac{l}{2}-\frac{1}{2p^-(1-\varepsilon_1)}\frac{(1-l)}{2}}{l}(B_1^2)^{\frac{-p^-}{p^--2}}-\epsilon(1-\varepsilon_1)p^-\frac{\left(\frac{p^-}{2}-1\right)\frac{l}{2}-\frac{1}{2p^-}\frac{(1-l)}{2}}{lp^-}(B_1^2)^{\frac{-p^-}{p^--2}}\\
\geq&0.
\end{aligned}
\tag{4.12}
$$

Therefore, we obtain by combining (4.11) and (4.12),

$$
\begin{aligned}
L'(t)\geq&(1-\alpha)H^{-\alpha}(t)\int_\Omega|u_t|^{m(x)}dx+\epsilon p^-(1-\varepsilon_1)H(t)+\left(\epsilon+\frac{\epsilon p^-(1-\varepsilon_1)}{2}\right)(\|u_t\|_2^2+\|\triangle u_t\|_2^2)\\
&+\epsilon\varepsilon_1\int_\Omega|u|^{p(x)}dx+\epsilon\left\{\left(\frac{p^-(1-\varepsilon_1)}{2}-1\right)\frac{l}{2}-\frac{1}{2p^-(1-\varepsilon_1)}\frac{1-l}{2}\right\}\|\triangle u\|_2^2\\
&-\epsilon\int_\Omega|u_t|^{m(x)-2}u_tudx.
\end{aligned}
\tag{4.13}
$$

Step 1.2: Estimate for $-\epsilon\int_\Omega|u_t|^{m(x)-2}u_tudx$. Applying Young's inequality with $\varepsilon_2>1$, the embedding $L^{p(x)}(\Omega)\hookrightarrow L^{m(x)}(\Omega)$, Lemma 2.4 and Lemma 4.2, we easily have

$$
\begin{aligned}
&\left|\int_\Omega|u_t|^{m(x)-2}u_tudx\right|\leq\int_\Omega|u_t|^{m(x)-1}H^{-\alpha\frac{m(x)-1}{m(x)}}(t)H^{\alpha\frac{m(x)-1}{m(x)}}(t)|u|dx\\
\leq&\varepsilon_2H^{-\alpha}(t)\int_\Omega|u_t|^{m(x)}dx+\frac{1}{\varepsilon_2^{m^--1}}\int_\Omega|u|^{m(x)}H^{\alpha(m(x)-1)}(t)dx\\
\leq&\varepsilon_2H^{-\alpha}(t)\int_\Omega|u_t|^{m(x)}dx+\frac{2C_1^{\alpha(m^--m^+)}}{\varepsilon_2^{m^--1}}H^{\alpha(m^+-1)}(t)\int_\Omega|u|^{m(x)}dx\\
\leq&\varepsilon_2H^{-\alpha}(t)\int_\Omega|u_t|^{m(x)}dx+\frac{C_2}{\varepsilon_2^{m^--1}}H^{\alpha(m^+-1)}(t)\max\left\{\|u\|_{p(x)}^{m^+},\|u\|_{p(x)}^{m^-}\right\},
\end{aligned}
\tag{4.14}
$$

where $C_1=\min\{H(0),1\}$, $C_2=2(1+|\Omega|)^{m^+}C_1^{\alpha(m^--m^+)}$. Next, we have

$$
\begin{aligned}
\|u\|_{p(x)}^{m^+}\leq&\max\left\{\left(\int_\Omega|u|^{p(x)}dx\right)^{\frac{m^+}{p^+}},\left(\int_\Omega|u|^{p(x)}dx\right)^{\frac{m^+}{p^-}}\right\}\\
\leq&\max\left\{[p^-H(t)]^{\frac{m^+}{p^+}-\frac{m^+}{p^-}},1\right\}\left(\int_\Omega|u|^{p(x)}dx\right)^{\frac{m^+}{p^-}},
\end{aligned}
$$

and

$$
\|u\|_{p(x)}^{m^-}\leq\max\left\{[p^-H(t)]^{\frac{m^-}{p^+}-\frac{m^+}{p^-}},[p^-H(t)]^{\frac{m^--m^+}{p^-}}\right\}\left(\int_\Omega|u|^{p(x)}dx\right)^{\frac{m^+}{p^-}},
$$

which illustrate

$$
\max\left\{\|u\|_{p(x)}^{m^+},\|u\|_{p(x)}^{m^-}\right\}\leq C_3\left(\int_\Omega|u|^{p(x)}dx\right)^{\frac{m^+}{p^-}},
$$

where $C_3 = 2\min\{p^-H(0),1\}^{\frac{m^-}{p^+}-\frac{m^+}{p^-}}$. Recalling $0<\alpha\le\frac{p^--m^+}{p^-(m^+-1)}$ and Lemma 4.2, apparently,

$$\begin{aligned}
&H^{\alpha(m^+-1)}(t)\ \max\left\{\|u\|_{p(x)}^{m^+},\|u\|_{p(x)}^{m^-}\right\}\le C_3H^{\alpha(m^+-1)}(t)\left(\int_\Omega|u|^{p(x)}dx\right)^{\frac{m^+}{p^-}}\\
\le& C_3\frac{H^{\alpha(m^+-1)+\frac{m^+}{p^-}-1}(t)}{H^{\alpha(m^+-1)+\frac{m^+}{p^-}-1}(0)}H^{1-\frac{m^+}{p^-}}(t)H^{\alpha(m^+-1)+\frac{m^+}{p^-}-1}(0)\left(\int_\Omega|u|^{p(x)}dx\right)^{\frac{m^+}{p^-}}\\
\le& C_3(\frac{1}{p^-})^{1-\frac{m^+}{p^-}}\left(\int_\Omega|u|^{p(x)}dx\right)^{1-\frac{m^+}{p^-}}H^{\alpha(m^+-1)+\frac{m^+}{p^-}-1}(0)\left(\int_\Omega|u|^{p(x)}dx\right)^{\frac{m^+}{p^-}}\\
\le& C_3(\frac{1}{p^-})^{1-\frac{m^+}{p^-}}C_1^{\alpha(m^+-1)+\frac{m^+}{p^-}-1}\int_\Omega|u|^{p(x)}dx,
\end{aligned}\tag{4.15}$$

it follows from (4.13), (4.14), and (4.15) that

$$\begin{aligned}
L'(t)\ge&(1-\alpha-\epsilon\varepsilon_2)H^{-\alpha}(t)\int_\Omega|u_t|^{m(x)}dx+\left(\epsilon+\frac{\epsilon p^-(1-\varepsilon_1)}{2}\right)(\|u_t\|_2^2+\|\triangle u_t\|_2^2)\\
&+\epsilon(1-\varepsilon_1)p^-H(t)+\epsilon\left(\varepsilon_1-\frac{C_1^{\alpha(m^+-1)+\frac{m^+}{p^-}-1}C_2C_3(\frac{1}{p^-})^{1-\frac{m^+}{p^-}}}{\varepsilon_2^{m^--1}}\right)\int_\Omega|u|^{p(x)}dx\\
&+\epsilon\left\{\left(\frac{p^-(1-\varepsilon_1)}{2}-1\right)\frac{l}{2}-\frac{1}{2p^-(1-\varepsilon_1)}\frac{1-l}{2}\right\}\|\triangle u\|_2^2,
\end{aligned}$$

let us fix the constant ε_2 so that

$$\varepsilon_1>\frac{C_1^{\alpha(m^+-1)+\frac{m^+}{p^-}-1}C_2C_3(\frac{1}{p^-})^{1-\frac{m^+}{p^-}}}{\varepsilon_2^{m^--1}},$$

and then choose ϵ so small that $1-\alpha>\epsilon\varepsilon_1$. Therefore, we obtain

$$L'(t)\ge M_1\left(H(t)+\|\triangle u\|_2^2+\|u_t\|_2^2+\|\triangle u_t\|_2^2+\int_\Omega|u|^{p(x)}dx\right),\tag{4.16}$$

where

$$\begin{aligned}
M_1=\epsilon\min\Bigg\{&\left(1+\frac{p^-(1-\varepsilon_1)}{2}\right),(1-\varepsilon_1)p^-,\varepsilon_1-\frac{C_1^{\alpha(m^+-1)+\frac{m^+}{p^-}-1}C_2C_3(\frac{1}{p^-})^{1-\frac{m^+}{p^-}}}{\varepsilon_2^{m^--1}},\\
&\left(\frac{p^-(1-\varepsilon_1)}{2}-1\right)\frac{l}{2}-\frac{1}{2p^-(1-\varepsilon_1)}\frac{1-l}{2}\Bigg\}.
\end{aligned}$$

Inequalities (4.16) and Lemma 4.2 imply $L(t)\ge L(0)$. Therefore, for a sufficiently small ϵ, we have

$$L(0)=H^{1-\alpha}(0)+\epsilon\int_\Omega u_1u_0dx+\epsilon\int_\Omega\triangle u_1\triangle u_0dx>0.$$

Step 2: A differential inequality for L(t). Applying Hölder's inequality, Young's inequality and the embedding $L^{p(x)}(\Omega) \hookrightarrow L^2(\Omega)$, we easily obtain

$$\begin{aligned}\left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\alpha}} &\le (\|u_t\|_2 \|u\|_2)^{\frac{1}{1-\alpha}} \le (1+|\Omega|)^{\frac{1}{1-\alpha}} \|u_t\|_2^{\frac{1}{1-\alpha}} \|u\|_{p(x)}^{\frac{1}{1-\alpha}} \\ &\le \frac{(1+|\Omega|)^{\frac{1}{1-\alpha}}}{\mu} \|u_t\|_2^{\frac{1}{1-\alpha}\mu} + \frac{(1+|\Omega|)^{\frac{1}{1-\alpha}}}{\nu} \|u\|_{p(x)}^{\frac{1}{1-\alpha}\nu},\end{aligned} \tag{4.17}$$

where $\frac{1}{\mu} + \frac{1}{\nu} = 1$. Choosing $\mu = 2(1-\alpha) > 1$, then $\nu = \frac{2(1-\alpha)}{2(1-\alpha)-1}$, further, (4.17) can be rewritten as

$$\left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\alpha}} \le \frac{(1+|\Omega|)^{\frac{1}{1-\alpha}}}{\mu} \|u_t\|_2^2 + \frac{(1+|\Omega|)^{\frac{1}{1-\alpha}}}{\nu} \|u\|_{p(x)}^{\frac{2}{2(1-\alpha)-1}}, \tag{4.18}$$

recalling $0 < \alpha < \frac{p^- - 2}{2p^-}$, we obtain

$$\begin{aligned}\|u\|_{p(x)}^{\frac{2}{2(1-\alpha)-1}} &\le \max\left\{ (\int_\Omega |u|^{p(x)} dx)^{\frac{2}{p^-[2(1-\alpha)-1]}}, (\int_\Omega |u|^{p(x)} dx)^{\frac{2}{p^+[2(1-\alpha)-1]}} \right\} \\ &\le \left\{ [p^- H(t)]^{\frac{2-p^-[2(1-\alpha)-1]}{p^-[2(1-\alpha)-1]}}, [p^- H(t)]^{\frac{2-p^+[2(1-\alpha)-1]}{p^+[2(1-\alpha)-1]}} \right\} \int_\Omega |u|^{p(x)} dx \\ &\le C_4 \int_\Omega |u|^{p(x)} dx,\end{aligned} \tag{4.19}$$

with $C_4 = \min\{p^- H(0), 1\}^{\frac{2-p^+[2(1-\alpha)-1]}{p^+[2(1-\alpha)-1]}}$. Inserting (4.19) into (4.18), we obtain

$$\left|\int_\Omega u_t u dx\right|^{\frac{1}{1-\alpha}} \le \frac{(1+|\Omega|)^{\frac{1}{1-\alpha}}}{\mu} \|u_t\|_2^2 + \frac{(1+|\Omega|)^{\frac{1}{1-\alpha}}}{\nu} C_4 \int_\Omega |u|^{p(x)} dx. \tag{4.20}$$

We now estimate

$$\left|\int_\Omega \triangle u_t \triangle u dx\right|^{\frac{1}{1-\alpha}} \le \|\triangle u_t\|_2^{\frac{1}{1-\alpha}} \|\Delta u\|_2^{\frac{1}{1-\alpha}} \le C_*^{\frac{1}{1-\alpha}} \le \frac{C_*^{\frac{1}{1-\alpha}}}{H(0)} H(t), \tag{4.21}$$

therefore, combining (4.20) and (4.21), we obtain

$$\begin{aligned}L^{\frac{1}{1-\alpha}}(t) &= \left(H^{1-\alpha}(t) + \epsilon \int_\Omega u_t u dx + \epsilon \int_\Omega \triangle u_t \triangle u dx \right)^{\frac{1}{1-\alpha}} \\ &\le M_2 \left(H(t) + \|u_t\|_2^2 + \|\triangle u_t\|_2^2 + \|\triangle u\|_2^2 + \int_\Omega |u|^{p(x)} dx \right),\end{aligned} \tag{4.22}$$

where

$$M_2 = \max\left\{ 2^{\frac{1}{1-\alpha}} (2^{\frac{1}{1-\alpha}} + \epsilon^{\frac{1}{1-\alpha}} \frac{C_*^{\frac{1}{1-\alpha}}}{H(0)}),\ 2^{\frac{2}{1-\alpha}} \epsilon^{\frac{1}{1-\alpha}} \frac{(1+|\Omega|)\frac{1}{1-\alpha}}{\mu},\ 2^{\frac{2}{1-\alpha}} \epsilon^{\frac{1}{1-\alpha}} \frac{(1+|\Omega|)\frac{1}{1-\alpha}}{\nu} C_4 \right\}.$$

Combining (4.16) and (4.22), we arrive at

$$L'(t) \ge \frac{M_1}{M_2} L^{\frac{1}{1-\alpha}}(t),\ \forall t \ge 0. \tag{4.23}$$

A simple integration of (4.23) over $(0, t)$ yields

$$L^{\frac{\alpha}{1-\alpha}}(t) \geq \frac{1}{L^{\frac{\alpha}{\alpha-1}}(0) - \frac{M_1}{M_2}\frac{\alpha}{1-\alpha}t},$$

this shows that $L(t)$ blows up in finite time

$$T^* \leq \frac{M_2}{M_1}\frac{1-\alpha}{\alpha}L^{\frac{\alpha}{\alpha-1}}(0),$$

furthermore, one gets from (4.22) that

$$\lim_{t\to T^{*-}}\left(H(t) + \|u_t\|_2^2 + \|\triangle u_t\|_2^2 + \|\triangle u\|_2^2 + \int_\Omega |u|^{p(x)}dx\right) = +\infty,$$

it easily follows that

$$\int_\Omega |u|^{p(x)}dx \leq \int_{\{|u|\geq 1\}} |u|^{p^+}dx + \int_{\{|u|<1\}} |u|^{p^-}dx \leq \|u\|_{p^+}^{p^+} + |\Omega|,$$

and using Lemma 4.2, we obtain

$$\lim_{t\to T^{*-}}\left(\|u_t\|_2^2 + \|\triangle u_t\|_2^2 + \|\triangle u\|_2^2 + \|u\|_{p^+}^{p^+}\right) = +\infty,$$

this leads to a contradiction with (4.9). Thus, the solution to Problem (1.1) blows up in finite time.

Author contributions

Ying Chu: Methodology, Wring-original draft, Writing-review editing; Bo Wen and Libo Cheng: Methodology, Writing-original draft.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors express their heartfelt thanks to the editors and referees who have provided some important suggestions. This work was supported by Science and Technology Development Plan Project of Jilin Province, China (20240101307JC).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, *Siam. J. Appl. Math.*, **4** (2006), 1383–1406. https://doi.org/10.1137/050624522
2. R. Aboulaich, D. Meskine, A. Souissi, New diffusion models in image processing, *Comput. Math. Appl.*, **56** (2008), 874–882. https://doi.org/10.1016/j.camwa.2008.01.017
3. S. Lian, W. Gao, C. Cao, H. Yuan, Study of the solutions to a model porous medium equation with variable exponent of nonlinearity, *J. Math. Anal. Appl.*, **342** (2008), 27–38. https://doi.org/10.1016/j.jmaa.2007.11.046
4. S. Antontsev, S. Shmarev, Blow up of solutions to parabolic equations with nonstandard growth conditions, *J. Comput. Appl. Math.*, **234** (2010), 2633–2645. https://doi.org/10.1016/j.cam.2010.01.026
5. S. Antontsev, S. Shmarev, Evolution PDEs with Nonstandard Growth Conditions: Existence, Uniqueness, Localization, Blow-up, Atlantis Press Paris, 2015.
6. B. Tahir, K. Mohamed, B. Masoud, Global existence, blow-up and asymptotic behavior of solutions for a class of $p(x)$-choquard diffusion equations in R^N, *J. Math. Anal. Appl.*, **506** (2021), 125720. https://doi.org/10.1016/j.jmaa.2021.125720
7. S. A. Messaoudi, Blow up in a nonlinearly damped wave equation, *Math. Nachr.*, **231** (2001), 105–111.
8. S. A. Messaoudi, Blow-up of positive-initial-energy solutions of a nonlinear viscoelastic hyperbolic equation, *J. Math. Anal. Appl.*, **320** (2006), 902–915. https://doi.org/10.1016/j.jmaa.2005.07.022
9. H. T. Song, Blow up of arbitrarily positive initial energy solutions for a viscoelastic wave equation, *Nonlinear Anal-Real*, **26** (2015), 306–314. https://doi.org/10.1016/j.nonrwa.2015.05.015
10. H. T. Song, Global nonexistence of positive initial energy solutions for a viscoelastic wave equation, *Nonlinear Anal*, **125** (2015), 260–269. https://doi.org/10.1016/j.na.2015.05.015
11. M. M. Cavalcanti, C. Domingos, J. Ferreira, Existence and uniform decay for a non-linear viscoelastic equation with strong damping, *Math. Meth. Appl. Sci.*, **24** (2001), 1043–1053. https://doi.org/10.1002/mma.250
12. S. A. Messaoudi, A. A.Talahmeh, J. H. Al-Smail, Nonlinear damped wave equation: existence and blow-up, *Comput. Math. Appl.*, **74** (2017), 3024–3041. https://doi.org/10.1016/j.camwa.2017.07.048
13. S. H. Park, J. R. Kang, Blow-up of solutions for a viscoelastic wave equation with variable exponents, *Math. Meth. Appl. Sci.*, **42** (2019), 2083–2097. https://doi.org/10.1002/mma.5501
14. M. Alahyane, A. A. Talahmeh, S. A. Messaoudi, Theoretical and numerical study of the blow up in a nonlinear viscoelastic problem with variable exponents and arbitrary positive energy, *Acta. Math. Sci.*, **42** (2022), 141–154. https://doi.org/10.1007/s10473-022-0107-y
15. A. Ouaoua, W. Boughamsa, Well-posedness and stability results for a class of nonlinear fourth-order wave equation with variable-exponents, *J. Nonlinear. Anal. Appl.*, **14** (2023), 1769–1785. https://doi.org/10.22075/ijnaa.2022.27129.3507

16. H. Ding, J. Zhou, Blow-up for the Timoshenko-type equation with variable exponentss, *Nonlinear Anal-Real*, **71** (2023), 103801. https://doi.org/10.1016/j.nonrwa.2022.103801

17. M. Liao, B. Guo, X Zhu, Bounds for blow-up time to a viscoelastic hyperbolic equation of Kirchhoff type with variable sources, *Acta. Appl. Math.*, **170** (2020), 755–772. https://doi.org/10.1007/s10440-020-00357-3

18. M. Liao, Study of a viscoelastic wave equation with a strong damping and variable exponents, *Mediterr. J. Math.*, **18** (2021), 186. https://doi.org/10.1007/s00009-021-01826-1

19. W. Lian, V. Radulescu, R. Xu, Global well-posedness for a class of fourth-order nonlinear strongly damped wave equations, *Adv. Calc. Var.*, **14** (2021), 589–611. https://doi.org/10.1515/acv-2019-0039

20. Y. Luo, R. Xu, C. Yang, Global well-posedness for a class of semilinear hyperbolic equations with singular potentials on manifolds with conical singularities, *Cal. Var. Partial Dif.*, **61** (2022), 210. https://doi.org/10.1007/s00526-022-02316-2

21. M. Liao, Z. Tan, Behavior of solutions to a Petrovsky equation with damping and variable-exponent sources, *Sci. China. Math.*, **66** (2023), 285–302. https://doi.org/10.1007/s11425-021-1926-x

22. Y. Pang, V. Radulescu, R. Xu, Global Existence and Finite Time Blow-up for the m-Laplacian Parabolic Problem, *Acta. Math. Sin.*, **39** (2023), 1497–1524. https://doi.org/10.1007/s10114-023-1619-7

23. L. Diening, P. Harjulehto, P. Hästö, M. Růžička, Lebesgue and Sobolev spaces with variable exponents, Springer Berlin, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-18363-8

24. X. L. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424–446. https://doi.org/10.1006/jmaa.2000.7617

25. J. L. Lions, Quelques methodes de resolution des problemes aux limites non lineaires, Dunod, 1969.

AIMS Press