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*Research article*

# Some estimates of multilinear operators on tent spaces

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**Abstract:** Let $0 < \alpha < mn$ and $0 < r, q < \infty$. In this paper, we obtain the boundedness of some multilinear operators by establishing pointwise inequalities and applying extrapolation methods on tent spaces $T^q_r(\mathbb{R}^{n+1}_+)$, where these multilinear operators include multilinear Hardy–Littlewood maximal operator $\mathcal{M}$, multilinear fractional maximal operator $\mathcal{M}_\alpha$, multilinear Calderón–Zygmund operator $\mathcal{T}$, and multilinear fractional integral operator $\mathcal{I}_\alpha$. Therefore, the results of Auscher and Prisuelos–Arribas [Math. Z. **286** (2017), 1575–1604] are extended to the general case.

**Keywords:** multilinear maximal operator; multilinear Calderón–Zygmund operator; multilinear fractional integral operator; tent space
**Mathematics Subject Classification:** 42B20, 42B25, 42B35

## 1. Introduction and main results

Tent spaces were first introduced by Coifman, Meyer, and Stein in [1] and started with the use of Lusin area functionals on harmonic functions. These spaces were extensively used in the recent theory of Hardy spaces associated with operators [2] and played an important role in harmonic analysis, as evidenced in [3]. They also appeared in the study of maximal regularity operators arising from some linear or nonlinear partial differential equations [4].

Recently, Auscher and Prisuelos–Arribas [5] showed how extrapolation allows us to conclude the boundedness of some operators on tent spaces, such as the Hardy–Littlewood maximal operator, the Calderón–Zygmund operator, the Riesz potential, the fractional maximal function, and the Riesz transform of elliptic operator. Moreover, many interesting results were also extensively investigated on tent spaces; we refer the readers to see [6, 7] and therein references. The purpose of this paper is to extend the boundedness of multilinear operators on tent spaces.

In the 1970s, Coifman and Meyer were among the first to adopt the multilinear point of view in their study of certain singular integral operators (see; for example, [8, 9]). The study of multilinear operators is not motivated by a mere quest to generalize the theory of linear operators but rather by their natural appearance in harmonic analysis. A series of papers on this topic enrich this theory, see [10, 11] and so on. In particular, we want to understand how some multilinear operators act on tent spaces. We assume $\mathcal{T}$ be a multilinear operator and $F_1(x,t),\ldots,F_m(x,t)$ be measurable functions on $\mathbb{R}^{n+1}_+$ and define $\mathcal{T}$ by the setting, for any $(x,t)\in\mathbb{R}^{n+1}_+$,

$$\mathcal{T}(F_1,\ldots,F_m)(x,t) := T_t(F_1(\cdot,t),\ldots,F_m(\cdot,t))(x).$$

The following definition of tent spaces, $T^q_r(\mathbb{R}^{n+1}_+)$, can be found in [1].

**Definition 1.1.** *For a measurable function* $F:\mathbb{R}^{n+1}_+ := \mathbb{R}^n\times(0,\infty)\to\mathbb{C}$ *and* $0<r<\infty$*, let*

$$\mathcal{A}_r(F)(x) := \left(\int_0^\infty \int_{B(x,t)} |F(y,t)|^r \frac{dydt}{t^{n+1}}\right)^{\frac{1}{r}}, \quad x\in\mathbb{R}^n.$$

*Tent space* $T^q_r := T^q_r(\mathbb{R}^{n+1}_+)$, $0<q,r<\infty$, *is defined as the set of all measurable functions F such that* $\mathcal{A}_r(F)\in L^q(\mathbb{R}^n)$.

For an m-tuple locally integrable function $\vec{f}=(f_1,\ \ldots,\ f_m)$, the multilinear Hardy–Littlewood maximal operator $\mathcal{M}$ is defined by, for any $x\in\mathbb{R}^n$,

$$\mathcal{M}(\vec{f})(x) := \sup_{B\ni x}\prod_{j=1}^m \frac{1}{|B|}\int_B |f_j(y_j)|\,dy_j,$$

where the supremum is taken over all the balls $B$ containing $x$.

Lerner, Ombrosi, Pérez, Torres, and Trujillo–González [12] first introduced the multilinear Hardy–Littlewood maximal operator and further obtained some mapping properties of the multilinear Hardy–Littlewood maximal operator on weighted Lebesgue spaces. In 2014, Iida [13] proved the boundedness of the multilinear Hardy–Littlewood maximal operator on weighted Morrey spaces.

Our first result can be stated as follows:

**Theorem 1.1.** *Let* $\mathcal{M}$ *be the multilinear Hardy–Littlewood maximal operator. If* $1<r,r_1,\ldots,r_m<\infty$, $1<q,q_1,\ldots,q_m<\infty$, $\frac{1}{r}=\frac{1}{r_1}+\cdots+\frac{1}{r_m}$ *and* $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$*, then, there exists a constant* $C>0$*, such that for all* $\vec{F}=(F_1,\cdots,F_m)\in T^{q_1}_{r_1}\times\cdots\times T^{q_m}_{r_m}$,

$$\|\mathcal{M}(\vec{F})\|_{T^q_r}\le C\prod_{j=1}^m \|F_j\|_{T^{q_j}_{r_j}}.$$

Let $\mathcal{T}$ be a multilinear operator initially defined on the $m$-fold product of Schwartz spaces and taking values into the space of tempered distributions,

$$\mathcal{T}: S(\mathbb{R}^n)\times\cdots\times S(\mathbb{R}^n)\to S'(\mathbb{R}^n).$$

We say that $\mathcal{T}$ is the $m$-linear Calderón–Zygmund operator, if for some $1\le q_j<\infty$, it extends to a bounded multilinear operator from $L^{q_1}\times\cdots\times L^{q_m}$ to $L^q$, where $\frac{1}{q}=\frac{1}{q_1}+\cdots+\frac{1}{q_m}$, and if there exists a function $K$, defined off the diagonal $x=y_1=\cdots=y_m$ in $(\mathbb{R}^n)^{m+1}$, satisfying

$$\mathcal{T}(\vec{f})(x)=\int_{(\mathbb{R}^n)^m} K(x,y_1,\ldots,y_m)\,f_1(y_1)\ldots f_m(y_m)\,dy_1\ldots dy_m$$

for all $x \notin \bigcap_{j=1}^{m} \operatorname{supp} f_j$,

$$|K(y_0, y_1, \ldots, y_m)| \le \frac{A}{\left(\sum_{k,l=0}^{m} |y_k - y_l|\right)^{mn}} \tag{1.1}$$

and

$$\left|K\left(y_0, \ldots, y_j, \ldots, y_m\right) - K\left(y_0, \ldots, y_j', \ldots, y_m\right)\right| \le \frac{A\left|y_j - y_j'\right|^{\varepsilon}}{\left(\sum_{k,l=0}^{m} |y_k - y_l|\right)^{mn+\varepsilon}}, \tag{1.2}$$

for some $\varepsilon > 0$ and all $0 \le j \le m$, whenever $|y_j - y_j'| \le \frac{1}{2} \max_{0\le k\le m} |y_j - y_k|$.

In 2002, Grafakos and Torres [14] obtained the boundedness of the multilinear Calderón–Zygmund operator on Lebesgue spaces. Lerner, Ombrosi, Pérez, Torres, and Trujillo–González [12] developed a multiple weight theory and obtained that the multilinear Calderón–Zygmund operator is bounded on weighted Lebesgue spaces. In 2014, Grafakos, Liu, Maldonado, and Yang [15] extended these results to the framework of metric spaces.

Here is the second result we obtained.

**Theorem 1.2.** *Let $\mathcal{T}$ be the m-linear Calderón–Zygmund operator. If $1 < r, r_1, \ldots, r_m < \infty$, $1 < q, q_1, \ldots, q_m < \infty$, $\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m}$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, then, there exists a constant $C > 0$, such that, for all $\vec{F} = (F_1, \cdots, F_m) \in T_{r_1}^{q_1} \times \cdots \times T_{r_m}^{q_m}$,*

$$\|\mathcal{T}(\vec{F})\|_{T_r^q} \le C \prod_{j=1}^{m} \|F_j\|_{T_{r_j}^{q_j}}.$$

Let $0 < \alpha < mn$. For $\vec{f} = (f_1, \ldots, f_m)$, the multilinear fractional integral operator $\mathcal{I}_\alpha$ is defined by

$$\mathcal{I}_\alpha(\vec{f})(x) := \int_{\mathbb{R}^{mn}} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1| + \cdots + |x-y_m|)^{mn-\alpha}} dy_1 \cdots dy_m.$$

The associated multilinear fractional maximal operator $\mathcal{M}_\alpha$ is defined by

$$\mathcal{M}_\alpha(\vec{f})(x) := \sup_{B\ni x} \prod_{j=1}^{m} \frac{1}{|B|^{1-\frac{\alpha}{mn}}} \int_B \left|f_j(y_j)\right| dy_j,$$

where the supremum is taken over all the balls $B$ containing $x$.

In 1992, Grafakos [16] first studied the multilinear fractional integral operator and obtained the boundedness of the multilinear fractional integral operator on Lebesgue spaces. In 2015, Li, Moen and Sun [17] extended this result to weighted Lebesgue spaces.

Next, our third result is as follows:

**Theorem 1.3.** *Let $0 < \alpha < mn$ and $\mathcal{I}_\alpha$ be the multilinear fractional integral operator. If $1 < r, r_1, \ldots, r_m < \infty$, $1 < q, q_1, \ldots, q_m < \infty$, $\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, and $\frac{1}{q} - \frac{1}{p} = \frac{\alpha}{n}$, then, there exists a constant $C > 0$, such that for all $\vec{F} = (F_1, \cdots, F_m) \in T_{r_1}^{q_1} \times \cdots \times T_{r_m}^{q_m}$,*

$$\|\mathcal{I}_\alpha(\vec{F})\|_{T_r^p} \le C \prod_{j=1}^{m} \|F_j\|_{T_{r_j}^{q_j}}.$$

We note that the multilinear fractional maximal operator has the same boundedness as that of the multilinear fractional integral since the pointwise inequality $\mathcal{M}_\alpha(\vec{f}) \lesssim \mathcal{I}_\alpha(\vec{f})$. Thus, we have the following result:

**Corollary 1.1.** *Let* $0 < \alpha < mn$ *and* $\mathcal{M}_\alpha$ *be the multilinear fractional maximal operator. If* $1 < r, r_1, \ldots, r_m < \infty$, $1 < q, q_1, \ldots, q_m < \infty$, $\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, *and* $\frac{1}{q} - \frac{1}{p} = \frac{\alpha}{n}$, *then, there exists a constant* $C > 0$, *such that, for all* $\vec{F} = (F_1, \cdots, F_m) \in T^{q_1}_{r_1} \times \cdots \times T^{q_m}_{r_m}$,

$$\|\mathcal{M}_\alpha(\vec{F})\|_{T^p_r} \leq C \prod_{j=1}^{m} \|F_j\|_{T^{q_j}_{r_j}}.$$

We end this section by explaining some notations. We write $A \lesssim B$ to mean that there is a positive constant C such that $A \leq CB$, and $A \sim B$ to suggest that there exists a positive constant $C$ such that $B \lesssim A$ and $A \lesssim B$. $⨍_B f(x)dx$ represents the average $\frac{1}{|B|}\int_B f(x)dx$ of $f$ over the set $B$. The letter C will be used for positive constants independent of relevant variables that may change from one occurrence to another.

## 2. Preliminaries

We present some necessary lemmas and definitions in this section, which are very important to prove our main results.

Let us recall the definitions of $A_p$ weights and reverse Hölder classes (see, for example, [18]). In what follows, for $x \in \mathbb{R}^n$ and $r > 0$, the symbol $B = B(x, r)$ denotes balls in $\mathbb{R}^n$.

**Definition 2.1.** *An* $A_p$ *weight* $\omega$ *is a non-negative locally integrable function on* $\mathbb{R}^n$ *that satisfies, when* $p \in (1, \infty)$,

$$[\omega]_{A_p} := \sup_{B\subset\mathbb{R}^n} \left(⨍_B \omega(x)\, dx\right)\left(⨍_B \omega(x)^{1-p'}\, dx\right)^{p-1} < \infty,$$

*and when* $p = 1$,

$$[\omega]_{A_1} := \sup_{B\subset\mathbb{R}^n} \left(⨍_B \omega(x)\, dx\right)\|\omega^{-1}\|_{L^\infty(B)} < \infty,$$

*where the quantity* $[\omega]_{A_p}$ *is called the* $A_p$ *constant of the weight* $\omega$.

In addition, we also need the reverse Hölder classes.

**Definition 2.2.** *For* $s \in (1, \infty]$, *we define the reverse Hölder class* $RH_s$ *as the collection of all weights* $\omega$ *such that*

$$[\omega]_{RH_s} := \sup_B \left(⨍_B \omega(x)^s dx\right)^{\frac{1}{s}} \left(⨍_B \omega(x)dx\right)^{-1} < \infty,$$

*when* $s = \infty$, $\left(⨍_B \omega(x)^s dx\right)^{\frac{1}{s}}$ *is understood as* $\operatorname{esssup}_B \omega$. *Define* $RH_1 := \bigcup_{1<s\leq\infty} RH_s$. *Then we see that* $RH_1 = A_\infty$.

Some properties about $A_p$ weights and reverse Hölder classes used later are summed up in the following; see [19, 20].

**Proposition 2.1.** *(1) $A_1 \subset A_p \subset A_q$ for $1 \le p \le q < \infty$.*
*(2) $RH_\infty \subset RH_q \subset RH_p$ for $1 < p \le q \le \infty$.*
*(3) $A_\infty = \bigcup_{1\le p<\infty} A_p = \bigcup_{1<s\le\infty} RH_s$.*
*(4) If $1 < p < \infty$, $\omega \in A_p$ if and only if $\omega^{1-p'} \in A_{p'}$.*
*(5) If $\omega \in A_p$, $1 < p < \infty$, then there exists $q \in (1, p)$ such that $\omega \in A_q$.*
*(6) If $\omega \in RH_s$, $1 < s < \infty$, then there exists $r \in (s, \infty)$ such that $\omega \in RH_r$.*
*(7) If $\omega \in A_p \cap RH_\infty$, $1 \le p < \infty$, then there exists $r \in (1, \infty)$ such that $\omega^r \in A_p$.*

For any $\alpha \in (0, \infty)$ and $r \in (1, \infty)$, we define the operator $\mathcal{A}_r^\alpha$ by

$$\mathcal{A}_r^\alpha(F)(x) := \left(\int_0^\infty \int_{B(x,\alpha t)} |F(y,t)|^r \frac{dy\,dt}{t^{n+1}}\right)^{\frac{1}{r}}, \quad x \in \mathbb{R}^n.$$

The following lemma (change of angles) comes from [21, Proposition 3.2].

**Lemma 2.1.** *Let $0 < \alpha \le \beta < \infty$.*

*(i) For every $\omega \in A_q$, $1 \le q < \infty$, there holds*

$$\|\mathcal{A}_r^\beta(f)\|_{L^p(\omega)} \le \left(\frac{\beta}{\alpha}\right)^{\frac{nq}{p}} \|\mathcal{A}_r^\alpha(f)\|_{L^p(\omega)}, \quad \textit{for all} \quad 0 < p \le rq.$$

*(ii) For every $\omega \in RH_{s'}$, $1 \le s < \infty$, there holds*

$$\|\mathcal{A}_r^\alpha(f)\|_{L^p(\omega)} \le \left(\frac{\alpha}{\beta}\right)^{\frac{n}{sp}} \|\mathcal{A}_r^\beta(f)\|_{L^p(\omega)}, \quad \textit{for all} \quad \frac{r}{s} \le p < \infty.$$

For $m$ exponents $p_1, \dots, p_m$, we write $p$ for the number given by $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\vec{p}$ for the vector $\vec{p} = (p_1, \dots, p_m)$.

**Definition 2.3.** *Let $1 \le p_1, \dots, p_m < \infty$. Given $\vec{\omega} = (\omega_1, \dots, \omega_m)$, set*

$$\nu_{\vec{\omega}} = \prod_{j=1}^m \omega_j^{\frac{p}{p_j}}.$$

*We say that $\vec{\omega}$ satisfies the $A_{\vec{p}}$ condition if*

$$\sup_B \left(\fint_B \nu_{\vec{\omega}}(x)\,dx\right)^{\frac{1}{p}} \prod_{j=1}^m \left(\fint_B \omega_j(x)^{1-p_j'}\,dx\right)^{\frac{1}{p_j'}} < \infty,$$

*when $p = 1$, $(\fint_B \omega_j(x)^{1-p_j'}\,dx)^{\frac{1}{p_j'}}$ is understood as $(\inf_B \omega_j)^{-1}$.*

By Hölder's inequality, we can check that $\prod_{j=1}^m A_{p_j} \subset A_{\vec{p}}$. Moreover, it is proved in [12] that, for $1 \le p_1, \dots, p_m < \infty$, $\vec{\omega} \in A_{\vec{p}}$ if and only if

$$\begin{cases} \omega^{1-p_j'} \in A_{mp_j'}, & j = 1, \dots, m, \\ \nu_{\vec{\omega}} \in A_{mp}, \end{cases}$$

where the condition $\omega_j^{1-p'_j} \in A_{mp'_j}$ in the case $p_j = 1$ is understood as $\omega_j^{\frac{1}{m}} \in A_1$.

We are going to present the definition of the multilinear Muckenhoupt classes $A_{\vec{p},\vec{r}}$ introduced in [22]. Given $\vec{p} = (p_1, \ldots, p_m)$ with $1 \le p_1, \ldots, p_m \le \infty$ and $\vec{r} = (r_1, \ldots, r_{m+1})$ with $1 \le r_1, \ldots, r_{m+1} < \infty$, we say that $\vec{r} \le \vec{p}$ whenever

$$r_i \le p_i, i = 1, \ldots, m, \text{ and } r'_{m+1} \ge p, \text{ where } \frac{1}{p} := \frac{1}{p_1} + \cdots + \frac{1}{p_m}.$$

Analogously, we say that $\vec{r} < \vec{p}$ if $r_i < p_i$ for each $i = 1, \ldots, m$ and $r'_{m+1} > p$.

**Definition 2.4.** *Let $\vec{p} = (p_1, \ldots, p_m)$ with $1 \le p_1, \ldots, p_m < \infty$ and let $\vec{r} = (r_1, \ldots, r_{m+1})$ with $1 \le r_1, \ldots, r_{m+1} < \infty$ such that $\vec{r} \le \vec{p}$. Suppose that $\vec{\omega} = (\omega_1, \ldots, \omega_m)$ and each $\omega_j$ is a weight on $\mathbb{R}^n$. We say that $\vec{\omega} \in A_{\vec{p},\vec{r}}$ if*

$$[\vec{\omega}]_{A_{\vec{p},\vec{r}}} := \sup_B \left( \fint_B \omega(x)^{\frac{r'_{m+1}}{r'_{m+1}-p}} dx \right)^{\frac{1}{p} - \frac{1}{r'_{m+1}}} \prod_{j=1}^m \left( \fint_B \omega_j(x)^{\frac{r_j p_j}{r_j - p_j}} dx \right)^{\frac{1}{r_j} - \frac{1}{p_j}} < \infty, \tag{2.1}$$

*where $\omega = \prod_{j=1}^m \omega_j$ and the supremum is taken over all balls $B \subset \mathbb{R}^n$. When $p = r'_{m+1}$, the term corresponding to $\omega$ needs to be replaced by* $\operatorname{esssup}_B \omega$, *and, analogously, when $p_j = r_j$, $\omega$ needs to be replaced by $\left( \fint_B \omega(x)^p dx \right)^{\frac{1}{p}}$.*

**Remark 2.1.** *If we take $\vec{r} = (1, \ldots, 1, r_{m+1})$ with $\frac{1}{r'_{m+1}} = \frac{1}{p} - \frac{1}{q}$ in (2.1), then we see that $A_{\vec{p},\vec{r}}$ is the same as $A_{\vec{p},r}$.*

The following lemma is proved in [12, Theorem 3.7].

**Lemma 2.2.** *Let $1 < p_1, \ldots, p_m < \infty$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. If $\omega := \omega_1 \omega_2 \cdots \omega_m \in \prod_{j=1}^m A_{p_j} \subset A_{\vec{p}}$, then, there exists a constant $C > 0$, such that, for all $\vec{f} = (f_1, \cdots, f_m) \in L^{p_1}(\omega_1^{p_1}) \times \cdots \times L^{p_m}(\omega_m^{p_m})$,*

$$\|\mathcal{M}(\vec{f})\|_{L^p(\omega^p)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j^{p_j})}.$$

Let $\vec{f} = (f_1, \ldots, f_m) \in L^1_{\text{loc}}(\mathbb{R}^n) \times \cdots \times L^1_{\text{loc}}(\mathbb{R}^n)$. The centered multilinear maximal operator $\mathcal{M}_c$ is defined by

$$\mathcal{M}_c(\vec{f})(x) := \sup_{r>0} \prod_{j=1}^m \fint_{B(x,r)} |f_j(y_j)| \, dy_j.$$

**Remark 2.2.** *For $\vec{f} = (f_1, \cdots, f_m) \in L^1_{\text{loc}}(\mathbb{R}^n) \times \cdots \times L^1_{\text{loc}}(\mathbb{R}^n)$, it is easy to see that $\mathcal{M}_c(\vec{f}) \sim \mathcal{M}(\vec{f})$.*

For the centered multilinear maximal operator $\mathcal{M}_c$, we have the following pointwise inequality:

**Lemma 2.3.** *For all $x \in \mathbb{R}^n$, $t > 0$ and $r$, $r_1$, $\ldots$, $r_m \in (1, \infty)$ with $\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m}$, and all $\vec{f} = (f_1, \cdots, f_m) \in L^{r_1}_{\text{loc}}(\mathbb{R}^n) \times \cdots \times L^{r_m}_{\text{loc}}(\mathbb{R}^n)$, we have*

$$\left( \fint_{B(x,t)} |\mathcal{M}_c(\vec{f})(y)|^r \, dy \right)^{\frac{1}{r}} \lesssim \prod_{j=1}^m \left( \fint_{B(x,2t)} |f_j(y_j)|^{r_j} \, dy_j \right)^{\frac{1}{r_j}} + \mathcal{M}\left( \prod_{j=1}^m \fint_{B(\cdot,t)} |f_j(y_j)| \, dy_j \right)(x). \tag{2.2}$$

*Proof.* Fix $x \in \mathbb{R}^n$ and $t > 0$, and split the supremum into $0 < \tau \le t$ and $t < \tau$. Then,

$$\begin{aligned}\left(\fint_{B(x,t)} |\mathcal{M}(\vec{f})(y)|^r\,dy\right)^{\frac{1}{r}} &\le \left\{\fint_{B(x,t)} \left[\sup_{0<\tau\le t}\prod_{j=1}^m \fint_{B(y,\tau)} |f(y_j)|\,dy_j\right]^r dy\right\}^{\frac{1}{r}}\\ &\quad + \left\{\fint_{B(x,t)} \left[\sup_{\tau>t}\prod_{j=1}^m \fint_{B(y,\tau)} |f(y_j)|\,dy_j\right]^r dy\right\}^{\frac{1}{r}}\\ &=: I + II.\end{aligned}$$

For $I$, since $B(y,\tau) \subset B(x,2t)$ for $0 < \tau \le t$ and $y \in B(x,t)$, it follows that

$$\begin{aligned}I &\le \left\{\fint_{B(x,t)} \left[\sup_{0<\tau\le t}\prod_{j=1}^m \fint_{B(y,\tau)} |f(y_j)\chi_{B(x,2t)}(y_j)|\,dy_j\right]^r dy\right\}^{\frac{1}{r}}\\ &\le \left(\fint_{B(x,t)} |\mathcal{M}(\vec{f}\chi_{B(x,2t)})(y)|^r\,dy\right)^{\frac{1}{r}}\\ &\lesssim \prod_{j=1}^m \left(\fint_{B(x,2t)} |f(y_j)|^{r_j}\,dy_j\right)^{\frac{1}{r_j}},\end{aligned}$$

where $\vec{f}\chi_{B(x,2t)} := (f_1\chi_{B(x,2t)}, \dots, f_m\chi_{B(x,2t)})$ and the last inequality we have used is $\mathcal{M}:\ L^{r_1}(\mathbb{R}^n) \times \cdots \times L^{r_m}(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ (see [12, Theorem 3.7]).

As for $II$, we note that, for $\xi_j,\ y_j \in \mathbb{R}^n$, $\xi \in B(y_j,t)$ if and only if $y_j \in B(\xi_j,t)$. If $y_j \in B(y,\tau)$, $\xi_j \in B(y_j,t)$ and $\tau > t$, then $\xi_j \in B(y,2\tau)$. Besides, we observe that the fact that $x \in B(y,t)$ and $\tau > t$ implies that $x \in B(y,2\tau)$. Hence, applying Fubini's theorem, we have

$$\begin{aligned}II &= \left\{\fint_{B(x,t)} \left[\sup_{\tau>t}\prod_{j=1}^m \fint_{B(y,\tau)} |f(y_j)| \fint_{B(y_j,t)} d\xi_j\,dy_j\right]^r dy\right\}^{\frac{1}{r}}\\ &\le \left\{\fint_{B(x,t)} \left[\sup_{\tau>t}\prod_{j=1}^m \fint_{B(y,2\tau)} \fint_{B(\xi_j,t)} |f(y_j)|\,dy_j\,d\xi_j\right]^r dy\right\}^{\frac{1}{r}}\\ &\lesssim \mathcal{M}\left(\prod_{j=1}^m \fint_{B(\cdot,t)} |f(y_j)|\,dy_j\right)(x).\end{aligned}$$

Combining with estimates for $I$ and $II$, we complete the proof of Lemma 2.3.

Next, we establish the pointwise inequality for the multilinear Calderón–Zygmund operator $\mathcal{T}$.

**Lemma 2.4.** *Let $\mathcal{T}$ be an $m$-linear Calderón–Zygmund operator. For $1 < r,\ r_1,\ \dots,\ r_m < \infty$ with $\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m}$ and $\vec{f} = (f_1, \cdots, f_m) \in L^{r_1}(\mathbb{R}^n) \times \cdots \times L^{r_m}(\mathbb{R}^n)$. Then, for all $x \in \mathbb{R}^n$ and $t > 0$, we have*

$$\left(\fint_{B(x,t)} |\mathcal{T}(\vec{f})(y)|^r\,dy\right)^{\frac{1}{r}} \lesssim \prod_{j=1}^m \left(\fint_{B(x,2t)} |f_j(y)|^{r_j}\,dy_j\right)^{\frac{1}{r_j}} + \mathcal{T}_*(\vec{f})(x) + \mathcal{M}(\vec{f})(x),$$

*where*

$$\mathcal{T}_*(\vec{f})(x) := \sup_{\epsilon>0}\left|\int_{|x-y_1|^2+\cdots+|x-y_m|^2>\epsilon^2} K(x,y_1,\ldots,y_m)f_1(y_1)\cdots f_m(y_m)\,dy_1\cdots dy_m\right|.$$

*Proof.* Fix $x \in \mathbb{R}^n$ and $t > 0$, consider the ball $B(x, 2t)$, and write $f_j = f_j\chi_{B(x,2t)} + f_j\chi_{B(x,2t)^c} =: f_j^0 + f_j^\infty$, $j = 1, \ldots, m$. Then

$$\begin{aligned}\prod_{j=1}^m f_j(y_j) &= \prod_{j=1}^m \left(f_j^0(y_j) + f_j^\infty(y_j)\right) = \sum_{\alpha_1,\ldots,\alpha_m\in\{0,\infty\}} f_1^{\alpha_1}(y_1)\cdots f_j^{\alpha_m}(y_m)\\ &= \prod_{j=1}^m f_j^0(y_j) + \sum{}' f_1^{\alpha_1}(y_1)\cdots f_j^{\alpha_m}(y_m),\end{aligned}$$

where each term of $\sum'$ contains at least one $\alpha_j \neq 0$. Then, we write that

$$\begin{aligned}\left(⨍_{B(x,t)} |\mathcal{T}(\vec{f})(y)|^r\,dy\right)^{\frac{1}{r}} &\lesssim \left(⨍_{B(x,t)} |\mathcal{T}(f_1^0,\ldots,f_m^0)(y)|^r\,dy\right)^{\frac{1}{r}}\\ &\quad + \sum{}' \left(⨍_{B(x,t)} |\mathcal{T}(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(y)|^r\,dy\right)^{\frac{1}{r}}\\ &=: I + II.\end{aligned}$$

Since $\mathcal{T} : L^{r_1}(\mathbb{R}^n)\times\cdots\times L^{r_m}(\mathbb{R}^n) \to L^r(\mathbb{R}^n)$ (see [12, Corollary 3.9]), we obtain

$$I \lesssim \prod_{j=1}^m \left(⨍_{B(x,2t)} |f_j(y_j)|^{r_j}\,dy_j\right)^{\frac{1}{r_j}}.$$

As for $II$, we consider each term in the sum $\sum'$ and get that

$$\begin{aligned}&\left(⨍_{B(x,t)} |\mathcal{T}(f_1^{\alpha_1},\ldots,f_m^{\alpha_m})(y)|^r\,dy\right)^{\frac{1}{r}}\\ &= \left(⨍_{B(x,t)} \left|\int_{(\mathbb{R}^n)^m} K(y,y_1,\ldots,y_m)f_1^{\alpha_1}(y_1),\ldots,f_m^{\alpha_m}(y_m)\,d\vec{y}\right|^r dy\right)^{\frac{1}{r}}\\ &= \left(⨍_{B(x,t)} \left|\int_{(\mathbb{R}^n)^m} [K(y,y_1,\ldots,y_m) - K(x,y_1,\ldots,y_m)]\, f_1^{\alpha_1}(y_1),\ldots,f_m^{\alpha_m}(y_m)\,d\vec{y}\right|^r dy\right)^{\frac{1}{r}}\\ &\quad + \left(⨍_{B(x,t)} \left|\int_{(\mathbb{R}^n)^m} K(x,y_1,\ldots,y_m)f_1^{\alpha_1}(y_1),\ldots,f_m^{\alpha_m}(y_m)\,d\vec{y}\right|^r dy\right)^{\frac{1}{r}}\\ &=: II_1^{\alpha_1,\ldots,\alpha_m} + II_2^{\alpha_1,\ldots,\alpha_m}.\end{aligned}$$

Observe that there exists $\alpha_{j_0} \neq 0$ such that $|y_{j_0} - x| > 2t$. Then, for $y \in B(x,t)$, we have $|x - y| < t < \frac{1}{2}|y_{j_0} - x| \leq \frac{1}{2}\max_{j\in\{1,\ldots,m\}} |y_j - x|$. By (1.1) and the same process as the proof of [12, (4.4)], we conclude that

$$II_1^{\alpha_1,\ldots,\alpha_m} \lesssim \mathcal{M}(\vec{f})(x),$$

and using $|x-y_1|^2+\cdots+|x-y_m|^2>|y_{j_0}-x|^2>(2t)^2$, it is easy to see

$$II_2^{\alpha_1,\dots,\alpha_m}\lesssim \mathcal{T}_*(\vec{f})(x).$$

Thus, we complete the proof of Lemma 2.4.

For the operator $\mathcal{T}^*$, we need the following lemma (see [15, Theorem 4.16]).

**Lemma 2.5.** *Let* $1<p_1,\dots,p_m<1$ *and* $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$. *If* $\omega:=\omega_1\omega_2\cdots\omega_m\in\prod_{j=1}^m A_{p_j}\subset A_{\vec{p}}$, *then, there exists a constant* $C>0$, *such that, for all* $\vec{f}=(f_1,\cdots,f_m)\in L^{p_1}(\omega_1^{p_1})\times\cdots\times L^{p_m}(\omega_m^{p_m})$,

$$\|\mathcal{T}^*(\vec{f})\|_{L^p(\omega^p)}\le C\prod_{j=1}^m\|f_j\|_{L^{p_1}(\omega_j^{p_j})}.$$

**Lemma 2.6.** *Let* $0<t<\infty, 0<\alpha<mn, \frac{1}{r}=\frac{1}{s}-\frac{\alpha}{n}$ *and* $\frac{1}{s}=\frac{1}{s_1}+\cdots+\frac{1}{s_m}$. *Then, for any* $x\in\mathbb{R}^n$, *if* $1<s_1,\dots,s_m<\infty$ *and* $\vec{f}=(f_1,\cdots,f_m)\in L^{s_1}_{\text{loc}}(\mathbb{R}^n)\times\cdots\times L^{s_m}_{\text{loc}}(\mathbb{R}^n)$, *we have*

$$\begin{aligned}\left(\fint_{B(x,t)}\left|\mathcal{I}_\alpha(\vec{f})(y)\right|^r dy\right)^{\frac{1}{r}}\lesssim & t^{n(\frac{1}{s}-\frac{1}{r})}\prod_{j=1}^m\left(\fint_{B(x,5t)}\left|f_j(y_j)\right|^{s_j}dy_j\right)^{\frac{1}{s_j}}\\ &+\mathcal{I}_\alpha\left[\left(\fint_{B(\cdot,t)}|f_1(z_1)|^{s_1}dz_1\right)^{\frac{1}{s_1}},\cdots,\left(\fint_{B(\cdot,t)}|f_m(z_m)|^{s_m}dz_m\right)^{\frac{1}{s_m}}\right](x).\end{aligned}$$

*Proof.* For the sake of brevity, we just consider $m=2$. For each $x\in\mathbb{R}^n$ and $t>0$, we split $f_j=f_j\chi_{B(x,5t)}+f_j\chi_{B(x,5t)^c}=:f_j^0+f_j^\infty$, $j=1,2$. Then,

$$\begin{aligned}\left(\fint_{B(x,t)}\left|\mathcal{I}_\alpha(\vec{f})(y)\right|^r dy\right)^{\frac{1}{r}}\le&\left(\fint_{B(x,t)}\left|\mathcal{I}_\alpha\left(f_1^0,f_2^0\right)(y)\right|^r dy\right)^{\frac{1}{r}}+\left(\fint_{B(x,t)}\left|\mathcal{I}_\alpha\left(f_1^0,f_2^\infty\right)(y)\right|^r dy\right)^{\frac{1}{r}}\\ &+\left(\fint_{B(x,t)}\left|\mathcal{I}_\alpha\left(f_1^\infty,f_2^0\right)(y)\right|^r dy\right)^{\frac{1}{r}}+\left(\fint_{B(x,t)}\left|\mathcal{I}_\alpha\left(f_1^\infty,f_2^\infty\right)(y)\right|^r dy\right)^{\frac{1}{r}}\\ =:&\ \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.\end{aligned}$$

Since $\mathcal{I}_\alpha: L^{s_1}(\mathbb{R}^n)\times L^{s_2}(\mathbb{R}^n)\to L^r(\mathbb{R}^n)$ in [23], we deduce that

$$\mathrm{I}\lesssim t^{n(\frac{1}{s}-\frac{1}{r})}\prod_{j=1}^m\left(\fint_{B(x,5t)}\left|f_j\left(y_j\right)\right|^{s_j}dy_j\right)^{\frac{1}{s_j}}.$$

As for II, for any $z_2\in\mathbb{R}^n\backslash B(x,5t)$, if $\xi\in B(z_2,t)$ and $y\in B(x,t)$, then $\max\{|y-z_2|,|x-\xi|\}\ge 4t$ and $\frac{|y-z_2|}{2}\le|x-\xi|\le 2|y-z_2|$. Furthermore, for any $z_1\in B(x,5t)$ and $\eta\in B(z_1,t)$,

$$\frac{|y-z_1|+|y-z_2|}{5}\le|x-\eta|+|x-\xi|\le 5\left(|y-z_1|+|y-z_2|\right).$$

Then, by Fubini's theorem, we have

$$\mathcal{I}_\alpha\left(\left|f_1^0\right|,\left|f_2^\infty\right|\right)(y)$$

$$= \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left|f_1^0(z_1)\right|\left|f_2^\infty(z_2)\right|}{(|y-z_1|+|y-z_2|)^{2n-\alpha}} \fint_{B(z_2,t)} d\xi \fint_{B(x_1,t)} d\eta dz_1 dz_2$$
$$= \frac{1}{|B(x,t)|^2}\int_{\{\xi:|\xi-x|>4t\}}\int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^2} \frac{\left|f_1^0(z_1)\right|\left|f_2^\infty(z_2)\right|\chi_{B(\xi,t)}(z_2)\chi_{B(\eta,t)}(z_1)}{(|y-z_1|+|y-z_2|)^{2n-\alpha}} dz_1dz_2d\xi d\eta$$
$$\lesssim \frac{1}{|B(x,t)|^2}\int_{\{\xi:|\xi-x|>4t\}}\int_{\mathbb{R}^n}\int_{(\mathbb{R}^n)^2} \frac{\left|f_1^0(z_1)\right|\left|f_2^\infty(z_2)\right|\chi_{B(\xi,t)}(z_2)\chi_{B(\eta,t)}(z_1)}{(|x-\eta|+|x-\xi|)^{2n-\alpha}} dz_1dz_2d\xi d\eta$$
$$\lesssim \mathcal{I}_\alpha\left(\fint_{B(\cdot,t)}|f_1(\eta)|\,d\eta, \fint_{B(\cdot,t)}|f_2(\xi)|\,d\xi\right)(x).$$

Therefore,

$$\mathrm{II} \lesssim \mathcal{I}_\alpha\left(\fint_{B(\cdot,t)}|f_1(\eta)|\,d\eta, \fint_{B(\cdot,t)}|f_2(\xi)|\,d\xi\right)(x).$$

Similarly, we also have

$$\mathrm{III} \lesssim \mathcal{I}_\alpha\left(\fint_{B(\cdot,t)}|f_1(\eta)|\,d\eta, \fint_{B(\cdot,t)}|f_2(\xi)|\,d\xi\right)(x).$$

It remains to estimate IV. For any $z_1, z_2 \in \mathbb{R}^n \backslash B(x,5t)$, if $\xi \in B(z_2,t), \eta \in B(z_1,t)$, and $y \in B(x,t)$, then we have

$$\max\{|y-z_1|, |y-z_2|, |x-\xi|, |x-\eta|\} \geq 4t$$

and

$$\frac{|y-z_1|}{2} \leq |x-\eta| \leq 2|y-z_1|, \quad \frac{|y-z_2|}{2} \leq |x-\xi| \leq 2|y-z_2|.$$

Furthermore, we get

$$\frac{|y-z_1|+|y-z_2|}{2} \leq |x-\eta|+|x-\xi| \leq 2(|y-z_1|+|y-z_2|).$$

Then, by Fubini's theorem, we obtain

$$\mathcal{I}_\alpha\left(\left|f_1^\infty\right|, \left|f_2^\infty\right|\right)(y)$$
$$= \int_{\mathbb{R}^n\times\mathbb{R}^n} \frac{\left|f_1^\infty(z_1)\right|\left|f_2^\infty(z_2)\right|}{(|y-z_1|+|y-z_2|)^{2n-\alpha}} \fint_{B(z_2,t)} d\xi \fint_{B(x_1,t)} d\eta dz_1 dz_2$$
$$= \frac{1}{|B(x,t)|^2}\int_{\{\xi:|\xi-x|>4t\}}\int_{\{\eta:|\eta-x|>4t\}}\int_{(\mathbb{R}^n)^2} \frac{\left|f_1^\infty(z_1)\right|\left|f_2^\infty(z_2)\right|\chi_{B(\xi,t)}(z_2)\chi_{B(\eta,t)}(z_1)}{(|y-z_1|+|y-z_2|)^{2n-\alpha}} dz_1dz_2d\xi d\eta$$
$$\lesssim \frac{1}{|B(x,t)|^2}\int_{\{\xi:|\xi-x|>4t\}}\int_{\{\eta:|\eta-x|>4t\}}\int_{(\mathbb{R}^n)^2} \frac{\left|f_1^\infty(z_1)\right|\left|f_2^\infty(z_2)\right|\chi_{B(\xi,t)}(z_2)\chi_{B(\eta,t)}(z_1)}{(|x-\eta|+|x-\xi|)^{2n-\alpha}} dz_1dz_2d\xi d\eta$$
$$\lesssim \mathcal{I}_\alpha\left(\fint_{B(\cdot,t)}|f_1(\eta)|\,d\eta, \fint_{B(\cdot,t)}|f_2(\xi)|\,d\xi\right)(x).$$

Thus, we have

$$\mathrm{IV} \leq \mathcal{I}_\alpha\left(\fint_{B(\cdot,t)}|f_1(\eta)|\,d\eta, \fint_{B(\cdot,t)}|f_2(\xi)|\,d\xi\right)(x).$$

Combining with the estimates for I-IV, we finish the proof of Lemma 2.6.

We need the following extrapolation theorem, which was introduced by Cruz-Uribe and Martell in [24].

**Lemma 2.7.** *Given $m \geq 1$, let $\mathcal{F}$ be a family of extrapolation $(m+1)$-tuples. For each $j, 1 \leq j \leq m$, suppose we have parameters $r_j^-$ and $r_j^+$, and an exponent $p_j \in (0, \infty)$, $0 \leq r_j^- \leq p_j \leq r_j^+ \leq \infty$, such that given any collection of weights $\omega_1, \ldots, \omega_m$ with $\omega_j^{p_j} \in A_{\frac{p_j}{r_j^-}} \cap RH_{\left(\frac{r_j^+}{p_j}\right)'}$ and $\omega = \omega_1 \cdots \omega_m$, we have the inequality*

$$\|f\|_{L^p(\omega^p)} \leq C \prod_{j=1}^{m} \left\|f_j\right\|_{L^{p_j}\left(\omega_j^{p_j}\right)},$$

*for all $(f, f_1, \ldots, f_m) \in \mathcal{F}$ such that $\|f\|_{L^p(\omega^p)} < \infty$, where $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j}$ and $C$ depends on $n, p_j, \left[\omega_j\right]_{\frac{p_j}{r_j^-}}, \left[\omega_j\right]_{RH_{\left(\frac{r_j^+}{q_j}\right)'}}$. Then for all exponents $q_j, r_j^- < q_j < r_j^+$, all weights $\omega_j^{q_j} \in A_{\frac{q_j}{r_j^-}} \cap RH_{\left(\frac{r_j^+}{q_j}\right)'}$ and $\omega = \omega_1 \cdots \omega_m$,*

$$\|f\|_{L^q(\omega^q)} \leq C \prod_{j=1}^{m} \left\|f_j\right\|_{L^{q_j}\left(\omega_j^{q_j}\right)},$$

*for all $(f, f_1, \ldots, f_m) \in \mathcal{F}$ such that $\|f\|_{L^q(\omega^q)} < \infty$, where $\frac{1}{q} = \sum_{j=1}^m \frac{1}{q_j}$ and $C$ depends on $n$, $p_j$, $q_j$, $\left[\omega_j\right]_{\frac{q_j}{r_j^-}}, \left[\omega_j\right]_{RH_{\left(\frac{r_j^+}{q_j}\right)'}}$.*

In order to show Theorem 1.3, we also need the off-diagonal extrapolation theorem (see [25, Theorem 4.5]).

**Lemma 2.8.** *Let $\mathcal{F}$ be a collection of $(m+1)$-tuples of non-negative functions. Let $\vec{r} = (r_1, \ldots, r_{m+1})$ with $1 \leq r_1, \ldots, r_{m+1} < \infty$. Assume that there exists $p_* \in (0, \infty)$ and $\vec{p} = (p_1, \ldots, p_m)$ with $\vec{r} \leq \vec{p}$ and $\frac{1}{p_*} \leq \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ such that for all $\vec{\omega} = (\omega_1, \ldots, \omega_m) \in A_{\vec{p}, \vec{r}}$,*

$$\|f\|_{L^{p_*}(\omega^{p_*})} \leq C \prod_{j=1}^{m} \left\|f_j\right\|_{L^{p_j}(\omega_j^{p_j})}, \quad (f, f_1, \ldots, f_m) \in \mathcal{F},$$

*where $\omega = \prod_{j=1}^m \omega_j$. Then, for all $q_* \in (0, \infty)$, for all $\vec{q} = (q_1, \ldots, q_m)$ with $\vec{r} < \vec{q}$, $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$ and $\frac{1}{q} - \frac{1}{q_*} = \frac{1}{p} - \frac{1}{p_*}$, and for all $\vec{v} = (v_1, \ldots, v_m) \in A_{\vec{q}, \vec{r}}$,*

$$\|f\|_{L^{q_*}(v^{q_*})} \leq C \prod_{j=1}^{m} \left\|f_j\right\|_{L^{q_j}(v_j^{q_j})}, \quad (f, f_1, \ldots, f_m) \in \mathcal{F},$$

*where $v = \prod_{j=1}^m v_j$.*

## 3. Proofs of Theorems 1.1–1.3

Now, we begin to prove Theorems 1.1–1.3 in this position.

*Proof of Theorem 1.1.* For $r, r_1, \ldots, r_m \in (1,\infty)$ and $\frac{1}{r} = \frac{1}{r_1} + \cdots + \frac{1}{r_m}$, we shall prove that for any $\omega = \omega_1\omega_2\cdots\omega_m \in \prod_{j=1}^m A_{r_j} \subset A_{\vec{r}}$ with $\omega_j^{r_j} \in A_{\frac{r_j}{r_j^-}} \cap RH_{\left(\frac{r_j^+}{r_j}\right)'}$ and $\vec{F} = (F_1, \cdots, F_m) \in T_{r_1}^{r_1} \times \cdots \times T_{r_m}^{r_m}$,

$$\left(\int_{\mathbb{R}^n} |\mathcal{A}_r(\mathcal{M}(\vec{F}))(x)|^r \omega^r(x)\,dx\right)^{\frac{1}{r}} \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^n} |\mathcal{A}_{r_j}(F_j)(x)|^{r_j} \omega_j^{r_j}(x)\,dx\right)^{\frac{1}{r_j}}. \tag{3.1}$$

From this and Lemma 2.7, it follows that, for any $q, q_1, \ldots, q_m \in (1,\infty)$ and $\frac{1}{q} = \frac{1}{q_1} + \cdots + \frac{1}{q_m}$, $\omega = \omega_1\omega_2\cdots\omega_m \in \prod_{i=1}^m A_{q_j} \subset A_{\vec{q}}$ with $\omega_j^{q_j} \in A_{\frac{q_j}{r_j^-}} \cap RH_{\left(\frac{r_j^+}{q_j}\right)'}$ and $\vec{F} = (F_1, \cdots, F_m) \in T_{r_1}^{q_1} \times \cdots \times T_{r_m}^{q_m}$,

$$\left(\int_{\mathbb{R}^n} |\mathcal{A}_r(\mathcal{M}(\vec{F}))(x)|^q \omega^q(x)\,dx\right)^{\frac{1}{q}} \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^n} |\mathcal{A}_r(F_j)(x)|^{q_j} \omega_j^{q_j}(x)\,dx\right)^{\frac{1}{q_j}}.$$

In particular, we take $\omega_j \equiv 1$ as above, then, for all $\vec{F} = (F_1, \cdots, F_m) \in (T_{r_1}^{q_1}, \cdots, T_{r_m}^{q_m})$,

$$\|\mathcal{M}(\vec{F})\|_{T_r^q} = \left(\int_{\mathbb{R}^n} |\mathcal{A}_r(\mathcal{M}(\vec{F}))(x)|^q\,dx\right)^{\frac{1}{q}} \leq C \prod_{j=1}^m \left(\int_{\mathbb{R}^n} |\mathcal{A}_{r_j}(F_j)(x)|^{q_j}\,dx\right)^{\frac{1}{q_j}} = C \prod_{j=1}^m \|F_j\|_{T_{r_j}^{q_j}}. \tag{3.2}$$

Since the set of compactly supported functions in $T_{r_j}^{r_j} \cap T_{r_j}^{q_j}$ is dense in $T_{r_j}^{q_j}$, by the monotone convergence theorem, we conclude that (3.2) holds true for functions $F_j \in T_{r_j}^{q_j},\ j = 1, \ldots, m$.

Therefore, to finish the proof of Theorem 1.1, it just remains to show (3.1). This follows from (2.2) applied to $f_j = F_j(\cdot, t),\ j = 1, \ldots, m$, and Lemma 2.2. Using Hölder's inequality and Lemma 2.3, for all $\omega = \omega_1\omega_2\cdots\omega_m \in \prod_{i=1}^m A_{p_i} \subset A_{\vec{p}}$, we have

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n} |\mathcal{A}_r(\mathcal{M}(\vec{F}))(x)|^r \omega^r(x)\,dx\right)^{\frac{1}{r}} = \left(\int_{\mathbb{R}^n}\int_0^\infty \fint_{B(x,t)} |\mathcal{M}(\vec{F}(\cdot,t))(y)|^r \frac{dy\,dt}{t}\omega^r(x)\,dx\right)^{\frac{1}{r}} \\
&\lesssim \left[\int_{\mathbb{R}^n}\int_0^\infty \prod_{j=1}^m \left(\fint_{B(x,2t)} |F_j(y_j,t)|^{r_j}\,dy_j\right)^{\frac{r}{r_j}} \frac{dt}{t}\omega^r(x)\,dx\right]^{\frac{1}{r}} \\
&\quad + \left[\int_{\mathbb{R}^n}\int_0^\infty \left|\mathcal{M}\left(\prod_{j=1}^m \fint_{B(\cdot,t)} |F(y_j,t)|\,dy_j\right)(x)\right|^r \frac{dt}{t}\omega^r(x)\,dx\right]^{\frac{1}{r}} \\
&\lesssim \left[\int_{\mathbb{R}^n}\prod_{j=1}^m \left(\int_0^\infty \fint_{B(x,2t)} |F_j(y_j,t)|^{r_j}\,dy_j\,\frac{dt}{t}\right)^{\frac{r}{r_j}} \omega^r(x)\,dx\right]^{\frac{1}{r}} \\
&\quad + \left[\int_0^\infty\int_{\mathbb{R}^n} \left|\mathcal{M}\left(\prod_{j=1}^m \fint_{B(\cdot,t)} |F(y_j,t)|\,dy_j\right)(x)\right|^r \omega^r(x)\,dx\,\frac{dt}{t}\right]^{\frac{1}{r}} \\
&\lesssim \left[\int_{\mathbb{R}^n}\prod_{j=1}^m |\mathcal{A}_{r_j}(F_j)(x)|^r \omega^r(x)\,dx\right]^{\frac{1}{r}} + \left[\int_0^\infty\int_{\mathbb{R}^n} \left|\prod_{j=1}^m \fint_{B(x,t)} |F(y_j,t)|\,dy_j\right|^r \omega^r(x)\,dx\,\frac{dt}{t}\right]^{\frac{1}{r}}
\end{aligned}$$

$$\lesssim \prod_{j=1}^{m}\left[\int_{\mathbb{R}^n}|\mathcal{A}_{r_j}(F_j)(x)|^{r_j}\omega_j^{r_j}(x)\,dx\right]^{\frac{1}{r_j}}+\left[\int_0^\infty\int_{\mathbb{R}^n}\prod_{j=1}^{m}\left(\fint_{B(x,t)}|F(y_j,t)|^{r_j}\,dy_j\right)^{\frac{r}{r_j}}\omega^r(x)\,dx\,\frac{dt}{t}\right]^{\frac{1}{r}}$$
$$\lesssim \prod_{j=1}^{m}\left[\int_{\mathbb{R}^n}|\mathcal{A}_{r_j}(F_j)(x)|^{r_j}\omega_j(x)\,dx\right]^{\frac{1}{r_j}}.$$

Thus, it finishes the proof of Theorem 1.1.

*Proof of Theorem 1.2.* We consider $\vec{F}=(F_1,\cdots,F_m)\in T_{r_1}^{q_1}\times\cdots\times T_{r_m}^{q_m}$ so that for almost every $t\in(0,\infty)$ and $j=1,2,\ldots,m$, $F(\cdot,t)\in L^{r_j}(\mathbb{R}^n)$ and all calculations make sense. For $\omega=\omega_1\omega_2\cdots\omega_m\in\prod_{j=1}^m A_{p_i}\subset A_{\vec{p}}$ with $\omega_j^{r_j}\in A_{\frac{r_j}{r_j^-}}\cap RH_{\left(\frac{r_j^+}{r_j}\right)'}$, by Lemma 2.4, Fubini's theorem, Lemma 2.2, and Lemma 2.5, we use Hölder's inequality and Lemma 2.1 and deduce that

$$\left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,t)}|\mathcal{T}(\vec{F}(\cdot,t))(y)|^r\,\frac{dy\,dt}{t^{n+1}}\omega^r(x)\,dx\right)^{\frac{1}{r}}$$
$$\lesssim\left(\int_{\mathbb{R}^n}\int_0^\infty\prod_{j=1}^{m}\left(\fint_{B(x,2t)}|F_j(y_j,t)|^{r_j}\,dy_j\right)^{\frac{r}{r_j}}\frac{dt}{t}\omega^r(x)\,dx\right)^{\frac{1}{r}}$$
$$+\left(\int_{\mathbb{R}^n}\int_0^\infty|\mathcal{T}^*(\vec{F}(\cdot,t))(x)|^r\,\frac{dt}{t}\omega^r(x)\,dx\right)^{\frac{1}{r}}+\left(\int_{\mathbb{R}^n}\int_0^\infty|\mathcal{M}(\vec{F}(\cdot,t))(x)|^r\,\frac{dt}{t}\omega^r(x)\,dx\right)^{\frac{1}{r}}$$
$$\lesssim\prod_{j=1}^{m}\left(\int_{\mathbb{R}^n}\int_0^\infty\fint_{B(x,2t)}|F_j(y_j,t)|^{r_j}\,dy_j\,\frac{dt}{t}\omega_j^{r_j}(x)\,dx\right)^{\frac{1}{r_j}}$$
$$+\left(\int_0^\infty\prod_{j=1}^{m}\left(\int_{\mathbb{R}^n}|F_j(y_j,t)|^{r_j}\omega_j^{r_j}(y_j)\,dy_j\right)^{\frac{r}{r_j}}\frac{dt}{t}\right)^{\frac{1}{r}}$$
$$\lesssim\prod_{j=1}^{m}\left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,t)}|F_j(y,t)|^{r_j}\,\frac{dy_j\,dt}{t^{n+1}}\omega_j^{r_j}(x)\,dx\right)^{\frac{1}{r_j}}+\prod_{j=1}^{m}\left(\int_{\mathbb{R}^n}\int_0^\infty|F_j(y_j,t)|^{r_j}\,\frac{dt}{t}\omega_j^{r_j}(y_j)\,dy_j\right)^{\frac{1}{r_j}}$$
$$\lesssim\prod_{j=1}^{m}\left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,t)}|F_j(y,t)|^{r_j}\,\frac{dy_j\,dt}{t^{n+1}}\omega_j^{r_j}(x)\,dx\right)^{\frac{1}{r_j}},$$

where the last inequality comes from the proof of [26, Proposition 2.3] in the case $\omega_j^{r_j}\in A_{r_j}\cap RH_\infty,\ j=1,\ldots,m$. Therefore, for all $\omega_j^{r_j}\in A_{r_j}\cap RH_\infty,\ j=1,\ldots,m$ and $\vec{F}=(F_1,\cdots,F_m)\in T_{r_1}^{r_1}\times\cdots\times T_{r_m}^{r_m}$,

$$\left(\int_{\mathbb{R}^n}\int_0^\infty\int_{B(x,t)}|\mathcal{T}(\vec{F})(y)|^r\,\frac{dy\,dt}{t^{n+1}}\omega^r(x)\,dx\right)^{\frac{1}{r}}\lesssim\prod_{j=1}^{m}\left(\int_{\mathbb{R}^n}\int_{B(x,t)}\int_0^\infty|F_j(y_j,t)|^{r_j}\,\frac{dy_j\,dt}{t^{n+1}}\omega_j^{r_j}(x)\,dx\right)^{\frac{1}{r_j}}.\quad(3.3)$$

Note now that in view of (3.3), we can apply Lemma 2.7, for $r_j^-=1$, $r_j^+=\infty$, $p_j=r_j$. Then, for all $1<q_j<\infty$, $\omega_j^{q_j}\in A_{q_j}\cap RH_\infty$ and $\vec{F}=(F_1,\cdots,F_m)\in T_{r_1}^{q_1}\times\cdots\times T_{r_m}^{q_m}$, we obtain

$$\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\int_{B(x,t)}|\mathcal{T}(\vec{F})(y)|^r\,\frac{dy\,dt}{t^{n+1}}\right)^{\frac{q}{r}}\omega^q(x)\,dx\right)^{\frac{1}{q}}$$

$$\lesssim \prod_{j=1}^{m}\left(\int_{\mathbb{R}^n}\left(\int_0^\infty \int_{B(x,t)} |F_j(y_j,t)|^{r_j}\,\frac{dy_j\,dt}{t^{n+1}}\right)^{\frac{q_j}{r_j}} \omega_j^{q_j}(x)\,dx\right)^{\frac{1}{q_j}}.$$

Thus, for $\omega_j^{q_j} \equiv 1 \in A_{q_j} \cap RH_\infty$ and $\vec{F} = (F_1, \cdots, F_m) \in T_{r_1}^{q_1} \times \cdots \times T_{r_m}^{q_m}$ as above, we have

$$\|\mathcal{T}(\vec{F})\|_{T_r^q} \lesssim \prod_{j=1}^{m} \|F_j\|_{T_{r_j}^{q_j}}.$$

This completes the proof of Theorem 1.2.

*Proof of Theorem 1.3.* Let $\vec{F} = (F_1, \cdots, F_m) \in T_{r_1}^{q_1} \times \cdots \times T_{r_m}^{q_m}$ and $\omega(x) = \omega_1(x)\omega_2(x)\cdots\omega_m(x)$. Taking $s_j = \frac{nr_j}{\alpha r_j + n}(j = 1, 2, \cdots, m)$ in Lemma 2.6, we can deduce that

$$\begin{aligned}
\left\|\mathcal{I}_\alpha(\vec{F})\right\|_{T_r^p} \lesssim& \left(\int_{\mathbb{R}^n}\left(\int_0^\infty t^{n(\frac{r}{s}-1)}\prod_{j=1}^{m}\left(\fint_{B(x,5t)} |F_j(y_j,t)|^{s_j} dy_j\right)^{\frac{r}{s_j}}\frac{dt}{t}\right)^{\frac{p}{r}} dx\right)^{\frac{1}{p}}\\
&+\left(\int_{\mathbb{R}^n}\left(\int_0^\infty \left(\mathcal{I}_\alpha\left(\fint_{B(\cdot,t)} |F_1(y_1,t)|dy_1, \cdots, \fint_{B(\cdot,t)} |F_m(y_m,t)|dy_m\right)(x)\right)^r \frac{dt}{t}\right)^{\frac{p}{r}} dx\right)^{\frac{1}{p}}\\
=:& I + II.
\end{aligned}$$

Since $r_j > s_j$, using Jensen's inequality, Hölder's inequality, Lemma 2.1, and [27, Theorem 2.19] for $s_1 = \frac{1}{m}(\frac{1}{r} - \frac{1}{s})$, $s_0 = 0, p_0 = q_j, p_1 = p_j$, and $q = r_j$, we have

$$\begin{aligned}
I &\lesssim \prod_{j=1}^{m}\left(\int_{\mathbb{R}^n}\left(\int_0^\infty \int_{B(x,5t)} \left|t^{\frac{n}{m}(\frac{1}{s}-\frac{1}{r})} F_j(y_j,t)\right|^{r_j} \frac{dy_j dt}{t^{n+1}}\right)^{\frac{p_j}{r_j}} dx\right)^{\frac{1}{p_j}}\\
&\lesssim \prod_{j=1}^{m}\left(\int_{\mathbb{R}^n}\left(\int_0^\infty \int_{B(x,5t)} |F(y,t)|^{r_j} \frac{dydt}{t^{n+1}}\right)^{\frac{q_j}{r_j}} dx\right)^{\frac{1}{q_j}}\\
&\lesssim \prod_{j=1}^{m} \|F_j\|_{T_{r_j}^{q_j}}.
\end{aligned}$$

Finally, to estimate $II$, we shall proceed by extrapolation. Since $0 < \alpha < nm$, $1 < s_1, \ldots, s_m < \infty$, $\frac{1}{s} = \frac{1}{s_1} + \cdots + \frac{1}{s_m}$ with $\frac{1}{r} = \frac{1}{s} - \frac{\alpha}{n}$ and $\vec{\omega} \in A_{\vec{s},r}$, we have the fact that $\mathcal{I}_\alpha : L^{s_1}(\omega_1^{s_1}) \times \cdots \times L^{s_m}(\omega_m^{s_m}) \to L^r(\omega^r)$ in [28, Theorem 2.3]. Applying this fact, Lemma 2.6, Minkowski's integral inequality, Hölder's inequality, we obtain

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n}\int_0^\infty \left(\mathcal{I}_\alpha\left(\fint_{B(\cdot,t)} |F_1(y_1,t)|dy_1, \cdots, \fint_{B(\cdot,t)} |F_m(y_m,t)|dy_m\right)(x)\right)^r \frac{dt}{t}\omega_1(x)^r\cdots\omega_m(x)^r dx\right)^{\frac{1}{r}}\\
&\lesssim \left(\int_0^\infty \int_{\mathbb{R}^n} \left(\mathcal{I}_\alpha\left(\fint_{B(\cdot,t)} |F_1(y_1,t)|dy_1, \cdots, \fint_{B(\cdot,t)} |F_m(y_m,t)|dy_m\right)(x)\right)^r \omega(x)^r dx\frac{dt}{t}\right)^{\frac{1}{r}}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left( \int_0^\infty \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \left( \fint_{B(x,t)} |F_j(y_j,t)| dy_j \right)^{s_j} \omega_j^{s_j}(x) dx \right)^{\frac{r}{s_j}} \frac{dt}{t} \right)^{\frac{1}{r}} \\
&\lesssim \prod_{j=1}^m \left( \int_0^\infty \left( \int_{\mathbb{R}^n} \left( \fint_{B(x,t)} |F_j(y_j,t)| dy_j \right)^{s_j} \omega_j^{s_j}(x) dx \right)^{\frac{r_j}{s_j}} \frac{dt}{t} \right)^{\frac{1}{r_j}} \\
&\lesssim \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \left( \fint_{B(x,t)} |F_j(y_j,t)| dy_j \right)^{r_j} \frac{dt}{t} \right)^{\frac{s_j}{r_j}} \omega_j^{s_j}(x) dx \right)^{\frac{1}{s_j}} \\
&\lesssim \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \fint_{B(x,t)} |F_j(y_j,t)|^{r_j} dy_j \frac{dt}{t} \right)^{\frac{s_j}{r_j}} \omega_j^{s_j}(x) dx \right)^{\frac{1}{s_j}}.
\end{aligned}
$$

Then, since $1 < \vartheta < q < \infty$ and $1 < s < r < \infty$ with $\frac{1}{s} - \frac{1}{r} = \frac{1}{q} - \frac{1}{\vartheta}$, applying Lemma 2.6, we have that, for all $\vec{v} \in A_{\vec{q},r}$, and $\vec{F} = (F_1, \cdots, F_m) \in T_{r_1}^{q_1} \times \cdots \times T_{r_m}^{q_m}$,

$$
\begin{aligned}
&\left( \int_{\mathbb{R}^n} \int_0^\infty \left( \mathcal{I}_\alpha \left( \fint_{B(\cdot,t)} |F_1(y_1,t)| dy_1, \cdots, \fint_{B(\cdot,t)} |F_m(y_m,t)| dy_m \right)(x) \right)^r \frac{dt}{t} \omega^r(x) dx \right)^{\frac{1}{r}} \\
&\lesssim \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \fint_{B(x,t)} |F_j(y_j,t)|^{r_j} dy_j \frac{dt}{t} \right)^{\frac{q_j}{r_j}} v_j^{q_j}(x) dx \right)^{\frac{1}{q_j}}.
\end{aligned}
$$

In particular for $v_j \equiv 1$, we have that $\vec{v} \in A_{\vec{q},r}$. Thus,

$$
II \lesssim \prod_{j=1}^m \left( \int_{\mathbb{R}^n} \left( \int_0^\infty \fint_{B(x,t)} |F_j(y_j,t)|^{r_j} dy_j \frac{dt}{t} \right)^{\frac{q_j}{r_j}} v_j^{q_j}(x) dx \right)^{\frac{1}{q_j}} = \prod_{j=1}^m \|F_j\|_{T_{r_j}^{q_j}}.
$$

The proof of Theorem 1.3 is finished.

## Author contributions

Heng Yang: methodology, writing-original draft, writing-review & editing; Jiang Zhou: writing-review & editing, supervision, formal analysis.

## Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.


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## Conflict of interest

The authors declare that there is no conflict of interest.

## References


1. R. Coifman, Y. Meyer, E. M. Stein, Un nouveal espace fonctionnel adapte a l'etude des operateurs definis par des integrales singulieres, in *Harmonic Analysis. Lecture Notes in Mathematics*, (eds. G. Mauceri, F. Ricci, G. Weiss), Springer, Berlin, Heidelberg, (1983), 1–15. https://doi.org/10.1007/BFb0069149
2. S. Hofmann, S. Mayboroda, Hardy and BMO spaces to divergence form elliptic operators, *Math. Ann.*, **344** (2009), 37–116. https://doi.org/10.1007/s00208-008-0295-3
3. R. Coifman, Y. Meyer, E. M. Stein, Some new function spaces and their applications to harmonic analysis, *J. Funct. Anal.*, **62** (1985), 304–335. https://doi.org/10.1016/0022-1236(85)90007-2
4. P. Auscher, C. Kriegler, S. Monniaux, P. Portal, Singular integral operators on tent spaces, *J. Evol. Equ.*, **12** (2012), 741–765. https://doi.org/10.1007/s00028-012-0152-4
5. P. Auscher, C. Prisuelos-Arribas, Tent space boundedness via extrapolation, *Math. Z.*, **286** (2017), 1575–1604. https://doi.org/10.1007/s00209-016-1814-7
6. J. Cao, D. Chang, Z. Fu, D. Yang, Real interpolation of weighted tent spaces, *Appl. Anal.*, **95** (2016), 2415–2443. https://doi.org/10.1080/00036811.2015.1091924
7. C. Cascante, J. M. Ortega, Imbedding potentials in tent spaces, *J. Funct. Anal.*, **198** (2003), 106–141. https://doi.org/10.1016/s0022-1236(02)00087-3
8. R. Coifman, Y. Meyer, On commutators of singular integral and bilinear singular integrals, *Trans. Amer. Math. Soc.*, **212** (1975), 315–331. https://doi.org/10.1090/S0002-9947-1975-0380244-8
9. R. Coifman, Y. Meyer, Commutateurs d'intégrales singulières et opérateurs multilinéaires, *Ann. Inst. Fourier (Grenoble)*, **28** (1978), 177–202. https://doi.org/10.5802/aif.708
10. G. Lu, Multiple weighted estimates for bilinear Calderón-Zygmund operator and its commutator on non-homogeneous spaces, *Bull. Sci. Math.*, **187** (2023), 103311. https://doi.org/10.1016/j.bulsci.2023.103311
11. S. He, J. Zhang, Endpoint estimates for multilinear fractional maximal operators, *Bull. Korean Math. Soc.*, **57** (2020), 383–391. https://doi.org/10.4134/BKMS.b190269
12. A. Lerner, S. Ombrosi, C. Pérez, R. H. Torres, R. Trujillo-González, New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory, *Adv. Math.*, **220** (2009), 1222–1264. https://doi.org/10.1016/j.aim.2008.10.014
13. T. Iida, The boundedness of the Hardy-Littlewood maximal operator and multilinear maximal operator in weighted Morrey type spaces, *J. Funct. Space.* **2014** (2014), 648251. https://doi.org/10.1155/2014/648251
14. L. Grafakos, R. H. Torres, Multilinear Calderón-Zygmund theory, *Adv. Math.*, **65** (2002), 124–164. https://doi.org/10.1006/aima.2001.2028

15. L. Grafakos, L. Liu, D. Maldonado, D. Yang, Multilinear analysis on metric spaces, *Diss. Math.*, **497** (2014), 1–121. https://doi.org/10.4064/dm497-0-1
16. L. Grafakos, On multilinear fractional integrals, *Studia Math.*, **102** (1992), 49–56. https://doi.org/10.4064/sm-102-1-49-56
17. K. Li, K. Moen, W. Sun, Sharp weighted inequalities for multilinear fractional maximal operators and fractional integrals, *Math. Nachr.*, **288** (2015), 619–632. https://doi.org/10.1002/mana.201300287
18. L. Grafakos, Classical Fourier Analysis, in *Graduate Texts in Mathematics.* 3 Eds., Springer, New York, 2014. https://doi.org/10.1007/978-1-4939-1194-3
19. J. García-Cuerva, J. Rubio de Francia, Weighted norm inequalities and related topics, Elsevier, 1985.
20. L. Grafakos, Modern Fourier Analysis, in *Graduate Texts in Mathematics.* 3 Eds., Springer, New York, 2014. https://doi.org/10.1007/978-1-4939-1230-8
21. J. M. Martell, C. Prisuelos-Arribas, Weighted Hardy spaces associated with elliptic operators Part: I. Weighted norm inequalities for conical square functions, *Trans. Am. Math. Soc.*, **369** (2017), 4193–4233. https://doi.org/10.1090/tran/6768
22. K. Li, J. M. Martell, S. Ombrosi, Extrapolation for multilinear Muckenhoupt classes and applications to the bilinear Hilbert transform, *Adv. Math.*, **373** (2020), 107286. https://doi.org/10.1016/j.aim.2020.107286
23. K. Moen, Weighted inequalities for multilinear fractional integral operators. *Collect. Math.*, **60** (2009), 213–238. https://doi.org/10.1007/bf03191210
24. D. Cruz-Uribe, J. M. Martell, Limited range multilinear extrapolation with applications to the bilinear Hilbert transform, *Math. Ann.*, **371** (2018), 615–653. https://doi.org/10.1007/s00208-018-1640-9
25. M. Cao, A. Olivo, K. Yabuta, Extrapolation for multilinear compact operators and applications, *Trans. Amer. Math. Soc.*, **375** (2022), 5011–5070. https://doi.org/10.1090/tran/8645
26. P. Auscher, S. Hofmann, J. M. Martell, Vertical versus conical square functions, *Trans. Am. Math. Soc.*, **364** (2012), 5469–5489. https://doi.org/10.1090/S0002-9947-2012-05668-6
27. A. Amenta, Interpolation and embeddings of weighted tent spaces, *J. Fourier. Anal. Appl.*, **24** (2018), 108–140. https://doi.org/10.1007/s00041-017-9521-2
28. X. Chen, Q. Xue, Weighted estimates for a class of multilinear fractional type operators, *J. Math. Anal. Appl.*, **362** (2010), 355–373. https://doi.org/10.1016/j.jmaa.2009.08.022


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