Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 16(3): 655–699.
DOI: 10.3934/cam.2024030
Received: 10 January 2024
Revised: 07 March 2024
Accepted: 01 August 2024
Published: 13 September 2024

Research article

Asymptotic behavior of a viscous incompressible fluid flow in a fractal network of branching tubes

Haifa El Jarroudi and Mustapha El Jarroudi*

Department of Mathematics, Abdelmalek Essaâdi University FST Tanger, Tangier, Morocco

* **Correspondence:** Email: meljarroudi@uae.ac.ma; eljarroudi@hotmail.com.

Abstract: We considered a viscous incompressible fluid flow in a varying bounded domain consisting of branching thin cylindrical tubes whose axes are line segments that form a network of pre-fractal curves constituting an approximation of the Sierpinski gasket. We supposed that the fluid flow is driven by volumic forces and governed by Stokes equations with boundary conditions for the velocity and the pressure on the wall of the tubes and inner continuity conditions for the normal velocity on the interfaces between the junction zones and the rest of the pipes. We constructed local perturbations, related to boundary layers in the junction zones, from solutions of Leray problems in semi-infinite cylinders representing the rescaled junctions. Using Γ-convergence methods, we studied the asymptotic behavior of the fluid as the radius of the tubes tends to zero and the sequence of the pre-fractal curves converges in the Hausdorff metric to the Sierpinski gasket. Based on the constructed local perturbations, we derived, according to a critical parameter related to a typical Reynolds number of the flow in the junction zones, three effective flow models in the Sierpinski gasket, consisting of a singular Brinkman flow, a singular Darcy flow, and a flow with constant velocity.

Keywords: viscous incompressible fluid flow; fractal branching tubes; asymptotic behavior; critical parameter; effective flow models
Mathematics Subject Classification: 35B40, 28A80, 35J75

1. Introduction

Fluid flows in branching tubes are common in many biological and industrial applications such as physiological branching flows and flows through pipe and duct networks (see, for instance, [1–8]). This subject is extensively studied in both theoretical and practical points of views. A mathematical model of fluid flows in a network of thin tubes has been derived in [9] from the asymptotic expansion of Navier–Stokes equations. Consistent asymptotic analysis of Navier–Stokes equations in thin tube structures, by letting the diameter of the tubes tend to zero, has been recently studied in a series of papers, such as [10] and [11]. The Navier–Stokes equations with pressure boundary conditions in the junctions of thin pipes are considered in [12] and [13], where approximations based on Leray and Poiseuille problems are constructed therein.

Let h be a positive integer. Let G_h be the pre-fractal polygonal curve obtained after h-iterations of the contractive similarities of the Sierpinski gasket G (see Figure 1). We consider a network of circular cylindrical pipes whose axes are the sides of the polygon G_h . We assume that these pipes are narrow axisymmetric tubes of radius ε_h very small with respect to the length 2^{-h} of each side of G_h. We consider an incompressible fluid flow in the bounded domain Ω^h consisting of these pipes connected, after local adjustments near the bifurcation points, through smooth thin regions centered at the vertices of G_h (see Figure 4). We suppose that each pipe is split into two principal regions: junction zones of length $\varepsilon_h \ln(1/\varepsilon_h) \ll 2^{-h}$ linked to the ends of the pipe and the rest of the pipe. We suppose that the fluid flow in Ω^h is driven by some volumic forces and governed by Stokes equations with boundary conditions for the velocity and the pressure on the external boundary of Ω^h and inner continuity conditions for the normal velocity on the interfaces between the junction zones and the rest of the pipes (see Section 2 for more details). We assume that the flow in the junction zones is controlled by a typical Reynolds number $\mathrm{Re}_{\mathbf{j},h}$.

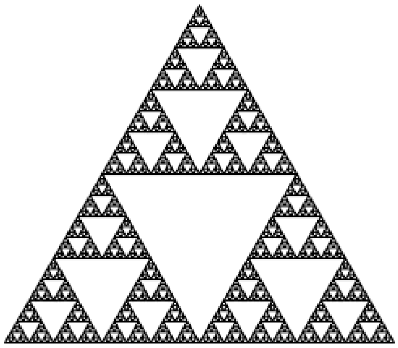

Figure 1. Representation of the Sierpinski gasket G.

The main focus of this paper is to study the asymptotic behavior of the fluid flowing through the branching pipes as the radius of the tubes tends to zero and the sequence of pre-fractal curves converges in the Hausdorff metric to the Sierpinski gasket G. Using Γ-convergence methods (see, for instance, [14] and [15]), we prove that the effective potential energy of the fluid turns out to be of the form

$$F_\infty(v) = \begin{cases} \dfrac{\mu\pi}{\mathfrak{m}(\Theta)\mathcal{H}^d(G)} \displaystyle\int_G v^2 d\mathcal{H}^d + \dfrac{2\mu\pi\mathfrak{m}(\Theta)}{3\sigma} \displaystyle\int_G \nabla v.Z\nabla v d\boldsymbol{\nu} \\ \qquad \text{if } v \in V^\infty, \\ +\infty \qquad \text{otherwise,} \end{cases} \tag{1.1}$$

where v is the fluid velocity, μ is the fluid viscosity, $\mathfrak{m}(\Theta)$ is the average value (see Eq. (6.10)) of the solution Θ of boundary value problem (6.5), $\dfrac{1}{\mathfrak{m}(\Theta)\mathcal{H}^d(G)}$ is the permeability of the Sierpinski gasket G, $\mathcal{H}^d$ being the d-dimensional Hausdorff measure on G where

$$d = \ln 3/\ln 2 \tag{1.2}$$

stands for the fractal dimension of G, Z is a random matrix given in Section 4 (see Eq. (4.15)–(4.18)), $\boldsymbol{\nu}$ is a singular measure with respect to the Hausdorff measure $\mathcal{H}^d$ on G called the Kusuoka measure (see Eq. (4.11)), which, according to [16], is a Gibbs measure of special kind, V^{∞} is the admissible velocities space (see Definition 2_3), and

$$\frac{1}{\sigma} = \lim_{h\to\infty} \frac{\varepsilon_h}{\mathrm{Re}_{\mathbf{j},h}}. \tag{1.3}$$

Depending on the values of σ, we obtain different asymptotic problems:

1. If $\sigma \in (0, +\infty)$, then $\mathrm{Re}_{\mathbf{j},h} = O(\varepsilon_h)$. In this case, the effective flow is described (see Theorem 3) by the following singular Brinkman equation in the Sierpinski gasket G:

$$\begin{aligned} -\frac{2\mu\pi \mathfrak{m}(\Theta)\mathcal{H}^d}{3\sigma\mathcal{H}^d(G)}\Delta_G(u) + \frac{\mu\pi\mathcal{H}^d}{\mathfrak{m}(\Theta)\mathcal{H}^d(G)}u + \boldsymbol{\nu}Z\nabla p.n \\ = \frac{\mathcal{H}^d}{\mathcal{H}^d(G)} f.n \text{ in } G, \end{aligned} \tag{1.4}$$

   where u is the fluid velocity, p is the pressure, Δ_G is the Laplace operator on the Sierpinski gasket (see Lemma 4), f is the effective source term, $n = (1, 0)$ on the horizontal part of G, $n = \left(1/2, \sqrt{3}/2\right)$ on the part of G which is perpendicular to the unit vector $\left(-\sqrt{3}/2, 1/2\right)$, and $n = \left(1/2, -\sqrt{3}/2\right)$ on the part of G which is perpendicular to the unit vector $\left(\sqrt{3}/2, 1/2\right)$. This equation includes the singular Brinkman viscous resistance term $-\frac{2\mu\pi\mathfrak{m}(\Theta)\mathcal{H}^d}{3\sigma\mathcal{H}^d(G)}\Delta_G(u)$, which is due to the viscous behavior of the fluid flow at the junction zones, and the singular Darcy resistance term $\frac{\mu\pi\mathcal{H}^d}{\mathfrak{m}(\Theta)\mathcal{H}^d(G)}u$.
2. If $\sigma = +\infty$, then $\mathrm{Re}_{\mathbf{j},h} = O(1)$ or $\mathrm{Re}_{\mathbf{j},h} \longrightarrow \infty$ as $h \longrightarrow \infty$. In this case, the term $\frac{\mu\pi\mathfrak{m}(\Theta)}{3\sigma}\int_G \nabla v.Z\nabla v d\boldsymbol{\nu}$ in (1.1) disappears and the flow is governed by singular Darcy's law in the Sierpinski gasket G.
3. If $\sigma = 0$, then $\mathrm{Re}_{\mathbf{j},h} = O\left(\varepsilon_h^{\alpha}\right)$ with $\alpha > 1$. In this case, the energy $F_{\infty}(v)$ is finite only if $\int_G \nabla v.Z\nabla v d\boldsymbol{\nu} = 0$, which implies that the velocity of the fluid flow is asymptotically constant in the Sierpinski gasket G.

The study of asymptotic analysis of boundary value problems in domains with fractal boundaries or containing thin inclusions developing a fractal geometry has been recently addressed in a series of papers (see, for instance, [17–29]). The problems obtained at the limit generally consist of singular forms containing fractal terms. The problem considered in this work is quite different from the previous ones, as we deal here with the determination of the fluid motion through branching tubes having a fractal structure. The overall effect of the pre-fractal branching networks on the fluid flow appears in the singular effective equation (1.4), according to the characteristics of the flow, as the radius of the tubes tends to zero and the sequence of pre-fractal curves converges in the Hausdorff metric to the Sierpinski gasket G. The asymptotic representation of the solution of the original singularly perturbed problem includes local perturbations representing the flow in the boundary layers in the junction zones. These local perturbations are solutions of Leray problems in semi-infinite cylinders representing the rescaled junctions. The main novelty of this paper lies in the construction of these local perturbations as well as the derivation of the effective flow described above by singular Brinkman and Darcy laws on the fractal G with divergence-free velocity in a fractal sense specified in Definition 2_2 in Section 5.

The problem considered in this work has some implications for modeling the behavior of fluid flows in various complex geometrical configurations of branching tubes. An important field to which this model is closely related is the behavior of fluid flows in some physiological structures such as lung airways (see, for instance, [1] and [30]) the cardiovascular system and cerebral arteriovenous (see, for instance, [30], [31], and [32]). It has been shown that physiological branching networks exhibit fractal structures for minimal energy dissipation (see, for instance, [33] and [34]). In particular, blood vessels have self-similar structures with optimal transport property of their fractal networks (see, for instance, [35]). Blood has been treated in [31] as a homogeneous, incompressible, Newtonian viscous fluid, making the assumptions that the flow is steady and axisymmetric with sufficiently small Reynolds number so that the flow is laminar. The authors observed that the overall effect of the non-Newtonian characteristics would be small.

The present investigation on fractal branching flows provides some motivations in the haemodynamics. The blood vessels can be illustrated, under some simplifying assumptions, by the network Ω^h of narrow branching tubes with laminar flow far ahead of the bifurcations and boundary layer flow near the bifurcations, where the local Reynolds number is the most effective factor controlling the flow throughout the whole network.

This paper is organized as follows. The statement of the problem is presented in Section 2, with a subsection reserved for the nomenclature and another devoted to the position of the problem. In Section 3, we formulate the main results of this work. In Section 4, we introduce the energy forms, the Kusuoka measures, and gradients on the Sierpinski gasket. Section 5 is devoted to some a priori estimates and compactness results. Section 6 is consecrated to the proof of the main results. A final conclusion is made in Section 7.

2. Statement of the problem

2.1. Nomenclature

$A_1A_2A_3$	equilateral triangle of vertices $A_1 = (0,0)$, $A_2 = (1,0)$, $A_3 = \left(1/2, \sqrt{3}/2\right)$
G	Sierpinski gasket built in the triangle $A_1A_2A_3$
G_h	prefractal polygonal curve obtained after h-iterations of contractive similarities of G
$\mathcal{V}_h$	set of vertices of G_h
E_h	set of edges of G_h
$\mathcal{V}_\infty$	set of all vertices of G
T_h^k	k^{th} triangle of G_h
$E_h^{i,k} = \left[a_h^{i,k}, b_h^{i,k}\right]$	i^{th} edge of T_h^k
2^{-h}	length of $E_h^{i,k}$
$y_{h,1}^{i,k}, y_{h,2}^{i,k}$	local variables on T_h^k
ε_h	small positive number

$\Pi_k^{h,i}$	i^{th} tube of radius ε_h and of length $2^{-h} - 2\varepsilon_h$ surrounding $E_h^{i,k}$
$B_k^h\left(a_h^{i,k}\right)$	small smooth branch junction of thickness of order $2\varepsilon_h$ centered at the vertex $a_h^{i,k}$
$B_k^h\left(b_h^{i,k}\right)$	small smooth branch junction of thickness of order $2\varepsilon_h$ centered at the vertex $b_h^{i,k}$
$\Sigma_{k,1}^{h,i}$	interface between $B_k^h\left(a_h^{i,k}\right)$ and $\Pi_k^{h,i}$
$\Sigma_{k,2}^{h,i}$	interface between $B_k^h\left(b_h^{i,k}\right)$ and $\Pi_k^{h,i}$
$\Omega_k^{h,i}$	pipe formed with $B_k^h\left(a_h^{i,k}\right)$, $B_k^h\left(b_h^{i,k}\right)$, $\Pi_k^{h,i}$, and the interfaces $\Sigma_{k,\alpha}^{h,i}$; $\alpha = 1, 2$, between them
Ω^h	network of the interconnected pipes $\Omega_k^{h,i}$
Γ^h	external boundary of Ω^h
$\mathcal{J}_k^{h,+,i}$	small junction zone of length $\varepsilon_h \ln(1/\varepsilon_h)$ located in the region $y_{h,1}^{i,k} > 0$
$\mathcal{J}_k^{h,-,i}$	small junction zone of length $\varepsilon_h \ln(1/\varepsilon_h)$ located in the region $y_{h,1}^{i,k} < 2^{-h}$
$\mathcal{J}^h$	union of the junction zones $\mathcal{J}_k^{h,\pm,i}$
μ	fluid viscosity
$\mathrm{Re}_{\mathbf{j},h}$	typical Reynolds number in $\mathcal{J}^h$
Re_h	characteristic Reynolds number in Ω^h
Eu_h	characteristic Euler number in Ω^h
Fr_h	characteristic Froude number in Ω^h
$\frac{5^h}{3^{h+1}}$	scaling factor associated to the ramification of the network Ω^h
d	the fractal dimension of G
$\mathcal{H}^d$	d-dimensional Hausdorff measure on G
$L^2_{\mathcal{H}^d}(G)$	space of square integrable L^2-functions with respect to the measure $\mathcal{H}^d$
$\mathcal{E}_G$	Dirichlet form in $L^2_{\mathcal{H}^d}(G)$
Z	random matrix
div_Z	divergence operator on G
$\boldsymbol{\nu}$	Kusuoka measure
$\mathcal{J}^{+\pm,i}$	semi infinite cylinders representing the rescaled junctions

2.2. *Position of the problem*

Let us consider the points of the plane xOy: $A_1 = (0,0)$, $A_2 = (1,0)$, and $A_3 = \left(1/2,\ \sqrt{3}/2\right)$. Let us denote $\{\psi_i\}_{i=1,2,3}$ as the family of contractive similitudes defined on $\mathbb{R}^2$ by

$$\psi_i(x) = \frac{x + A_i}{2},\ \forall x = (x_1, x_2) \in \mathbb{R}^2. \tag{2.1}$$

Let $\mathcal{V}_0 = \{A_1, A_2, A_3\}$ be the set of vertices of the equilateral triangle $A_1A_2A_3$. We define inductively

$$\mathcal{V}_{h+1} = \bigcup_{i=1,2,3} \psi_i(\mathcal{V}_h), \tag{2.2}$$

for every $h \in \mathbb{N}$, and set

$$\mathcal{V}_\infty = \bigcup_{h\in\mathbb{N}} \mathcal{V}_h. \tag{2.3}$$

The Sierpinski gasket, which is denoted here by G, is defined as the closure of the set $\mathcal{V}_\infty$

$$G = \overline{\mathcal{V}_\infty}. \tag{2.4}$$

We consider the graph $G_h = (\mathcal{V}_h, E_h)$, where E_h is the set of edges $[a_h, b_h]$; $a_h, b_h \in \mathcal{V}_h$, such that $|a_h - b_h| = 2^{-h}$; $|a_h - b_h|$ being the Euclidean distance between a_h and b_h (see Figure 2). The graph G_h is then the standard approximation of the Sierpinski gasket, which means that the sequence $(G_h)_h$ converges, as h tends to ∞, in the Hausdorff metric, to the Sierpinski gasket G.

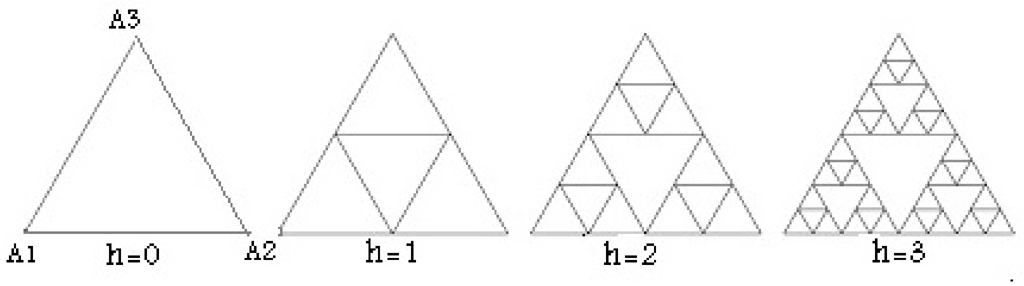

Figure 2. The graph G_h for $h = 0, 1, 2, 3$.

We denote $Card\,(\mathcal{V}_h)$ as the number of vertices of $\mathcal{V}_h$. We can easily check that

$$Card\,(\mathcal{V}_h) = \frac{3^{h+1} + 3}{2}, \forall h \in \mathbb{N}. \tag{2.5}$$

Let $k \in \left\{1, 2, ..., 3^h\right\}$. We denote T_h^k as the k^{th} triangle of the graph G_h obtained at the step h. Let n^k be the unit normal to T_h^k. Then, $n^k = \left(-\sqrt{3}/2, 1/2\right)$, $n^k = \left(\sqrt{3}/2, 1/2\right)$, or $n^k = (0, 1)$. We denote $E_h^{1,k} = \left[a_h^{1,k}, b_h^{1,k}\right]$ as the edge of T_h^k, which is normal to $n^k = (0, 1)$, $E_h^{2,k} = \left[a_h^{2,k}, b_h^{2,k}\right]$ as the edge of T_h^k, which is normal to $n^k = \left(-\sqrt{3}/2, 1/2\right)$, and $E_h^{3,k} = \left[a_h^{3,k}, b_h^{3,k}\right]$ as the edge of T_h^k which is normal to $n^k = \left(\sqrt{3}/2, 1/2\right)$ (see Figure 3).

Let us consider the following rotation matrices:

$$\begin{cases} \mathcal{R}_1 &= Id_{\mathbb{R}^3}, \\ \mathcal{R}_2 &= \begin{pmatrix} 1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ \mathcal{R}_3 &= \mathcal{R}_2^t, \end{cases} \tag{2.6}$$

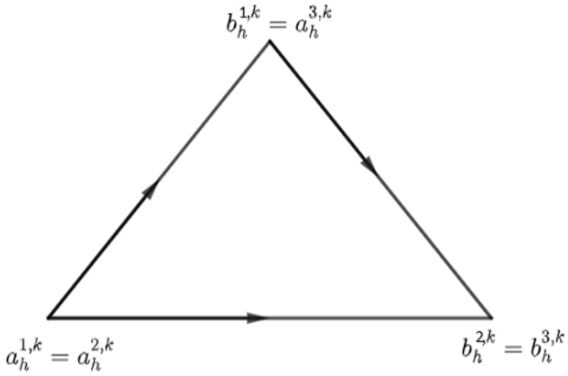

Figure 3. Orientation of the segments $E_h^{1,k}$, $E_h^{2,k}$, and $E_h^{3,k}$.

$Id_{\mathbb{R}^3}$ being the 3×3 identity matrix. We also define the change of variables $y_{h,1}^{i,k}$, $y_{h,2}^{i,k}$, x_3; $i = 1, 2, 3$, for every $h \in \mathbb{N}$, every $k \in \left\{1, 2, ..., 3^h\right\}$, and every $x = (x_1, x_2, x_3) \in \left[a_h^{i,k}, b_h^{i,k}\right] \times \mathbb{R}$, by

$$\begin{pmatrix} y_{h,1}^{i,k}(x) \\ y_{h,2}^{i,k}(x) \\ x_3 \end{pmatrix} = \mathcal{R}_i \begin{pmatrix} x_1 - a_{h,1}^{i,k} \\ x_2 - a_{h,2}^{i,k} \\ x_3 \end{pmatrix}. \tag{2.7}$$

Let S be the unit disk of $\mathbb{R}^2$ centred at the origin. Let $(\varepsilon_h)_{h \in \mathbb{N}}$ be a decreasing sequence of positive numbers, such that

$$\lim_{h\to\infty} \varepsilon_h = \lim_{h\to\infty} 2^h \varepsilon_h \ln(1/\varepsilon_h) = 0. \tag{2.8}$$

We define, for $h \in \mathbb{N}$, $k \in \left\{1, 2, ..., 3^h\right\}$, and $i = 1, 2, 3$, the tube $\Pi_k^{h,i}$ by

$$\Pi_k^{h,i} = \left\{ \begin{array}{l} (x_1, x_2, x_3) \in \mathbb{R}^3;\ \varepsilon_h < y_{h,1}^{i,k}(x) < 2^{-h} - \varepsilon_h, \\ \left(y_{h,2}^{i,k}(x), x_3\right) \in \varepsilon_h S \end{array} \right\}. \tag{2.9}$$

We define the interfaces

$$\left\{ \begin{array}{lcl} \Sigma_{k,1}^{h,i} & = & \left\{ \begin{array}{l} (x_1, x_2, x_3) \in \mathbb{R}^3;\ \left(y_{h,2}^{i,k}(x), x_3\right) \in \varepsilon_h S, \\ y_{h,1}^{i,k}(x) = \varepsilon_h \end{array} \right\}, \\ \Sigma_{k,2}^{h,i} & = & \left\{ \begin{array}{l} (x_1, x_2, x_3) \in \mathbb{R}^3;\ \left(y_{h,2}^{i,k}(x), x_3\right) \in \varepsilon_h S, \\ y_{h,1}^{i,k}(x) = 2^{-h} - \varepsilon_h \end{array} \right\}, \\ \Sigma_k^{h,i} & = & \Sigma_{k,1}^{h,i} \cup \Sigma_{k,2}^{h,i}. \end{array} \right. \tag{2.10}$$

We then set

$$\left\{ \begin{array}{lcl} \Pi^h & = & \bigcup_{\substack{k=1 \\ i=1,2,3}}^{3^h} \Pi_k^{h,i}, \\ \Sigma_\alpha^h & = & \bigcup_{\substack{k=1 \\ i=1,2,3}}^{3^h} \Sigma_{k,\alpha}^{h,i};\ \alpha = 1, 2, \\ \Sigma^h & = & \Sigma_1^h \cup \Sigma_2^h. \end{array} \right. \tag{2.11}$$

We now define thin, smooth regions which ensure the junctions between the tubes $\Pi_k^{h,i}$. Let $B_k^h\left(a_h^{i,k}\right)$ and $B_k^h\left(b_h^{i,k}\right)$ be bounded open sets of thickness of order $2\varepsilon_h$ and centered at the points $\left(a_{h,1}^{i,k}, a_{h,2}^{i,k}, 0\right)$ and $\left(b_{h,1}^{i,k}, b_{h,2}^{i,k}, 0\right)$, respectively, such that $\partial B_k^h\left(a_h^{i,k}\right)$ and $\partial B_k^h\left(b_h^{i,k}\right)$ are C^2-surfaces with

$$\begin{cases} \partial B_k^h\left(a_h^{i,k}\right) \cap \partial\Pi^h &= \varepsilon_h S, \\ \partial B_k^h\left(b_h^{i,k}\right) \cap \partial\Pi^h &= \varepsilon_h S, \end{cases} \tag{2.12}$$

(see Figure 4).

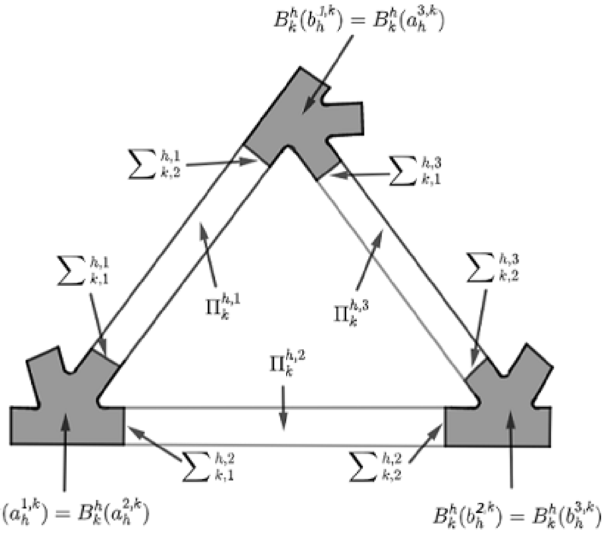

Figure 4. Smooth, thin zones $B_k^h\left(a_h^{i,k}\right)$ and $B_k^h\left(b_h^{i,k}\right)$, which ensure the junctions between the tubes $\Pi_k^{h,i}$.

We set

$$B^h = \bigcup_{\substack{k=1 \\ i=1,2,3}}^{3^h} B_k^h\left(a_h^{i,k}\right) \cup B_k^h\left(b_h^{i,k}\right). \tag{2.13}$$

Let us define the pipe $\Omega_k^{h,i}$; $h \in \mathbb{N}$, $k \in \left\{1, 2, ..., 3^h\right\}$, and $i = 1, 2, 3$, by

$$\Omega_k^{h,i} = \Pi_k^{h,i} \cup \Sigma_k^{h,i} \cup B_k^h\left(a_h^{i,k}\right) \cup B_k^h\left(b_h^{i,k}\right). \tag{2.14}$$

We consider the network Ω^h of interconnected pipes and its external boundary Γ^h defined by

$$\begin{aligned} \Omega^h &= \Sigma^h \cup \bigcup_{\substack{k=1 \\ i=1,2,3}}^{3^h} \Omega_k^{h,i}, \\ \Gamma^h &= \partial\Omega^h. \end{aligned} \tag{2.15}$$

We consider a viscous incompressible fluid flow in Ω^h. We suppose that this flow is essentially laminar except in the set $\mathcal{J}^h$ of the junction zones, where the main characteristics of the flow and their influence on the fluid motion will be analyzed. On the basis of works [12] and [13], we define the set $\mathcal{J}^h$ as

$$\mathcal{J}^h = \bigcup_{\substack{k=1 \\ i=1,2,3}}^{3^h} \mathcal{J}_k^{h,+,i} \cup \mathcal{J}_k^{h,-,i}, \tag{2.16}$$

where, for every $k \in \left\{1, 2, ..., 3^h\right\}$ and $i = 1, 2, 3$,

$$\begin{array}{lcl} \mathcal{J}_k^{h,+,i} & = & \left\{x = (x_1, x_2, x_3) \in \Omega^h;\, 0 < y_{h,1}^{i,k}(x) < \varepsilon_h \ln(1/\varepsilon_h)\right\}, \\ \mathcal{J}_k^{h,-,i} & = & \left\{\begin{array}{l} x = (x_1, x_2, x_3) \in \Omega^h; \\ 2^{-h} - \varepsilon_h \ln(1/\varepsilon_h) < y_{h,1}^{i,k}(x) < 2^{-h} \end{array}\right\}. \end{array} \tag{2.17}$$

Taking into account the typical scales in $\Omega^h \backslash \mathcal{J}^h$, we suppose that the characteristic Reynolds number in these regions is of order $\dfrac{2^{-h}}{\mu}$. The characteristic Reynolds number in Ω^h can be then defined as

$$\mathrm{Re}_h = \left\{\begin{array}{ll} \mathrm{Re}_{\mathbf{j},h} & \text{in } \mathcal{J}^h, \\ \dfrac{2^{-h}}{\mu} & \text{in } \Omega^h \backslash \mathcal{J}^h, \end{array}\right. \tag{2.18}$$

where $\mathrm{Re}_{\mathbf{j},h}$ is assumed to be a typical Reynolds number of the flow in the region $\mathcal{J}^h$. According to [36], the product $\mathrm{Eu}_h \mathrm{Re}_h$ of the characteristic Euler number Eu_h and the characteristic Reynolds number Re_h is the ratio between the caracteristic pressure and viscosity. Then, assuming that the characteristic pressure is the ratio between a constant normal force and the surface of the disk $\varepsilon_h S$, we may write

$$\mathrm{Re}_h \mathrm{Eu}_h = \frac{1}{\mu \pi \varepsilon_h^2}. \tag{2.19}$$

According to the above equality, we suppose that the characteristic Euler number Eu_h in the network Ω^h takes the form

$$\mathrm{Eu}_h = \frac{2^h}{\pi \varepsilon_h^2}. \tag{2.20}$$

On the other hand, as the diameter of any tube of the network Ω^h is $2\varepsilon_h$, we deduce, according to [37, page 98], that the ratio of the characteristic Froude number Fr_h to the characteristic Reynolds number Re_h is of order ε_h^2. Accordingly, we suppose that the characteristic Froude number in Ω^h has the following scaling:

$$Fr_h = 2^{-h} \pi \varepsilon_h^2. \tag{2.21}$$

Since the characteristic Reynolds number is small in $\Omega^h \backslash \mathcal{J}^h$, we suppose that the inertia effects are negligible in the whole Ω^h and the flow is governed by the following Stokes equations:

$$\left\{\begin{array}{rcll} -\dfrac{1}{\mathrm{Re}_h} \dfrac{5^h}{3^{h+1}} \Delta u^h + \mathrm{Eu}_h \dfrac{5^h}{3^{h+1}} \nabla p_h & = & \dfrac{1}{Fr_h} \dfrac{5^h}{3^{h+1}} f_h & \text{in } \Omega^h, \\ \operatorname{div} u^h & = & 0 & \text{in } \Omega^h, \end{array}\right. \tag{2.22}$$

where $\frac{5^h}{3^{h+1}}$ is a scaling factor, which is associated to the ramification of the pre-fractal network Ω^h and determined by the decimation principle (see [38] for more details on scaling exponents governing some physical phenomena in fractal media), the source term f_h is the solution of the following problem posed in each tube $\Omega_k^{h,i}$; $k \in \left\{1, 2, ..., 3^h\right\}$ and $i = 1, 2, 3$,

$$\begin{cases} \operatorname{div} f_h &= g_h \quad \text{in } \Omega_k^{h,i}, \\ f_h.n &= 0 \quad \text{on } \partial\Omega_k^{h,i}, \end{cases} \tag{2.23}$$

where n is the outward unit normal on $\partial\Omega_k^{h,i}$ and g_h is a $L^2\left(\Omega^h\right)$ function such that

$$\begin{cases} \displaystyle\int_{\Omega_k^{h,i}} g_h dx &= 0, \\ \displaystyle\sup_h \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} g_h^2 dx &< +\infty, \end{cases} \tag{2.24}$$

$|A|$ being the Lebesgue measure of the measurable and bounded subset A of $\mathbb{R}^3$. The boundary conditions (2.25) are given, for every $i = 1, 2, 3$, by

$$\begin{cases} u^h &= 0 & \text{on } \Gamma^h, \\ u^h \mid_{\Sigma_1^h} .\mathcal{R}_i e_1 &= u^h \mid_{\Sigma_2^h} .\mathcal{R}_i e_1 & \text{on } \Sigma^h, \\ \dfrac{\partial p_h}{\partial n} &= 0 & \text{on } \Gamma^h, \end{cases} \tag{2.25}$$

where, in accordance with the divergence free of the velocity, the condition $(2.25)_2$ ensures that the outward normal velocities are the same on the two interfaces $\Sigma_{k,1}^{h,i}$ and $\Sigma_{k,2}^{h,i}$, $e_1 = (1, 0, 0)$, and $\frac{\partial p_h}{\partial n}$ is the normal derivative of the pressure on Γ^h; n being the outward unit normal on Γ^h.

Remark 1. *The homogeneous Neumann boundary condition $(2.25)_3$ on Γ^h is justified as follows. According to [39, Chapter II], thin boundary layers are concentrated in the immediate neighborhood of the wall Γ^h due to the homogeneous Dirichlet boundary condition for the velocity on Γ^h. The characteristic Reynolds number in these boundary layers, denoted here by $\mathrm{Re}_{w,h}$, is sufficiently large so that the viscous term $\frac{1}{\mathrm{Re}_{w,h}}\Delta u^h$ is negligible when one gets too close to the wall Γ^h. We deduce, according to [40, Remarks page 1119], that the boundary condition*

$$\mathrm{Eu}_h \frac{\partial p_h}{\partial n} = \frac{1}{\mathrm{Re}_{w,h}} \Delta u^h.n \quad \text{on } \Gamma^h,$$

obtained by taking into account equation $(2.22)_1$ and the fact that $f_h.n = 0$ on Γ^h, can ostensibly be approximated by $\mathrm{Eu}_h \frac{\partial p_h}{\partial n} = 0$ on Γ^h, which implies that $\frac{\partial p_h}{\partial n} = 0$ on Γ^h.

Let us introduce the space V^h defined by

$$V^h = \left\{ \begin{array}{l} v \in H^1\left(\Omega^h, \mathbb{R}^3\right); v \mid_{\Sigma_1^h} .\mathcal{R}_i e_1 = v \mid_{\Sigma_2^h} .\mathcal{R}_i e_1; i = 1, 2, 3, \\ \operatorname{div} v = 0 \text{ in } \Omega^h, v = 0 \text{ on } \Gamma^h \end{array} \right\}. \tag{2.26}$$

We state here a result of existence and uniqueness of a solution for problem (2.22) with boundary conditions (2.25).

Lemma 1. *Problem (2.22)–(2.25) has a unique velocity solution* $u^h \in V^h$ *and pressure solution* $p_h \in H^1\left(\Omega^h\right)$*, which is unique up to an additive constant.*

Proof. Applying the divergence operator to the first equation of problem (2.22), using (2.23)–$(2.24)_1$ and the boundary condition $(2.25)_3$, we deduce that the pressure verifies the Neumann boundary value problem

$$\begin{cases} \Delta p_h = g_h & \text{in } \Omega^h, \\ \dfrac{\partial p_h}{\partial n} = 0 & \text{on } \Gamma^h. \end{cases} \tag{2.27}$$

This problem has a solution $p_h \in H^1\left(\Omega^h\right)$, which is unique up to an additive constant. On the other hand, as

$$\mathrm{Eu}_h \frac{5^h}{3^{h+1}} \int_{\Omega^h} v.\nabla p_h = 0, \tag{2.28}$$

for every $v \in V^h$, the weak formulation of problem (2.22) can be written as, for every $v \in V^h$,

$$\frac{5^h}{3^{h+1}\,\mathrm{Re}_h} \int_{\Omega^h} \nabla u^h.\nabla v dx = \frac{1}{Fr_h} \frac{5^h}{3^{h+1}} \int_{\Omega^h} f_h.v dx. \tag{2.29}$$

Using the Poincaré inequality, we have

$$\left| \int_{\Omega^h} f_h.v dx \right| \leq C_h \left\{ \int_{\Omega^h} |\nabla v|^2 \, dx \right\}^{1/2},$$

where C_h is a positive constant. Then, according to the Lax–Milgram theorem, we infer that problem (2.29) has a unique solution $u^h \in V^h$.

Let us consider the functional F_h defined by

$$F_h(v) = \begin{cases} \dfrac{5^h}{3^{h+1}\,\mathrm{Re}_h} \displaystyle\int_{\Omega^h} |\nabla v|^2 \, dx & \text{if } v \in V^h, \\ +\infty & \text{otherwise.} \end{cases} \tag{2.30}$$

The velocity u^h, solution of problem (2.29), is then the solution of the minimization problem

$$\min_{v \in V^h} \left\{ F_h(v) - 2 \frac{1}{Fr_h} \frac{5^h}{3^{h+1}} \int_{\Omega^h} f_h.v dx \right\}. \tag{2.31}$$

One of the main purposes of this paper is to prove the Γ-convergence of the sequence of functionals $(F_h)_h$ to the functional F_∞ defined in (1.1).

3. The main results

In this section we state our main results in this work. Let $\mathcal{M}\left(\mathbb{R}^3\right)$ be the space of Borel regular measures on $\mathbb{R}^3$. According to Proposition 8 in Section 5, we introduce the following topology τ:

Definition 1. *We say that a sequence* $\left(v^h\right)_h$*;* $v^h \in V^h$*,* τ*-converges to* (v, v^*, v^{**}) *if*

$$\sqrt{5^h} v^h \frac{\pi \mathbf{1}_{\Omega^h}(x)}{3\left|\Omega^h\right|} dx \underset{h\to\infty}{\overset{*}{\rightharpoonup}} (v, v^*, v^{**}) \frac{d\mathcal{H}^d(s) \otimes \delta_0(x_3)}{\mathcal{H}^d(G)} \text{ in } \mathcal{M}\left(\mathbb{R}^3\right),$$

where the symbol $\overset{*}{\rightharpoonup}$ *stands for the weak*-convergence of measures.*

We formulate our result on the Γ-convergence of the sequence of functionals $(F_h)_h$ in the following

Theorem 2. *We suppose that* $\sigma \in (0, +\infty)$*. Then*

1. ($\limsup$ *inequality) For every* $v \in V^\infty$*, there exists a sequence* $\left(v^h\right)_h$*, with* $v^h \in V^h$ *and* $\left(v^h\right)_h$ τ*-converges to* (v, v^*, v^{**})*, where* $v^{**} = 0$*,* $v^* = 0$ *on the part of G which is perpendicular to* $(0, 1)$*,* $v^* = v\sqrt{3}$ *on the part of G which is perpendicular to* $\left(-\sqrt{3}/2, 1/2\right)$*, and* $v^* = -v\sqrt{3}$ *on the part of G which is perpendicular to* $\left(\sqrt{3}/2, 1/2\right)$*, such that*

$$\limsup_{h\to\infty} F_h\left(v^h\right) \le F_\infty(v),$$

   where V^∞ *is defined in Definition* 2_3 *of Section 5 and* F_∞ *is the functional energy defined in (1.1),*
2. ($\liminf$ *inequality) For every sequence* $\left(v^h\right)_h$*, such that* $v^h \in V^h$ *and* $\left(v^h\right)_h$ τ*-converges to* (v, v^*, v^{**})*, we have* $v \in V^\infty$*,* $v^{**} = 0$ *on G,* $v^* = 0$ *on the part of G which is perpendicular to* $(0, 1)$*,* $v^* = v\sqrt{3}$ *on the part of G which is perpendicular to* $\left(-\sqrt{3}/2, 1/2\right)$*,* $v^* = -v\sqrt{3}$ *on the part of G which is perpendicular to* $\left(\sqrt{3}/2, 1/2\right)$*, and*

$$\liminf_{h\to\infty} F_h\left(v^h\right) \ge F_\infty(v).$$

We are now in a position to formulate the asymptotic problem.

Theorem 3. *Let* $\left(u^h, p_h\right)$ *be the solution of problem (2.22) with boundary conditions (2.25). Under the hypothesis of Theorem 2, we have*

1. *The sequence* $\left(u^h\right)_h$ τ*-converges to* $(u, u^*, 0)$*, with* $u \in V^\infty$*,* $u^* = 0$ *on the part of G which is perpendicular to* $(0, 1)$*,* $u^* = u\sqrt{3}$ *on the part of G which is perpendicular to* $\left(-\sqrt{3}/2, 1/2\right)$*, and* $u^* = -u\sqrt{3}$ *on the part of G which is perpendicular to* $\left(\sqrt{3}/2, 1/2\right)$*. There exists* $p \in H_Z(G)$*;* $H_Z(G)$ *being the space defined in Definition* 2_1 *of Section 5, and* $f = (f_1, f_2, 0) \in L^2_{\mathcal{H}^d}\left(G, \mathbb{R}^3\right)$*, such that*

$$\begin{cases} \sqrt{5^h}\widehat{p}_h \dfrac{\pi \mathbf{1}_{\Omega^h}(x)}{3\left|\Omega^h\right|} dx & \underset{h\to\infty}{\overset{*}{\rightharpoonup}} & p\dfrac{d\mathcal{H}^d \otimes \delta_0(x_3)}{\mathcal{H}^d(G)} & \text{in } \mathcal{M}\left(\mathbb{R}^3\right), \\ \sqrt{5^h} f_h \dfrac{\pi \mathbf{1}_{\Omega^h}(x)}{3\left|\Omega^h\right|} dx & \underset{h\to\infty}{\overset{*}{\rightharpoonup}} & f\dfrac{d\mathcal{H}^d \otimes \delta_0(x_3)}{\mathcal{H}^d(G)} & \text{in } \mathcal{M}\left(\mathbb{R}^3\right), \\ \lim\limits_{h\to\infty} \dfrac{5^h}{\left|\Omega^h\right|} \displaystyle\int_{\Omega^h} u^h.\nabla p_h & = & \displaystyle\int_G uZ\nabla p.nd\boldsymbol{\nu} = 0, & \end{cases}$$

   where $n = (1, 0)$ *on the horizontal part* G_1 *of G,* $n = \left(1/2, \sqrt{3}/2\right)$ *on the part* G_2 *of G which is perpendicular to* $\left(-\sqrt{3}/2, 1/2\right)$*, and* $n = \left(1/2, -\sqrt{3}/2\right)$ *on the part* G_3 *of G which is perpendicular to* $\left(\sqrt{3}/2, 1/2\right)$*,*
2. *The couple* (u, p) *is the solution of equation (1.4) stated in the Introduction.*

4. Energy forms on the Sierpinski gasket

4.1. Standard Dirichlet forms

In this subsection we introduce the notion of Dirichlet forms on the Sierpinski gasket. For the definition and properties of Dirichlet forms, we refer to [41] and [42].

For any function $w : \mathcal{V}_\infty \longrightarrow \mathbb{R}$, we define

$$\mathcal{E}_G^h(w) = \left(\frac{5}{3}\right)^h \sum_{\substack{r,s\in\mathcal{V}_h \\ |r-s|=2^{-h}}} (w(r) - w(s))^2 . \tag{4.1}$$

We then define the energy $\mathcal{E}_G$ on G by

$$\mathcal{E}_G(w) \;=\; \lim_{h\to\infty} \mathcal{E}_G^h(w), \tag{4.2}$$

with domain $\mathcal{D}_\infty = \{w : \mathcal{V}_\infty \longrightarrow \mathbb{R} : \mathcal{E}_G(w) < \infty\}$. According to [42, Theorem 2.2.6], every function $w \in \mathcal{D}_\infty$ can be uniquely extended to be an element of $C(G)$ still denoted by w. Let us set

$$\mathcal{D} = \{w \in C(G) : \mathcal{E}_G(w) < \infty\}, \tag{4.3}$$

where $\mathcal{E}_G(w) = \mathcal{E}_G(w\,|_{\mathcal{V}_\infty})$. Then, $\mathcal{D} \subset C(G) \subset L^2_{\mathcal{H}^d}(G)$. We define the space $\mathcal{D}_\mathcal{E}$ as

$$\mathcal{D}_\mathcal{E} = \overline{\mathcal{D}}^{\|.\|_{\mathcal{D}_\mathcal{E}}}, \tag{4.4}$$

where $\|.\|_{\mathcal{D}_\mathcal{E}}$ is the intrinsic norm

$$\|w\|_{\mathcal{D}_\mathcal{E}} = \left\{\mathcal{E}_G(w) + \|w\|^2_{L^2_{\mathcal{H}^d}(G)}\right\}^{1/2}. \tag{4.5}$$

We denote $\mathcal{E}_G(.,.)$ as the bilinear form defined on $\mathcal{D}_\mathcal{E} \times \mathcal{D}_\mathcal{E}$ by

$$\mathcal{E}_G(w,z) = \frac{1}{2}\left(\mathcal{E}_G(w+z) - \mathcal{E}_G(w) - \mathcal{E}_G(z)\right),\ \forall w,z \in \mathcal{D}_\mathcal{E}, \tag{4.6}$$

from which we deduce, according to (4.2), that

$$\mathcal{E}_G(w,z) = \lim_{h\to\infty} \mathcal{E}_G^h(w,z), \tag{4.7}$$

where

$$\mathcal{E}_G^h(w,z) = \left(\frac{5}{3}\right)^h \sum_{\substack{r,s\in\mathcal{V}_h \\ |r-s|=2^{-h}}} (w(r) - w(s))(z(r) - z(s)). \tag{4.8}$$

The form $\mathcal{E}_G(.,.)$ is a closed Dirichlet form in the Hilbert space $L^2_{\mathcal{H}^d}(G)$ and, according to [43, Theorem 4.1], $\mathcal{E}_G(.,.)$ is a local regular Dirichlet form in $L^2_{\mathcal{H}^d}(G)$. This means that

1. (local property) $w, z \in \mathcal{D}_\mathcal{E}$ with supp[w] and supp[z] are disjoint compact sets $\Longrightarrow \mathcal{E}_G(w,z) = 0$,
2. (regularity) $\mathcal{D}_\mathcal{E} \cap C_0(G)$ is dense both in $C_0(G)$ (the space of functions of $C(G)$ with compact support) with respect to the uniform norm and in $\mathcal{D}_\mathcal{E}$ with respect to the intrinsic norm (4.5).

We deduce that $\mathcal{D}_{\mathcal{E}}$ is injected in $L^2_{\mathcal{H}^d}(G)$ and is a Hilbert space with the scalar product associated to the norm (4.5). The second property implies that $\mathcal{D}_{\mathcal{E}}$ is not trivial (that is, $\mathcal{D}_{\mathcal{E}}$ is not made by only the constant functions). Moreover every function of $\mathcal{D}_{\mathcal{E}}$ possesses a continuous representative. Indeed, according to [44, Theorem 6.3. and example 7_1], the space $\mathcal{D}_{\mathcal{E}}$ is continuously embedded in the space $C^{\beta}(G)$ of Hölder continuous functions with $\beta = \ln \frac{5}{3} / \ln 4$.

Now, applying [45, Chap. 6], we have the following result:

Lemma 4. *There exists a unique self-adjoint nonpositive operator Δ_G on $L^2_{\mathcal{H}^d}(G)$ with domain*

$$\mathcal{D}_{\Delta_G} = \left\{ w \in L^2_{\mathcal{H}^d}(G)\,;\, \Delta_G w \in L^2_{\mathcal{H}^d}(G) \right\} \subset \mathcal{D}_{\mathcal{E}}$$

dense in $L^2_{\mathcal{H}^d}(G)$, such that, for every $w \in \mathcal{D}_{\Delta_G}$ and $z \in \mathcal{D}_{\mathcal{E}}$,

$$\mathcal{E}_G(w, z) = -\int_G (\Delta_G w)\, z \frac{d\mathcal{H}^d}{\mathcal{H}^d(G)}.$$

4.2. Kusuoka measures and gradients

In this subsection we define the Kusuoka measure and the gradient on the Sierpinski gasket G. For the definitions and properties of Kusuoka measures and gradients on fractals, we refer to [46–49].

Let $\varrho : \mathcal{V}_\infty \longrightarrow \mathbb{R}$. Then, according to [42, Proposition 3.2.1], there exists a unique $\mathfrak{h} \in \mathcal{D}_\infty$ such that $\mathfrak{h}|_{\mathcal{V}_0} = \varrho$ and

$$\mathcal{E}_G(\mathfrak{h}) = \inf \left\{ \mathcal{E}_G(v)\,;\, v \in \mathcal{D}_\infty, v|_{\mathcal{V}_0} = \varrho \right\},$$

where $\mathfrak{h}$ is called the harmonic function in G with boundary value $\mathfrak{h}|_{\mathcal{V}_0} = \varrho$. On each $\mathcal{V}_h$, $h \in \mathbb{N}^*$, a harmonic function $\mathfrak{h}$ verifies

$$\left(\mathfrak{h} \circ \psi_{i_1 \ldots i_h}\right)|_{\mathcal{V}_0} = T_{i_1 \ldots i_h}\left(\mathfrak{h}|_{\mathcal{V}_0}\right);\, i_1, ..., i_h \in \{1, 2, 3\}, \tag{4.9}$$

(see [42, Proposition 3.2.1]), where $\psi_{i_1 \ldots i_h} = \psi_{i_1} \circ \cdots \circ \psi_{i_h}$ and $T_{i_1 \ldots i_h} = T_{i_1} \ldots T_{i_h}$ with

$$T_1 = \frac{1}{5}\begin{bmatrix} 5 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}, T_2 = \frac{1}{5}\begin{bmatrix} 2 & 2 & 1 \\ 0 & 5 & 0 \\ 1 & 2 & 2 \end{bmatrix}, T_3 = \frac{1}{5}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 5 \end{bmatrix}.$$

Let $M_0 = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3;\, x_1 + x_2 + x_3 = 0\right\}$. Kigami [46] introduced the map $\Phi : G \longrightarrow M_0$ defined by

$$\Phi(x) = \frac{1}{\sqrt{2}}\left(\begin{pmatrix} \mathfrak{h}_1(x) \\ \mathfrak{h}_2(x) \\ \mathfrak{h}_3(x) \end{pmatrix} - \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right),$$

with $\mathfrak{h}_i\left(A_j\right) = \delta_{ij}$ for $A_j \in \mathcal{V}_0$, where δ_{ij} is, for $i, j = 1, 2, 3$, the Kronecker delta symbol. We have the following.

Proposition 5. *[47, Proposition 4.4] If $G_H = \Phi(G)$, then Φ is a homeomorphism between G and G_H. Moreover, define $H_i : M_0 \longrightarrow M_0$; $i = 1, 2, 3$, by*

$$H_i(x) = T_i^t\,(x - \Phi(A_i)) + \Phi(A_i),$$

then $G_H = \bigcup_{i=1,2,3} H_i(G_H)$ and $\Phi \circ \psi_i = H_i \circ \Phi$ for any $i = 1, 2, 3$.

G_H is called the harmonic Sierpinski gasket, which is the self-similar set associated with the collection of contractions $\{H_1, H_2, H_3\}$ on M_0. Let P be the projection from $\mathbb{R}^3$ into M_0 defined, for every $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, by

$$Px = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} - \frac{(x_1 + x_2 + x_3)}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{4.10}$$

According to [48], the Kusuoka measure $\boldsymbol{\nu}$ on G is the unique Borel probability measure defined by

$$\boldsymbol{\nu}(G_{i_1 \dots i_h}) = \frac{1}{2}\left(\frac{5}{3}\right)^h tr\left(T^t_{i_1 \dots i_h} P T_{i_1 \dots i_h}\right), \tag{4.11}$$

where $G_{i_1 \dots i_h} = \psi_{i_1 \dots i_h}(G)$. Let us define

$$I = \{\omega = i_1 i_2 \dots / \ i_n \in \{1, 2, 3\} \text{ for any } n \in \mathbb{N}^*\}, \tag{4.12}$$

and $\pi : I \longrightarrow G$ such that $\psi_j \circ \pi(\omega) = \pi(j\omega)$, for $j = 1, 2, 3$. For any $\omega \in I$, there exists a unique $x \in G$ such that

$$\{x\} = \bigcap_{h \in \mathbb{N}^*} G_{i_1 \dots i_h} \text{ and } \pi(\omega) = x. \tag{4.13}$$

We now define, by abuse of notation, the Kusuoka measure $\boldsymbol{\nu}$ on I (see, for instance, [49]) as the pullback of the Kusuoka measure $\boldsymbol{\nu}$ on G under the projection map π, that is

$$\boldsymbol{\nu}\left(\pi^{-1}(.)\right) = \boldsymbol{\nu}(.). \tag{4.14}$$

Let us set

$$Z(i_1 \dots i_h) = \frac{T^t_{i_1 \dots i_h} P T_{i_1 \dots i_h}}{tr\left(T^t_{i_1 \dots i_h} P T_{i_1 \dots i_h}\right)}. \tag{4.15}$$

Then, according to [48], for $\boldsymbol{\nu}$-almost all ω, there exists a limit

$$Z(\omega) = \lim_{h \to \infty} Z(i_1 \dots i_h). \tag{4.16}$$

Let $Z(x) \equiv Z\left(\pi^{-1}(x)\right)$. Then, $Z(x)$ is well defined on $\mathcal{V}_\infty$ (see for instance [47]). Indeed, according to [49, Theorem 3.6], for $\boldsymbol{\nu}-$almost all $x \in G$,

$$\begin{aligned} Z\left(\pi^{-1}(x)\right) &= Z(\omega) \\ &= \lim_{h \to \infty} Z_h(i_1 \dots i_h), \end{aligned} \tag{4.17}$$

where

$$Z_h(i_1 \dots i_h) = \frac{1}{2}\left(\frac{5}{3}\right)^h \frac{T^t_{i_1 \dots i_h} P T_{i_1 \dots i_h}}{\boldsymbol{\nu}(G_{i_1 \dots i_h})}. \tag{4.18}$$

Let U be an open subset of M_0 containing G_H. Let us define

$$C^1(G) = \left\{u ; u = (v\,|_{G_H}) \circ \Phi, \, v \in C^1(U) \right\}. \tag{4.19}$$

According to [47], if we fix an orthonormal basis of M_0 and regard M_0 as $\mathbb{R}^2$, then, for any $u \in C^1(G)$,

$$\nabla u = \begin{pmatrix} \dfrac{\partial u}{\partial x_1} \\ \dfrac{\partial u}{\partial x_2} \end{pmatrix}. \tag{4.20}$$

We have the following.

Theorem 6. *[47, Theorem 4.8] $C^1(G)$ is a dense subset of $\mathcal{D}_{\mathcal{E}}$ under the norm*

$$\|u\| = \sqrt{\mathcal{E}_G(u,u)} + \|u\|_\infty,$$

and, for any $u, v \in C^1(G)$,

$$\mathcal{E}_G(u,v) = \int_G \nabla u.Z\nabla v d\boldsymbol{\nu}.$$

5. A priori estimates and compactness results

In this section, we establish some a priori estimates and compactness results which will be useful for the proof of the main results.

Lemma 7. *Let $v^h \in V^h$, such that $\sup_h F_h\left(v^h\right) < \infty$. If $\sigma \in (0,+\infty)$ then*

$$\sup_h \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} \left|v^h\right|^2 dx < +\infty.$$

Proof. The proof follows from the Poincaré inequality in a bounded domain with the Dirichlet boundary condition on a part of the boundary and a scaling argument. Let us define, for every $k \in \left\{1,2,...,3^h\right\}$ and $i = 1,2,3$,

$$U_k^{h,i} = \left\{ \begin{array}{l} \left(y_{h,1}^{i,k}, y, z\right) \in \mathbb{R}^3;\ y_{h,1}^{i,k} \in \left(\varepsilon_h \ln(1/\varepsilon_h), 2^{-h} - \varepsilon_h \ln(1/\varepsilon_h)\right) \\ (y,z) \in S \end{array} \right\}.$$

Let $\varphi \in C^1\left(U_k^{h,i}\right)$, such that $\varphi = 0$ on $\partial U_k^{h,i} \cap \partial S$. Using the Poincaré inequality, we infer that, for every $y_{h,1}^{i,k} \in \left(\varepsilon_h \ln(1/\varepsilon_h), 2^{-h} - \varepsilon_h \ln(1/\varepsilon_h)\right)$,

$$\int_S \varphi^2\left(y_{h,1}^{i,k}, y, z\right) dydz \leq C \int_S \left|\nabla_{y,z}\varphi\left(y_{h,1}^{i,k}, y, z\right)\right|^2 dydz,$$

where C is a positive constant independent of h and

$$\nabla_{y,z}\varphi\left(y_{h,1}^{i,k}, y, z\right) = \begin{pmatrix} \dfrac{\partial \varphi}{\partial y}\left(y_{h,1}^{i,k}, y, z\right) \\ \\ \dfrac{\partial \varphi}{\partial z}\left(y_{h,1}^{i,k}, y, z\right) \end{pmatrix}.$$

Now, introducing the scaling $y_{h,2}^{i,k} = \varepsilon_h y$, $x_3 = \varepsilon_h z$, and integrating with respect to $y_{h,1}^{i,k}$ between $\varepsilon_h \ln(1/\varepsilon_h)$ and $2^{-h} - \varepsilon_h \ln(1/\varepsilon_h)$, we get

$$\int_{\varepsilon_h \ln(1/\varepsilon_h)}^{2^{-h}-\varepsilon_h \ln(1/\varepsilon_h)} \int_{\varepsilon_h S} \varphi^2 dy_{h,1}^{i,k} dy_{h,2}^{i,k} dx_3 \leq C\varepsilon_h^2 \int_{\varepsilon_h \ln(1/\varepsilon_h)}^{2^{-h}-\varepsilon_h \ln(1/\varepsilon_h)} \int_{\varepsilon_h S} |\nabla\varphi|^2 \, dy_{h,1}^{i,k} dy_{h,2}^{i,k} dx_3,$$

from which we deduce, using the change of variables (2.7), that, for every $v^h \in V^h$,

$$\int_{\Omega_k^{h,i} \setminus \mathcal{J}_k^{h,+,i} \cup \mathcal{J}_k^{h,-,i}} \left|v^h\right|^2 dx \leq C\varepsilon_h^2 \int_{\Omega_k^{h,i} \setminus \mathcal{J}_k^{h,+,i} \cup \mathcal{J}_k^{h,-,i}} \left|\nabla v^h\right|^2 dx. \tag{5.1}$$

We can use the same method in $\mathcal{J}_k^{h,+,i} \cup \mathcal{J}_k^{h,-,i}$ to obtain

$$\int_{\mathcal{J}_k^{h,+,i} \cup \mathcal{J}_k^{h,-,i}} \left|v^h\right|^2 dx \leq C\varepsilon_h^2 \int_{\mathcal{J}_k^{h,+,i} \cup \mathcal{J}_k^{h,-,i}} \left|\nabla v^h\right|^2 dx. \tag{5.2}$$

The combination of (5.1) and (5.2) implies that

$$\int_{\Omega_k^{h,i}} \left|v^h\right|^2 dx \leq C\varepsilon_h^2 \int_{\Omega_k^{h,i}} \left|\nabla v^h\right|^2 dx. \tag{5.3}$$

Then, summing over i and k in (5.3), we obtain that

$$\frac{5^h}{3^{h+1}\varepsilon_h^2 \operatorname{Re}_h} \int_{\Omega^h} \left|v^h\right|^2 dx \leq C \frac{5^h}{3^{h+1} \operatorname{Re}_h} \int_{\Omega^h} \left|\nabla v^h\right|^2 dx. \tag{5.4}$$

As $\sigma \in (0, +\infty)$, we have that

$$\operatorname{Re}_{\mathbf{j},h} \leq C\varepsilon_h \leq C2^{-h},$$

from which we deduce that $2^h \leq C \dfrac{1}{\operatorname{Re}_{\mathbf{j},h}}$ in $\mathcal{J}^h$. Thus, using (5.4),

$$\begin{aligned} \frac{5^h}{3^{h+1}\varepsilon_h^2 2^{-h}} \int_{\Omega^h} \left|v^h\right|^2 dx &\leq \frac{5^h}{3^{h+1}\varepsilon_h^2 \operatorname{Re}_h} \int_{\Omega^h} \left|v^h\right|^2 dx \\ &\leq CF_h\left(v^h\right). \end{aligned} \tag{5.5}$$

Observing that $3^h \varepsilon_h^2 2^{-h} \approx \dfrac{\left|\Omega^h\right|}{\pi}$, we conclude that

$$\sup_h \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} \left|v^h\right|^2 dx \leq C \sup_h F_h\left(v^h\right) < +\infty. \tag{5.6}$$

We have the following result:

Proposition 8. *Let $\mathbf{1}_{\Omega^h}$ be the characteristic function of the set Ω^h. Let $v^h \in V^h$, such that* $\sup_h F_h\left(v^h\right) < +\infty$*. If $\sigma \in (0,+\infty)$, then there exists a subsequence of $\left(v^h\right)_h$, still denoted as $\left(v^h\right)_h$, such that*

$$\sqrt{5^h} v^h \frac{\pi \mathbf{1}_{\Omega^h}(x)}{3\left|\Omega^h\right|} dx \underset{h\to\infty}{\overset{*}{\rightharpoonup}} v \frac{d\mathcal{H}^d(s) \otimes \delta_0(x_3)}{\mathcal{H}^d(G)} \text{ in } \mathcal{M}\left(\mathbb{R}^3\right),$$

where $v = (v_1, v_2, v_3) \in L^2_{\mathcal{H}^d}\left(G, \mathbb{R}^3\right)$ with $v_3 = 0$ on G, $v_2 = 0$ on the part of G which is perpendicular to $(0,1)$, $v_2 = v_1\sqrt{3}$ on the part of G which is perpendicular to $\left(-\sqrt{3}/2, 1/2\right)$, and $v_2 = -v_1\sqrt{3}$ on the part of G which is perpendicular to $\left(\sqrt{3}/2, 1/2\right)$.

Proof. Let us consider the sequence of measures $(\vartheta_h)_h$ on $\mathbb{R}^3$ defined by

$$\vartheta_h = \frac{\pi \mathbf{1}_{\Omega^h}(x)}{3\left|\Omega^h\right|} dx.$$

Using an ergodicity argument (see, for instance, [50, Theorem 6.1]), we deduce that, for every $\varphi \in C_0\left(\mathbb{R}^3\right)$,

$$\begin{aligned} \lim_{h\to\infty} \int_{\mathbb{R}^3} \varphi(x)\, d\vartheta_h &= \lim_{h\to\infty} \sum_{\substack{k=1 \\ i=1,2,3}}^{3^h} \frac{1}{3^{h+1}} \varphi\left(\frac{a_h^{i,k} + b_h^{i,k}}{2}, 0\right) \\ &= \frac{1}{\mathcal{H}^d(G)} \int_G \varphi(s,0)\, d\mathcal{H}^d(s), \end{aligned}$$

from which we deduce that

$$\vartheta_h \underset{h\to\infty}{\overset{*}{\rightharpoonup}} \vartheta = \mathbf{1}_G(s) \frac{d\mathcal{H}^d(s) \otimes \delta_0(x_3)}{\mathcal{H}^d(G)}.$$

Let $v^h \in L^2\left(\Omega^h, \mathbb{R}^3\right)$, such that $\sup_h F_h\left(v^h\right) < +\infty$. If $\sigma \in (0,+\infty)$ then, according to Lemma 7,

$$\sup_h \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} \left|v^h\right|^2 dx < +\infty. \tag{5.7}$$

Observing that, for some positive constant C independent of h,

$$\begin{aligned} \left|\int_{\mathbb{R}^3} \sqrt{5^h} v^h d\vartheta_h\right|^2 &\le C5^h \int_{\mathbb{R}^3} \left|v^h\right|^2 d\vartheta_h \\ &\le \frac{C5^h}{\left|\Omega^h\right|} \int_{\Omega^h} \left|v^h\right|^2 dx, \end{aligned}$$

and, by taking into account (5.7), we deduce that the sequence $\left(\sqrt{5^h} v^h \vartheta_h\right)_h$ is uniformly bounded in variation, hence $*$-weakly relatively compact. Possibly passing to a subsequence, we can suppose that the sequence $\left(\sqrt{5^h} v^h \vartheta_h\right)_h$ $*$-weakly converges to some χ. Let $\varphi \in C_0\left(\mathbb{R}^3, \mathbb{R}^3\right)$. By using Fenchel's inequality, we have

$$\begin{aligned} &\liminf_{h\to\infty} \frac{1}{2} \int_{\mathbb{R}^3} \left|\sqrt{5^h} v^h\right|^2 d\vartheta_h \\ &\ge \liminf_{h\to\infty} \left(\int_{\mathbb{R}^3} \sqrt{5^h} v^h . \varphi d\vartheta_h - \frac{1}{2} \int_{\mathbb{R}^3} |\varphi|^2 d\vartheta_h\right) \\ &\ge \langle \chi, \varphi \rangle - \frac{1}{2} \int_{\mathbb{R}^3} |\varphi|^2 d\vartheta. \end{aligned}$$

As the left-hand side of this inequality is bounded, we deduce that

$$\sup\left\{\langle\chi,\varphi\rangle\,;\,\varphi\in C_0\left(\mathbb{R}^3,\mathbb{R}^3\right),\ \int_G|\varphi|^2(s,0)\,d\mathcal{H}^d(s)\leq 1\right\}<+\infty,$$

from which we deduce, according to Riesz' representation Theorem, that there exists v such that $v(s,0)\in L^2_{\mathcal{H}^d}\left(G,\mathbb{R}^3\right)$ and $\chi=v(s,x_3)\,\vartheta$.

Let us introduce the function $v^{h,i}$; $i=1,2,3$, related to v^h by

$$v^{h,i}\left(y^{i,k}_{h,1},y^{i,k}_{h,2},x_3\right)=\mathcal{R}_i v^h\circ\mathcal{R}^t_i\left(\begin{pmatrix}y^{i,k}_{h,1}\\ y^{i,k}_{h,2}\\ x_3\end{pmatrix}+\mathcal{R}_i\begin{pmatrix}a^{i,k}_{h,1}\\ a^{i,k}_{h,2}\\ 0\end{pmatrix}\right),\tag{5.8}$$

where $y^{i,k}_{h,1},y^{i,k}_{h,2},x_3$ are the variables defined in (2.7). We can easily prove, after some computations that for every $i=1,2,3$,

$$\operatorname{div}_y v^{h,i}=\operatorname{div} v^h,\tag{5.9}$$

where div_y is the divergence operator in the variables $y^{i,k}_{h,1},y^{i,k}_{h,2},x_3$. On the other hand, as $\Pi^{h,i}_k$ is a cylinder of revolution, we can introduce the cylindrical coordinates $y^{i,k}_{h,1}\equiv y^{i,k}_{h,1}$, $y^{i,k}_{h,2}=r\cos\theta$, $x_3=r\sin\theta$, and the polar components of $v^{h,i}$ defined by

$$\left\{\begin{array}{lcl}v^{h,i}_1\left(y^{i,k}_{h,1},r,\theta\right)&=&v^{h,i}_1\left(y^{i,k}_{h,1},r\cos\theta,r\sin\theta\right),\\ v^{h,i}_r\left(y^{i,k}_{h,1},r,\theta\right)&=&\left(v^{h,i}_2\cos\theta+v^{h,i}_3\sin\theta\right)\left(y^{i,k}_{h,1},r\cos\theta,r\sin\theta\right),\\ v^{h,i}_\theta\left(y^{i,k}_{h,1},r,\theta\right)&=&\left(-v^{h,i}_2\sin\theta+v^{h,i}_3\cos\theta\right)\left(y^{i,k}_{h,1},r\cos\theta,r\sin\theta\right).\end{array}\right.\tag{5.10}$$

Let $\widetilde{v}^{h,i}=\left(v^{h,i}_1,v^{h,i}_r,v^{h,i}_\theta\right)$. The divergence of $\widetilde{v}^{h,i}$ in cylindrical coordinates is given by

$$\operatorname{div}_r\left(\widetilde{v}^{h,i}\right)=\frac{\partial v^{h,i}_1}{\partial y^{i,k}_{h,1}}+\frac{v^{h,i}_r}{r}+\frac{\partial v^{h,i}_r}{\partial r}+\frac{1}{r}\frac{\partial v^{h,i}_\theta}{\partial\theta}.\tag{5.11}$$

Since $\operatorname{div} v^h=0$, we deduce from (5.9) and (5.11) that

$$\operatorname{div}_y v^{h,i}=\operatorname{div}_r\left(\widetilde{v}^{h,i}\right)=0.\tag{5.12}$$

Using the boundary condition $(2.25)_2$, we have, for every $h\in\mathbb{N}$,

$$v^{h,i}_1(\varepsilon_h,r,\theta)-v^{h,i}_1\left(2^{-h}-\varepsilon_h,r,\theta\right)=0,\tag{5.13}$$

from which we deduce, using Green's formula, that, for $\psi\in C^\infty_c(0,2\pi)$ and $\varphi(\theta)=\displaystyle\int_0^\theta\psi(\xi)\,d\xi$ with $\varphi(2\pi)=0$,

$$\begin{aligned}&\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\varepsilon_h}^{2^{-h}-\varepsilon_h}\int_0^{\varepsilon_h}\int_0^{2\pi}\frac{\partial v^{h,i}_1}{\partial y^{i,k}_{h,1}}\varphi(\theta)\,rdy^{i,k}_{h,1}drd\theta\\ &=-\int_0^{\varepsilon_h}\int_0^{2\pi}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\left(v^{h,i}_1(\varepsilon_h,r,\theta)-v^{h,i}_1\left(2^{-h}-\varepsilon_h,r,\theta\right)\right)\varphi(\theta)\,rdrd\theta\\ &=0.\end{aligned}\tag{5.14}$$

Since $\mathrm{div}_r\left(\widetilde{v}^{h,i}\right) = 0$, we deduce from formula (5.11), according to (5.14), that

$$\begin{aligned}&\frac{2^h\sqrt{5^h}}{3^{h+1}\varepsilon_h}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\varepsilon_h}^{2^{-h}-\varepsilon_h}\int_0^{\varepsilon_h}\int_0^{2\pi} v_r^{h,i}\varphi(\theta)\,dy_{h,1}^{i,k}drd\theta\\ &+\frac{2^h\sqrt{5^h}}{3^{h+1}\varepsilon_h}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\varepsilon_h}^{2^{-h}-\varepsilon_h}\int_0^{\varepsilon_h}\int_0^{2\pi}\frac{\partial v_r^{h,i}}{\partial r}\varphi(\theta)\,rdy_{h,1}^{i,k}drd\theta\\ &+\frac{2^h\sqrt{5^h}}{3^{h+1}\varepsilon_h}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\varepsilon_h}^{2^{-h}-\varepsilon_h}\int_0^{\varepsilon_h}\int_0^{2\pi}\frac{\partial v_\theta^{h,i}}{\partial\theta}\varphi(\theta)\,dy_{h,1}^{i,k}drd\theta\\ &=0.\end{aligned}\tag{5.15}$$

Using Green's formula, we deduce that

$$\begin{aligned}&\int_{\varepsilon_h}^{2^{-h}-\varepsilon_h}\int_0^{\varepsilon_h}\int_0^{2\pi}\frac{\partial v_r^{h,i}}{\partial r}\varphi(\theta)\,rdy_{h,1}^{i,k}drd\theta\\ &=-\int_{\varepsilon_h}^{2^{-h}-\varepsilon_h}\int_0^{\varepsilon_h}\int_0^{2\pi} v_r^{h,i}\varphi(\theta)\,dy_{h,1}^{i,k}drd\theta,\end{aligned}\tag{5.16}$$

and

$$\begin{aligned}&\int_{\varepsilon_h}^{2^{-h}-\varepsilon_h}\int_0^{\varepsilon_h}\int_0^{2\pi}\frac{\partial v_\theta^{h,i}}{\partial\theta}\varphi(\theta)\,dy_{h,1}^{i,k}drd\theta\\ &=-\int_{\varepsilon_h}^{2^{-h}-\varepsilon_h}\int_0^{\varepsilon_h}\int_0^{2\pi} v_\theta^{h,i}\psi(\theta)\,dy_{h,1}^{i,k}drd\theta.\end{aligned}\tag{5.17}$$

Combining with (5.15), we deduce that

$$\frac{2^h\sqrt{5^h}}{3^{h+1}\varepsilon_h}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\varepsilon_h}^{2^{-h}-\varepsilon_h}\int_0^{\varepsilon_h}\int_0^{2\pi} v_\theta^{h,i}\psi(\theta)\,dy_{h,1}^{i,k}drd\theta=0.\tag{5.18}$$

Recalling that $v_\theta^{h,i} = -v_2^{h,i}\sin\theta + v_3^{h,i}\cos\theta$ and $v_3^{h,i} = v_3^h$, and using the first part of this Lemma, we obtain that

$$\begin{aligned}&\lim_{h\to\infty}\frac{2^h\sqrt{5^h}}{3^{h+1}\varepsilon_h}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\varepsilon_h}^{2^{-h}-\varepsilon_h}\int_0^{\varepsilon_h}\int_0^{2\pi} v_\theta^{h,i}\psi(\theta)\,dy_{h,1}^{i,k}drd\theta\\ &=\frac{1}{\mathcal{H}^d(G)}\int_G\int_0^{2\pi}\left(-w(s)\sin\theta+v_3(s)\cos\theta\right)\psi(\theta)\,dsd\theta=0,\end{aligned}\tag{5.19}$$

where

$$w(s)=\begin{cases}v_2(s) & \text{on } G_1,\\ -v_1(s)\sqrt{3}+v_2(s) & \text{on } G_2,\\ v_1(s)\sqrt{3}+v_2(s) & \text{on } G_3,\end{cases}\tag{5.20}$$

where G_1 is the part of G which is perpendicular to $(0,1)$, G_2 is the part of G which is perpendicular to $\left(-\sqrt{3}/2, 1/2\right)$, and G_3 is the part of G which is perpendicular to $\left(\sqrt{3}/2, 1/2\right)$. We deduce from (5.19) that $-w(s)\sin\theta + v_3(s)\cos\theta = 0$ for every $\theta \in (0, 2\pi)$, thus $w = v_3 = 0$ on G. Therefore, combining with (5.20), $v_2 = 0$ on G_1, $v_2 = v_1\sqrt{3}$ on G_2, and $v_2 = -v_1\sqrt{3}$ on G_3.

Proposition 9. *We suppose that $\sigma \in (0, +\infty)$. Let $v^h \in V^h \cap H^2\left(\Omega^h, \mathbb{R}^3\right)$, such that $\sup_h F_h\left(v^h\right) < +\infty$. Then, for every sequence $(\varphi_h)_h$, such that $\varphi_h \in H^1\left(\Omega^h\right)$ and*

$$\begin{gathered}\sup_h \frac{5^h}{\left|\Omega^h\right|}\int_{\Omega^h}\left|\nabla\varphi_h\right|^2 dx < +\infty\,,\\ \sqrt{5^h}\varphi_h \frac{\pi\mathbf{1}_{\Omega^h}(x)}{3\left|\Omega^h\right|}dx \underset{h\to\infty}{\overset{*}{\rightharpoonup}} \varphi\frac{d\mathcal{H}^d(s)\otimes\delta_0(x_3)}{\mathcal{H}^d(G)} \text{ in } \mathcal{M}\left(\mathbb{R}^3\right),\end{gathered} \tag{5.21}$$

we have

1. *$\varphi(s,0) \in \mathcal{D}_{\mathcal{E}}$ and $\int_G \nabla\varphi.Z\nabla\varphi d\boldsymbol{\nu} < +\infty$,*
2. *there exists a subsequence of $\left(v^h\right)_h$, still denoted as $\left(v^h\right)_h$, and $v \in L^2_{\mathcal{H}^d}(G)$, such that*

$$\lim_{h\to\infty}\frac{5^h}{\left|\Omega^h\right|}\int_{\Omega^h} v^h.\nabla\varphi_h dx = \int_G vn.Z\nabla\varphi d\boldsymbol{\nu} = 0,$$

where $n = (1,0)$ on the horizontal part of G, $n = \left(1/2, \sqrt{3}/2\right)$ on the part of G which is perpendicular to $\left(-\sqrt{3}/2, 1/2\right)$, and $n = \left(1/2, -\sqrt{3}/2\right)$ on the part of G which is perpendicular to $\left(\sqrt{3}/2, 1/2\right)$.

Proof. 1. Let us define, for every $k \in \left\{1, 2, ..., 3^h\right\}$ and $i = 1, 2, 3$,

$$\varphi_h^i\left(y_{h,1}^{i,k}, y_{h,2}^{i,k}, x_3\right) = \varphi_h \circ \mathcal{R}_i^t\left(\begin{pmatrix} y_{h,1}^{i,k}\\ y_{h,2}^{i,k}\\ x_3\end{pmatrix} + \mathcal{R}_i\begin{pmatrix} a_{h,1}^{i,k}\\ a_{h,2}^{i,k}\\ 0\end{pmatrix}\right), \tag{5.22}$$

and

$$\begin{aligned}\widetilde{\varphi}_h^i\left(y_{h,1}^{i,k}\right) &= \frac{1}{\pi\varepsilon_h^2}\int_{\varepsilon_h S}\varphi_h^i\left(y_{h,1}^{i,k}, y_{h,2}^{i,k}, x_3\right)dy_{h,2}^{i,k}dx_3\\ &= \frac{1}{\pi\varepsilon_h^2}\int_{\varepsilon_h S}\varphi_h\left(\mathcal{R}_i^t\begin{pmatrix} y_{h,1}^{i,k}\\ y_{h,2}^{i,k}\\ x_3\end{pmatrix} + \begin{pmatrix} a_{h,1}^{i,k}\\ a_{h,2}^{i,k}\\ 0\end{pmatrix}\right)dy_{h,2}^{i,k}dx_3,\end{aligned} \tag{5.23}$$

where $y_{h,1}^{i,k}, y_{h,2}^{i,k}, x_3$ are the change of variables defined in (2.7). Then

$$\begin{aligned}
\frac{5^h}{|\Omega^h|}\int_{\Omega^h}|\nabla\varphi_h|^2\,dx &= \frac{5^h}{|\Omega^h|}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\Omega_k^{h,i}}|\nabla\varphi_h|^2\,dx\\
&\geq \frac{5^h}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_0^{2^{-h}}\frac{2^h}{\pi\varepsilon_h^2}\int_{\varepsilon_h S}\left(\frac{\partial\varphi_h^i}{\partial y_{h,1}^{i,k}}\right)^2 dy_{h,1}^{i,k}dy_{h,2}^{i,k}dx_3\\
&\geq \frac{5^h}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\frac{1}{\pi\varepsilon_h^2}\int_{\varepsilon_h S}\left(\int_0^{2^{-h}}\frac{\partial\varphi_h^i}{\partial y_{h,1}^{i,k}}dy_{h,1}^{i,k}\right)^2 dy_{h,2}^{i,k}dx_3\\
&= \frac{5^h}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\frac{1}{\pi\varepsilon_h^2}\int_{\varepsilon_h S}\left(\varphi_h^i\left(2^{-h},y_{h,2}^{i,k},x_3\right)-\varphi_h^i\left(0,y_{h,2}^{i,k},x_3\right)\right)^2 dy_{h,2}^{i,k}dx_3\\
&\geq \frac{5^h}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\left(\frac{1}{\pi\varepsilon_h^2}\int_{\varepsilon_h S}\left(\varphi_h^i\left(2^{-h},y_{h,2}^{i,k},x_3\right)-\varphi_h^i\left(0,y_{h,2}^{i,k},x_3\right)\right)dy_{h,2}^{i,k}dx_3\right)^2\\
&= \frac{5^h}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\left(\widetilde{\varphi}_h^i\left(2^{-h}\right)-\widetilde{\varphi}_h^i(0)\right)^2\\
&= \frac{5^h}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\left(\widetilde{\varphi}_h\left(a_h^{i,k}\right)-\widetilde{\varphi}_h\left(b_h^{i,k}\right)\right)^2\\
&= \mathcal{E}_G(\widetilde{\varphi}_h),
\end{aligned}\tag{5.24}$$

where $\widetilde{\varphi}_h(x_1,x_2)=\widetilde{\varphi}_h^i\left(y_{h,1}^{i,k}\right)$ for $(x_1,x_2)\in\left[a_h^{i,k},b_h^{i,k}\right]$. We now introduce the harmonic extension of $\widetilde{\varphi}_h\,|_{\mathcal{V}_h}$ obtained by the so-called *decimation* procedure (see, for instance, [51, Corollary1]). We define the function $H_{h+1}\widetilde{\varphi}_h:\mathcal{V}_{h+1}\longrightarrow\mathbb{R}$ as the unique minimizer of the problem

$$\min\left\{\mathcal{E}_G^{h+1}(w)\,;\,w:\mathcal{V}_{h+1}\longrightarrow\mathbb{R},\,w=\widetilde{\varphi}_h\text{ on }\mathcal{V}_h\right\}.\tag{5.25}$$

Then $\mathcal{E}_G^{h+1}(H_{h+1}\widetilde{\varphi}_h)=\mathcal{E}_G^h(\widetilde{\varphi}_h)$. For $m>h$, we define the function $H_m\widetilde{\varphi}_h$ from $\mathcal{V}_m$ into $\mathbb{R}^2$ by

$$H_m\widetilde{\varphi}_h=H_m\left(H_{m-1}\left(...\left(H_{h+1}\widetilde{\varphi}_h\right)\right)\right).$$

We have, for every $m>h$, $H_m\widetilde{\varphi}_h\,|_{\mathcal{V}_h}=\widetilde{\varphi}_h\,|_{\mathcal{V}_h}$ and

$$\mathcal{E}_G^m(H_m\widetilde{\varphi}_h)=\mathcal{E}_G^h(\widetilde{\varphi}_h).\tag{5.26}$$

We define now, for fixed $h\in\mathbb{N}$, the function $H\widetilde{\varphi}_h$ on $\mathcal{V}_\infty$ as follows. For $a\in\mathcal{V}_\infty$, we choose $m\geq h$ such that $a\in\mathcal{V}_m$ and set

$$H\widetilde{\varphi}_h(a)=H_m\widetilde{\varphi}_h(a).\tag{5.27}$$

As $\sup_h\dfrac{5^h}{|\Omega^h|}\displaystyle\int_{\Omega^h}|\nabla\varphi_h|^2\,dx<+\infty$, we have, according to (5.24), (5.26), and (5.27),

$$\sup_h\mathcal{E}_G(H\widetilde{\varphi}_h)=\sup_h\mathcal{E}_G^h(\widetilde{\varphi}_h)<+\infty,\tag{5.28}$$

from which we deduce, using Section 4, that $H\widetilde{\varphi}_h$ has a unique continuous extension on G, still denoted as $H\widetilde{\varphi}_h$, and that the sequence $(H\widetilde{\varphi}_h)_h$ is bounded in $\mathcal{D}_{\mathcal{E}}$. Therefore, there exists a subsequence, still denoted as $(H\widetilde{\varphi}_h)_h$, weakly converging in the Hilbert space $\mathcal{D}_{\mathcal{E}}$ to some $\varphi^* \in \mathcal{D}_{\mathcal{E}}$, such that

$$\mathcal{E}_G(\varphi^*) \leq \liminf_{h\to\infty} \mathcal{E}_G(H\widetilde{\varphi}_h) \leq \liminf_{h\to\infty} \mathcal{E}_G^h(\widetilde{\varphi}_h). \tag{5.29}$$

On the other hand, using the hypothesis $(5.21)_2$, we have that

$$\widetilde{\varphi}_h \sqrt{5^h} \frac{2^h \mathbf{1}_{T^h}(x)}{3^{h+1}} dx \underset{h\to\infty}{\overset{*}{\rightharpoonup}} \varphi(s,0) \frac{d\mathcal{H}^d(s)}{\mathcal{H}^d(G)} \text{ in } \mathcal{M}\left(\mathbb{R}^2\right), \tag{5.30}$$

where $T^h = \bigcup_{k=1}^{3^h} T_h^k$; T_h^k being the k^{th} triangle obtained at the step k in the construction of the fractal G. We deduce from this that, for every $\psi \in C_0(G)$,

$$\begin{aligned} \lim_{h\to\infty} \frac{1}{\mathcal{H}^d(G)} \int_G H\widetilde{\varphi}_h \psi d\mathcal{H}^d(s) &= \lim_{h\to\infty} \int_{\mathbb{R}^3} \widetilde{\varphi}_h \psi d\upsilon_h \\ &= \frac{1}{\mathcal{H}^d(G)} \int_G \varphi(s,0) \psi d\mathcal{H}^d(s), \end{aligned} \tag{5.31}$$

where $(\upsilon_h)_h$ is the sequence of measures defined by

$$\upsilon_h = \frac{1}{Card(\mathcal{V}_h)} \sum_{a\in\mathcal{V}_h} \delta_a, \tag{5.32}$$

δ_a being the Dirac measure at the point a. Thus, $\varphi^*(s) = \varphi(s,0)$, $\varphi(s,0) \in \mathcal{D}_{\mathcal{E}}$, and, according to (5.24) and (5.29),

$$\begin{aligned} \mathcal{E}_G(\varphi) &\leq \liminf_{h\to\infty} \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} \left|\nabla\varphi_h\right|^2 dx \\ &\leq \sup_h \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} \left|\nabla\varphi_h\right|^2 dx < +\infty, \end{aligned} \tag{5.33}$$

from which we deduce, using Theorem 6, that

$$\mathcal{E}_G(\varphi) = \int_G \nabla\varphi . Z \nabla\varphi d\boldsymbol{\nu} < +\infty. \tag{5.34}$$

2. As div $v^h = 0$, we can write

$$\begin{aligned} &\frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} v^h.\nabla\varphi_h dx = \frac{5^h}{\left|\Omega^h\right|} \int_{B^h} v^h.\nabla\varphi_h dx + \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h\setminus B^h} v^h.\nabla\varphi_h dx \\ &= \frac{5^h}{\left|\Omega^h\right|} \int_{B^h} v^h.\nabla\varphi_h dx \\ &+ \frac{5^h}{\left|\Omega^h\right|} \sum_{\substack{k=1 \\ i=1,2,3}}^{3^h} \int_{\varepsilon_h S} v^h \,|_{\Sigma_{k,1}^{h,i}} .n^i \left(\varphi_h \,|_{\Sigma_{k,1}^{h,i}} - \varphi_h \,|_{\Sigma_{k,2}^{h,i}}\right) = 0, \end{aligned} \tag{5.35}$$

where $n^i = \mathcal{R}_i e_1$. Since $\left|B^h\right| \longrightarrow 0$ as $h \to \infty$, using the proof of Lemma 7 and the hypothesis (5.21), we have that

$$\lim_{h\to\infty} \frac{5^h}{\left|\Omega^h\right|} \int_{B^h} v^h.\nabla\varphi_h dx = 0.$$

Thus, passing to the limit in (5.35), we get

$$\begin{aligned}
&\lim_{h\to\infty} \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} v^h.\nabla\varphi_h dx \\
&= \lim_{h\to\infty} \frac{5^h}{\left|\Omega^h\right|} \sum_{\substack{k=1\\ i=1,2,3}}^{3^h} \int_{\varepsilon_h S} v^h \mid_{\Sigma^{h,i}_{k,1}} .n^i \left(\varphi_h \mid_{\Sigma^{h,i}_{k,1}} - \varphi_h \mid_{\Sigma^{h,i}_{k,2}}\right) \\
&= \lim_{h\to\infty} \frac{5^h}{\left|\Omega^h\right|} \sum_{\substack{k=1\\ i=1,2,3}}^{3^h} \int_{\varepsilon_h S} v^h \mid_{\Sigma^{h,i}_{k,1}} .n^i \left(\varphi \mid_{\Sigma^{h,i}_{k,1}} - \varphi \mid_{\Sigma^{h,i}_{k,2}}\right) \\
&= 0.
\end{aligned} \tag{5.36}$$

As $\varphi(s,0) \in \mathcal{D}_{\mathcal{E}}$, using some density argument, we may suppose that $\varphi(s,0) \in C^1(G)$. As $v^h \in H^2\left(\Omega^h, \mathbb{R}^3\right)$, we may write

$$\begin{aligned}
&\lim_{h\to\infty} \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} v^h.\nabla\varphi_h dx \\
&= \lim_{h\to\infty} \frac{5^h}{\pi 3^{h+1}} \sum_{\substack{k=1\\ i=1,2,3}}^{3^h} \int_{\varepsilon_h S} v^h \left(a^{i,k}_h + \epsilon^i_h\right).n^i 2^h \left(\varphi\left(a^{i,k}_h\right) - \varphi\left(b^{i,k}_h\right)\right),
\end{aligned} \tag{5.37}$$

where $\epsilon^i_h = \varepsilon_h \mathcal{R}^t_i e_1$. On the other hand, there exists a function $r_h \in C^1\left(\Omega^h\right)$ such that $v^h = \nabla r_h$. Indeed, as $\operatorname{div} v^h = 0$, r_h is a solution of the equation $\Delta r_h = 0$ in Ω^h with some boundary conditions on $\partial\Omega^h$. Using the smoothness of φ, we infer that

$$\begin{aligned}
&\varphi\left(a^{i,k}_h\right) - \varphi\left(b^{i,k}_h\right) \\
&= \sum_{l=1,2} \int_0^1 \frac{\partial\varphi}{\partial x_l} \left(t\left(a^{i,k}_h - b^{i,k}_h\right) + b^{i,k}_h\right)\left(a^{i,k}_h - b^{i,k}_h\right)_l dt \\
&= 2^{-h}\nabla\varphi\left(a^{i,k}_h\right).n^i + O\left(2^{-2h}\right).
\end{aligned} \tag{5.38}$$

Then, replacing in (5.37), taking into account the fact that $\sup_h F_h\left(v^h\right) < +\infty$ and the estimates on v^h given in Lemma 7, we obtain that

$$\frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} v^h.\nabla\varphi_h = \frac{5^h}{3^h} \sum_{\substack{k=1\\ i=1,2,3}}^{3^h} \nabla r_h\left(a^{i,k}_h\right).n^i \nabla\varphi\left(a^{i,k}_h\right).n^i + O\left(\varepsilon_h 2^h\right). \tag{5.39}$$

As for the fractal G, we can construct, according to Proposition 5, a graph approximation $G_{H,h}$ of the harmonic Sierpinski gasket G_H and a sequence $\left(\Omega^h_H\right)_h$ of thin branching tubes whose axes are iterated curves of the graph $G_{H,h}$. As $r_h \in C^1(\Omega^h)$, there exists $\mathfrak{r}_h \in C^1(\Omega_H^h)$, such that $r_h\mid_{G_h} = \mathfrak{r}_h\mid_{G_{H,h}} \circ\Phi$. Similarly, there exists $\varsigma \in C^1(U)$, U being an open subset of M_0 containing G_H, such that $\varphi\mid_G = \varsigma|_{G_H} \circ \Phi$. Let us set, for $k_1, ..., k_h \in \{1, 2, 3\}$,

$$\begin{array}{rcl} \Xi_{k_1...k_h}(A_i) & = & \nabla\mathfrak{r}_h \circ \Phi\left(\psi_{k_1...k_h}(A_i)\right).n^i T^t_{k_1...k_h} P\mathfrak{H}(A_i), \\ F_{k_1...k_h}(A_i) & = & \mathfrak{H}(A_i) P T_{k_1...k_h}\nabla\varsigma \circ \Phi\left(\psi_{k_1...k_h}(A_i)\right).n^i, \end{array} \tag{5.40}$$

where $\mathfrak{H}(A_i) = (\mathfrak{h}_1(A_i), \mathfrak{h}_2(A_i), \mathfrak{h}_3(A_i)) = (\delta_{1i}, \delta_{2i}, \delta_{3i})$. Then, observing that, there exist $k_1, ..., k_h \in \{1, 2, 3\}$ such that $a_h^{i,k} = \psi_{k_1...k_h}(A_i)$, using (5.40), the fact that $P^t = P$, $T^t_{k_1...k_h}P = T^t_{k_1...k_h}$, and [46, Lemma 3.2], we deduce that

$$\begin{array}{l} \nabla r_h\left(a_h^{i,k}\right).n^i\nabla\varphi\left(a_h^{i,k}\right).n^i \\ = \Xi_{k_1...k_h}(A_i).F_{k_1...k_h}(A_i) \\ = \nabla\mathfrak{r}_h \circ \Phi\left(\psi_{k_1...k_h}(A_i)\right).n^i Z_h \nabla\varsigma \circ \Phi\left(\psi_{k_1...k_h}(A_i)\right).n^i \boldsymbol{\nu}(G_{k_1...k_h}). \end{array} \tag{5.41}$$

Using Lemma 8, there exists a subsequence of $\left(v^h\right)_h$, still denoted as $\left(v^h\right)_h$, and $v \in L^2_{\mathcal{H}^d}(G)$, such that

$$\sqrt{5^h} v^h \frac{\pi \mathbf{1}_{\Omega^h}(x)}{3\left|\Omega^h\right|}dx \underset{h\to\infty}{\overset{*}{\rightharpoonup}} (v, v^*, 0)\frac{d\mathcal{H}^d(s) \otimes \delta_0(x_3)}{\mathcal{H}^d(G)} \text{ in } \mathcal{M}\left(\mathbb{R}^3\right),$$

where $v^* = 0$ on the part of G which is perpendicular to $(0, 1)$, $v^* = v\sqrt{3}$ on the part of G which is perpendicular to $\left(-\sqrt{3}/2, 1/2\right)$, and $v^* = -v\sqrt{3}$ on the part of G which is perpendicular to $\left(\sqrt{3}/2, 1/2\right)$. The corresponding subsequence of gradients $(\nabla r_h\mid_{G_h} = \nabla\mathfrak{r}_h\mid_{G_H} \circ\Phi)_h$ converges to the same limit. Thus, using the limits (5.36)–(5.37), the relations (5.38)–(5.41), and the smoothness of φ and v^h, we obtain that

$$\begin{array}{l} \displaystyle\lim_{h\to\infty}\frac{5^h}{\left|\Omega^h\right|}\int_{\Omega^h} v^h.\nabla\varphi_h dx \\ \displaystyle = \lim_{h\to\infty}\frac{5^h}{3^h}\sum_{\substack{k=1 \\ i=1,2,3}}^{3^h}\nabla r_h\left(a_h^{i,k}\right).n^i\nabla\varphi\left(a_h^{i,k}\right).n^i \\ \displaystyle = \int_G (v, v^*).nZ\nabla\varphi.n d\boldsymbol{\nu} \\ \displaystyle = \int_G v n.Z\nabla\varphi d\boldsymbol{\nu} \\ = 0, \end{array} \tag{5.42}$$

where we have used the fact that $(v, v^*).n = v$.

According to the above proposition, we introduce the following

Definition 2. *1. We define the space $H_Z(G)$ by*

$$H_Z(G) = \left\{\varphi \in L^2_{\mathcal{H}^d}(G); \int_G \nabla\varphi.Z\nabla\varphi d\boldsymbol{\nu} < +\infty\right\}. \tag{5.43}$$

2. *Let $n = (1,0)$ on the horizontal part of G, $n = \left(1/2,\ \sqrt{3}/2\right)$ on the part of G which is perpendicular to $\left(-\sqrt{3}/2, 1/2\right)$, and $n = \left(1/2, -\sqrt{3}/2\right)$ on the part of G which is perpendicular to $\left(\sqrt{3}/2, 1/2\right)$. Let $v \in \mathcal{D}_{\mathcal{E}}$. We define the divergence of v on G by the relation*

$$\langle \mathrm{div}_Z(v), \varphi \rangle = \int_G vn.Z\nabla\varphi d\boldsymbol{\nu},$$

for every $\varphi \in H_Z(G)$.

3. *We define the space V^∞ by*

$$V^\infty = \{v \in \mathcal{D}_{\mathcal{E}};\ \langle \mathrm{div}_Z(v), \varphi\rangle = 0,\ \forall\varphi \in H_Z(G)\}. \tag{5.44}$$

We introduce the following useful result which is due to Bogovskiĭ [52]:

Lemma 10. *Let $D \subset \mathbb{R}^3$ be a bounded domain with Lipschitz continuous boundary ∂D. There exists a linear operator $\mathcal{B} : L^2(D) \to H_0^1\left(D;\mathbb{R}^3\right)$, such that, for every $\varpi \in L^2(D)$ satisfying $\int_D \varpi dx = 0$,*

$$\begin{cases} \mathrm{div}\,(\mathcal{B}(\varpi)) &= \varpi \text{ in } D, \\ \|\nabla\mathcal{B}(\varpi)\|_{L^2(D;\mathbb{R}^9)} &\le C(D)\,\|\varpi\|_{L^2(D)}, \end{cases}$$

where $C(D)$ is a constant which only depends on D.

Let us define $D = S \times (0,1)$. As a consequence, we have the following result:

Lemma 11. *Let $D_h = \varepsilon_h S \times \left(0, 2^{-h}\right)$. There exists a linear operator $\mathcal{B}_h : L^2(D_h) \to H_0^1\left(D_h;\mathbb{R}^3\right)$, such that, for every $\varpi \in L^2(D_h)$ with $\int_{D_h} \varpi dx = 0$,*

$$\begin{cases} \mathrm{div}\,(\mathcal{B}_h) &= \varpi \text{ in } D_h, \\ \|\nabla\mathcal{B}_h\|_{L^2(D_h;\mathbb{R}^9)} &\le \dfrac{C(D)}{\varepsilon_h}\,\|\varpi\|_{L^2(D_h)}, \end{cases}$$

where $C(D)$ is a constant which still only depends on D.

Proof. For every $\varpi \in L^2(D_h)$ satisfying $\int_{D_h} \varpi dx = 0$, we define

$$\varpi_h(y) = \varpi\left(\varepsilon_h y_1, \varepsilon_h y_2, 2^{-h} y_3\right),\ \forall y = (y_1, y_2, y_3) \in D.$$

Then, since $\int_{D_h} \varpi dy = 0$, we can apply Lemma 10 in D to obtain

$$\begin{cases} \mathrm{div}\,(\mathcal{B}(\varpi_h)) &= \varpi_h \text{ in } D, \\ \|\nabla\mathcal{B}(\varpi_h)\|_{L^2(D;\mathbb{R}^9)} &\le C(D)\,\|\varpi_h\|_{L^2(D)}. \end{cases} \tag{5.45}$$

Let us define, for every $x \in D_h$,

$$\mathcal{B}_h(\varpi)(x) = \left(\varepsilon_h\mathcal{B}_1(\varpi_h), \varepsilon_h\mathcal{B}_2(\varpi_h), 2^{-h}\mathcal{B}_3(\varpi_h)\right)\left(\frac{x_1}{\varepsilon_h}, \frac{x_2}{\varepsilon_h}, \frac{x_3}{2^{-h}}\right). \tag{5.46}$$

Then

$$
\begin{array}{rcl}
\operatorname{div}\mathcal{B}_h(\varpi)(x) &=& \operatorname{div}(\mathcal{B}(\varpi_h))\left(\dfrac{x_1}{\varepsilon_h},\dfrac{x_2}{\varepsilon_h},\dfrac{x_3}{2^{-h}}\right)\\
&=& \varpi_h\left(\dfrac{x_1}{\varepsilon_h},\dfrac{x_2}{\varepsilon_h},\dfrac{x_3}{2^{-h}}\right)\\
&=& \varpi(x).
\end{array}
\tag{5.47}
$$

On the other hand, observing that

$$
\nabla\mathcal{B}_h(\varpi)(x) = M^h(\mathcal{B}(\varpi_h))\left(\frac{x_1}{\varepsilon_h},\frac{x_2}{\varepsilon_h},\frac{x_3}{2^{-h}}\right),
$$

where

$$
M^h(\mathcal{B}(\varpi_h)) = \begin{pmatrix}
\dfrac{\partial\mathcal{B}_1(\varpi_h)}{\partial x_1} & \dfrac{\partial\mathcal{B}_1(\varpi_h)}{\partial x_2} & \dfrac{\varepsilon_h}{2^{-h}}\dfrac{\partial\mathcal{B}_1(\varpi_h)}{\partial x_3}\\
\dfrac{\partial\mathcal{B}_2(\varpi_h)}{\partial x_1} & \dfrac{\partial\mathcal{B}_2(\varpi_h)}{\partial x_2} & \dfrac{\varepsilon_h}{2^{-h}}\dfrac{\partial\mathcal{B}_2(\varpi_h)}{\partial x_3}\\
\dfrac{\varepsilon_h}{2^{-h}}\dfrac{\partial\mathcal{B}_3(\varpi_h)}{\partial x_1} & \dfrac{\varepsilon_h}{2^{-h}}\dfrac{\partial\mathcal{B}_3(\varpi_h)}{\partial x_2} & \dfrac{\partial\mathcal{B}_3(\varpi_h)}{\partial x_3}
\end{pmatrix},
$$

we deduce that

$$
\begin{array}{rcl}
\displaystyle\int_{D_h}|\nabla\mathcal{B}_h(\varpi)|^2\,dx &=& \displaystyle 2^{-h}\sum_{\alpha,\beta=1,2}\int_D\left|\frac{\partial\mathcal{B}_\alpha((\varpi_h))}{\partial x_\beta}\right|^2 dx\\
&& \displaystyle +\sum_{\alpha=1,2}2^h\varepsilon_h^4\int_D\left|\frac{\partial\mathcal{B}_\alpha((\varpi_h))}{\partial x_3}\right|^2 dx\\
&& \displaystyle +\sum_{\alpha=1,2}2^h\varepsilon_h^2\int_D\left|\frac{\partial\mathcal{B}_3(\varpi_h)}{\partial x_\alpha}\right|^2 dx\\
&& \displaystyle +2^{-h}\varepsilon_h^2\int_D\left|\frac{\partial\mathcal{B}_3(\varpi_h)}{\partial x_3}\right|^2 dx\\
&\leq& \displaystyle 2^{-h}\int_D|\nabla\mathcal{B}(\varpi_h)|^2\,dx.
\end{array}
\tag{5.48}
$$

Last, according to (5.45), we have

$$
\begin{array}{rcl}
\displaystyle 2^{-h}\int_D|\nabla\mathcal{B}(\varpi_h)|^2\,dx &\leq& \displaystyle C(D)\,2^{-h}\int_D|\varpi_h|^2\,dx\\
&\leq& \displaystyle \frac{C(D)}{\varepsilon_h^2}\int_{D_h}|\varpi|^2\,dx.
\end{array}
\tag{5.49}
$$

Therefore, combining (5.48) and (5.49), we infer that

$$
\int_{D_h}|\nabla\mathcal{B}_h(\varpi)|^2\,dx \leq \frac{C(D)}{\varepsilon_h^2}\int_{D_h}|\varpi|^2\,dx. \tag{5.50}
$$

Let $\left(u^h, p_h\right)$ be the solution of problem (2.22) with boundary conditions (2.25). Let us define, for every $h\in\mathbb{N}$, $i=1,2,3$, and $k\in\left\{1,2,...,3^h\right\}$, the zero average-value pressure $\widehat{p}_k^{h,i}$ by

$$
\widehat{p}_k^{h,i} = p_h - \frac{1}{\left|\Omega_k^{h,i}\right|}\int_{\Omega_k^{h,i}} p_h dx \text{ in } \Omega_k^{h,i}, \tag{5.51}
$$

and the pressure $\widehat{p}_h$ by

$$\widehat{p}_h \equiv \widehat{p}_k^{h,i} \text{ on each } \Omega_k^{h,i}. \tag{5.52}$$

The following estimates hold true:

Lemma 12. *If* $\sigma \in (0, +\infty)$ *then*

1. $\sup_h F_h\left(u^h\right) < +\infty$, $\sup_h \dfrac{5^h}{\left|\Omega^h\right|} \displaystyle\int_{\Omega^h} \left|u^h\right|^2 dx < +\infty$,
2. $\sup_h \dfrac{5^h}{\left|\Omega^h\right|} \displaystyle\int_{\Omega^h} (\widehat{p}_h)^2 dx < +\infty$, $\sup_h \dfrac{5^h}{\left|\Omega^h\right|} \displaystyle\int_{\Omega^h} |\nabla p_h|^2 dx < +\infty$.

Proof. 1. Applying Lemma 11 for the solution f_h of problem (2.23), we deduce that, for every $k \in \left\{1, 2, ..., 3^h\right\}$ and $i = 1, 2, 3$,

$$\int_{\Omega_k^{h,i}} |\nabla f_h|^2 dx \leq \frac{C}{\varepsilon_h^2} \int_{\Omega_k^{h,i}} |g_h|^2 dx. \tag{5.53}$$

Additionally, using the inequality (5.3), we have

$$\int_{\Omega_k^{h,i}} |f_h|^2 dx \leq C\varepsilon_h^2 \int_{\Omega_k^{h,i}} |\nabla f_h|^2 dx. \tag{5.54}$$

We deduce from (5.53) and (5.54), that

$$\frac{5^h}{\left|\Omega^h\right|} \int_{\Omega_k^{h,i}} |f_h|^2 dx \leq C \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega_k^{h,i}} |g_h|^2 dx, \tag{5.55}$$

then, using the hypothesis $(2.24)_2$, we conclude that

$$\sup_h \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} |f_h|^2 dx < +\infty. \tag{5.56}$$

Multiplying $(2.22)_1$ by u^h and integrating by parts, we obtain that

$$\frac{5^h}{\mathrm{Re}_h} \int_{\Omega^h} \left|\nabla u^h\right|^2 dx = \frac{1}{Fr_h} \frac{5^h}{3^{h+1}} \int_{\Omega^h} f_h . u^h dx, \tag{5.57}$$

from which we deduce, in virtue of the fact that $\dfrac{1}{Fr_h}\dfrac{5^h}{3^{h+1}} \approx \dfrac{5^h}{\left|\Omega^h\right|}$, by using inequality (5.6) and estimate (5.56),

$$\sup_h \frac{5^h}{\mathrm{Re}_h} \int_{\Omega^h} \left|\nabla u^h\right|^2 dx < +\infty , \tag{5.58}$$

and, as $\sigma \in (0, +\infty)$, according to Lemma 7,

$$\sup_h \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} \left|u^h\right|^2 dx < +\infty. \tag{5.59}$$

2. According to Lemma 11, there exists $\phi_k^{h,i} \in H_0^1\left(\Omega_k^{h,i}, \mathbb{R}^3\right)$ such that

$$\begin{cases} -\operatorname{div} \phi_k^{h,i} &= \widehat{p}_k^{h,i} \quad \text{in } \Omega_k^{h,i}, \\ \phi_k^{h,i} &= 0 \quad \text{on } \partial\Omega_k^{h,i}, \end{cases} \tag{5.60}$$

and

$$\left\|\nabla \phi_k^{h,i}\right\|_{L^2\left(\Omega_k^{h,i},\mathbb{R}^9\right)} \le \frac{C}{\varepsilon_h} \left\|\widehat{p}_k^{h,i}\right\|_{L^2\left(\Omega_k^{h,i}\right)}. \tag{5.61}$$

Let us define ϕ^h on Ω^h by $\phi^h = \phi_k^{h,i}$ on each $\Omega_k^{h,i}$, for every $k \in \left\{1, 2, ..., 3^h\right\}$ and $i = 1, 2, 3$. Then, according to inequality (5.61), we have that

$$\left\|\nabla \phi^h\right\|_{L^2\left(\Omega^h,\mathbb{R}^9\right)} \le \frac{C}{\varepsilon_h} \left\|\widehat{p}_h\right\|_{L^2\left(\Omega^h\right)}. \tag{5.62}$$

Multiplying $(2.22)_1$ by ϕ^h and integrating by parts, we deduce that

$$\begin{aligned} \frac{5^h}{\mathrm{Re}_h} \int_{\Omega^h} \nabla u^h.\nabla \phi^h dx + \mathrm{Eu}_h \frac{5^h}{3^{h+1}} \int_{\Omega^h} \left(\widehat{p}_h\right)^2 dx \\ = \frac{1}{Fr_h} \frac{5^h}{3^{h+1}} \int_{\Omega^h} f_h.\phi^h dx. \end{aligned} \tag{5.63}$$

Using the fact that $\mathrm{Eu}_h \dfrac{5^h}{3^{h+1}} = \dfrac{1}{Fr_h}\dfrac{5^h}{3^{h+1}} \approx \dfrac{5^h}{\left|\Omega^h\right|}$, inequality (5.62), and the uniform boundedness (5.56) and (5.58), we deduce that

$$\frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} \left(\widehat{p}_h\right)^2 dx \le C \left\{ \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} \left(\widehat{p}_h\right)^2 dx \right\}^{1/2}, \tag{5.64}$$

which implies that

$$\sup_h \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} \left(\widehat{p}_h\right)^2 dx < +\infty. \tag{5.65}$$

On the other hand, multiplying $(2.27)_1$ by p_h, integrating by parts, and, using the hypothesis $(2.24)_1$, we get

$$\begin{aligned} \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} \left|\nabla p_h\right|^2 dx &= -\frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} g_h p_h dx \\ &= -\frac{5^h}{\left|\Omega^h\right|} \sum_{\substack{k=1 \\ i=1,2,3}}^{3^h} \int_{\Omega_k^{h,i}} g_h p_h dx \\ &= -\frac{5^h}{\left|\Omega^h\right|} \sum_{\substack{k=1 \\ i=1,2,3}}^{3^h} \int_{\Omega_k^{h,i}} g_h \widehat{p}_k^{h,i} dx \\ &= -\frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} g_h \widehat{p}_h dx, \end{aligned} \tag{5.66}$$

from which we deduce by using $(2.24)_2$ and the uniform boundedness (5.65):

$$\sup_h \frac{5^h}{\left|\Omega^h\right|} \int_{\Omega^h} \left|\nabla p_h\right|^2 dx < +\infty.$$

6. Proof of the main result

6.1. Proof of Theorem 2

6.1.1. Local problems

Let us define new orthonormal basis systems $\left(e^i_m\right)_{m=1,2,3}$; $i = 1, 2, 3$, by

$$e^i_m = \mathcal{R}_i e_m, \tag{6.1}$$

where $e_m = (\delta_{1m}, \delta_{2m}, \delta_{3m})$. We define the rescaled junctions $\mathcal{J}^{+,i}$ and $\mathcal{J}^{-,i}$, for $i = 1, 2, 3$, by

$$\begin{array}{rcl} \mathcal{J}^{+,i} & = & \left\{y = y_1 e^i_1 + y_2 e^i_2 + y_3 e^i_3;\, y_1 > 0,\ (y_2, y_3) \in S\right\}, \\ \mathcal{J}^{-,i} & = & \left\{y = y_1 e^i_1 + y_2 e^i_2 + y_3 e^i_3;\, y_1 < 0,\ (y_2, y_3) \in S\right\}. \end{array} \tag{6.2}$$

We consider the following Leray problems:

$$(\mathcal{P}^+_i)\left\{\begin{array}{rcll} -\mu\Delta w^{+,i} + \nabla\pi^{+,i} & = & 0 & \text{in } \mathcal{J}^{+,i}, \\ \operatorname{div} w^{+,i} & = & 0 & \text{in } \mathcal{J}^{+,i}, \\ w^{+,i} & = & 0 & \text{on } \partial\mathcal{J}^{+,i}, \\ \lim\limits_{y_1\to+\infty} w^{+,i}(y) & = & \Theta(y_2, y_3)\, e^i_1 & \text{in } \mathcal{J}^{+,i}, \end{array}\right. \tag{6.3}$$

and

$$(\mathcal{P}^-_i)\left\{\begin{array}{rcll} -\mu\Delta w^{-,i} + \nabla\pi^{-,i} & = & 0 & \text{in } \mathcal{J}^{-,i}, \\ \operatorname{div} w^{-,i} & = & 0 & \text{in } \mathcal{J}^{-,i}, \\ w^{-,i} & = & 0 & \text{on } \partial\mathcal{J}^{-,i}, \\ \lim\limits_{y_1\to-\infty} w^{-,i}(y) & = & \Theta(y_2, y_3)\, e^i_1 & \text{in } \mathcal{J}^{-,i}, \end{array}\right. \tag{6.4}$$

where Θ is the solution of the auxiliary problem

$$\left\{\begin{array}{rcll} -\mu\Delta\Theta & = & 1 & \text{in } S, \\ \Theta & = & 0 & \text{on } \partial S. \end{array}\right. \tag{6.5}$$

We define, for every $k \in \left\{1, 2, ..., 3^h\right\}$ and $i = 1, 2, 3$, the sequence of functions $\left(w^{h,\pm,i}\right)_h$ by

$$\left\{\begin{array}{rcll} w^{h,+,i}(x) & = & \mathcal{R}_i w^{+,i}\left(\frac{y^{i,k}_{h,1}(x)}{\varepsilon_h}, \frac{y^{i,k}_{h,2}(x)}{\varepsilon_h}, \frac{x_3}{\varepsilon_h}\right) & \text{for } x \in \mathcal{J}^{h,+,i}_k, \\ w^{h,-,i}(x) & = & \mathcal{R}_i w^{-,i}\left(\frac{y^{i,k}_{h,1}(x)-2^{-h}}{\varepsilon_h}, \frac{y^{i,k}_{h,2}(x)}{\varepsilon_h}, \frac{x_3}{\varepsilon_h}\right) & \text{for } x \in \mathcal{J}^{h,-,i}_k, \end{array}\right. \tag{6.6}$$

where the sets $\mathcal{J}^{h,+,i}_k$ and $\mathcal{J}^{h,-,i}_k$ are defined in (2.17) and the coordinates $y^{i,k}_{h,1}$, $y^{i,k}_{h,2}$, x_3; $i = 1, 2, 3$, are related to the variable x through the relations (2.7). Let us define, for every $k \in \left\{1, 2, ..., 3^h\right\}$ and $i = 1, 2, 3$, the intermediate tubes

$$\omega^{h,i}_k = \left\{\begin{array}{c} x = (x_1, x_2, x_3) \in \mathbb{R}^3;\ \left(y^{i,k}_{h,2}(x), x_3\right) \in \varepsilon_h S, \\ \varepsilon_h \ln(1/\varepsilon_h) < y^{i,k}_{h,1}(x) < 2^{-h} - \varepsilon_h \ln(1/\varepsilon_h), \end{array}\right\} \tag{6.7}$$

and their upper and lower bases, respectively,

$$
\begin{aligned}
\gamma_k^{h,+,i} &= \left\{ \begin{array}{l} (x_1, x_2, x_3) \in \mathbb{R}^3;\ \left(y_{h,2}^{i,k}(x), x_3\right) \in \varepsilon S, \\ y_{h,1}^{i,k}(x) = \varepsilon_h \ln(1/\varepsilon_h) \end{array} \right\}, \\
\gamma_k^{h,-,i} &= \left\{ \begin{array}{l} (x_1, x_2, x_3) \in \mathbb{R}^3;\ \left(y_{h,2}^{i,k}(x), x_3\right) \in \varepsilon S, \\ y_{h,1}^{i,k}(x) = 2^{-h} - \varepsilon_h \ln(1/\varepsilon_{h,}) \end{array} \right\}.
\end{aligned} \tag{6.8}
$$

6.1.2. Limit sup inequality

Let $v \in C^1(G)$. Let $x \in \Omega_k^{h,i}$. Then, $(x_1, x_2) \in \left[a_h^{i,k}, b_h^{i,k}\right]$; $i = 1, 2, 3$, for every $h \in \mathbb{N}$ and every $k \in \left\{1, 2, ..., 3^h\right\}$. Let $x_h^{i,k} = \dfrac{a_h^{i,k} + b_h^{i,k}}{2}$. We define the sequence of vector functions $\left(v_k^{0,h,i}\right)_h$ by

$$
v_k^{0,h,i}(x) = \frac{v\left(x_h^{i,k}\right)}{\mathfrak{m}(\Theta)\ \sqrt{5^h}} \Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h}, \frac{x_3}{\varepsilon_h}\right) e_1^i + \psi_k^{h,i}(x), \tag{6.9}
$$

where

$$
\mathfrak{m}(\Theta) = \frac{1}{\pi} \int_S \Theta(y)\, dy \tag{6.10}
$$

and

$$
\psi_k^{h,i}(x) = r_k^{h,i}(v)\, \Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h}, \frac{x_3}{\varepsilon_h}\right) e_1^i, \tag{6.11}
$$

with

$$
r_k^{h,i}(v) = \frac{1}{\sqrt{\ln(1/\varepsilon_h)}} \left(v\left(a_h^{i,k}\right) - v\left(b_h^{i,k}\right)\right). \tag{6.12}
$$

We introduce the function $\phi_h^{i,\pm,k}$ defined by

$$
\left\{ \begin{aligned}
\phi_h^{i,+,k}(x) &= r_k^{h,i}(v)\, w^{i,+}\left(\ln(1/\varepsilon_h), \frac{y_{h,2}^{i,k}(x)}{\varepsilon_h}, \frac{x_3}{\varepsilon_h}\right), \\
\phi_h^{i,-,k}(x) &= r_k^{h,i}(v)\, w^{i,-}\left(-\ln(1/\varepsilon_h), \frac{y_{h,2}^{i,k}(x)}{\varepsilon_h}, \frac{x_3}{\varepsilon_h}\right),
\end{aligned} \right. \tag{6.13}
$$

and the function $\theta_k^{h,\pm,1}$ defined by

$$
\left\{ \begin{aligned}
\theta_k^{h,+,1}(x) &= \varepsilon_h^2 \left(\phi_h^{i,+,k}(x) - \psi_k^{h,i}(x)\right), \\
\theta_k^{h,-,i}(x) &= \varepsilon_h^2 \left(\phi_h^{i,-,k}(x) - \psi_k^{h,i}(x)\right).
\end{aligned} \right. \tag{6.14}
$$

Let $\eta_k^{h,i}$ be the solution of the problem

$$
\left\{ \begin{aligned}
\operatorname{div} \eta_k^{h,i} &= 0 && \text{in } \omega_k^{h,i}, \\
\eta_k^{h,i} &= \varepsilon_h^{-2} \theta^{h,+,i} && \text{on } \gamma_{k,}^{h,+,i}, \\
\eta_k^{h,i} &= \varepsilon_h^{-2} \theta^{h,-,i} && \text{on } \gamma_{k,}^{h,-,i}, \\
\eta_k^{h,i} &= 0 && \text{on } \partial\omega_k^{h,i} \backslash \gamma_k^{h,+,i} \cup \gamma_k^{h,-,i}.
\end{aligned} \right. \tag{6.15}
$$

We define the sequence of test-functions $\left(v_k^{h,i}\right)_h$; $v_k^{h,i} = \left(v_{k,j}^{h,i}\right)_{j=1,2,3}$, by

$$v_k^{h,i} = \begin{cases} v_k^{0,h,i} + \eta_k^{h,i} & \text{in } \omega_k^{h,i}, \\ r_k^{h,i}(v)\, w^{h,\pm,i} & \\ +\dfrac{v\left(x_h^{i,k}\right)}{\mathfrak{m}(\Theta)\sqrt{5^h}}\Theta\left(\dfrac{y_{h,2}^{i,k}(x)}{\varepsilon_h}, \dfrac{x_3}{\varepsilon_h}\right) e_1^i & \text{in } \mathcal{J}_k^{h,\pm,i}. \end{cases} \tag{6.16}$$

We then define the test function v^h in Ω^h by

$$v^h(x) = v_k^{h,i}(x) \text{ for } x \in \Omega_k^{h,i},\, k \in \left\{1, 2, ..., 3^h\right\}, \text{ and } i = 1, 2, 3. \tag{6.17}$$

We have the following results:

Proposition 13. *We have*

1. *$v^h \in V^h$ for ε_h small enough,*
2. *$\left(v^h\right)_h$ τ-converges to $(v, v^*, 0)$, where $v^* = 0$ on the part of G which is perpendicular to $(0, 1)$, $v^* = v\sqrt{3}$ on the part of G which is perpendicular to $\left(-\sqrt{3}/2, 1/2\right)$, and $v^* = -v\sqrt{3}$ on the part of G which is perpendicular to $\left(\sqrt{3}/2, 1/2\right)$,*
3. *if $\sigma \in (0, \infty)$, then*

$$\lim_{h\to\infty} F_h\left(v^h\right) = \frac{\mu\pi}{\mathfrak{m}(\Theta)\mathcal{H}^d(G)}\int_G v^2 d\mathcal{H}^d + \frac{2\mu\pi\mathfrak{m}(\Theta)}{3\sigma}\int_G d\mathcal{L}_G(v).$$

Proof. 1. Introducing the variables $y_2 = \dfrac{y_{h,2}^{i,k}(x)}{\varepsilon_h}$ and $y_3 = \dfrac{x_3}{\varepsilon_h}$, we have, for ε_h small enough, that

$$\begin{aligned} \int_{\gamma_k^{h,+,i}} \theta^{h,+,i}.e_1^i &= -r_k^{h,i}(v)\,\varepsilon_h^2 \int_S \begin{pmatrix} w^{+,i}\left(\ln(1/\varepsilon_h), y_2, y_3\right) \\ -\Theta(y_2, y_3)\, e_1^i \end{pmatrix}.e_1^i dy \\ &= -r_k^{h,i}(v)\,\varepsilon_h^2 \int_S \left(\Theta(y_2, y_3) - \Theta(y_2, y_3)\right) dy \\ &= 0, \end{aligned} \tag{6.18}$$

and

$$\begin{aligned} \int_{\gamma_k^{h,-,i}} \theta^{h,-,i}.e_1^i &= -r_k^{h,i}(v)\,\varepsilon_h^2 \int_S \begin{pmatrix} w^{-,i}\left(-\ln(1/\varepsilon_h), y_2, y_3\right) \\ -\Theta(y_2, y_3)\, e_1^i \end{pmatrix}.e_1^i dy \\ &= -r_k^{h,i}(v)\,\varepsilon_h^2 \int_S \left(\Theta(y_2, y_3) - \Theta(y_2, y_3)\right) dy \\ &= 0. \end{aligned} \tag{6.19}$$

This implies that problem (6.15) is solvable. On the other hand, using [53, Theorem VI.1.2], there exists $\tau > 0$ such that, for any $i = 1, 2, 3$ and every $y \in \mathcal{J}^{\pm,i}$,

$$\left|w^{\pm,i}(y) - \Theta(y_2, y_3)\, e_1^i\right| + \left|\nabla w^{\pm,i}(y) - \nabla\left(\Theta(y_2, y_3)\, e_1^i\right)\right| \le Ce^{-\tau|y|}, \tag{6.20}$$

from which we deduce that

$$\left|\theta_k^{h,\pm,i}(x)\right| \le \frac{C\varepsilon_h^3\sqrt{\varepsilon_h}}{\sqrt{\ln(1/\varepsilon_h)}},\; \left|\nabla\theta_k^{h,\pm,i}(x)\right| \le \frac{C\varepsilon_h^2\sqrt{\varepsilon_h}}{\sqrt{\ln(1/\varepsilon_h)}}, \tag{6.21}$$

which implies that

$$\begin{array}{rcl}\left\|\theta_k^{h,\pm,i}\right\|_{H^{1/2}\left(\gamma_k^{h,\pm,i}\right)} & \leq & C\sqrt{\left\|\theta_k^{h,\pm,i}\right\|_{L^2\left(\gamma_k^{h,i}\right)}\left\|\nabla\theta_k^{h,\pm,i}\right\|_{L^2\left(\gamma_k^{h,\pm,i}\right)}} \\ & \leq & \dfrac{C\varepsilon_h^{7/2}\sqrt{\varepsilon_h}}{\sqrt{\ln\left(1/\varepsilon_h\right)}},\end{array} \tag{6.22}$$

and, using [54, Lemma 9],

$$\begin{array}{rcl}\left\|\nabla\eta_k^{h,i}\right\|_{L^2\left(\omega_k^{h,i}\right)} & \leq & \dfrac{C}{\varepsilon_h^2\sqrt{\varepsilon_h}}\left\|\theta_k^{h,\pm,i}\right\|_{H^{1/2}\left(\gamma_k^{h,\pm,i}\right)} \\ & \leq & \dfrac{C\varepsilon_h\sqrt{\varepsilon_h}}{\sqrt{\ln\left(1/\varepsilon_h\right)}}.\end{array} \tag{6.23}$$

Since $\operatorname{div}\eta_k^{h,i} = 0$, $\operatorname{div}_y w^{+,i} = \operatorname{div}_y w^{-,i} = 0$, for every $i = 1, 2, 3$, and Θ is independent of y_1, we have

$$\operatorname{div} v_k^{h,i} = 0, \text{ for every } i = 1, 2, 3.$$

Therefore, for ε_h small enough, $v^h \in V^h$.

2. Let $\varphi \in C_0\left(\mathbb{R}^3\right)$. We have

$$\begin{array}{l}\lim\limits_{h\to\infty}\displaystyle\int_{\mathbb{R}^3}\varphi(x)\,\psi_k^{h,i}(x)\ \sqrt{5^h}\frac{\mathbf{1}_{\Omega^h}(x)}{3\left|\Omega^h\right|}dx \\ = \lim\limits_{h\to\infty}\dfrac{\sqrt{5^h}}{3^{h+1}\pi\sqrt{\ln\left(1/\varepsilon_h\right)}}\displaystyle\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\left\{\begin{array}{l}\left(v\left(a_h^{i,k}\right)-v\left(b_h^{i,k}\right)\right) \\ \times\varphi\left(x_h^{i,k},0\right)\displaystyle\int_S\Theta\left(y_2,y_3\right)dy_2dy_3\end{array}\right\} \\ = 0,\end{array} \tag{6.24}$$

and

$$\begin{array}{l}\lim\limits_{h\to\infty}\dfrac{\sqrt{5^h}}{3\mathfrak{m}(\Theta)\left|\Omega^h\right|\ \sqrt{5^h}}\displaystyle\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\mathcal{J}_k^{h,\pm,i}}\left(\begin{array}{c}\varphi(x)\,v\left(x_h^{i,k}\right) \\ \times\Theta\left(\dfrac{y_{h,2}^{i,k}(x)}{\varepsilon_h},\dfrac{x_3}{\varepsilon_h}\right).e_1^i\end{array}\right)dx \\ = \lim\limits_{h\to\infty}\dfrac{\varepsilon_h\ln\left(1/\varepsilon_h\right)}{3^{h+1}\pi\mathfrak{m}(\Theta)}\displaystyle\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\varphi\left(x_h^{i,k}\right)v\left(x_h^{i,k}\right).e_1^i\int_S\Theta\left(y_2,y_3\right)dy_2dy_3 \\ = 0.\end{array} \tag{6.25}$$

Then, using the estimate (6.20) for $w^{\pm,i}(y)$, the estimates (6.21)–(6.22) for $\theta_k^{h,\pm,i}$, the estimate (6.23) for $\eta_k^{h,i}$, and the inequality (5.3) applied to $\eta_k^{h,i}$, we deduce that

$$\begin{aligned}
&\lim_{h\to\infty}\int_{\mathbb{R}^3}\varphi(x)\ \sqrt{5^h}v^h\frac{\mathbf{1}_{\Omega^h}(x)}{3\left|\Omega^h\right|}dx\\
&=\lim_{h\to\infty}\frac{1}{3\left|\Omega^h\right|\mathfrak{m}(\Theta)}\int_{\Omega^h}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\left(\begin{array}{c}\varphi\left(x_h^{i,k},0\right)v\left(x_h^{i,k}\right)\\ \times\Theta\left(\dfrac{y_{h,2}^{i,k}(x)}{\varepsilon_h},\dfrac{x_3}{\varepsilon_h}\right)e_1^i\end{array}\right)dx\\
&=\lim_{h\to\infty}\frac{1}{\pi\mathfrak{m}(\Theta)3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\left(\begin{array}{c}\varphi\left(x_h^{i,k},0\right)v\left(x_h^{i,k}\right)\\ \times\left(\int_S\Theta(y_2,y_3)\,dy_2dy_3\right)e_1^i\end{array}\right)\\
&=\frac{1}{\mathcal{H}^d(G)}\int_G\varphi(s,0)(v(s),v^*(s),0)\,d\mathcal{H}^d(s).
\end{aligned}\tag{6.26}$$

3. Let us suppose that $\sigma\in(0,\infty)$. Then, in virtue of the estimates (6.20)–(6.23), we have that

$$\begin{aligned}
&\lim_{h\to\infty}\frac{5^h}{3^{h+1}\,\mathrm{Re}_h}\int_{\Omega^h}\left|\nabla v^h\right|^2dx\\
&=\lim_{h\to\infty}\frac{\mu 2^h}{\mathfrak{m}^2(\Theta)3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\omega_k^{h,i}}\left|\begin{array}{c}\nabla\left(\Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h},\frac{x_3}{\varepsilon_h}\right)e_1^i\right)\\ \times v\left(x_h^{i,k}\right)\end{array}\right|^2dx\\
&+\lim_{h\to\infty}\frac{\mu 2^h5^h}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\omega_k^{h,i}}\left|\nabla\psi_k^{h,i}(x)\right|^2dx\\
&+\lim_{h\to\infty}\frac{5^h}{3^{h+1}\,\mathrm{Re}_{\mathbf{j},h}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\mathcal{J}_k^{h,+,i}}\left(r_k^{h,i}(v)\right)^2\left|\nabla w^{h,+,i}\right|^2dx\\
&+\lim_{h\to\infty}\frac{5^h}{3^{h+1}\,\mathrm{Re}_{\mathbf{j},h}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\mathcal{J}_k^{h,-,i}}\left(r_k^{h,i}(v)\right)^2\left|\nabla w^{h,-,i}\right|^2dx\\
&+\lim_{h\to\infty}\frac{1}{3^{h+1}\mathfrak{m}^2(\Theta)\,\mathrm{Re}_{\mathbf{j},h}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\mathcal{J}_k^{h,i}}\left|\begin{array}{c}\nabla\left(\Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h},\frac{x_3}{\varepsilon_h}\right)e_1^i\right)\\ \times v\left(x_h^{i,k}\right)\end{array}\right|^2dx.
\end{aligned}\tag{6.27}$$

where $\mathcal{J}_k^{h,i}=\mathcal{J}_k^{h,+,i}\cup\mathcal{J}_k^{h,-,i}$. Then, as

$$\left|\nabla\left(\Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h},\frac{x_3}{\varepsilon_h}\right)e_1^i\right)\right|^2=\frac{1}{\varepsilon_h^2}\left|\nabla\Theta(y_2,y_3)\right|^2,\tag{6.28}$$

and $\int_S |\nabla\Theta(y_2,y_3)|^2\,dy_2dy_3 = \pi\mathfrak{m}(\Theta)$, we deduce that

$$\begin{aligned}
&\lim_{h\to\infty}\frac{\mu 2^h}{3^{h+1}\mathfrak{m}^2(\Theta)}\sum_{\substack{k=1\\i=1,2,3}}^{3^h}\int_{\omega_k^{h,i}}\left|\begin{array}{c} v\left(x_h^{i,k}\right)\\ \times\nabla\left(\Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h},\frac{x_3}{\varepsilon_h}\right)e_1^i\right)\end{array}\right|^2 dx\\
&=\lim_{h\to\infty}\frac{\mu}{\mathfrak{m}^2(\Theta)\,3^{h+1}}\sum_{\substack{k=1\\i=1,2,3}}^{3^h}v^2\left(x_h^{i,k}\right)\int_S|\nabla\Theta(y_2,y_3)|^2\,dy_2dy_3\\
&=\lim_{h\to\infty}\frac{\mu\pi}{\mathfrak{m}(\Theta)}\frac{1}{3^{h+1}}\sum_{\substack{k=1\\i=1,2,3}}^{3^h}v^2\left(x_h^{i,k}\right)\\
&=\frac{\mu\pi}{\mathfrak{m}(\Theta)\,\mathcal{H}^d(G)}\int_G v^2\,d\mathcal{H}^d.
\end{aligned}\tag{6.29}$$

After some computations, we infer that

$$\begin{aligned}
&\lim_{h\to\infty}\frac{\mu 2^h 5^h}{3^{h+1}}\sum_{\substack{k=1\\i=1,2,3}}^{3^h}\int_{\omega_k^{h,i}}\left|\nabla\psi_k^{h,i}(x)\right|^2 dx\\
&=\lim_{h\to\infty}\frac{\mu 5^h}{3^{h+1}\ln(1/\varepsilon_h)}\sum_{\substack{k=1\\i=1,2,3}}^{3^h}\left\{\begin{array}{c}\left(v\left(a_h^{i,k}\right)-v\left(b_h^{i,k}\right)\right)^2\\ \times\int_S|\nabla\Theta(y_2,y_3)|^2\,dy_2dy_3\end{array}\right\}\\
&=0,
\end{aligned}\tag{6.30}$$

and, for the last limit in (6.27),

$$\begin{aligned}
&\lim_{h\to\infty}\frac{1}{3^{h+1}\mathfrak{m}^2(\Theta)\operatorname{Re}_{\mathbf{j},h}}\sum_{\substack{k=1\\i=1,2,3}}^{3^h}\int_{\mathcal{J}_k^{h,i}}\left|\begin{array}{c} v\left(x_h^{i,k}\right)\\ \times\nabla\left(\Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h},\frac{x_3}{\varepsilon_h}\right)e_1^i\right)\end{array}\right|^2 dx\\
&=0.
\end{aligned}\tag{6.31}$$

Using once again the estimate (6.20), we deduce that

$$\begin{aligned}
&\lim_{h\to\infty}\frac{5^h}{3^{h+1}\operatorname{Re}_{\mathbf{j},h}}\sum_{\substack{k=1\\i=1,2,3}}^{3^h}\int_{\mathcal{J}_k^{h,+,i}}\left(r_k^{h,i}(v)\right)^2\left|\nabla w^{h,+,i}\right|^2 dx\\
&=\lim_{h\to\infty}\frac{5^h}{3^{h+1}\operatorname{Re}_{\mathbf{j},h}\ln(1/\varepsilon_h)}\sum_{\substack{k=1\\i=1,2,3}}^{3^h}\left(\begin{array}{c}\left(v\left(a_h^{i,k}\right)-v\left(b_h^{i,k}\right)\right)^2\\ \times\int_{\mathcal{J}_k^{h,+,i}}\left|\nabla w^{h,+,i}\right|^2\end{array}\right)dx\\
&=\lim_{h\to\infty}\frac{\varepsilon_h}{3\operatorname{Re}_{\mathbf{j},h}}\left(\frac{5}{3}\right)^h\sum_{\substack{k=1\\i=1,2,3}}^{3^h}\left\{\begin{array}{c}\left(v\left(a_h^{i,k}\right)-v\left(b_h^{i,k}\right)\right)^2\\ \times\int_S|\nabla\Theta(y_2,y_3)|^2\,dy_2dy_3\end{array}\right\}\\
&=\frac{\pi\mathfrak{m}(\Theta)}{3\sigma}\int_G d\mathcal{L}_G(v),
\end{aligned}\tag{6.32}$$

and

$$\lim_{h\to\infty}\frac{5^h}{3^{h+1}\,\mathrm{Re}_{\mathbf{j},h}}\sum_{k=1}^{3^h}\sum_{i=1,2,3}\int_{\mathcal{J}_k^{h,-,i}}\left(r_k^{h,i}(v)\right)^2\left|\nabla w^{h,-,i}\right|^2 dx = \frac{\pi\mathfrak{m}(\Theta)}{3\sigma}\int_G d\mathcal{L}_G(v)\,. \tag{6.33}$$

Now, combining (6.27)–(6.33), we get the result.

Proposition 14. *If $\sigma\in(0,+\infty)$, then for every $v\in V^\infty$, there exists a sequence $\left(v^h\right)_h$, with $v^h\in V^h$ and $\left(v^h\right)_h$ τ-converges to (v,v^*,v^{**}), where $v^{**}=0$, $v^*=0$ on the part of G which is perpendicular to $(0,1)$, $v^*=v\sqrt{3}$ on the part of G which is perpendicular to $\left(-\sqrt{3}/2,1/2\right)$, and $v^*=-v\sqrt{3}$ on the part of G which is perpendicular to $\left(\sqrt{3}/2,1/2\right)$, such that*

$$\limsup_{h\to\infty}F_h\left(v^h\right)\le F_\infty(v)\,.$$

Proof. Let $v\in V^\infty$. Let $(v_m)_m\subset C^1(G)$ such that $v_m \underset{m\to\infty}{\longrightarrow} v$ with respect to the norm (4.5). We define the sequence $\left(v^{m,h}\right)_{m,h}$ by replacing in (6.9), (6.16), and (6.17) v by v_m. Then, according to Proposition 13, the sequence $\left(v^{m,h}\right)_{m,h}$ τ-converges to $(v_m,v_m^*,0)$, where $v_m^*=0$ on the part of G which is perpendicular to $(0,1)$, $v_m^*=v_m\sqrt{3}$ on the part of G which is perpendicular to $\left(-\sqrt{3}/2,1/2\right)$, $v_m^*=-v_m\sqrt{3}$ on the part of G which is perpendicular to $\left(\sqrt{3}/2,1/2\right)$, and

$$\lim_{h\to\infty}F_h\left(v^{m,h}\right)\le F_\infty(v_m)\,.$$

The continuity of F_∞ implies that $\lim_{m\to\infty}\lim_{h\to\infty}F_h\left(v^{m,h}\right)=F_\infty(v)$. The topology τ being metrizable, we deduce, using a diagonalization argument (see [14, Corollary 1.18]), that the sequence $\left(v^h\right)_h=\left(v^{h,m(h)}\right)_h$; $\lim_{h\to\infty}m(h)=+\infty$, τ-converges to $(v,v^*,0)$, with $v^*=0$ on the part of G which is perpendicular to $(0,1)$, $v^*=v\sqrt{3}$ on the part of G which is perpendicular to $\left(-\sqrt{3}/2,1/2\right)$, $v^*=-v\sqrt{3}$ on the part of G which is perpendicular to $\left(\sqrt{3}/2,1/2\right)$, and

$$\limsup_{h\to\infty}F_h\left(v^h\right)\le F_\infty(v)\,.$$

6.1.3. Limit inf inequality

Proposition 15. *If $\sigma\in(0,+\infty)$, then for every sequence $\left(v^h\right)_h$, such that $v^h\in V^h$ and $\left(v^h\right)_h$ τ-converges to (v,v^*,v^{**}), we have $v\in V^\infty$, $v^{**}=0$ on G, $v^*=0$ on the part of G which is perpendicular to $(0,1)$, $v^*=v\sqrt{3}$ on the part of G which is perpendicular to $\left(-\sqrt{3}/2,1/2\right)$, $v^*=-v\sqrt{3}$ on the part of G which is perpendicular to $\left(\sqrt{3}/2,1/2\right)$, and*

$$\liminf_{h\to\infty}F_h\left(v^h\right)\ge F_\infty(v)\,.$$

Proof. Observe that if $\sup_h F_h\left(v^h\right) = +\infty$, then the lim inf inequality is trivial. We suppose that $\sup_h F_h\left(v^h\right) < +\infty$ and, using some regularity argument, we may suppose that $v^h \in V^h \cap H^2\left(\Omega^h, \mathbb{R}^3\right)$. Then, according to Proposition 8, we have that $v \in L^2_{\mathcal{H}^d}(G)$, $v^{**} = 0$ on G, $v^* = 0$ on the part of G which is perpendicular to $(0, 1)$, $v^* = v\sqrt{3}$ on the part of G which is perpendicular to $\left(-\sqrt{3}/2, 1/2\right)$, $v^* = -v\sqrt{3}$ on the part of G which is perpendicular to $\left(\sqrt{3}/2, 1/2\right)$, and, according to Proposition 9,

$$\langle \operatorname{div}_Z(v), \varphi\rangle = 0, \forall\varphi \in H_Z(G), \tag{6.34}$$

where $H_Z(G)$ is the space defined in Definition 2_1. Let $(v_m)_m \subset C^1(G)$ such that $v_m \underset{m\to\infty}{\longrightarrow} v$ with respect to the norm $L^2_{\mathcal{H}^d}(G)$-strong. We define the sequence $\left(v^{m,h}\right)_{m,h}$ by replacing v by v_m in test-functions (6.9), (6.16), and (6.17). We deduce from the definition of the subdifferentiability of convex functionals that

$$\begin{aligned}\frac{5^h}{3^{h+1}\operatorname{Re}_h}\int_{\Omega^h}\left|\nabla v^h\right|^2 dx \geq \frac{5^h}{3^{h+1}\operatorname{Re}_h}\int_{\Omega^h}\left|\nabla v^{m,h}\right|^2 dx\\ +2\frac{5^h}{3^{h+1}\operatorname{Re}_h}\int_{\Omega^h}\nabla\left(v^{m,h}\right).\nabla\left(v^h - v^{m,h}\right)dx.\end{aligned} \tag{6.35}$$

We then compute

$$\begin{aligned}
&\lim_{h\to\infty}\frac{5^h}{3^{h+1}\operatorname{Re}_h}\int_{\Omega^h}\nabla\left(v^{m,h}\right).\nabla\left(v^h - v^{m,h}\right)dx\\
&= \lim_{h\to\infty}\frac{5^h\operatorname{Re}_h^{-1}}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\omega_k^{h,i}}\nabla\eta_k^{h,i}.\nabla\left(v^h - v^{m,h}\right)dx\\
&+\lim_{h\to\infty}\frac{5^h\operatorname{Re}_h^{-1}}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\omega_k^{h,i}}\nabla\psi_k^{h,i}(x).\nabla\left(v^h - v^{m,h}\right)dx\\
&-\lim_{h\to\infty}\frac{\sqrt{5^h}\operatorname{Re}_h^{-1}}{\mathfrak{m}(\Theta)\,3^{h+1}\varepsilon_h^2}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\omega_k^{h,i}}\left(\begin{array}{c}\Delta\Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h}, \frac{x_3}{\varepsilon_h}\right)\\ \times v_m\left(x_h^{i,k}\right)\left(v^h - v^{m,h}\right).e_1^i\end{array}\right)dx\\
&+\lim_{h\to\infty}\frac{5^h\operatorname{Re}_{\mathbf{j},h}^{-1}}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h} r_k^{h,i}(v)\int_{\mathcal{J}_k^{h,+,i}}\nabla w^{h,+,i}.\nabla\left(v^h - v^{m,h}\right)dx\\
&+\lim_{h\to\infty}\frac{5^h\operatorname{Re}_{\mathbf{j},h}^{-1}}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h} r_k^{h,i}(v)\int_{\mathcal{J}_k^{h,-,i}}\nabla w^{h,-,i}.\nabla\left(v^h - v^{m,h}\right)dx\\
&+\lim_{h\to\infty}\frac{\sqrt{5^h}\operatorname{Re}_{\mathbf{j},h}^{-1}}{\mathfrak{m}(\Theta)\,3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\mathcal{J}_k^{h,i}}\left(\begin{array}{c}\left(\nabla v^h - \nabla v^{m,h}\right)\\ .\nabla\left(\Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h}, \frac{x_3}{\varepsilon_h}\right)e_1^i\right)\end{array}\right)dx.
\end{aligned} \tag{6.36}$$

Using the estimate (6.23), we deduce that

$$\begin{aligned}
&\frac{5^h}{3^{h+1}\operatorname{Re}_h}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\omega_k^{h,i}}\nabla\eta_k^{h,i}.\nabla\left(v^h-v^{m,h}\right)dx\\
&\leq\frac{C\varepsilon_h\sqrt{\varepsilon_h}}{\sqrt{\ln\left(1/\varepsilon_h\right)}}\sqrt{\frac{5^h}{3^{h+1}\operatorname{Re}_h}}\left\{\frac{5^h}{3^{h+1}\operatorname{Re}_h}\int_{\Omega^h}\left|\nabla\left(v^h-v^{m,h}\right)\right|^2dx\right\}^{1/2}\\
&\leq\frac{C\varepsilon_h\sqrt{\varepsilon_h}}{\sqrt{\ln\left(1/\varepsilon_h\right)}}\sqrt{\frac{5^h}{3^{h+1}\operatorname{Re}_h}},
\end{aligned}\tag{6.37}$$

from which we deduce that

$$\lim_{h\to\infty}\frac{5^h}{3^{h+1}\operatorname{Re}_h}\sum_{k=1}^{3^h}\sum_{i=1,2,3}\int_{\omega_k^{h,i}}\nabla\eta_k^{h,i}.\nabla\left(v^h-v^{m,h}\right)dx=0.\tag{6.38}$$

On the other hand, using the fact that $\varepsilon_h^2 3^h\operatorname{Re}_h\approx\frac{\left|\Omega^h\right|}{\pi\mu}$ in $\omega_k^{h,i}$ and according to the problem (6.5) of which Θ is the solution, we deduce that

$$\begin{aligned}
&\lim_{h\to\infty}\frac{\sqrt{5^h}\operatorname{Re}_h^{-1}}{\mathfrak{m}(\Theta)\,3^{h+1}\varepsilon_h^2}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\omega_k^{h,i}}\begin{pmatrix}\Delta\Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h},\frac{x_3}{\varepsilon_h}\right)\\ \times v_n\left(x_h^{i,k}\right)\left(v^h-v^{m,h}\right).e_1^i\end{pmatrix}dx\\
&=\lim_{h\to\infty}\frac{\mu\pi\sqrt{5^h}}{\mathfrak{m}(\Theta)\,3\left|\Omega^h\right|}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\omega_k^{h,i}}\begin{pmatrix}\Delta\Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h},\frac{x_3}{\varepsilon_h}\right)\\ \times v_n\left(x_h^{i,k}\right)\left(v^h-v^{m,h}\right).e_1^i\end{pmatrix}dx\\
&=\frac{-\pi}{\mathcal{H}^d(G)}\int_G v_m\left(v-v_m\right)d\mathcal{H}^d.
\end{aligned}\tag{6.39}$$

Using the limits (6.24) and (6.30), and the fact that

$$\sup_h F_h\left(v^h-v^{m,h}\right)<+\infty,\tag{6.40}$$

we deduce that

$$\lim_{h\to\infty}\frac{5^h\operatorname{Re}_h^{-1}}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\omega_k^{h,i}}\nabla\psi_k^{h,i}(x).\nabla\left(v^h-v^{m,h}\right)dx=0.\tag{6.41}$$

Analogously, using the estimate (6.20), the equations (6.5), the expression (6.12) of $r_k^{h,i}(v)$, and the estimate (6.40), we get

$$\begin{aligned}
&\lim_{h\to\infty}\frac{5^h\operatorname{Re}_{\mathbf{j},h}^{-1}}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}r_k^{h,i}(v)\int_{\mathcal{J}_k^{h,+,i}}\nabla w^{h,+,i}.\nabla\left(v^h-v^{m,h}\right)dx\\
&=-\lim_{h\to\infty}\frac{5^h\operatorname{Re}_{\mathbf{j},h}^{-1}}{3^{h+1}\varepsilon_h^2}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\mathcal{J}_k^{h,+,i}}\begin{pmatrix}r_k^{h,i}(v)\,\Delta\Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h},\frac{x_3}{\varepsilon_h}\right)\\ \times\left(v^h-v^{m,h}\right).e_1^i\end{pmatrix}dx\\
&=0,
\end{aligned}\tag{6.42}$$

and, similarly,

$$\lim_{h\to\infty}\frac{5^h \operatorname{Re}_{\mathbf{j},h}^{-1}}{3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h} r_k^{h,i}(v)\int_{\mathcal{J}_k^{h,-,i}}\nabla w^{h,-,i}.\nabla\left(v^h - v^{m,h}\right)dx \\ = 0. \tag{6.43}$$

As $\sup_h \frac{5^h}{3^{h+1}\operatorname{Re}_h}\int_{\Omega^h}\left|\nabla\left(v^h - v^{m,h}\right)\right|^2 dx < +\infty$, we have

$$\lim_{h\to\infty}\frac{\sqrt{5^h}\operatorname{Re}_{\mathbf{j},h}^{-1}}{\mathfrak{m}(\Theta)\,3^{h+1}}\sum_{\substack{k=1\\ i=1,2,3}}^{3^h}\int_{\mathcal{J}_k^{h,i}}\left(\begin{array}{c}\left(\nabla v^h - \nabla v^{m,h}\right)\\ .\nabla\left(\Theta\left(\frac{y_{h,2}^{i,k}(x)}{\varepsilon_h},\frac{x_3}{\varepsilon_h}\right)e_1^i\right)\end{array}\right)dx \\ = 0. \tag{6.44}$$

In addition, owing to Proposition 13, we have

$$\lim_{h\to\infty}\frac{5^h}{3^{h+1}\operatorname{Re}_h}\int_{\Omega^h}\left|\nabla v^{m,h}\right|^2 dx = F_\infty(v_m). \tag{6.45}$$

Thus, combining (6.35)–(6.45), we deduce that

$$\begin{array}{l}\liminf_{h\to\infty} F_h\left(v^h\right) \geq F_\infty(v_m)\\ +\frac{2\mu\pi}{\mathcal{H}^d(G)}\int_G v_m(v - v_m)\,d\mathcal{H}^d.\end{array} \tag{6.46}$$

Then, letting m tend to ∞, we obtain

$$\liminf_{h\to\infty} F_h\left(v^h\right) \geq F_\infty(v),$$

and, as a consequence, $\mathcal{E}_G(v) < +\infty$. Thus, $v \in \mathcal{D}_{\mathcal{E}}$ and, taking into account (6.34), we have that $v \in V^\infty$.

6.2. *Proof of Theorem 3*

Proof. 1. Let $\left(u^h, p_h\right)$ be a solution of problem (2.22) with boundary conditions (2.25). According to Lemma 12 and Proposition 8 there exists a subsequence of $\left(u^h\right)_h$, still denoted as $\left(u^h\right)_h$, such that

$$\sqrt{5^h}u^h\frac{\pi\mathbf{1}_{\Omega^h}(x)}{3\left|\Omega^h\right|}dx \underset{h\to\infty}{\overset{*}{\rightharpoonup}} (u, u^*, 0)\frac{d\mathcal{H}^d(s)\otimes\delta_0(x_3)}{\mathcal{H}^d(G)} \text{ in } \mathcal{M}\left(\mathbb{R}^3\right), \tag{6.47}$$

with $u^* = 0$ on the part of G which is perpendicular to $(0,1)$, $u^* = u\sqrt{3}$ on the part of G which is perpendicular to $\left(-\sqrt{3}/2, 1/2\right)$, and $u^* = -u\sqrt{3}$ on the part of G which is perpendicular to $\left(\sqrt{3}/2, 1/2\right)$. As the boundary $\partial\Omega^h$ is C^2, the velocity u^h is at least in $H^2\left(\Omega^h\right)$. Thus, according to Proposition 9, we have that

$$\langle \operatorname{div}_Z(u), \varphi\rangle = 0, \ \forall\varphi \in H_Z(G). \tag{6.48}$$

On the other hand, since u^h is the unique velocity solution of problem (2.31), we deduce from Theorem 2 and [15, Theorem 7.8], that the whole sequence $\left(u^h\right)_h$ verifies the convergence (6.47),

$$\lim_{h\to\infty} F_h\left(u^h\right) = F_\infty(u), \tag{6.49}$$

and, taking into account (6.48), we deduce that $u \in V^\infty$. In addition, using Lemma 12 and the proof of Proposition 8, we have that

$$\sqrt{5^h}\widehat{p}_h \frac{\pi \mathbf{1}_{\Omega^h}(x)}{3\left|\Omega^h\right|}dx \underset{h\to\infty}{\overset{*}{\rightharpoonup}} p\frac{d\mathcal{H}^d(s)\otimes\delta_0(x_3)}{\mathcal{H}^d(G)} \text{ in } \mathcal{M}\left(\mathbb{R}^3\right), \tag{6.50}$$

with $p \in H_Z(G)$, and, using the uniform boundedness (5.56),

$$\sqrt{5^h} f_h \frac{\pi \mathbf{1}_{\Omega^h}(x)}{3\left|\Omega^h\right|}dx \underset{h\to\infty}{\overset{*}{\rightharpoonup}} f\frac{d\mathcal{H}^d(s)\otimes\delta_0(x_3)}{\mathcal{H}^d(G)} \text{ in } \mathcal{M}\left(\mathbb{R}^3\right), \tag{6.51}$$

with $f \in L^2_{\mathcal{H}^d}\left(G,\mathbb{R}^3\right)$. Using Proposition 9 and Lemma 12_2, we deduce that, for every $v \in V^\infty$,

$$\begin{aligned}\int_G (v,v^*).nZ\nabla p.nd\boldsymbol{v} &= \int_G vn.Z\nabla pd\boldsymbol{v}\\ &= \int_G vZ\nabla p.nd\boldsymbol{v}\\ &= 0,\end{aligned} \tag{6.52}$$

where $n = (1,0)$ on the horizontal part of G, $n = \left(1/2,\ \sqrt{3}/2\right)$ on the part of G which is perpendicular to $\left(-\sqrt{3}/2, 1/2\right)$, and $n = \left(1/2, -\sqrt{3}/2\right)$ on the part of G which is perpendicular to $\left(\sqrt{3}/2, 1/2\right)$.

2. According to Theorem 2 and [15, Theorem 7.8], u is the solution of the problem

$$\min_{v\in V^\infty}\left\{\begin{array}{l}\dfrac{\mu\pi}{\mathfrak{m}(\Theta)\mathcal{H}^d(G)}\displaystyle\int_G v^2 d\mathcal{H}^d + \dfrac{2\mu\pi\mathfrak{m}(\Theta)}{3\sigma}\displaystyle\int_G \nabla v.Z\nabla vd\boldsymbol{v}\\ -\dfrac{2}{\mathcal{H}^d(G)}\displaystyle\int_G f.(v,v^*,0)\,d\mathcal{H}^d\end{array}\right\}. \tag{6.53}$$

Then, using Lemma 4 and the fact that $\int_G vZ\nabla p.nd\boldsymbol{v} = 0$ and $(v,v^*) = v.n$, for every $v \in V^\infty$, we deduce from (6.53) that, for every $v \in V^\infty$,

$$\begin{aligned}&-\frac{4\mu\pi\mathfrak{m}(\Theta)}{3\sigma\mathcal{H}^d(G)}\int_G \Delta_G(u)\,vd\mathcal{H}^d(s)\\ &+\frac{2\mu\pi}{\mathfrak{m}(\Theta)\mathcal{H}^d(G)}\int_G uvd\mathcal{H}^d(s) + 2\int_G vZ\nabla p.nd\boldsymbol{v}\\ &= \frac{2\mathcal{H}^d}{\mathcal{H}^d(G)}\int_G vf.nd\mathcal{H}^d,\end{aligned} \tag{6.54}$$

where, by abuse of notation, $f.n = (f_1, f_2).n$. Therefore, (u,p) is the solution (with p up to an additive constant) of the following problem:

$$\begin{aligned}-\frac{2\mu\pi\mathfrak{m}(\Theta)\mathcal{H}^d}{3\sigma\mathcal{H}^d(G)}\Delta_G(u) + \frac{\mu\pi\mathcal{H}^d}{\mathfrak{m}(\Theta)\mathcal{H}^d(G)}u + \boldsymbol{v}Z\nabla p.n&\\ = \frac{\mathcal{H}^d}{\mathcal{H}^d(G)}f.n \text{ in } G,&\end{aligned}$$

which completes the proof of Theorem 3.

7. Conclusion

In this paper, we considered the motion of a viscous incompressible fluid in a varying bounded domain consisting of branching cylindrical pipes whose axes are line segments that form a network of pre-fractal polygonal curves G_h obtained after h-iterations of the contractive similarities of the standard Sierpinski gasket. We assumed that these pipes are narrow axisymmetric tubes of radius ε_h very small with respect to the length 2^{-h} of each side of G_h. We supposed that the fluid flow is driven by some volumic forces and governed by Stokes equations with continuity of the velocity at the interfaces separating the junction zones from the rest of the pipes, homogeneous Dirichlet boundary condition for the velocity, and homogeneous Neumann boundary condition for the pressure on the wall of the tubes. The flow in each pipe is split into two streams: boundary layers flow in junction zones of length $\varepsilon_h \ln(1/\varepsilon_h) \ll 2^{-h}$ and laminar flow in the rest of the pipe. We assumed that the flow in the junction zones is controlled by a typical Reynolds number $\mathrm{Re}_{\mathbf{j},h}$. Using Γ-convergence methods, we studied the asymptotic behavior of the fluid flowing in the branching tubes as the radius of the tubes tends to zero and the sequence of the pre-fractal curves converges in the Hausdorff metric to the Sierpinski gasket. According to critical values taken by $\mathrm{Re}_{\mathbf{j},h}$, we derived three uncommon effective models of fluid flows in the Sierpinski gasket:

1. a singular Brinkman equation if $\mathrm{Re}_{\mathbf{j},h} = O(\varepsilon_h)$,
2. a singular Darcy flow if $\mathrm{Re}_{\mathbf{j},h} = O(1)$ or $\mathrm{Re}_{\mathbf{j},h} \longrightarrow \infty$ as $h \longrightarrow \infty$,
3. a flow with constant velocity if $\mathrm{Re}_{\mathbf{j},h} = O\left(\varepsilon_h^{\alpha}\right)$ with $\alpha > 1$.

As far as the modeling is concerned, fractal branching pipe networks have to be considered to describe fluid flows in various complex geometrical configurations. An important field to which this model is closely related is the behavior of fluid flows in some physiological structures such as the blood circulation through arterial networks. Our model may serve as a starting point for further investigations in this area.

Author contributions

Haifa El Jarroudi: Writing-original draft, Writing-review and editing, Methodology, Formal Analysis; Mustapha El Jarroudi: Writing-original draft, Writing-review and editing, Methodology, Supervision.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors wish to express their gratitude to the anonymous referee for giving a number of valuable comments and helpful suggestions, which improve the presentation of sentation of the manuscript significantly.

Conflict of interest

The authors declare there is no conflict of interest.

References

1. T. J. Pedley, R. C. Schroter, M. F. Sudlow, Flow and pressure drop in systems of repeatedly branching tubes, *J. Fluid Mech.*, **46** (1971), 365–383. https://doi.org/10.1017/S0022112071000594
2. F. Durst, T. Loy, Investigations of laminar flow in a pipe with sudden contraction of cross sectional area, *Comp. Fluids*, **13** (1985), 15–36. https://doi.org/10.1016/0045-7930(85)90030-1
3. S. Mayer, On the pressure and flow-rate distributions in tree-like and arterial-venous networks, *Bltn. Mathcal. Biology*, **58** (1996), 753–785. https://doi.org/10.1007/BF02459481
4. M. Blyth, A. Mestel, Steady flow in a dividing pipe, *J. Fluid Mech.*, **401** (1999), 339–364. https://doi.org/10.1017/S0022112099006904
5. T. J. Pedley, Arterial and venous fluid dynamics, In: G. Pedrizzetti, K. Perktold (eds), *Cardiovascular Fluid Mechanics*. International Centre for Mechanical Sciences. Springer, Vienna, **446** (2003), 1–72. https://doi.org/10.1007/978-3-7091-2542-7_1
6. F. T. Smith, R. Purvis, S. C. R. Dennis, M. A. Jones, N. C. Ovenden, M. Tadjfar, Fluid flow through various branching tubes, *J. Eng. Math.*, **47** (2003), 277–298. https://doi.org/10.1023/B:ENGI.0000007981.46608.73
7. M. Tadjfar, F. Smith, Direct simulations and modelling of basic three-dimensional bifurcating tube flows, *J. Fluid Mech.*, **519** (2004), 1–32. https://doi.org/10.1017/S0022112004000606
8. R. I. Bowles, S. C. R. Dennis, R. Purvis, F. T. Smith, Multi-branching flows from one mother tube to many daughters or to a network, *Phil. Trans. R. Soc. A.*, **363** (2005), 1045–1055. https://doi.org/10.1098/rsta.2005.1548
9. G. Panasenko, Partial asymptotic decomposition of domain: Navier-Stokes equation in tube structure, *C. R. Acad. Sci., Ser. IIB, Mech. Phys. Astron.*, **326** (1998), 893–898. https://doi.org/10.1016/S1251-8069(99)80045-3
10. G. Panasenko, K. Pileckas, Asymptotic analysis of the non-steady Navier-Stokes equations in a tube structure. I. The case without boundary-layer-in-time, *Nonlinear Anal.*, **122** (2015), 125–168. https://doi.org/10.1016/j.na.2015.03.008
11. G. Panasenko, K. Pileckas, Asymptotic analysis of the non-steady Navier-Stokes equations in a tube structure. II. General case, *Nonlinear Anal.*, **125** (2015), 582–607. https://doi.org/10.1016/j.na.2015.05.018
12. E. Marusic-Paloka, Rigorous justification of the Kirchhoff law for junction of thin pipes filled with viscous fluid, *Asymptot. Anal.*, **33** (2003), 51–66.
13. M. Lenzinger, Corrections to Kirchhoff's law for the flow of viscous fluid in thin bifurcating channels and pipes, *Asymp. Anal.*, **75** (2011), 1–23. https://doi.org/10.3233/ASY-2011-1048
14. H. Attouch, *Variational convergence for functions and operators*, Appl. Math. Series, London, Pitman, 1984.

15. G. Dal Maso, *An introduction to Γ-convergence*, PNLDEA 8, Birkhäuser, Basel, 1993. https://doi.org/10.1007/978-1-4612-0327-8

16. U. Bessi, Another point of view on Kusuoka's measure, *Discrete Contin. Dyn. Syst.*, **41** (2021), 3241–3271. https://doi.org/10.3934/dcds.2020404

17. M. R. Lancia, M. A. Vivaldi, Asymptotic convergence of transmission energy forms, *Adv. Math. Sc. Appl.*, **13** (2003), 315–341.

18. U. Mosco, M. A. Vivaldi, An example of fractal singular homogenization, *Georgian Math. J.*, **14** (2007), 169–194. https://doi.org/10.1515/GMJ.2007.169

19. U. Mosco, M. A. Vivaldi, Fractal reinforcement of elastic membranes, *Arch. Rational Mech. Anal.*, **194** (2009), 49–74. https://doi.org/10.1007/s00205-008-0145-1

20. R. Capitanelli, M. A. Vivaldi, Insulating layers and Robin problems on Koch mixtures, *J. Differential Equations*, **251** (2011), 1332–1353 .

21. U. Mosco, M. A. Vivaldi, Thin fractal fibers, *Math. Meth. Appl. Sci.*, **36** (2013), 2048–2068. https://doi.org/10.1002/mma.1621

22. R. Capitanelli, M. R. Lancia, M. A. Vivaldi, Insulating layers of fractal type, *Differ. Integ. Equs.*, **26** (2013), 1055–1076. https://doi.org/10.57262/die/1372858561

23. U. Mosco, M. A. Vivaldi, Layered fractal fibers and potentials, *J. Math. Pures Appl.*, **103** (2015), 1198–1227. https://doi.org/10.1016/j.matpur.2014.10.010

24. R. Capitanelli, M. A. Vivaldi, Reinforcement problems for variational inequalities on fractal sets, *Calc. Var.*, **54** (2015), 2751–2783. https://doi.org/10.1007/s00526-015-0882-6

25. R. Capitanelli, M. A. Vivaldi, Dynamical quasi-filling fractal layers, *Siam J. Math. Anal.*, **48** (2016), 3931–3961. https://doi.org/10.1137/15M1043893

26. S. Creo, Singular p-homogenization for highly conductive fractal layers, *Z. Anal. Anwend.*, **40** (2021), 401–424. https://doi.org/10.4171/ZAA/1690

27. M. El Jarroudi, Homogenization of a quasilinear elliptic problem in a fractal-reinforced structure, *SeMA*, **79** (2022), 571–592. https://doi.org/10.1007/s40324-021-00250-5

28. M. El Jarroudi, Y. Filali, A. Lahrouz, M. Er-Riani, A. Settati, Asymptotic analysis of an elastic material reinforced with thin fractal strips, *Netw. Heterog. Media*, **17** (2022), 47–72. https://doi.org/10.3934/nhm.2021023

29. M. El Jarroudi, M. El Merzguioui, M. Er-Riani, A. Lahrouz, J. El Amrani, Dimension reduction analysis of a three-dimensional thin elastic plate reinforced with fractal ribbons, *Eur. J. Appl. Math.*, **34** (2023), 838–869. https://doi.org/10.1017/s0956792523000025

30. M. J. Lighthill, Physiological fluid dynamics: a survey, *J. Fluid Mech.* **52** (1972), 475–497. https://doi.org/10.1017/s0022112072001557

31. J. S. Lee, Y. C. Fung, Flow in nonuniform small blood vessels, *Microvascular Res.*, **3** (1971), 272–287. https://doi.org/10.1016/0026-2862(71)90053-7

32. M. R. Roach, S. Scott, G. G. Ferguson, The hemodynamic importance of the geometry of birfurcations in the circle of Willis (glass model studies), *Stroke*, **3** (1972), 255–267. https://doi.org/10.1161/01.STR.3.3.255

33. B. B. Mandelbrot, *The Fractal Geometry of Nature*, Macmillan, New York, 1983.

34. G. B. West, J. H. Brown, B. J. Enquist, A general model for the origin of allometric scaling laws in biology, *Science*, **276** (1997), 122–126. https://doi.org/10.1126/science.276.5309.122

35. Y. Chen, X. Zhang, L. Ren, Y. Geng, G. Bai, Analysis of blood flow characteristics in fractal vascular network based on the time fractional order, *Phys. Fluids*, **33** (2021), 041902. https://doi.org/10.1063/5.0046622

36. M. C. Ruzicka, On dimensionless numbers, *Chem. Eng. Res. Desi.*, **86** (2008), 835–868. https://doi.org/10.1016/j.cherd.2008.03.007

37. E. Marušic-Paloka, A. Mikelic, The derivation of a nonlinear filtration law including the inertia effects via homogenization, *Nonl. Anal.*, **42** (2000), 97–137. https://doi.org/10.1016/S0362-546X(98)00346-0

38. U. Mosco, Energy functionals on certain fractal structures, *J. Conv. Anal.*, **9** (2000), 581–600.

39. H. Schlichting, *Boundary Layer Theory*, 7th edition, McGraw–Hill, New York, 1979.

40. P. M. Gresho, R. L. Sani, On pressure boundary conditions for the incompressible Naviers-Stokes equations, *Int. J. Num. Meth. Fluids*, **7** (1987), 1111–1145. https://doi.org/10.1002/fld.1650071008

41. M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, de Gruyter Studies in Mathematics: Vol. 19, Berlin: Eds. Bauer Kazdan, Zehnder, 1994. https://doi.org/10.1515/9783110889741

42. J. Kigami, *Analysis on Fractals*, volume 143 of Cambridge Tracts in Mathematics. Cambridge University Press, Cambridge, 2001.

43. M. Fukushima, T. Shima, On a spectral analysis for the Sierpinski gasket, *Potential Anal.*, **1** (1992), 1–35. https://doi.org/10.1007/BF00249784

44. U. Mosco, Variational fractals, *Ann. Scuola Norm. Sup. Pisa*, **25** (1997), 683–712.

45. T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1966. https://doi.org/10.1007/978-3-642-53393-8

46. J. Kigami, Harmonic metric and Dirichlet form on the Sierpinski gasket, In: K. D. Elworthy, N. Ikeda (eds.), *Asymptotic Problems in Probability Theory: Stochastic Models and Diffusions on Fractals*, Pitman Research Notes in Math., Longman, London, **283** (1993), 201–218.

47. J. Kigami, Measurable Riemannian geometry on the Sierpinski gasket: the Kusuoka measure and the Gaussian heat kernel estimate, *Math. Ann.*, **340** (2008), 781–804 . https://doi.org/10.1007/s00208-007-0169-0

48. S. Kusuoka, Dirichlet forms on fractals and products of random matrices, *Publ. Res. Inst. Math. Sci.*, **25** (1989), 659–680. https://doi.org/10.2977/prims/1195173187

49. A. Teplyaev, Harmonic coordinates on fractals with finitely ramified cell structure, *Canad. J. Math.*, **60** (2008), 457–480. https://doi.org/10.4153/CJM-2008-022-3

50. K. Falconer, *Techniques in fractal geometry*, J. Wiley and sons, Chichester, 1997. https://doi.org/10.2307/2533585

51. B. E. Breckner, C. Varga, Elliptic problems on the Sierpinski gasket, In: T. Rassias, L. Tóth (eds), *Topics in mathematical analysis and applications*, Springer Optimization and Its Applications, **94** (2014), 119–173. https://doi.org/10.1007/978-3-319-06554-0_6

52. M. E. Bogovskii, Solutions of some problems of vector analysis associated with the operators div and grad, *Trudy Sem. S. L. Sobolev*, **80** (1980), 5–40.

53. G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier–Stokes Equations, I, II*, Springer–Verlag, Berlin, 1994.

54. E. Marusic-Paloka, The effects of flexion and torsion for the fluid flow through a curved pipe, *Appl. Math. Optim.*, **44** (2001), 245–272. https://doi.org/10.1007/s00245-001-0021-y

AIMS Press