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Research article

Multiplicity and concentration of normalized solutions for a Kirchhoff type problem with L^2-subcritical nonlinearities

Yangyu Ni, Jijiang Sun*and Jianhua Chen

Department of Mathematics, Nanchang University, Nanchang, Jiangxi, 330031, P. R. China

* **Correspondence:** jijiang@ncu.edu.cn, sunjijiang2005@163.com

Abstract: In this paper, we studied the existence of multiple normalized solutions to the following Kirchhoff type equation:

$$\begin{cases} -\left(a\varepsilon^2 + b\varepsilon \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u = \mu u + f(u) \quad \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = m\varepsilon^3, u \in H^1(\mathbb{R}^3), \end{cases}$$

where a, b, $m > 0$, ε is a small positive parameter, V is a nonnegative continuous function, f is a continuous function with L^2-subcritical growth and $\mu \in \mathbb{R}$ will arise as a Lagrange multiplier. Under the suitable assumptions on V and f, the existence of multiple normalized solutions was obtained by using minimization techniques and the Lusternik-Schnirelmann theory. We pointed out that the number of normalized solutions was related to the topological richness of the set where the potential V attained its minimum value.

Keywords: Kirchhoff type equation; normalized solutions; multiplicity; mass subcritical; Lusternik-Schnirelman category
Mathematics Subject Classification: 35J50, 35J93, 35Q60

1. Introduction

In 1883, Kirchhoff [1] first proposed the following time-dependent wave equation:

$$\rho \frac{\partial^2 u}{\partial^2 t} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial^2 x} = 0 \tag{1.1}$$

as an extension of the classical D'Alembert's wave equations for the free vibration of elastic strings, where u is the transverse displacement, ρ is the mass density, h is the cross-sectional area, L is the length, E is Young's modulus, and P_0 is the initial axial tension. The Kirchhoff equation (1.1) has attracted the attention of many researchers since Lions proposed an abstract functional analysis framework. Some interesting results can be referenced, for example, in [2–4].

In the past several years, there have been a lot of interesting results for the following Kirchhoff type problem:

$$\begin{cases} -\left(a\varepsilon^2 + b\varepsilon \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(x)u = f(u), \quad x \in \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3). \end{cases} \tag{1.2}$$

Li and Ye [5] studied (1.2) with $f(u) = |u|^{p-1}u$, $2 < p < 5$ under some suitable assumptions on V. By employing a monotonicity trick and a new version of the global compactness lemma, they proved a positive ground state solution. In [6], combining the nondegeneracy result and Lyapunov-Schmidt reduction method, for $\varepsilon > 0$ sufficiently small, Li et al. [6] obtained the existence of solutions to problem (1.2) with $f(u) = |u|^{p-1}u$, $p \in (1, 5)$. If $V(x)$ and $\lambda f(u)+|u|^4u$ are replaced by $M(x)$ and $f(u)$, respectively, where $\lambda > 0$ is a parameter, f is a continuous superlinear and subcritical nonlinearity. Using minimax theorems and the Ljusternik-Schnirelmann theory, Wang et al. [7] proved that for $\lambda > 0$ enough large and $\varepsilon > 0$ enough small, there exists a positive ground state solution, and they also verified the number of positive solutions in connection with the topology of the set of the global minima of the potentials.

He and Zou considered the Kirchhoff equation (1.2) in [8] with $f(u)$ satisfying the Ambrosetti-Rabinowitz condition and $V(x)$ satisfying

(V_1) $\inf_{x\in\mathbb{R}^3} V(x) < \liminf_{|x|\to\infty} V(x)$.

Through Ljusternik-Schnirelmann theory and minimax methods, He and Zou [8] first obtained the abstract framework and some compactness properties of the functional associated to (1.2), then proved the number of solutions with the topology of the set where V attains its minimum. Under general conditions of f, the potential function $V(x)$ is nonnegative and has k sets of local minima in $\mathbb{R}^3$. By variational methods, Hu and Shuai [9] bear out the existence of multi-peak solutions to singularly prturbed Kirchhoff problems (1.2). Besides, readers can find some interesting results about (1.2) in [10–14] and the references therein.

In the past decade, normalized solutions, that is, solutions with the prescribed L^2 norm, to several problems have been received plenty of attention. From the point of view of physics, this approach seems to be more meaningful since it offers a better insight into the dynamical properties of the stationary solutions, for example, stability or instability. Recently, the study on the normalized solutions to the following Kirchhoff type equation involving an L^2 constraint:

$$\begin{cases} -\left(a + b \int_{\mathbb{R}^3} |\nabla v|^2 dx\right) \Delta v + \lambda v = |u|^{p-2}u, \ x \in \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |v|^2 dx = m, \ v \in H^1(\mathbb{R}^3), \end{cases} \tag{1.3}$$

has also been the purpose of very active research, where $a, b, m > 0$ are the given constants, $\lambda \in \mathbb{R}$ appears as a Lagrange multiplier, and $p \in (2, 6)$. It is clear that solutions to (1.3) correspond to critical points of the functional $\Phi : H^1(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\Phi(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \frac{1}{p} \int_{\mathbb{R}^3} |u|^p dx$$

constrained to the sphere

$$S_m := \left\{u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = m\right\}.$$

Moreover, it is well-known that the study of (1.3) and the type of results one can expect depend on p. In particular, the range of p determines whether the functional I is bounded from below on S_m and impacts on the choice of the approaches to search for constrained critical points. Roughly speaking, in the L^2-subcritical case, i.e., $p \in \left(2, \frac{14}{3}\right)$, one may use a minimization on S_m in order to obtain the existence of a global minimizer; in the L^2-supercritical case, i.e., $p \in \left(\frac{14}{3}, 6\right)$, I is unbounded from below on S_m for any $m > 0$ and more efforts are always needed. We refer the reader to [15–26] and references therein.

A strong motivation to study multiplicity and concentration of normalized solutions to some Kirchhoff type equations mainly comes from the concentration phenomena for the following constrained singularly perturbed nonlinear Schrödinger equation:

$$\begin{cases} -\varepsilon^2\Delta v + V(x)v = \lambda v + f(v) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |v|^2 dx = a^2\varepsilon^N. \end{cases} \tag{1.4}$$

Setting $u(x) = v(\varepsilon x)$, equation (1.4) will become the following equivalent equation:

$$\begin{cases} -\Delta u + V(\varepsilon x)u = \lambda u + f(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = a^2. \end{cases} \tag{1.5}$$

In [27], Alves and Thin supposed V satisfies the following conditions:

(V) $V \in C(\mathbb{R}^N, \mathbb{R}) \cap L^\infty(\mathbb{R}^N)$, $V(0) = 0$, and

$$0 = \inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to+\infty} V(x) = V_\infty,$$

and $f \in C^1(\mathbb{R}, \mathbb{R})$ satisfying the following assumptions:

$(\tilde{f}_1)$ f is odd and there are $q \in \left(2, 2 + \frac{4}{N}\right)$ and $\alpha \in (0, +\infty)$ such that $\lim_{s\to 0} \frac{|f(s)|}{|s|^{q-1}} = \alpha > 0$.

$(\tilde{f}_2)$ There exist constants $c_1, c_2, c_3, c_4 > 0$, and $p \in \left(2, 2 + \frac{4}{N}\right)$ such that

$$|f(s)| \le c_1 + c_2|s|^{p-1} \quad \forall s \in \mathbb{R} \;\text{ and }\; |f'(s)| \le c_3 + c_4|s|^{p-2} \quad \forall s \in \mathbb{R}.$$

$(\tilde{f}_3)$ There is $q_1 \in \left(2, 2 + \frac{4}{N}\right)$ such that $\frac{f(s)}{s^{q_1-1}}$ is an increasing function of s on $(0, +\infty)$.

Alves and Thin [27] demonstrated the existence of multiple normalized solutions to the class of elliptic problems (1.5) and the relation between the numbers of normalized solutions and the topology of the set where the potential V attains its minimum value by minimization techniques and the Lusternik-Schnirelmann category.

After that, Alves and Thin [28] studied the class of elliptic problems (1.4) by assuming the conditions that $V \in C(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3)$ satisfies

(AT) $V(x) \ge 0$ for all $x \in \mathbb{R}^3$ and there exists a bounded set $\Lambda \subset \mathbb{R}^3$ such that

$$\min_{x\in\bar{\Lambda}} V(x) < \min_{x\in\partial\Lambda} V(x),$$

and the nonlinearity f is a continuous function with an L^2-subcritical growth and satisfies the following assumptions:

(f_1') f is odd and $\lim_{t\to 0}\frac{|f(s)|}{|s|^{q_0-1}} = \alpha > 0$ for some $q_0 \in (2, 2+\frac{4}{N})$;

(f_2') there are constants $c_1, c_2 > 0$, and $p \in \left(2, 2+\frac{4}{N}\right)$ such that

$$|f(s)| \le c_1 + c_2|s|^{p-1} \quad \forall s \in \mathbb{R};$$

(f_3') there is $q \in [q_0, 2+\frac{4}{N})$ such that $\frac{f(s)}{s^{q-1}}$ is an increasing function of s on $(0, +\infty)$.

Through minimization techniques, the Lusternik-Schnirelmann category, and the penalization method, Alves and Thin [28] showed the existence of multiple normalized solutions to problem (1.4). Besides, they also proved the concentration of solutions. We mention that the geometry (AT) considered in [27] does imply that potential V has a global minimum. For other research results on (1.4), we refer readers to [29–35] and the references therein.

To the best of our knowledge, so far few results on the existence and multiplicity of normalized solutions are known to the singularly perturbed Kirchhoff problems involving an L^2 constraint. Inspired by Alves and Thin [27], we are interested in investigating the multiplicity and concentration of solutions to the following Kirchhoff type equation with L^2-constraint:

$$\begin{cases} -\left(a\varepsilon^2 + b\varepsilon\int_{\mathbb{R}^3}|\nabla w|^2dx\right)\Delta w + V(x)w = \mu w + f(w) \quad \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3}|w|^2dx = m\varepsilon^3, w \in H^1(\mathbb{R}^3), \end{cases} \tag{1.6}$$

where $a, b, m, \varepsilon > 0$, μ is an unknown parameter that appears as a Lagrange multiplier, and $V \in C(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3)$ satisfies (V) and $V(x) = V(|x|)$. In the following, we will suppose that the nonlinearity f satisfies the L^2-subcritical growth assumptions. More precisely, we introduce the following assumptions:

(f_1) $f \in C(\mathbb{R}, \mathbb{R})$, $\lim_{s\to 0}\frac{f(s)}{s} = 0$;

(f_2) $\limsup_{|s|\to\infty}\frac{|f(s)|}{|s|^{11/3}} = 0$;

(f_3) there exists $\zeta \ne 0$ such that $F(\zeta) > 0$;

(f_4) $\liminf_{s\to 0}\frac{F(s)}{|s|^{10/3}} = +\infty$;

(f_4') $\limsup_{s\to 0}\frac{F(s)}{|s|^{10/3}} < +\infty$.

We would like to point out that in [36], under the so-called Berestycki-Lions type mass subcritical growth conditions assumptions: (f_3) and

(HS_1) $f \in C(\mathbb{R}, \mathbb{R})$, $\lim_{s\to 0}\frac{f(s)}{s} = 0$, and $\limsup_{|t|\to\infty}\frac{|f(t)|}{|t|^5} < \infty$;

(HS_2) $\limsup_{t\to\infty}\frac{F(t)}{|t|^{14/3}} \le 0$,

which are weaker than (f_1) and (f_2), if f satisfies (f_4) or (f_4'), for given mass $m > 0$, Hu and Sun studied the existence and nonexistence of constrained minimizers of the energy functional

$$I(u) := \frac{a}{2}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + \frac{b}{4}\left(\int_{\mathbb{R}^3}|\nabla u|^2\,dx\right)^2 - \int_{\mathbb{R}^3}F(u)dx$$

on $S_m = \left\{u \in H^1(\mathbb{R}^3) : \|u\|_2^2 = m\right\}$, where $a, b > 0$. They also established the relationship between the normalized ground state solutions and the ground state to the action functional $I(u) - \frac{\lambda}{2}\|u\|_2^2$.

To illustrate our results, we provide some notations. Define

$$M = \{x \in \mathbb{R}^3 : V(x) = 0\}$$

and

$$M_\delta = \{x \in \mathbb{R}^3 : dist(x, M) \leq \delta\},$$

where $\delta > 0$ and $dist(x, M)$ denotes the usual distance in $\mathbb{R}^3$ between x and M.

Now we state our main result.

Theorem 1.1. *Suppose that f is odd and satisfies the conditions $(f_1) - (f_3)$ and $V \in C(\mathbb{R}^3, \mathbb{R}) \cap L^\infty(\mathbb{R}^3)$ satisfies (V) and $V(x) = V(|x|)$. If (f_4) holds, we set $m > 0$. If (f_4') holds, we assume $m > m^*$ for some $m^* > 0$. Then, for each $\delta > 0$ small enough, the following properties holds:*

(1) *There exist $\varepsilon_0 > 0$ and $\Theta_m > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and $\|V\|_\infty \leq \Theta_m$, (1.6) admits at least $cat_{M_\delta}(M)$ couples $(u_j, \mu_j) \in H_r^1(\mathbb{R}^3) \times \mathbb{R}$ of weak solutions with $\int_{\mathbb{R}^3} |u_j|^2 dx = m\varepsilon^3$ and $\mu_j < 0$.*
(2) *Let u_ε denote one of these solutions and ξ_ε is the global maximum of $|u_\varepsilon|$, then*

$$\lim_{\varepsilon \to 0} V(\xi_\varepsilon) = 0.$$

Remark 1.1. *(i) As in [37], if Y is a closed subset of a topological space X, the Lusternik-Schnirelmann category $cat_X(Y)$ is the least number of closed and contractible sets in X which cover Y. If $X = Y$, we use the notation $cat(X)$.*

(ii) It is worth mentioning that our assumptions are much weaker than [27] and we do not need the monotonicity condition $(\widetilde{f_3})$, which plays a crucial role in verifying the compactness of the Palais-Smale sequences. However, due to the existence of nonlocal term, the arguments to prove the compactness of certain bounded Palais-Smale sequences in [27] cannot be used directly even if the monotonicity condition holds. To overcome this difficulty, in the present paper, we work in the radial subspace of $H^1(\mathbb{R}^3)$ and more subtle analyses are required. It is an open question whether problem (1.6) admits a solution without the radially symmetric assumption on V. Moreover, it is interesting to study the existence and concentration phenomena of solutions under the local assumption (AT).

(iii) Due to the existence of the nonlocal term, in contrast to the mass constrained nonlinear Schrödinger equations in [27], the behavior of f near 0 for the Kirchhoff type equation depends heavily on the growth rate $\frac{10}{3}$ and not the mass critical exponent $\frac{14}{3}$. Thus, in the present paper, we discuss the two different cases (f_4) and (f_4') separately.

To begin with, in order to prove our Theorem 1.1, we set $u(x) = w(\varepsilon x)$, and equation (1.6) is equivalent to the following equation:

$$\begin{cases} -\left(a + b \int_{\mathbb{R}^3} |\nabla u|^2 dx\right) \Delta u + V(\varepsilon x) u = \mu u + f(u) \quad \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = m. \end{cases} \tag{1.7}$$

We also show the energy functional:

$$I_\varepsilon(u) = \frac{a}{2} \int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^2 + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon x) |u|^2 dx - \int_{\mathbb{R}^3} F(x) dx,$$

where $F(t) = \int_0^t f(s)ds$. In addition, we denote by $I_0, I_\infty : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ the following functionals

$$I_0(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 - \int_{\mathbb{R}^3} F(u)dx,$$

and

$$I_\infty(u) = \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^3} V_\infty |u|^2 dx - \int_{\mathbb{R}^3} F(u)dx.$$

It is clear that we need to prove at least that $cat_{M_\delta}(M)$ couples $(u_i, \mu_i) \in H_r^1(\mathbb{R}^3) \times \mathbb{R}$ solutions to (1.7) correspond to critical points of the energy functional I constrained to the sphere

$$S_m = \left\{u \in H_r^1(\mathbb{R}^3) : \|u\|_2^2 = m\right\}. \tag{1.8}$$

The paper is organized as follows. In Section 2, we study some technique results. In Section 3, we prove the energy functional I_ε on the sphere S_m satisfies the Palais-Smale condition at some negative level, and then prove Theorem 1.1 via the Lusternik-Schnirelmann category theory.

Notation. *Throughout this paper, we denote by $c, c_i, C, C_i, C_i', C', i = 1, 2, \cdots$ for various positive constants whose exact value may change from lines to lines but are not essential to the analysis of the problem. $\|u\|_q = \left(\int_{\mathbb{R}^3} |u|^q dx\right)^{\frac{1}{q}}$ denotes the usual norm of $L^q(\mathbb{R}^3)$ for $q \in [2, \infty)$, and $\|u\| = \left(\int_{\mathbb{R}^3}(|\nabla u|^2 + |u|^2)dx\right)^{\frac{1}{2}}$ denotes the usual norm in the Sobolev space $H^1(\mathbb{R}^3)$. $H_r^1(\mathbb{R}^3)$ denotes the radial subspace of $H^1(\mathbb{R}^3)$. We use "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function space, respectively. We will write $o(1)$ to denote quantity that tends to 0 as $n \to \infty$.*

2. Some technical results

In this section, inspired by [27, 30], we will give some technical results which are useful to study our problem. First of all, we consider the existence of the normalized solution for the autonomous problem:

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + \mathcal{V}u = \mu u + f(u), & \text{in } \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = m, \end{cases} \tag{2.1}$$

where $a, b, m > 0, \mathcal{V} \geq 0$, and $\mu \in \mathbb{R}$ is represented as a Lagrange multiplier, f is a continuous function satisfying $(f_1) - (f_3)$, and either (f_4) or (f_4') holds. As is known to all, solutions to (2.1) correspond to critical points of the functional $I_{\mathcal{V}} : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ defined by

$$\begin{aligned} I_{\mathcal{V}}(u) &= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + \frac{\mathcal{V}}{2}\int_{\mathbb{R}^3} |u|^2 dx - \int_{\mathbb{R}^3} F(u)dx \\ &=: I(u) + \frac{\mathcal{V}}{2}\int_{\mathbb{R}^3} |u|^2 dx \end{aligned}$$

restricted to the sphere S_m which is defined in (1.8). Set

$$E_{\mathcal{V},m} := \inf_{u \in S_m} I_{\mathcal{V}}(u), \quad E_m := \inf_{u \in S_m} I(u). \tag{2.2}$$

We will summarize some properties of E_m under our assumptions. Before the proof, we introduce the well-known Gagliardo-Nirenberg inequality [38], which is very useful in the subsequent proof, for some positive constant $C(l)$,

$$\|u\|_l^l \le C(l)\|u\|_2^{(1-\beta_l)l}\|\nabla u\|_2^{\beta_l l}, \tag{2.3}$$

where $\beta_l = \frac{3}{2} - \frac{3}{l}$ and $l \in [2, 6]$.

As in [36, Lemma 2.2], we have the following result.

Lemma 2.1. *Assume that $(f_1) - (f_3)$ are satisfied. Then, the following conclusions hold.*

(i) *For any $m > 0$, I is coercive and bounded from below on S_m, and, thus, E_m is well-defined. Moreover, $E_m \le 0$.*
(ii) *There exists $m_0 > 0$ such that $E_m < 0$ for any $m > m_0$.*
(iii) *If (f_4) holds, then one has $E_m < 0$ for any $m > 0$.*
(iv) *If (f_4') holds, then one has $E_m = 0$ for $m > 0$ small enough.*
(v) *The function $m \to E_m$ is continuous and nonincreasing.*

Proof. The proof can be found in [36, Lemma 2.2]. For the reader's convenience, we state their proofs.

(i) Note that (f_1) and (f_2) imply that for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ such that

$$|f(t)| \le \varepsilon|t| + C_\varepsilon|t|^{\frac{11}{3}} \quad \text{and} \quad |F(t)| \le \varepsilon|t|^2 + C_\varepsilon|t|^{\frac{14}{3}}, \quad \text{for all } t \in \mathbb{R}. \tag{2.4}$$

Then, for any $u \in H_r^1(\mathbb{R}^3)$, from (2.4) and (2.3), we deduce that

$$\int_{\mathbb{R}^3} F(u)dx \le C_\varepsilon \int_{\mathbb{R}^3} |u|^2\, dx + \varepsilon \int_{\mathbb{R}^3} |u|^{\frac{14}{3}}\, dx \le C_\varepsilon\, \|u\|_2^2 + \varepsilon C_{\frac{14}{3}}\, \|\nabla u\|_2^4\, \|u\|_2^{\frac{2}{3}}.$$

Then, choosing $\varepsilon = \frac{b}{8C_{\frac{14}{3}} m^{\frac{1}{3}}}$, for $u \in S_m$, we have

$$I_{\mathcal{V}}(u) \ge \frac{a}{2}\, \|\nabla u\|_2^2 + \frac{b}{8}\, \|\nabla u\|_2^4 - C_\varepsilon m, \tag{2.5}$$

which implies $I_{\mathcal{V}}$ is coercive and bounded from below on S_m, and, thus, E_m is well-defined.

For any $u \in H_r^1(\mathbb{R}^3)$ and $s \in \mathbb{R}$, we define $(s * u)(x) := e^{3s/2}u(e^s x)$ for a.e. $x \in \mathbb{R}^3$. Fixing $u \in S_m \cap L^\infty(\mathbb{R}^3)$, it is clear that $s * u \in S_m$ and

$$\|\nabla(s * u)\|_2 \to 0 \quad \text{and} \quad \|s * u\|_\infty \to 0, \quad \text{as } s \to -\infty.$$

Then, by (f_1), we derive that

$$\lim_{s\to-\infty} I(s * u) = \lim_{s\to-\infty} \left(\frac{a}{2}\, \|\nabla(s * u)\|_2^2 + \frac{b}{4}\, \|\nabla(s * u)\|_2^4 - \int_{\mathbb{R}^3} F(s * u)dx\right) = 0.$$

Thus, $E_m \le 0$ for any $m > 0$.

(ii) In view of (f_3) and arguing as in [39, Theorem 2], we can find a function $u \in H_r^1(\mathbb{R}^3)$ such that $\int_{\mathbb{R}^3} F(u)dx > 0$. For any $m > 0$, we set $u_m(x) := u\left(\left(\frac{\|u\|_2^2}{m}\right)^{\frac{1}{3}} x\right)$. Clearly, $u_m \in S_m$. Then, it follows that

$$I(u_m) = \frac{am^{\frac{1}{3}}}{2\|u\|_2^{\frac{2}{3}}}\, \|\nabla u\|_2^2 + \frac{bm^{\frac{2}{3}}}{4\|u\|_2^{\frac{4}{3}}}\, \|\nabla u\|_2^4 - \frac{m}{\|u\|_2^2} \int_{\mathbb{R}^3} F(u)dx,$$

which implies that $E_m \le I(u_m) < 0$ for $m > 0$ large enough.

(iii) For any $m > 0$, we choose $u \in S_m \cap L^\infty(\mathbb{R}^3)$. By (f_3), for $M := \frac{a\|\nabla u\|_2^2}{\|u\|_{\frac{10}{3}}^{\frac{10}{3}}} > 0$, there exists $\delta > 0$ such that $F(t) \ge M|t|^{\frac{10}{3}}$ for any $|t| \le \delta$. Then, for any $s < 0$ small enough such that $\|s * u\|_\infty \le \delta$ and $e^{2s}\|\nabla u\|_2^2 < \frac{2a}{b}$, we have

$$\begin{aligned} E_m \le I(s * u) &\le \frac{ae^{2s}}{2}\|\nabla u\|_2^2 + \frac{be^{4s}}{4}\|\nabla u\|_2^4 - Me^{2s}\int_{\mathbb{R}^3}|u|^{\frac{10}{3}}dx \\ &= \frac{be^{4s}}{4}\|\nabla u\|_2^4 - \frac{ae^{2s}}{2}\|\nabla u\|_2^2 < 0, \end{aligned}$$

as required.

(iv) Fix $p \in (\frac{10}{3}, \frac{14}{3})$. By (f_2) and (f_4'), there exists $C > 0$ such that

$$F(t) \le C\left(|t|^{\frac{10}{3}} + |t|^{\frac{14}{3}} + |t|^p\right), \quad \text{for all } t \in \mathbb{R}.$$

For any $u \in H_r^1(\mathbb{R}^3)$, from (2.3), we have

$$\begin{aligned} \int_{\mathbb{R}^3} F(u)dx &\le C\int_{\mathbb{R}^3}\left(|u|^{\frac{10}{3}} + |u|^{\frac{14}{3}} + |u|^p\right)dx \\ &\le C\left(C_{\frac{10}{3}}\|\nabla u\|_2^2\|u\|_2^{\frac{4}{3}} + C_{\frac{14}{3}}\|\nabla u\|_2^4\|u\|_2^{\frac{2}{3}} + C_p\|\nabla u\|_2^{\frac{3(p-2)}{2}}\|u\|_2^{\frac{6-p}{2}}\right). \end{aligned} \tag{2.6}$$

Taking m small enough such that

$$CC_{\frac{10}{3}}m^{\frac{2}{3}} \le \frac{a}{4} \text{ and } CC_{\frac{14}{3}}m^{\frac{1}{3}} \le \frac{b}{8}, \tag{2.7}$$

for any $u \in S_m$, by (2.6), we conclude that

$$\begin{aligned} I(u) &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \int_{\mathbb{R}^3}F(u)dx \\ &\ge \|\nabla u\|_2^2\left(\frac{a}{2} + \frac{b}{4}\|\nabla u\|_2^2\right) \\ &\quad - C\|\nabla u\|_2^2\left(C_{\frac{10}{3}}m^{\frac{2}{3}} + C_{\frac{14}{3}}m^{\frac{1}{3}}\|\nabla u\|_2^2 + C_pm^{\frac{6-p}{4}}\|\nabla u\|_2^{\frac{3p-10}{2}}\right) \\ &\ge \|\nabla u\|_2^2\left(\frac{a}{4} + \frac{b}{8}\|\nabla u\|_2^2 - CC_pm^{\frac{6-p}{4}}\|\nabla u\|_2^{\frac{3p-10}{2}}\right). \end{aligned} \tag{2.8}$$

By Young's inequality and (2.8), one has

$$\begin{aligned} &CC_pm^{\frac{6-p}{4}}\|\nabla u\|_2^{\frac{3p-10}{2}} \\ &= \left[\frac{b}{2(3p-10)}\right]^{\frac{3p-10}{4}}\|\nabla u\|_2^{\frac{3p-10}{2}}\left[\frac{2(3p-10)}{b}\right]^{\frac{3p-10}{4}}CC_pm^{\frac{6-p}{4}} \\ &\le \frac{b}{8}\|\nabla u\|_2^2 + \frac{14-3p}{4}(CC_p)^{\frac{4}{14-3p}}\left[\frac{2(3p-10)}{b}\right]^{\frac{3p-10}{14-3p}}m^{\frac{6-p}{14-3p}} \\ &\le \frac{b}{8}\|\nabla u\|_2^2 + \frac{a}{4}, \end{aligned} \tag{2.9}$$

and if we choose $m > 0$, it satisfies

$$m^{\frac{6-p}{14-3p}} \leq (CC_p)^{\frac{4}{3p-14}} \frac{a}{14-3p}\left[\frac{b}{2(3p-10)}\right]^{\frac{3p-10}{14-3p}}. \tag{2.10}$$

Therefore, from (2.8) and (2.9), we deduce $I(u) \geq 0$ for any $u \in S_m$ if we choose $m > 0$ small enough such that (2.7) and (2.10) hold. Therefore, from (i), we infer that $E_m = 0$ for $m > 0$ small enough.

(v) To show the continuity, it is equivalent to prove that for a given $m > 0$, and any positive sequence m_k such that $m_k \to m$ as $k \to \infty$, one has

$$\lim_{k\to\infty} E_{m_k} = E_m. \tag{2.11}$$

In view of the definition of E_{m_k}, for every $k \in \mathbb{N}$, let $u_k \in S_{m_k}$ such that

$$I(u_k) \leq E_{m_k} + \frac{1}{k} \leq \frac{1}{k}. \tag{2.12}$$

From (2.5), it follows that $\{u_k\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Noting that $\sqrt{\frac{m}{m_k}}u_k \in S_m$, from $m_k \to m$ as $k \to \infty$, (2.4), and (2.12), similar to the proof of [40, Lemma 2.4], we obtain that

$$E_m \leq I\left(\sqrt{\frac{m}{m_k}}u_k\right) = I(u_k) + o(1) \leq E_{m_k} + o(1). \tag{2.13}$$

On the other hand, choosing a minimization sequence $\{v_n\} \in S_m$ for I, we can follow the same line as in (2.13) to obtain that $E_{m_k} \leq E_m + o(1)$. Therefore, we obtain (2.11).

To show that E_m is nonincreasing in $m > 0$, we first claim that for any $m > 0$,

$$E_{tm} \leq tE_m, \quad \text{for any } t > 1. \tag{2.14}$$

Indeed, for any $u \in S_m$ and $t > 1$, set $v(x) := u(t^{-\frac{1}{3}}x)$. Then, $v \in S_{tm}$ and we deduce that

$$\begin{aligned} E_{tm} \leq I(v) &= \frac{at^{\frac{1}{3}}}{2}\|\nabla u\|_2^2 + \frac{bt^{\frac{2}{3}}}{4}\|\nabla u\|_2^4 - t\int_{\mathbb{R}^3} F(u)dx \\ &= tI(u) + \frac{at^{\frac{1}{3}}(1-t^{\frac{2}{3}})}{2}\|\nabla u\|_2^2 + \frac{bt^{\frac{2}{3}}(1-t^{\frac{1}{3}})}{4}\|\nabla u\|_2^4 \\ &< tI(u). \end{aligned} \tag{2.15}$$

Since $u \in S_m$ is arbitrary, we obtain the inequality (2.14). As a consequence, from (i) and (2.14), it follows that E_m is nonincreasing.

In view of Lemma 2.1,

$$m^* := \inf\{m \in (0, +\infty) : E_m < 0\} \tag{2.16}$$

is well-defined and it is easy to obtain the following property of m^*.

Lemma 2.2. *Assume that* $(f_1) - (f_3)$. *Then, the following statements are true.*

(i) *If* (f_4) *holds, then* $m^* = 0$.

(ii) *If* (f_4') *holds, then* $m^* > 0$*; in addition,* $E_m = 0$ *for* $m \in (0, m^*]$ *and* $E_m < 0$ *for* $m \in (m^*, +\infty)$.

Noting that $E_m < 0$ for any $m > m_*$ and $E_{\mathcal{V},m} = E_m + \frac{\mathcal{V}m}{2}$, an immediate consequence of Lemma 2.2 is the following corollary.

Corollary 2.3. *Assume that* f *satisfies the conditions* $(f_1) - (f_3)$*. Then, the following conclusions hold.*

(i) *For any* $m > 0$, $I_{\mathcal{V}}$ *is coercive and bounded from below on* S_m*, and, thus,* $E_{\mathcal{V},m}$ *is well-defined.*
(ii) *If* (f_4) *or* (f_4') *holds, then, for any* $m > m^*$*, there exists* $\Theta_m > 0$ *such that* $E_{\mathcal{V},m} < 0$ *for any* $0 \leq \mathcal{V} \leq \Theta_m$.

Lemma 2.4. *Assume that* f *satisfies the conditions* $(f_1) - (f_3)$*, and either* (f_4) *or* (f_4') *holds. Then, for any* $m > m^*$*, fix* $\mathcal{V} \in [0, \Theta_m]$*, where* Θ_m *is defined in Corollary 2.3, we have* $\frac{k}{m} E_{\mathcal{V},m} < E_{\mathcal{V},k}$ *for all* $k \in (0, m)$.

Proof. If (f_4') holds, from Lemma 2.2 (ii), we conclude that $E_{\mathcal{V},k} = \frac{\mathcal{V}k}{2} > 0$ for all $k \in (0, m^*]$, which implies $\frac{k}{m} E_{\mathcal{V},m} < 0 < E_{\mathcal{V},k}$ for all $k \in (0, m^*]$. Therefore, either (f_4) or (f_4') holds, and we assume $k \in (m^*, m)$.

Let $t = \frac{m}{k}$ and $\{u_n\} \subset S_k$ such that $I_{\mathcal{V}}(u_n) \to E_{\mathcal{V},k}$. We claim that there exists $C > 0$ such that

$$\liminf_{n\to\infty} \|\nabla u_n\|_2^2 \geq C. \tag{2.17}$$

Indeed, if (2.17) is not true, then passing to a subsequence, $\|\nabla u_n\|_2^2 \to 0$. Then, by (2.4) and (2.3), we obtain

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(u_n) dx = 0.$$

Then, recalling $k > m^*$, by Corollary 2.3, we deduce that for $\mathcal{V} \in [0, \Theta_m]$,

$$\begin{aligned} 0 > E_{\mathcal{V},k} = \lim_{n\to\infty} I_{\mathcal{V}}(u_n) &= \lim_{n\to\infty} \left(\frac{a}{2} \|\nabla u_n\|_2^2 + \frac{b}{4} \|\nabla u_n\|_2^4 + \frac{\mathcal{V}k}{2} - \int_{\mathbb{R}^3} F(u_n) dx \right) \\ &= \frac{\mathcal{V}k}{2} \geq 0, \end{aligned}$$

a contradiction.

Since $t > 1$, noting that $v_n(x) =: u_n(t^{-\frac{1}{3}} x) \in S_m$, from (2.17), we deduce that

$$\begin{aligned} E_{\mathcal{V},m} \leq I_{\mathcal{V}}(v_n) &= \frac{a t^{\frac{1}{3}}}{2} \|\nabla u_n\|_2^2 + \frac{b t^{\frac{2}{3}}}{4} \|\nabla u_n\|_2^4 + \frac{t\mathcal{V}k}{2} - t \int_{\mathbb{R}^3} F(u_n) dx \\ &= t I_{\mathcal{V}}(u_n) + \frac{a t^{\frac{1}{3}}(1 - t^{\frac{2}{3}})}{2} \|\nabla u_n\|_2^2 + \frac{b t^{\frac{2}{3}}(1 - t^{\frac{1}{3}})}{4} \|\nabla u_n\|_2^4 \\ &\leq t E_{\mathcal{V},k} + \frac{a t^{\frac{1}{3}}(1 - t^{\frac{2}{3}}) C}{2} + \frac{b t^{\frac{2}{3}}(1 - t^{\frac{1}{3}}) C^2}{4} + o(1), \end{aligned}$$

which implies

$$E_{\mathcal{V},m} < \frac{m}{k} E_{\mathcal{V},k}. \tag{2.18}$$

The proof is complete.

In the following, we always assume that f satisfies the conditions $(f_1)-(f_3)$ and either (f_4) or (f_4') holds. Now, we give the following compactness result for $I_{\mathcal{V}}$ on S_m, which will play a crucial role in our proof.

Lemma 2.5. *Fix $m > m^*$. Let $\mathcal{V} \in [0, \Theta_m]$, and $\{u_n\} \subset S_m$ be a minimizing sequence with respect to $I_{\mathcal{V}}$. Then, $\{u_n\}$ has a strongly convergent subsequence.*

Proof. Since $E_{\mathcal{V}}$ is coercive on S_m, the sequence $\{u_n\}$ is bounded. Then, up to a subsequence, there exists $u \in H_r^1(\mathbb{R}^3)$ such that $u_n \rightharpoonup u$. Moreover, since $H_r^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$ $(2 < q < 6)$ is compact, one has $u_n \to u$ in $L^q(\mathbb{R}^3)$ $(2 < q < 6)$.

To begin, we suppose $u \neq 0$ and $\|u\|_2^2 = \bar{m} < m$. Set

$$v_n = u_n - u \quad \text{and} \quad \|v_n\|_2^2 = d_n \to d.$$

By the Brezis-Lieb Lemma (see [37]),

$$\|u_n\|_2^2 = \|v_n\|_2^2 + \|u\|_2^2 + o_n(1),$$

we infer that $m = \bar{m} + d$ and $\bar{m}, d_n \in (0, m)$ for n large enough. Furthermore, it follows from (2.4) and $u_n \to u$ in $L^q(\mathbb{R}^3)$ $(2 < q < 6)$ that

$$\int_{\mathbb{R}^3} F(v_n)dx = \int_{\mathbb{R}^3} F(u_n)dx - \int_{\mathbb{R}^3} F(u)dx = o_n(1). \tag{2.19}$$

Hence, from Lemma 2.4, we deduce that

$$\begin{aligned} E_{\mathcal{V},m} + o_n(1) = I_{\mathcal{V}}(u_n) &\geq I_{\mathcal{V}}(v_n) + I_{\mathcal{V}}(u) + o_n(1) \\ &\geq E_{\mathcal{V},d_n} + E_{\mathcal{V},\bar{m}} + o_n(1) \\ &\geq \frac{d_n}{m} E_{\mathcal{V},m} + E_{\mathcal{V},\bar{m}} + o_n(1). \end{aligned}$$

Letting $n \to +\infty$ and using Lemma 2.4 again, we derive that

$$E_{\mathcal{V},m} \geq \frac{d}{m} E_{\mathcal{V},m} + E_{\mathcal{V},\bar{m}} > \frac{d}{m} E_{\mathcal{V},m} + \frac{\bar{m}}{m} E_{\mathcal{V},m} = \left(\frac{d}{m} + \frac{\bar{m}}{m}\right) E_{\mathcal{V},m} = E_{\mathcal{V},m},$$

a contradiction. Thus, we get $\|u\|_2^2 = m$. Consequently,

$$u_n \to u \quad \text{in} \quad L^2(\mathbb{R}^3). \tag{2.20}$$

Noticing $E_{\mathcal{V},m} = \lim\limits_{n\to+\infty} I_{\mathcal{V}}(u_n)$, $\|u\|_2^2 = m$ and (2.19), we conclude that

$$\begin{aligned} E_{\mathcal{V},m} &\leq I_{\mathcal{V}}(u) \\ &= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + \frac{\mathcal{V}m}{2} - \int_{\mathbb{R}^3} F(u)dx \\ &\leq \varliminf_{n\to\infty} \left\{\frac{a}{2}\int_{\mathbb{R}^3} |\nabla u_n|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx\right)^2 + \frac{\mathcal{V}m}{2} - \int_{\mathbb{R}^3} F(u_n)dx\right\} \\ &= \varliminf_{n\to\infty} I_{\mathcal{V}}(u_n) = E_{\mathcal{V},m}. \end{aligned}$$

As a consequence,

$$\liminf_{n\to+\infty} I_{\mathcal{V}}(u_n) = I_{\mathcal{V}}(u).$$

Therefore, from (2.20) and (2.19), we deduce that

$$\|u_n\|^2 \to \|u\|^2,$$

which ensures $u_n \to u$ in $H_r^1(\mathbb{R}^3)$.

Now, we assume $u = 0$. Clearly, by (2.4) and $u_n \to 0$ in $L^q(\mathbb{R}^3)$ $(2 < q < 6)$, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(u_n)dx = 0,$$

which implies

$$0 > E_{\mathcal{V},m} = \lim_{n\to\infty} I_{\mathcal{V}}(u_n) \geq -\lim_{n\to\infty} \int_{\mathbb{R}^3} F(u_n)dx = 0,$$

a contradiction. This proves the lemma.

Theorem 2.6. *Fix $m > m^*$. Then, for any $\mathcal{V} \in [0, \Theta_m]$, problem (2.1) has a couple (u, μ) solutions, where u is positive, radial, and $\mu < 0$.*

Proof. We divide our proof into two steps.

Step 1. By Lemma 2.3 and Lemma 2.5, there exists a bounded minimizing sequence $\{u_n\} \subset S_m$ with respect to $E_{\mathcal{V},m}$ and $u \in S_m$ such that $u_n \to u$ in $H_r^1(\mathbb{R}^3)$ and $I_{\mathcal{V}}(u_n) \to E_{\mathcal{V},m} = I_{\mathcal{V}}(u)$. We define $\psi : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$\psi(u) = \frac{1}{2} \int_{\mathbb{R}^3} |u|^2 dx.$$

From the Lagrange multiplier, there exists $\mu \in \mathbb{R}$ such that

$$I'_{\mathcal{V}}(u) = \mu\psi'(u) \quad \text{in} \quad (H_r^1(\mathbb{R}^3))^*, \tag{2.21}$$

where $(H_r^1(\mathbb{R}^3))^*$ denotes the dual space of $H_r^1(\mathbb{R}^3)$. Hence,

$$-\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u + \mathcal{V}u = \mu u + f(u) \quad \text{in} \quad \mathbb{R}^3.$$

Therefore,

$$\begin{aligned} \frac{\mu}{2}\int_{\mathbb{R}^3} |u|^2 dx &= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)^2 + \frac{\mathcal{V}}{2}\int_{\mathbb{R}^3} |u|^2 dx - \int_{\mathbb{R}^3} F(u)dx \\ &= I_{\mathcal{V}}(u) = E_{\mathcal{V},m} < 0, \end{aligned}$$

that is, $\mu < 0$.

Step 2. By the fact $I_{\mathcal{V}}(u) = E_{\mathcal{V},m}$ and the definition of the functional $I_{\mathcal{V}}$ and S_m, clearly, $I_{\mathcal{V}}(|u|) = I_{\mathcal{V}}(u) = E_{\mathcal{V},m}$ and $|u| \in S_m$. So, we can replace u by $|u|$.

Now we prove u is positive. Arguing indirectly, we assume that there exists $x_0 \in \mathbb{R}^3$ such that $u(x_0) = 0$. Since $u \neq 0$, there exists $x_1 \in \mathbb{R}^3$ such that $u(x_1) > 0$. Fix $R_1 > 0$ large enough such that $x_0, x_1 \in B_{R_1}(0)$. By the Harnack inequality [41, Theorem 8.20], there exists $C_5 > 0$ such that

$$\sup_{y\in B_{R_1}(0)} u(y) \leq C_5 \inf_{y\in B_{R_1}(0)} u(y).$$

Combining this, $u(x_1) > 0$, and $u(x_0) = 0$, we get the contradiction from

$$0 < u(x_1) \leq \sup_{y\in B_{R_1}(0)} u(y) \leq C_5 \inf_{y\in B_{R_1}(0)} u(y) \leq C_5 u(x_0) = 0.$$

The proof is complete.

From the result of Theorem 2.6, we can get the following corollary:

Corollary 2.7. *Fix $m > m^*$ and let $0 \leq \mathcal{V}_1 < \mathcal{V}_2 \leq \Theta_m$. Then, $E_{\mathcal{V}_1,m} < E_{\mathcal{V}_2,m} < 0$.*

Proof. Let $u \in S_m$ satisfy $I_{\mathcal{V}_2}(u) = E_{\mathcal{V}_2,m}$. Then, $E_{\mathcal{V}_1,m} \leq I_{\mathcal{V}_1}(u) < I_{\mathcal{V}_2}(u) = E_{\mathcal{V}_2,m} < 0$.

Remark 2.1. *We denote $E_{0,m}$ and $E_{\infty,m}$ by the following real numbers:*

$$E_{0,m} = \inf_{u\in S_m} I_0(u) \quad \text{and} \quad E_{\infty,m} = \inf_{u\in S_m} I_\infty(u).$$

An immediate result of Corollary 2.7 and condition (V) is

$$E_{0,m} < E_{\infty,m} < 0.$$

3. Proof of Theorem 1.1

In this section, we will prove our main result. From now on, we always assume that f is odd and satisfies the conditions $(f_1) - (f_3)$. Moreover, we assume that either (f_4) or (f_4') holds, $m > m^*$, and $||V||_\infty \leq \Theta_m$, where m^* and Θ_m are given in (2.16) and Corollary 2.3, respectively.

To start, we manage to study the convergence of the Palais-Smale sequence for I_ε at some negative level. Denote

$$E_{\varepsilon,m} = \inf_{u\in S_m} I_\varepsilon(u).$$

From Remark 2.1, we fix $0 < \rho = \frac{1}{2}(E_{\infty,m} - E_{0,m})$. We need following result which describes the relation between the levels $E_{\varepsilon,m}$ and $E_{\infty,m}$ playing an important role in our proof.

Lemma 3.1. *There is $\varepsilon_0 > 0$ such that $E_{\varepsilon,m} < E_{\infty,m}$ for all $\varepsilon \in (0, \varepsilon_0)$.*

Proof. Let $u_0 \in S_m$ with $I_0(u_0) = E_{0,m}$. Then, by the definition of $E_{\varepsilon,m}$,

$$\begin{aligned} E_{\varepsilon,m} &\leq I_\varepsilon(u_0) \\ &= \frac{a}{2}\int_{\mathbb{R}^3} |\nabla u_0|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^3} |\nabla u_0|^2 dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^3} V(\varepsilon x)|u_0|^2 dx - \int_{\mathbb{R}^3} F(u_0)dx. \end{aligned}$$

Therefore,

$$\limsup_{\varepsilon\to 0^+} E_{\varepsilon,m} \leq \lim_{\varepsilon\to 0^+} I_\varepsilon(u_0) = I_0(u_0) = E_{0,m}.$$

From this and Remark 2.1, the estimate $E_{\varepsilon,m} < E_{\infty,m}$ is established for ε small enough.

From now on, fix $\varepsilon \in (0, \varepsilon_0)$, where ε_0 is given in Lemma 3.1.

Lemma 3.2. *If $\{u_n\} \subset S_m$ satisfies $I_\varepsilon(u_n) \to c$ with $c < E_{0,m} + \rho < 0$ and $u_n \rightharpoonup u$ in $H_r^1(\mathbb{R}^3)$, then, $u \neq 0$.*

Proof. Assume by contradiction that $u = 0$. By (V), for any given $\alpha > 0$, there is $R > 0$ such that for any $|x| \geq R$,

$$V(x) \geq V_\infty - \alpha.$$

Based on assumptions and the boundedness of $\{u_n\}$ in $H_r^1(\mathbb{R}^3)$ and $u_n \to 0$ in $L^2_{loc}\left(\mathbb{R}^3\right)$, we deduce that for some $C > 0$,

$$\begin{aligned}
E_{0,m} + \rho + o_n(1) &> I_\varepsilon(u_n) \\
&= I_\infty(u_n) + \frac{1}{2}\int_{\mathbb{R}^3} (V(\varepsilon x) - V_\infty)|u_n|^2 dx \\
&\geq I_\infty(u_n) + \frac{1}{2}\int_{B_{\frac{R}{\varepsilon}}(0)} (V(\varepsilon x) - V_\infty)|u_n|^2 dx - \frac{\alpha}{2}\int_{B^c_{\frac{R}{\varepsilon}}(0)} |u_n|^2 dx \\
&\geq I_\infty(u_n) - \alpha C \\
&\geq E_{\infty,m} - \alpha C.
\end{aligned}$$

Due to the arbitrariness of α, we have $E_{0,m} + \rho \geq E_{\infty,m}$, which is absurd. The proof is complete.

Lemma 3.3. *For each $\varepsilon \in (0, \varepsilon_0)$, the functional I_ε satisfies the $(PS)_c$ condition restricted to S_m for $c < E_{0,m} + \rho$, namely, if any sequence $\{u_n\} \subset S_m$ such that*

$$I_\varepsilon(u_n) \to c \quad as \quad n \to +\infty \quad and \quad \left\| I_\varepsilon|'_{S_m}(u_n) \right\| \to 0 \quad as \quad n \to +\infty,$$

then $\{u_n\}$ has a convergent subsequence.

Proof. Similar to Corollary 2.3 (i), we can verify that I_ε is coercive on S_m. Thus, $\{u_n\}$ is bounded in $H_r^1(\mathbb{R}^3)$. Up to a subsequence, we assume that $u_n \rightharpoonup u_\varepsilon$ in $H^1(\mathbb{R}^3)$ and $u_n \to u_\varepsilon$ in $L^q(\mathbb{R}^3)$ $(2 < q < 6)$. From Lemma 3.2, $u_\varepsilon \neq 0$.

We define $\psi : H_r^1(\mathbb{R}^3) \to \mathbb{R}$ by

$$\psi(u) = \frac{1}{2}\int_{\mathbb{R}^3} |u|^2 dx.$$

Then, by Willem [37, Prosition 5.12], there exists $\{\mu_n\} \subset \mathbb{R}$ such that

$$\left\| I'_\varepsilon(u_n) - \mu_n \psi'(u_n) \right\|_{(H_r^1(\mathbb{R}^3))^*} \to 0 \quad \text{as} \quad n \to +\infty.$$

From $\psi'(u_n)u_n = m$, the boundedness of $\{u_n\}$ in $H_r^1(\mathbb{R}^3)$, (f_1), (f_2), and (2.3), it follows that

$$\begin{aligned}
|\mu_n| &= \frac{1}{m}|\mu_n \psi'(u_n)u_n| = \frac{1}{m}\left| I'_\varepsilon(u_n)u_n \right| + o_n(1) \\
&\leq \frac{1}{m}\left(\left| a\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right| + \left| b\left(\int_{\mathbb{R}^3} |\nabla u_n|^2 dx \right)^2 \right| + \left| \int_{\mathbb{R}^3} V(\varepsilon x) u_n^2 dx \right| \right) \\
&\quad + \frac{1}{m}\left| \int_{\mathbb{R}^3} f(u_n)u_n dx \right| + o_n(1) \\
&\leq C_1\left(\|u_n\|^2 + \|u_n\|^4 + \|u_n\|^{\frac{14}{3}} + 1 \right) \\
&\leq C_2,
\end{aligned} \tag{3.1}$$

where C_1, C_2 are positive constants independent of ε and n. This implies that $\{\mu_n\}$ is a bounded sequence. As a consequence, we assume, up to a subsequence, $\mu_n \to \mu_\varepsilon$ as $n \to +\infty$, and, thereby,

$$\left\| I'_\varepsilon(u_n) - \mu_\varepsilon \psi'(u_n) \right\|_{(H^1_r(\mathbb{R}^3))^*} \to 0 \quad \text{as} \quad n \to +\infty. \tag{3.2}$$

Meanwhile,

$$-\,(a + bA_\varepsilon)\,\Delta u_\varepsilon + V(\varepsilon x)u_\varepsilon = \mu_\varepsilon u_\varepsilon + f(u_\varepsilon), \tag{3.3}$$

where $A_\varepsilon = \lim_{n\to\infty} \int_{\mathbb{R}^3} |\nabla u_n|^2 dx \geq 0$.

Now, we verify that there exists $\mu_* < 0$ independent of ε such that

$$\mu_\varepsilon \leq \mu_* < 0, \qquad \forall\, \varepsilon \in (0, \varepsilon_0). \tag{3.4}$$

Noting that $u_n \rightharpoonup u_\varepsilon$ in $H^1_r(\mathbb{R}^3)$, one has $u_n \to u_\varepsilon$ in $L^q(\mathbb{R}^3)$ $(2 < q < 6)$. Then, by (2.4), we conclude that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} F(u_n)dx = \int_{\mathbb{R}^3} F(u_\varepsilon)dx.$$

Therefore, from (3.3), we deduce that

$$\begin{aligned}\frac{\mu_\varepsilon}{2} \int_{\mathbb{R}^3} |u_\varepsilon|^2 dx &= \frac{a + bA_\varepsilon}{2} \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx + \frac{1}{2} \int_{\mathbb{R}^3} V(\varepsilon x) u_\varepsilon^2 dx - \int_{\mathbb{R}^3} F(u_\varepsilon)dx \\ &\leq \varliminf_{n\to\infty} I_\varepsilon(u_n) < E_{0,m} + \rho + o_n(1) < 0,\end{aligned}$$

which implies $\mu_\varepsilon < 0$. In addition, we also have

$$0 > E_{0,m} + \rho \geq \frac{\mu_\varepsilon}{2} \int_{\mathbb{R}^3} |u_\varepsilon|^2 dx \geq \frac{\mu_\varepsilon}{2} \varliminf_{n\to\infty} \int_{\mathbb{R}^3} |u_n|^2 dx = \frac{\mu_\varepsilon}{2} m.$$

Therefore,

$$\limsup_{\varepsilon\to 0} \mu_\varepsilon \leq \frac{2\,(E_{0,m} + \rho)}{m} < 0. \tag{3.5}$$

Hence, (3.4) holds.

Now we prove that $u_n \to u_\varepsilon$ in $H^1_r(\mathbb{R}^3)$. Set $v_n := u_n - u_\varepsilon$. From (3.2) and (3.3), we infer that

$$\begin{aligned}(a + bA_\varepsilon) \int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} V(\varepsilon x)|v_n|^2 dx - \mu_\varepsilon \int_{\mathbb{R}^3} |v_n|^2 dx& \\ = \int_{\mathbb{R}^3} f(v_n) v_n dx + o_n(1).&\end{aligned}$$

Then, using (3.4), (2.4), and the fact that $v_n \to 0$ in $L^q(\mathbb{R}^3)$ $(2 < q < 6)$, we conclude that

$$C\left(\int_{\mathbb{R}^3} |\nabla v_n|^2 dx + \int_{\mathbb{R}^3} |v_n|^2 dx\right) \leq o_n(1), \tag{3.6}$$

where positive constant C does not rely on ε. Thus, $v_n \to 0$ in $H^1_r(\mathbb{R}^3)$, i.e., $u_n \to u_\varepsilon$ in $H^1_r(\mathbb{R}^3)$. Therefore, $\|u_\varepsilon\|_2^2 = m$ and

$$-\left(a + b \int_{\mathbb{R}^3} |\nabla u_\varepsilon|^2 dx\right) \Delta u_\varepsilon + V(\varepsilon x)u_\varepsilon = \mu_\varepsilon u_\varepsilon + f(u_\varepsilon), \quad \text{in } \mathbb{R}^3.$$

The proof is finished.

Fix $\delta > 0$ and define η as a smooth nonincreasing cutoff function in $[0, +\infty)$ by

$$\eta(s) = \begin{cases} 1, & 0 \le s \le \frac{\delta}{2}, \\ [0,1], & \frac{\delta}{2} < s < \delta, \\ 0, & s \ge \delta. \end{cases}$$

Recall that $M = \{x \in \mathbb{R}^3 : V(x) = 0\}$. For any $y \in M$, let us define

$$\phi_{\varepsilon,y}(x) = \eta(|\varepsilon x - y|)\, w_0\left(\frac{\varepsilon x - y}{\varepsilon}\right),$$

where w_0 is a positive radial solution of the problem

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^3} |\nabla u|^2 dx\right)\Delta u = \mu u + f(u), & \text{in} \quad \mathbb{R}^3, \\ \int_{\mathbb{R}^3} |u|^2 dx = m, \end{cases}$$

with $I_0(w_0) = E_{0,m}$. Then, let

$$\tilde{\phi}_{\varepsilon,y}(x) = \frac{\sqrt{m}\phi_{\varepsilon,y}(x)}{\|\phi_{\varepsilon,y}\|_2},$$

and denote $\Phi_\varepsilon : M \to S_m$ by $\Phi_\varepsilon(y) = \tilde{\phi}_{\varepsilon,y}$. Obviously, it has compact support for any $y \in M$. In addition, let $R = R(\delta) > 0$ be such that $M_\delta \subset B_R(0)$. Define $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ as

$$\chi(x) = \begin{cases} x, & |x| \le R, \\ \frac{Rx}{|x|}, & |x| \ge R. \end{cases}$$

Finally, let us consider $\omega_\varepsilon : S_m \to \mathbb{R}^3$ given by

$$\omega_\varepsilon(u) = \frac{\int_{\mathbb{R}^3} \chi(\varepsilon x)|u|^2 dx}{m}.$$

Lemma 3.4. *The function Φ_ε has the following two limits:*

(1) $\lim_{\varepsilon \to 0} I_\varepsilon(\Phi_\varepsilon(y)) = E_{0,m}$, *uniformly in* $y \in M$,
(2) $\lim_{\varepsilon \to 0} \omega_\varepsilon(\Phi_\varepsilon(y)) = y$, *uniformly in* $y \in M$.

Proof. (1) Assume that $\{y_n\} \subset M$. From Lebesgue's dominated convergence theorem, it follows that

$$\begin{aligned} \lim_{n\to+\infty} \int_{\mathbb{R}^3} |\Phi_{\varepsilon_n}(y_n)|^2 dx &= \lim_{n\to+\infty} \int_{\mathbb{R}^3} |\eta(|\varepsilon_n x|) w_0(x)|^2 dx \\ &= \lim_{n\to+\infty} \left[\int_{B_{\frac{\delta}{2\varepsilon_n}}(0)} |w_0(x)|^2 dx + \int_{B_{\frac{\delta}{\varepsilon_n}}(0) \setminus B_{\frac{\delta}{2\varepsilon_n}}(0)} |\eta(|\varepsilon_n x|) w_0(x)|^2 dx \right] \\ &= \int_{\mathbb{R}^3} |w_0(x)|^2 dx. \end{aligned}$$

Likewise, we also have

$$\begin{aligned}\lim_{n\to+\infty}\int_{\mathbb{R}^3} F\left(\Phi_{\varepsilon_n}(y_n)\right)dx &= \lim_{n\to+\infty}\int_{\mathbb{R}^3} F\left(\frac{\sqrt{m}\eta(|\varepsilon_n x|)w_0(x)}{\|\phi_{\varepsilon_n,y_n}\|_2}\right)dx\\ &= \int_{\mathbb{R}^3} F(w_0)dx,\\ \lim_{n\to+\infty}\int_{\mathbb{R}^3} |\nabla\Phi_{\varepsilon_n}(y_n)|^2dx &= \lim_{n\to+\infty}\int_{\mathbb{R}^3}\frac{m}{\|\phi_{\varepsilon_n,y_n}\|_2^2}|\nabla\left(\eta(|\varepsilon_n x|)w_0(x)\right)|^2dx\\ &= \int_{\mathbb{R}^3}|\nabla w_0|^2dx,\end{aligned}$$

and

$$\lim_{n\to+\infty}\int_{\mathbb{R}^3} V(\varepsilon_n x)|\Phi_{\varepsilon_n}(y_n)|^2dx = 0.$$

Consequently,

$$\lim_{n\to+\infty} I_{\varepsilon_n}(\Phi_{\varepsilon_n}(y_n)) = I_0(w_0) = E_{0,m},$$

and this proves the first limit.

(2) Suppose by contradiction that there is $\delta_0 > 0$, $\{y_n\} \subset M$ with $y_n \to y \in M$ and $\varepsilon \to 0$ such that

$$\left|\omega_{\varepsilon_n}\left(\Phi_{\varepsilon_n}(y_n)\right) - y_n\right| \geq \delta_0, \quad \forall n \in \mathbb{N}. \tag{3.7}$$

Using the definitions of $\Phi_{\varepsilon_n}(y_n)$ and ω_{ε_n}, combined with $\{y_n\} \subset M \subset B_R(0)$ and Lebesgue's dominated convergence theorem, we deduce that

$$\left|\omega_{\varepsilon_n}\left(\Phi_{\varepsilon_n}(y_n)\right) - y_n\right| = \left|\frac{\int_{\mathbb{R}^3}\left(\chi(\varepsilon_n x + y_n) - y_n\right)|\eta(|\varepsilon_n x|)w_0(x)|^2\,dx}{m}\right| \to 0,$$

which contradicts (3.7), and this proves the desired result.

Let $\gamma : [0, +\infty) \to [0, +\infty)$ be a positive function such that $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$, then define $\tilde{S}_m$ as

$$\tilde{S}_m = \{u \in S_m : I_\varepsilon(u) \leq E_{0,m} + \gamma(\varepsilon)\}. \tag{3.8}$$

Thanks to (1) of Lemma 3.4, the function

$$\gamma(\varepsilon) = \sup_{y\in M}\left|I_\varepsilon(\Phi_\varepsilon(y)) - E_{0,m}\right|$$

satisfies $\gamma(\varepsilon) \to 0$ as $\varepsilon \to 0$. Hence, $\Phi_\varepsilon(y) \in \tilde{S}_m$ for all $y \in M$.

Proposition 3.5. *Let $\varepsilon_n \to 0$ and $\{u_n\} \subset S_m$ with $I_{\varepsilon_n}(u_n) \to E_{0,m}$. Then, $\{u_n\}$ has a convergent subsequence in $H_r^1(\mathbb{R}^3)$.*

Proof. Since $\{u_n\} \subset S_m$, from (V), we deduce that

$$I_{\varepsilon_n}(u_n) \geq I_0(u_n) \geq E_{0,m},$$

which implies $I_0(u_n) \to E_{0,m}$ as $n \to +\infty$. From Lemma 2.5, $\{u_n\}$ has a convergent subsequence in $H_r^1(\mathbb{R}^3)$.

Lemma 3.6.

$$\lim_{\varepsilon\to 0}\sup_{u\in\tilde{S}_m}\inf_{z\in M}|\omega_\varepsilon(u)-z|=0.$$

Proof. Let $\varepsilon_n\to 0$ and $u_n\in\tilde{S}_m$ such that

$$\inf_{z\in M_\delta}\left|\omega_{\varepsilon_n}(u_n)-z\right|=\sup_{u\in\tilde{S}_m}\inf_{z\in M_\delta}\left|\omega_{\varepsilon_n}(u_n)-z\right|+o_n(1).$$

Since $u_n\in\tilde{S}_m$, by the definition of $\tilde{S}_m$, we deduce that $u_n\in S_m$, and as $\varepsilon_n\to 0$,

$$E_{0,m}\le I_0(u_n)\le I_{\varepsilon_n}(u_n)\le E_{0,m}+\gamma(\varepsilon_n),\quad \forall n\in\mathbb{N},$$

from which it follows that $I_{\varepsilon_n}(u_n)\to E_{0,m}$. From Proposition 3.5, $\{u_n\}$ is strongly convergent to some $u\in H^1_r(\mathbb{R}^3)$. Then, due to the definition of ω_{ε_n} and $u_n\in S_m$, using the Lebesgue's dominated convergence theorem, we obtain that

$$\omega_{\varepsilon_n}(u_n)=\frac{\int_{\mathbb{R}^3}\chi(\varepsilon_n x)\,|u_n|^2\,dx}{m}\to 0\quad\text{as}\quad n\to+\infty,$$

that is, $\omega_{\varepsilon_n}(u_n)=o_n(1)$. Noting that $0\in M$, we conclude that

$$\lim_{n\to+\infty}\inf_{z\in M}\left|\omega_{\varepsilon_n}(u_n)-z\right|=0.$$

The proof is complete.

Proof of Theorem 1.1. We will divide the proof into two parts:

Step 1: Multiplicity of solutions.

Set $\varepsilon\in(0,\varepsilon_0)$ and fix $\delta>0$. By Lemmas 3.4 and 3.6, we can obtain that the diagram $M\overset{\Phi_\varepsilon}{\to}S_m\overset{\omega_\varepsilon}{\to}M_\delta$ is well-defined. For ε small enough, we denote $\omega_\varepsilon(\Phi_\varepsilon(y)):=y+\varsigma(y)$ for $y\in M$ and $Q(t,y):=y+(1-t)\varsigma(y)$. By Lemma 3.7, $\|\varsigma(y)\|\le\frac{\delta}{2}$ uniformly for $y\in M$. Obviously, the continuous function $Q:[0,1]\times M\to M_\delta$ satisfies $Q(0,y)=\omega_\varepsilon(\Phi_\varepsilon(y))$ and $Q(1,y)=y$ for any $y\in M$. Therefore, $\omega_\varepsilon\circ\Phi_\varepsilon$ is homotopic to the inclusion map $id:M\to M_\delta$. In view of [42], we arrive at

$$cat(\tilde{S}_m)\ge cat_{M_\delta}(M).$$

Recall that I_ε is bounded from below on S_m. Moreover, from Lemma 3.3, I_ε satisfies the $(PS)_c$ condition for $c\in(E_{0,m},E_{0,m}+\gamma(\varepsilon))$. Then, due to the Lusternik-Schnirelmann category of critical points (see [37,43]), we infer that I_ε admits at least $cat_{M_\delta}(M)$ critical points on S_m.

Step 2: Concentration phenomena of the solutions.

Let u_ε be a solution of (1.7) with $I_\varepsilon(u_\varepsilon)\le E_{0,m}+\gamma(\varepsilon)$, where γ was given in (3.8). From Proposition 3.5, for any $\varepsilon_n\to 0$, there exists $u\in H^1_r(\mathbb{R}^3)\setminus\{0\}$ such that $u_{\varepsilon_n}\to u$ in $H^1_r(\mathbb{R}^3)$. Clearly, as in (3.5), $u_n:=u_{\varepsilon_n}$ satisfies

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u_n|^2dx\right)\Delta u_n+V(\varepsilon_n x)u_n=\mu_n u_n+f(u_n),\quad\text{in }\ \mathbb{R}^3,$$

with

$$\limsup_{\varepsilon\to 0}\mu_n\le\frac{2(\rho+E_{0,m})}{m}<0.$$

Since $u_n \to u$ in $H^1_r(\mathbb{R}^3)$, similar to [8, Lemma 4.5], we obtain

$$\lim_{|x|\to+\infty} u_n(x) = 0, \quad \text{uniformly in } n \in \mathbb{N}.$$

As a consequence, given $\theta > 0$, there exist $R > 0$ and $n_0 \in \mathbb{N}$ such that

$$|u_n(x)| \leq \theta,$$

for $|x| \geq R$ and $n \geq n_0$. We claim that $\|u_n\|_\infty \nrightarrow 0$; otherwise we will have $u_n \to 0$ in $H^1_r(\mathbb{R}^3)$, contrary to $u \neq 0$. Now, we fix $\theta > 0$ small such that $\|u_n\|_\infty \geq 2\theta$ and choose $\xi_n \in \mathbb{R}^3$ such that $|u_n(\xi_n)| = \|u_n\|_\infty$ for all $n \in \mathbb{N}$. It follows that $|\xi_n| \leq R$ for all $n \in \mathbb{N}$. Therefore,

$$\lim_{n\to+\infty} V(\varepsilon_n \xi_n) = V(0) = 0,$$

as required.

Author contributions

Yangyu Ni: Writing-original draft, Writing-review & editing; Jijiang Sun: Supervision, Writing-review & editing, Methodology, Validation; Jianhua Chen: Formal Analysis, Validation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1883.
2. A. Arosio, S. Panizzi, On the well-posedness of the Kirchhoff string, *Trans. Amer. Math. Soc.*, **348** (1996), 305–330.
3. S. Bernstein, Sur une classe d'*é*quations fonctionelles aux d*é*riv*é*es partielles, *Izv. Akad. Nauk SSSR Ser. Mat*, **4** (1940), 17–26.
4. S. I. Pohožaev, On a class of quasilinear hyperbolic equations, *sb. Math.*, **25** (1975), 145–158.

5. G. Li, H. Ye, Existence of positive ground state solutions for the nonlinear Kirchhoff type equations in $\mathbb{R}^3$, *J. Differ. Equ.*, **257** (2014), 566–600. https://doi.org/10.1016/j.jde.2014.04.011
6. G. Li, P. Luo, S. Peng, C. Wang, C. L. Xiang, A singularly perturbed Kirchhoff problem revisited, *J. Differ. Equ.*, **268** (2020), 541–589. https://doi.org/10.1016/j.jde.2019.08.016
7. J. Wang, L. Tian, J. Xu, F. Zhang, Multiplicity and concentration of positive solutions for a Kirchhoff type problem with critical growth, *J. Differ. Equ.*, **253** (2012), 2314–2351. https://doi.org/10.1016/j.jde.2012.05.023
8. X. He, W. Zou, Existence and concentration behavior of positive solutions for a Kirchhoff equation in $\mathbb{R}^3$, *J. Differ. Equ.*, **252** (2012), 1813–1834. https://doi.org/10.1016/J.JDE.2011.08.035
9. T. Hu, W. Shuai, Multi-peak solutions to Kirchhoff equations in $\mathbb{R}^3$ with general nonlinearity, *J. Differ. Equ.*, **265** (2018), 3587–3617. https://doi.org/10.1016/j.jde.2018.05.012
10. G. M. Figueiredo, N. Ikoma, J. R. Santos Júnior, Existence and concentration result for the Kirchhoff type equations with general nonlinearities, *Arch. Ration. Mech. Anal.*, **213** (2014), 931–979. https://doi.org/10.1007/s00205-014-0747-8
11. Y. He, G. Li, Standing waves for a class of Kirchhoff type problems in $\mathbb{R}^3$ involving critical Sobolev exponents, *Calc. Var.*, **54** (2015), 3067–3106. https://doi.org/10.1007/s00526-015-0894-2
12. Q. Xie, X. Zhang, Semi-classical solutions for Kirchhoff type problem with a critical frequency, *Proc. Roy. Soc. Edinburgh Sect. A.*, **151** (2021), 761–798. https://doi.org/10.1017/prm.2020.37
13. L. Kong, H. Chen, Normalized ground states for fractional Kirchhoff equations with Sobolev critical exponent and mixed nonlinearities, *J. Math. Phys.* **64** (2023), 061501. https://doi.org/10.1063/5.0098126
14. L. Kong, L. Zhu, Y. Deng, Normalized solutions for nonlinear Kirchhoff type equations with low-order fractional Laplacian and critical exponent, *Appl. Math. Lett.*, **147** (2023), 108864. https://doi.org/10.1016/j.aml.2023.108864
15. S. Chen, V. Rădulescu, X. Tang, Normalized solutions of nonautonomous Kirchhoff equations: sub- and super-critical cases, *Appl. Math. Optim.*, **84** (2021), 773–806. https://doi.org/10.1007/s00245-020-09661-8
16. J. Hu, J. Sun, Normalized ground states for Kirchhoff type equations with general nonlinearities, *Adv. Differential Equ.*, **29** (2024), 111–152. https://doi.org/10.57262/ade029-0102-111
17. T. Hu, C. L. Tang, Limiting behavior and local uniqueness of normalized solutions for mass critical Kirchhoff equations, *Calc. Var.*, **60** (2021), 210. https://doi.org/10.1007/s00526-021-02018-1
18. Q. Li, J. Nie, W. Zhang, Existence and asymptotics of normalized ground states for a Sobolev critical Kirchhoff equation, *J. Geom. Anal.*, **33** (2023), 126. https://doi.org/10.1007/s12220-022-01171-z
19. Q. Li, V. D. Radulescu, W. Zhang, Normalized ground states for the Sobolev critical Schrödinger equation with at least mass critical growth, *Nonlinearity*, **37** (2024), 025018. https://doi.org/10.1088/1361-6544/ad1b8b
20. G. Li, H. Ye, On the concentration phenomenon of L^2-subcritical constrained minimizers for a class of Kirchhoff equations with potentials, *J. Differ. Equ.*, **266** (2019), 7101–7123. https://doi.org/10.1016/j.jde.2018.11.024

21. S. Qi, W. Zou, Exact Number of Positive Solutions for the Kirchhoff Equation, *SIAM J. Math. Anal.*, **54** (2022), 5424–5446. https://doi.org/10.1137/21M1445879

22. H. Ye, The existence of normalized solutions for L^2-critical constrained problems related to Kirchhoff equations, *Z. Angew. Math. Phys.*, **66** (2015), 1483–1497. https://doi.org/10.1007/s00033-014-0474-x

23. H. Ye, The sharp existence of constrained minimizers for a class of nonlinear Kirchhoff equations, *Math. Methods Appl. Sci.*, **38** (2015), 2663–2679. https://doi.org/10.1002/mma.3247

24. H. Ye, The mass concentration phenomenon for L^2-critical constrained problems related to Kirchhoff equations, *Z. Angew. Math. Phy.*, **67** (2016), 29. https://doi.org/10.1007/s00033-016-0624-4

25. X. Zeng, J. Zhang, Y. Zhang, X. Zhong, On the Kirchhoff equation with prescribed mass and general nonlinearities, *Discrete Contin. Dyn. Syst. Ser. S*, **16** (2023), 3394–3409. https://doi.org/10.3934/dcdss.2023160

26. X. Zeng, Y. Zhang, Existence and uniqueness of normalized solutions for the Kirchhoff equation, *Appl. Math. Lett.*, **74** (2017), 52–59. https://doi.org/10.1016/j.aml.2017.05.012

27. C. O. Alves, N. V. Thin, On existence of multiple normalized solutions to a class of elliptic problems in whole $\mathbb{R}^N$ via Lusternik-Schnirelmann Category, *SIAM J. Math. Anal.*, **55** (2023), 1264–1283. https://doi.org/10.1137/22M1470694

28. C. O. Alves, N. V. Thin, On existence of multiple normalized solutions to a class of elliptic problems in whole $\mathbb{R}^N$ via penalization method, *Potential Anal.*, 2023. https://doi.org/10.1007/s11118-023-10116-2

29. N. Ackermann, T. Weth, Unstable normalized standing waves for the space periodic NLS, *Anal. PDE.*, **12** (2018), 1177–1213. https://doi.org/10.2140/apde.2019.12.1177

30. C. O. Alves, On existence of multiple normalized solutions to a class of elliptic problems in whole $\mathbb{R}^N$, *Z. Angew. Math. Phys.*, **73** (2022), 97. https://doi.org/10.1007/s00033-022-01741-9

31. B. Pellacci, A. Pistoia, G. Vaira, G. Verzini, Normalized concentrating solutions to nonlinear elliptic problems, *J. Differ. Equ.*, **275** (2021), 882–919. https://doi.org/10.1016/j.jde.2020.11.003

32. N. S. Papageorgiou, J. Zhang, W. Zhang, Solutions with sign information for noncoercive double phase equations, *J. Geom. Anal.*, **34** (2024), 14. https://doi.org/10.1007/s12220-023-01463-y

33. Z. Tang, C. Zhang, L. Zhang, L. Zhou, Normalized multibump solutions to nonlinear Schrödinger equations with steep potential well, *Nonlinearity*, **35** (2022), 4624. https://doi.org/10.1088/1361-6544/ac7b61

34. C. Zhang, X. Zhang, Normalized multi-bump solutions of nonlinear Schrödinger equations via variational approach, *Calc. Var.*, **61** (2022), 57. https://doi.org/10.1007/s00526-021-02166-4

35. J. Zhang, W. Zhang, Semiclassical states for coupled nonlinear Schrödinger system with competing potentials, *J. Geom. Anal.*, **32** (2022), 114. https://doi.org/10.1007/s12220-022-00870-x

36. J. Hu, J. Sun, On constrained minimizers for Kirchhoff type equations with Berestycki-Lions type mass subcritical conditions, *Electron. Res. Arch.*, **31** (2023), 2580–2594. https://doi.org/10.3934/era.2023131

37. M. Willem, *Minimax Theorems*, Birkhäser, Boston, 1996. https://doi.org/10.1007/978-1-4612-4146-1

38. M. I. Weinstein, Nonlinear Schrödinger equations and sharp interpolation estimates, *Commu. Math. Phys.*, **87** (1983), 567–576. https://doi.org/10.1007/BF01208265

39. H. Berestycki, P. L. Lions, Nonlinear scalar field equations I: Existence of a ground state, *Arch. Rat. Mech. Anal.*, **82** (1983), 313–346. https://doi.org/10.1007/BF00250555

40. M. Shibata, Stable standing waves of nonlinear Schrödinger equations with a general nonlinear term, *Manuscripta Math.*, **143** (2014), 221–237. https://doi.org/10.1007/s00229-013-0627-9

41. D. Gilbarg, N. S. Trudinger, *Elliptic partial differential equations of second order*, springer, Berlin, 1977.

42. V. Benci, G. Cerami, Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology, *Calc. Var.*, **2** (1994), 29–48. https://doi.org/10.1007/BF01234314

43. N. Ghoussoub, *Duality and perturbation methods in critical point theory*, Cambridge University Press, 1993.