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Research article

Large-time behavior of cylindrically symmetric Navier-Stokes equations with temperature-dependent viscosity and heat conductivity

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Abstract: In this study, the initial-boundary value problem for cylindrically symmetric Navier-Stokes equations was considered with temperature-dependent viscosity and heat conductivity. Firstly, we established the existence and uniqueness of a strong solution when the viscosity and heat conductivity were both power functions of temperature. Moreover, the large-time behavior of the strong solution was obtained with large initial data, since all of the estimates in this paper were independent of time. It is worth noting that we identified the relationship between the initial data and the power of the temperature in the viscosity for the first time.

Keywords: cylindrically symmetric Navier-Stokes equations; existence; uniqueness; large-time behavior
Mathematics Subject Classification: 35Q35, 76N10

1. Introduction

As it is well-known that the cylindrically symmetric Navier-Stokes equations take the form

$$\rho_t + \frac{(r\rho u)_r}{r} = 0, \tag{1.1}$$

$$\rho(u_t + uu_r) - \frac{\rho v^2}{r} + P_r = \left(\frac{\lambda(ru)_r}{r}\right)_r - \frac{2u\mu_r}{r}, \tag{1.2}$$

$$\rho(v_t + uv_r) + \frac{\rho uv}{r} = (\mu v_r)_r + \frac{2\mu v_r}{r} - \frac{(\mu v)_r}{r} - \frac{uv}{r^2}, \tag{1.3}$$

$$\rho(w_t + uw_r) = (\mu w_r)_r + \frac{\mu w_r}{r}, \tag{1.4}$$

$$\rho(e + ue_r) + \frac{P(ru)_r}{r} = \frac{(\kappa r\theta_r)_r}{r} + Q, \tag{1.5}$$

where $\rho(r,t)$ is the density, $u(r,t)$, $v(r,t)$, and $w(r,t)$ are velocities in different directions, $\theta(r,t)$ is the temperature, the pressure P and the internal energy e are related with the density and temperature

$$P = P(\rho,\theta) = R\rho\theta \quad \text{and} \quad e = e(\rho,\theta) = c_v\theta, \tag{1.6}$$

the specific gas constant R and the specific heat at constant volume c_v are positive constants, respectively; the symbol Q denotes

$$Q = \frac{\lambda(ru)_r^2}{r^2} - \frac{4\mu uu_r}{r} + \mu w_r^2 + \mu(v_r - \frac{v}{r})^2, \tag{1.7}$$

μ and λ are viscosity coefficients, and κ is the heat conductivity coefficient.

Without loss of generality, we shall consider the system (1.1) with the following initial-boundary data:

$$\begin{cases}(\rho,u,v,w,\theta)|_{t=0} = (\rho_0,u_0,v_0,w_0,\theta_0)(r), & 0 < a \le r \le b < \infty,\\ (u,v,w,\partial_r\theta)|_{r=a} = (u,v,w,\partial_r\theta)|_{r=b} = 0, & t \ge 0.\end{cases} \tag{1.8}$$

Our main goal is to show the large-time behavior of global solutions to the initial-boundary value problem (1.1)–(1.8) with large initial data. For this purpose, it is convenient to transform the initial-boundary value problem (1.1)–(1.8) into Lagrangian coordinates. We introduce the Lagrangian coordinates (t,x) and denote $(\tilde{\rho},\tilde{u},\tilde{v},\tilde{w},\tilde{\theta})(t,x) = (\rho,u,v,w,\theta)(t,r)$, where

$$r = r(t,x) = r_0(x) + \int_0^t u(s, r(s,x))\mathrm{d}s, \tag{1.9}$$

and

$$r_0(x) := f^{-1}(x), \qquad f(r) := \int_a^r y\rho_0(y)\mathrm{d}z.$$

Note that the function f is invertible on $[a,b]$ provided that $\rho_0(y) > 0$ for each $y \in [a,b]$ (which will be assumed in Theorem 1.1). Due to $(1.1)_1$ and (1.8), we see

$$\frac{\partial}{\partial t}\int_a^{r(t,x)} y\rho(t,y)\mathrm{d}y = 0.$$

Then it is easy to check

$$\int_a^{r(t,x)} y\rho(t,y)\mathrm{d}y = f(r_0(x)) = x \quad \text{and} \quad \int_b^{r(t,1)} y\rho(t,y)\mathrm{d}y = 0, \tag{1.10}$$

which translates the domain $[0,T]\times[a,b]$ into $[0,T]\times[0,1]$. Hereafter, we denote $(\tilde{\rho},\tilde{u},\tilde{v},\tilde{w},\tilde{\theta})$ by (ρ,u,v,w,θ) for simplicity. The identities (1.9) and (1.10) imply

$$r_t(t,x) = u(t,x), \qquad r_x(t,x) = r^{-1}\tau(t,x), \tag{1.11}$$

where $\tau := \rho^{-1}$ is the specific volume. By means of identities (1.11), system (1.1)–(1.8) is changed to

$$\tau_t = (ru)_x, \tag{1.12}$$

$$u_t - \frac{v^2}{r} + rP_x = r\left(\frac{\lambda(ru)_x}{\tau}\right)_x - 2u\mu_x, \tag{1.13}$$

$$v_t - \frac{uv}{r} = r\left(\frac{\mu r v_x}{\tau}\right)_x + 2\mu v_x - (\mu v)_x - \frac{\mu\tau v}{r^2}, \tag{1.14}$$

$$w_t = r\left(\frac{\mu r w_x}{\tau}\right)_x + \mu w_x, \tag{1.15}$$

$$e_t + P(ru)_x = \left(\frac{\kappa r^2\theta_x}{\tau}\right)_x + Q, \tag{1.16}$$

where $t > 0$, $x \in \Omega = (0, 1)$, $P = \frac{R\theta}{\tau}$, $e = \frac{R\theta}{\gamma-1}$, and $Q = \frac{\lambda(ru)_x^2}{\tau} - 4\mu u u_x + \frac{\mu r^2 w_x^2}{\tau} + \mu\tau\left(\frac{r v_x}{\tau} - \frac{v}{r}\right)^2$.

Throughout this paper, we assume that μ, λ, and κ are power functions of absolute temperature as follows:

$$\mu = \tilde{\mu}\theta^{\alpha}, \qquad \lambda = \tilde{\lambda}\theta^{\alpha}, \qquad \kappa = \tilde{\kappa}\theta^{\beta}, \tag{1.17}$$

where constants $\tilde{\mu}, \tilde{\lambda}, \tilde{\kappa}, \alpha$, and β are positive constants.

The objective of this paper is to study the global existence and stability of the solutions to an initial-boundary value problem of (1.12)–(1.16) with the initial data:

$$(\tau, u, v, w, \theta)(x, 0) = (\tau_0, u_0, v_0, w_0, \theta_0), x \in (0, 1), \tag{1.18}$$

and the boundary conditions:

$$(u, v, w, \theta_x)(0, t) = (u, v, w, \theta_x)(1, t) = 0, \qquad t \geq 0. \tag{1.19}$$

Using Navier-Stokes equations as a model for describing fluid motion has been widely accepted by the physics community. In recent years, some significant progress has been made in the study of Navier-Stokes equations with constant viscosity coefficients. When the initial value has a certain small property and vacuum state does not exist, the global existence, uniqueness, and large-time behavior of the solutions can be easily calculated [1–8]. However, solving the problem of large initial values is very challenging, and the first significant breakthrough was achieved by Lions [9]. Besides, by assuming that the initial value is only sufficiently small in the energy space, Hoff [10, 11] confirmed the existence of global weak solutions. In the process of studying fluid motion, a vacuum state is often involved, which makes calculations far more complex. The results in [12, 13] indicate the Cauchy problem of Navier-Stokes equations with constant coefficients containing vacuum state is not appropriate. This uncertainty is reflected by the fact that the solutions of the system have no continuous dependence on the initial values. Based on physical considerations, Liu-Xin-Yang [12] studied the Cauchy problem of the Navier-Stokes equations with density dependent viscosity, and proved its local suitability. However, only when the temperature and density change within a suitable range, real fluids can be considered as ideal fluids (viscosity coefficients are constants). In the case of large changes in temperature or density, the viscosity of the real fluid will vary greatly [14].

On the other hand, Navier-Stokes equations can be developed using the Chapman-Enskog expansion of the microscopic particle collision model Boltzman equation. Consequently, it can be determined that the viscosity depends on the temperature. However, compared to the abundant research using classical models, the studies on the physical case using the temperature-dependent viscosity model are lacking. Because the viscosity and heat conductivity are both temperature-dependent, degeneracy and strong non-linearity may appear. Pan-Zhang [15] and Huang-Shi [16] obtained global strong solutions and large-time behavior in bounded domains for one-dimensional Navier-Stokes equations, when $\alpha = 0$ and $0 < \beta < 1$. The studies of Liu-Yang-Zhao-Zhou [17] and Wan-Wang [18] also acquired global solutions of Navier-Stokes equations in one dimensional and cylindrically symmetrical cases, respectively, with the requirement that $|\gamma - 1|$ was small enough. Wang-Zhao [19] removed the smallness condition of $|\gamma - 1|$, and established global classical solutions to Navier-Stokes equations in the one-dimensional whole space when μ and κ satisfy:

$$\mu = \tilde{\mu} h(\tau)\theta^{\alpha}, \qquad \kappa = \tilde{\kappa} h(\tau)\theta^{\alpha},$$

where α is small enough. In their calculations, the viscosity and heat-conductivity were dependent on temperature and density, and to overcome the difficulties caused by density, the following conditions could not be removed:

$$\|h(\tau)^{-1}\tau^{-1}\|_{L^\infty(\Omega)} + \|h(\tau)^{-1}\tau\|_{L^\infty(\Omega)} \le C.$$

This means that estimate of $\|\tau_x\|_{L^2(\Omega)}$ can be directly obtained without the upper and lower bounds of density, as long as the coefficient μ^{-1} or κ^{-1} appears. However, if $h(\tau)$ is constant, then the constants $l_1 = l_2 = 0$ and the result of this case cannot be established using the model in [19]. Recently, Sun-Zhang-Zhao [20] considered an initial-boundary value problem of the compressible Navier-Stokes equations for one-dimensional viscous and heat-conducting ideal polytropic fluids with temperature-dependent transport coefficients, and discovered the global-in-time existence of strong solutions. In that paper, the initial data could be large if $\alpha \ge 0$ is small and the growth exponent $\beta \ge 0$ is arbitrarily large. It is worth mentioning that the smallness of $\alpha > 0$ depends on the size of the initial data. However, unfortunately the study did not provide a specific relationship between α and the initial data in [20]. Our main results are concluded as follows.

Theorem 1.1. *For given positive constants* $M_0, V_0 > 0$, *assume that*

$$\|(\tau_0, u_0, v_0, w_0, \theta_0)\|_{H^2(\Omega)} \le M_0, \ \inf_{x\in(0,1)}\{\tau_0, \theta_0\} \ge V_0. \tag{1.20}$$

Then there exist $\epsilon_0 > 0$ *and* C_0 *which depend only on* β, M_0, *and* V_0, *such that the initial-boundary value problem* (1.12)–(1.19) *with* $0 \le \alpha \le \epsilon_0 := \min\{|\alpha_1|, |\alpha_2|\}$ *and* $\beta > 0$ *admit a unique global-in-time strong solution* (τ, u, v, w, θ) *on* $[0, 1] \times [0, +\infty)$ *satisfying*

$$C_0^{-1} \le \tau(x,t) \le C_0, \qquad C_1^{-1} \le \theta(x,t) \le C_1,$$

and

$$(\tau - \bar{\tau}, u, v, w, \theta - E_0) \in C([0, +\infty); H^2(\Omega)),$$

where α_1, α_2 defined in what follows are dependent only on β, M_0, and V_0 (see details in (3.2),(3.5)*, and* (3.6)*). Moreover, for any $t > 0$, the exponential decay rate is*

$$\|(\tau - \bar{\tau}, u, v, w, \theta - E_0)\|_{H^1}^2 + \|r - \bar{r}\|_{H^2}^2 \le Ce^{-\gamma_0 t}, \tag{1.21}$$

where

$$\bar{\tau} = \int_0^1 \tau dx, \quad E_0 = \int_0^1 \left(\theta_0 + \frac{u_0^2 + v_0^2 + w_0^2}{2c_v}\right) dx, \quad \bar{r} = [a^2 + 2\bar{\tau}x]^{\frac{1}{2}}.$$

A few remarks are in order.

Remark 1. For $k = 1, 2$ and $1 \le p \le \infty$, we adopt the simplified nations for the standard Sobolev space as follows:

$$\|\cdot\| := \|\cdot\|_{L^2(\Omega)}, \quad \|\cdot\|_k := \|\cdot\|_{H^k(\Omega)}, \quad \|f\|_\infty := \max_{x\in\Omega} |f(x)|, \quad \|\cdot\|_{L^p} := \|\cdot\|_{L^p(\Omega)}.$$

Remark 2. We remark here that the growth exponent $\beta \in (0, +\infty)$ can be arbitrarily large, and the choice of $\epsilon_0 > 0$ depends only on β, V_0, and the H^2-norm of the initial data. An outline of this paper is as follows. We devote Section 2 to a discussion of a number of *a priori* estimates independent of time, which are needed to extend the local solution to all time. Based on the previous estimates, the main results, Theorem 1.1 are proved in Section 3.

Remark 3. In this paper, the positive c , C, and $C_i (i = 0, 1, \cdots, 16)$ are some positive constants which depend only on β, M_0, and V_0, but not on the time t. Furthermore, c and C are different from line to line.

2. A priori estimates

First of all, define

$$\begin{aligned} &X(t_1, t_2; m_1, m_2; N) \\ &\quad := \{(\tau, u, v, w, \theta) \in C([t_1, t_2]; H^2(\Omega)), \tau_x \in L^2(t_1, t_2; H^1(\Omega)) \\ &\qquad (u_x, v_x, w_x, \theta_x) \in L^2(t_1, t_2; H^2(\Omega)), \\ &\qquad \tau_t \in C([t_1, t_2]; H^1(\Omega)) \cap L^2(t_1, t_2; H^1(\Omega)), \\ &\qquad (u_t, v_t, w_t, \theta_t) \in C([t_1, t_2]; L^2(\Omega)) \cap L^2(t_1, t_2; H^1(\Omega)), \\ &\qquad \tau \ge m_1, \theta \ge m_2, \mathcal{E}(t_1, t_2) \le N^2, \forall (x, t) \in [0, 1] \times [t_1, t_2]\}, \end{aligned}$$

where N, m_1, m_2, and $t_1, t_2 (t_2 > t_1)$ are constants and

$$\mathcal{E}(t_1, t_2) := \sup_{t_1 \le t \le t_2} \|(\tau_x, u_x, \theta_x)\|_1^2 + \|\theta_t\|^2 + \int_{t_1}^{t_2} \|\theta_t\|^2 dt$$

with

$$\theta_t|_{t=t_1} := \frac{1}{c_v}\left[-P(ru)_x + \left(\frac{\kappa r^2 \theta_x}{\tau}\right)_x + Q\right]\Big|_{t=t_1},$$

$$\theta_{xt}|_{t=t_1} := \frac{1}{c_v}\left[-P(ru)_x + \left(\frac{\kappa r^2 \theta_x}{\tau}\right)_x + Q\right]_x\Big|_{t=t_1}.$$

The main purpose of this section is to derive the global t-independent estimates of the solutions $(\tau, u, v, w, \theta) \in X(0, T; m_1, m_2, N)$.

We start with the following basic energy estimate.

Lemma 2.1. *Assume that the conditions listed in Theorem 1.1 hold. Then there exists a constant* $0 < \epsilon_1 \le 1$ *depending only on* M_0 *and* V_0*, such that if*

$$m_2^{-\alpha} \le 2, \qquad N^\alpha \le 2, \qquad \alpha H(m_1, m_2, N) \le \epsilon_1, \tag{2.1}$$

where

$$H(m_1, m_2, N) := (m_1 + m_2 + N + 1)^5,$$

then for $T \ge 0$ *,*

$$\begin{aligned} &\int_0^1 \eta_{\hat\theta}(\tau, u, v, w, \theta)(x, t)\mathrm{d}x \\ &+ \int_0^T \int_0^1 \left[\frac{\tau u^2}{\theta} + \frac{u_x^2 + w_x^2}{\tau\theta} + \frac{\theta^\beta \theta_x^2}{\tau\theta^2} + \frac{\tau}{\theta}\left(\frac{r v_x}{\tau} - \frac{v}{r}\right)^2 \right] \mathrm{d}x\mathrm{d}s \le C, \end{aligned} \tag{2.2}$$

where

$$\begin{aligned} &\eta_{\hat\theta}(\tau, u, v, w, \theta) := \hat\theta\phi\left(\frac{\tau}{\bar\tau}\right) + \frac{u^2 + v^2 + w^2}{2} + c_v \hat\theta \phi\left(\frac{\theta}{\hat\theta}\right), \\ &\phi(z) := z - \log z - 1. \end{aligned}$$

Proof. Multiplying (1.12)–(1.16) by $R\hat\theta(\bar\tau^{-1} - \tau^{-1})$, u, v, w, and $(1 - \hat\theta\theta^{-1})$, respectively, integrating over $[0, 1]$, and adding them together, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \eta_{\hat\theta}(\tau, u, v, w, \theta)\mathrm{d}x + \int_0^1 \left[\frac{\tilde\kappa r^2 \theta^\beta \theta_x^2}{\tau\theta^2} + \frac{Q}{\theta} \right] \mathrm{d}x = 0, \tag{2.3}$$

where $Q = \frac{\lambda(ru)_x^2}{\tau} - 4\mu u u_x + \frac{\mu r^2 w_x^2}{\tau} + \mu\tau\left(\frac{r v_x}{\tau} - \frac{v}{r}\right)^2$.

Apparently, by means of $\lambda = 2\mu + \lambda'$, one has

$$\begin{aligned} \lambda(ru)_x^2 - 4\tau\mu u u_x &= (2\mu + \lambda')r^2 u_x^2 + (2\mu + \lambda')\frac{\tau^2 u^2}{r^2} + 2\lambda'\tau u u_x \\ &= 2\mu r^2 u_x^2 + 2\mu\frac{\tau^2 u^2}{r^2} + \frac{2\mu + 3\lambda'}{3}\left[r u_x + \frac{\tau u}{r}\right]^2 - \frac{2\mu}{3}\left[r u_x + \frac{\tau u}{r}\right]^2 \\ &= \frac{2}{3}\mu\left(r^2 u_x^2 + \frac{\tau^2 u^2}{r^2}\right) + \frac{2\mu + 3\lambda'}{3}\left[r u_x + \frac{\tau u}{r}\right]^2 + \frac{2\mu}{3}\left[r u_x - \frac{\tau u}{r}\right]^2 \\ &\ge \frac{2}{3}\mu\left(r^2 u_x^2 + \frac{\tau^2 u^2}{r^2}\right). \end{aligned}$$

Thus, one has

$$Q \ge C\frac{u_x^2}{\tau} + C\tau u^2 + C\frac{w_x^2}{\tau} + C\tau\left(\frac{r v_x}{\tau} - \frac{v}{r}\right)^2,$$

which combined with (2.1) and (2.3) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \eta_{\hat{\theta}}(\tau,u,v,w,\theta)(t,x)\mathrm{d}x + c\int_0^1 \left(\frac{\tau u^2}{\theta} + \frac{u_x^2+w_x^2}{\tau\theta} + \frac{\theta^\beta\theta_x^2}{\tau\theta^2} + \frac{\tau}{\theta}\left(\frac{rv_x}{\tau} - \frac{v}{r}\right)^2\right)\mathrm{d}x \le 0. \tag{2.4}$$

Integrating (2.4) over $(0,T)$, we can obtain (2.2) by the initial conditions $(\tau_0,u_0,v_0,\theta_0)$. □

Next, by means of Lemma 2.1, we derive the upper and lower bounds of τ.

Lemma 2.2. *Assume that the conditions of Lemma 2.1 hold. Then for* $(x,t)\in\Omega\times[0,\infty)$,

$$C_0^{-1} \le \tau(x,t) \le C_0.$$

Proof. The proof is divided into three steps.

Step 1 (Representation of the formula for τ).

It follows from (1.13) that

$$\left(\frac{u}{r}\right)_t + \frac{u^2-v^2}{r^2} + \frac{2u\mu_x}{r} + P_x = \Big(\lambda(\ln\tau)_t\Big)_x = \lambda(\ln\tau)_{xt} + \lambda_x\frac{(ru)_x}{\tau}.$$

that is

$$\left(\frac{u}{\lambda r}\right)_t + g + \left(\lambda^{-1}P\right)_x = (\ln\tau)_{xt}, \tag{2.5}$$

where

$$g := \frac{u^2-v^2}{\lambda r^2} + \frac{2u\mu_x}{\lambda r} - \left(\lambda^{-1}\right)_x P - \frac{\lambda_x(ru)_x}{\lambda\tau} - (\lambda^{-1})_t\frac{u}{r}.$$

Integrating (2.5) over $[0,t]\times[x_1(t),x]$, we have

$$\begin{aligned}\int_{x_1(t)}^x \left(\frac{u}{\lambda r} - \frac{u_0}{\lambda_0 r_0}\right)\mathrm{d}\xi + \int_0^t\int_{x_1(t)}^x g\mathrm{d}\xi\mathrm{d}s + \int_0^t \lambda^{-1}P(x) - \lambda^{-1}P(x_1)\mathrm{d}s \\ = \ln\tau(x,t) - \ln\tau(x_1(t),t) - \Big[\ln\tau_0(x) - \ln\tau(x_1(t),0)\Big],\end{aligned} \tag{2.6}$$

where $x_1(t)\in[0,1]$ is determined by the following progresses. Next, for convenience, we define

$$\begin{aligned}F &:= \frac{(ru)_x}{\tau} - \lambda^{-1}P - \int_0^x g(\xi)\mathrm{d}\xi,\\ \varphi &:= \int_0^t F(x,s)\mathrm{d}s + \int_0^x \frac{u_0}{\lambda_0 r_0}\mathrm{d}\xi.\end{aligned}$$

It follows from the definitions above that

$$\varphi_x = \frac{u}{\lambda r}, \qquad \varphi_t = F. \tag{2.7}$$

By the definition of F and (1.12), one has

$$\begin{aligned}&\int_0^t \left[\lambda^{-1}P(x_1(t),s) + \int_0^{x_1(t)} g(\xi,s)\mathrm{d}\xi\right]\mathrm{d}s\\ &= \int_0^t \left(\frac{(ru)_x}{\tau} - F\right)(x_1(t),s)\mathrm{d}s\\ &= \ln\tau(x_1(t),t) - \ln\tau(x_1(t),0) - \int_0^t F(x_1(t),s)\mathrm{d}s.\end{aligned} \tag{2.8}$$

Due to (1.12) and (2.7), we have

$$\begin{aligned}&(\tau\varphi)_t - (ru\varphi)_x \\ &= \tau\varphi_t - ru\varphi_x = \tau F - \frac{u^2}{\lambda} \\ &= (ru)_x - \frac{\tau P}{\lambda} - \tau\int_0^x g(\xi)\mathrm{d}\xi - \frac{u^2}{\lambda}.\end{aligned} \tag{2.9}$$

Integrating (2.9) over $[0,t]\times\Omega$, one has

$$\int_0^1 \varphi\tau\mathrm{d}x = \int_0^1 \tau_0\int_0^x\left(\frac{u_0}{\lambda_0 r_0}\right)(\xi)\mathrm{d}\xi\mathrm{d}x - \int_0^t\int_0^1\left[\frac{\tau}{\lambda}P + \tau\int_0^x g\mathrm{d}\xi + \frac{u^2}{\lambda}\right]\mathrm{d}x\mathrm{d}s. \tag{2.10}$$

Hence, by virtue of the mean value theorem, there exits $x_1(t)\in[0,1]$ such that $\varphi(x_1(t),t) = \int_0^1 \varphi\tau\mathrm{d}x$. By the definition of φ, (2.8), and (2.10), one obtains

$$\begin{aligned}&\int_0^t F(x_1(t),s)\mathrm{d}s = \varphi(x_1(t),t) - \int_0^{x_1(t)}\frac{u_0}{\lambda_0 r_0}(\xi)\mathrm{d}\xi \\ &= \int_0^1\tau_0\int_0^x\frac{u_0}{\lambda_0 r_0}(\xi)\mathrm{d}\xi\mathrm{d}x - \int_0^t\int_0^1\left(\frac{\tau}{\lambda}P + \tau\int_0^x g\mathrm{d}\xi + \frac{u^2}{\lambda}\right)\mathrm{d}x\mathrm{d}s \\ &\quad - \int_0^{x_1(t)}\frac{u_0}{\lambda_0 r_0}(\xi)\mathrm{d}\xi.\end{aligned} \tag{2.11}$$

Putting (2.11) into (2.8), it follows that

$$\begin{aligned}&\int_0^t\left(\frac{P}{\lambda}(x_1(t),s) + \int_0^{x_1(t)} g(\xi,s)\mathrm{d}\xi\right)\mathrm{d}s \\ &= \ln\tau(x_1(t),t) - \ln\tau(x_1(t),0) - \int_0^1\tau_0\int_0^x\frac{u_0}{\lambda_0 r_0}(\xi)\mathrm{d}\xi\mathrm{d}x \\ &\quad + \int_0^{x_1(t)}\frac{u_0}{\lambda_0 r_0}(\xi)\mathrm{d}\xi + \int_0^t\int_0^1\left(\frac{\tau}{\lambda}P + \tau\int_0^x g\mathrm{d}\xi + \frac{u^2}{\lambda}\right)\mathrm{d}x\mathrm{d}s.\end{aligned} \tag{2.12}$$

Inserting (2.12) into (2.6), we derive

$$\begin{aligned}&\int_0^t\frac{P}{\lambda}\mathrm{d}s + \int_0^t\int_0^x g\mathrm{d}\xi\mathrm{d}s - \int_0^t\int_0^1\left(\frac{\tau}{\lambda}P + \tau\int_0^x g\mathrm{d}\xi + \frac{u^2}{\lambda}\right)\mathrm{d}x\mathrm{d}s \\ &+ \int_{x_1(t)}^x\left(\frac{u}{\lambda r} - \frac{u_0}{\lambda_0 r_0}\right)\mathrm{d}\xi + \int_0^1\tau_0\int_0^x\frac{u_0}{\lambda_0 r_0}\mathrm{d}\xi\mathrm{d}x - \int_0^{x_1(t)}\frac{u_0}{\lambda_0 r_0}\mathrm{d}\xi \\ &= \ln\tau - \ln\tau_0.\end{aligned} \tag{2.13}$$

Let

$$g = \frac{u^2 - v^2}{\lambda r^2} + g_1,$$

where

$$g_1 := \frac{2u\mu_x}{\lambda r} - \left(\lambda^{-1}\right)_x P - \frac{\lambda_x(ru)_x}{\lambda\tau} - (\lambda^{-1})_t\frac{u}{r}.$$

It follows from (2.13) that

$$\tau = B^{-1}AD, \tag{2.14}$$

where

$$\begin{aligned}
A &:= \exp\left\{\int_0^t \left[\frac{P}{\lambda}(x,s) + \int_0^x \left(g_1(\xi,s) + \frac{u^2}{\lambda r^2}\right)\mathrm{d}\xi + \int_0^1 \tau \int_0^x \left(\frac{v^2}{\lambda r^2} - g_1\right)\mathrm{d}\xi\mathrm{d}x\right]\mathrm{d}s\right\},\\
B &:= \exp\left\{\int_0^t \left[\int_0^1 \left(\frac{\tau}{\lambda}P + \tau\int_0^x \frac{u^2}{\lambda r^2}(\xi)\mathrm{d}\xi + \frac{u^2}{\lambda}\right)\mathrm{d}x + \int_0^x \frac{v^2}{\lambda r^2}\mathrm{d}\xi\right]\mathrm{d}s\right\},\\
D &:= \tau_0 \exp\left\{\int_0^1 \tau_0 \int_0^x \frac{u_0}{\lambda_0 r_0}\mathrm{d}\xi\mathrm{d}x - \int_0^{x_1(t)} \frac{u_0}{\lambda_0 r_0}\mathrm{d}\xi + \int_{x_1(t)}^x \left(\frac{u}{\lambda r} - \frac{u_0}{\lambda_0 r_0}\right)(\xi)\mathrm{d}\xi\right\}.
\end{aligned}$$

By (2.14), one has

$$\tau D^{-1}B = A. \tag{2.15}$$

Define that

$$J := \frac{P}{\lambda}(x,s) + \int_0^x \left(g_1(\xi,s) + \frac{u^2}{\lambda r^2}\right)\mathrm{d}\xi + \int_0^1 \tau \int_0^x \left(\frac{v^2}{\lambda r^2} - g_1\right)\mathrm{d}\xi\mathrm{d}x.$$

Then, multiplying (2.15) by J gives

$$\tau D^{-1}BJ = \frac{\mathrm{d}}{\mathrm{d}t}A.$$

Since $A(0) = 1$, integrating the above equality over $(0,t)$ about time, one has

$$\begin{aligned}
\tau =& DB^{-1} + \frac{1}{\lambda}\int_0^t \frac{B(s)}{B(t)}\frac{D(t)}{D(s)}\tau\left[\frac{P}{\lambda}(x,s) + \int_0^x \left(g_1(\xi,s) + \frac{u^2}{\lambda r^2}\right)\mathrm{d}\xi\right.\\
&\left. + \int_0^1 \tau \int_0^x \left(\frac{v^2}{\lambda r^2} - g_1\right)\mathrm{d}\xi\mathrm{d}x\right]\mathrm{d}s.
\end{aligned} \tag{2.16}$$

Step 2 (Lower bound for τ). First of all, by means of (2.1) and (2.2), one has

$$C^{-1} \le D \le C. \tag{2.17}$$

Next, we estimate B. Employing Jensen's inequality to the convex function ϕ, we have

$$\int_0^1 z\mathrm{d}x - \log\int_0^1 z\mathrm{d}x - 1 \le \int_0^1 \phi(z)\mathrm{d}x. \tag{2.18}$$

By (2.18) and Lemma 2.1, one obtains

$$C^{-1} \le \int_0^1 \tau\mathrm{d}x, \quad \bar{\theta} := \int_0^1 \theta\mathrm{d}x \le C, \tag{2.19}$$

which means that

$$C^{-1} \le \int_0^1 \frac{\tau}{\lambda}P\mathrm{d}x \le C. \tag{2.20}$$

Hence, by means of the definition of B and (2.20), choosing ε_1 suitably small, there exist two constants C_1 and C_2, such that

$$e^{c_1 t} \leq B(t) \leq e^{c_2 t}. \tag{2.21}$$

That is,

$$e^{-c_1(t-s)} \leq \frac{B(s)}{B(t)} \leq e^{-c_2(t-s)}. \tag{2.22}$$

Apparently, by means of (2.1) and (2.19), we deduce

$$\begin{aligned} &\left|\tau \int_0^1 \tau \int_0^x g_1 d\xi dx\right| \\ &\leq C|\alpha|\|\tau\|_\infty^2 \big(\|\theta^{-1}\|_\infty \|\theta_x\|\|u\| + \|\theta^{-\alpha}\tau^{-1}\|_\infty \|\theta_x\| \\ &\quad + \|\theta^{-1}\tau^{-1}\|_\infty \|\theta_x\|\|u\|_1 + \|\theta^{-1}\tau^{-1}\|_\infty \|\theta_t\|\|u\|\big) \\ &\leq C\varepsilon_1. \end{aligned} \tag{2.23}$$

Similarly, one also has

$$\left\|\int_0^x g_1 d\xi\right\|_\infty \leq C\varepsilon_1. \tag{2.24}$$

Thus, for $t \leq t_0 < \infty$,

$$\begin{aligned} \tau &\geq DB^{-1} - C\varepsilon_1 \int_0^t e^{-c_2(t-s)} ds \\ &= DB^{-1} - \frac{C\varepsilon_1}{c_2}(1 - e^{-c_2 t}) \\ &\geq Ce^{-ct_0} - \varepsilon_2(1 - e^{-c_2 t_0}). \end{aligned}$$

For a enough large t, we have

$$\inf_{x\in\Omega} \tau(x,t) \geq C \int_0^t \frac{B(s)}{B(t)} \theta ds - \varepsilon_2(1 - e^{-c_2 t}). \tag{2.25}$$

So, we need the estimates of θ and $\frac{B(s)}{B(t)}$. By the mean value theorem and (2.19), there exits $x_2(t) \in [0,1]$, such that

$$C^{-1} \leq \theta(x_2(t), t) \leq C. \tag{2.26}$$

By Cauchy-Schwarz's inequality and (2.19), one has

$$\begin{aligned} &\left|[\ln(\theta+1)]^{\frac{\beta}{2}+1} - [\ln(\theta(x_2(t),t)+1)]^{\frac{\beta}{2}+1}\right| \\ &= \left|\int_{x_2}^x \frac{(\ln(\theta+1))^{\frac{\beta}{2}} \theta_x}{\sqrt{\tau}(\theta+1)} \sqrt{\tau}(\xi) d\xi\right| \\ &\leq \left(\int_0^1 \frac{(\ln(\theta+1))^\beta \theta_x^2}{\tau(\theta+1)^2} dx\right)^{\frac{1}{2}} \left(\int_0^1 \tau dx\right)^{\frac{1}{2}} \\ &\leq C\left(\int_0^1 \frac{\theta^\beta \theta_x^2}{\tau\theta^2} dx\right)^{1/2}, \end{aligned}$$

which means that

$$\theta \geq C - C\int_0^1 \frac{\theta^\beta\theta_x^2}{\tau\theta^2}\mathrm{d}x. \tag{2.27}$$

By (2.16)–(2.17), (2.23)–(2.24), (2.21), Lemma 2.1, and (2.19), one has

$$\int_0^1 \tau\mathrm{d}x \leq C\mathrm{e}^{-ct} + C\int_0^t \frac{B(s)}{B(t)}\mathrm{d}s,$$

that is

$$\int_0^t \frac{B(s)}{B(t)}\mathrm{d}s \geq C - C\mathrm{e}^{-ct}. \tag{2.28}$$

Putting (2.27) into (2.25), by (2.22), (2.28), and Lemma 2.1, for a enough large t, one has

$$\begin{aligned}
&\int_0^t \frac{B(s)}{B(t)}\theta\mathrm{d}s\\
&\geq C\int_0^t \frac{B(s)}{B(t)}\left(1 - \int_0^1 \frac{\theta^\beta\theta_x^2}{\tau\theta^2}\mathrm{d}x\right)\mathrm{d}s\\
&\geq C - C\mathrm{e}^{-ct} - C\left(\int_0^{t/2} + \int_{t/2}^t\right)\frac{B(s)}{B(t)}\int_0^1 \frac{\theta^\beta\theta_x^2}{\tau\theta^2}\mathrm{d}x\mathrm{d}s\\
&\geq C - C\mathrm{e}^{-ct} - C\int_0^{t/2} \mathrm{e}^{-c(t-s)}\int_0^1 \frac{\theta^\beta\theta_x^2}{\tau\theta^2}\mathrm{d}x\mathrm{d}s - C\int_{t/2}^t\int_0^1 \frac{\theta^\beta\theta_x^2}{\tau\theta^2}\mathrm{d}x\mathrm{d}s\\
&\geq C - C\mathrm{e}^{-ct} - C\mathrm{e}^{-ct/2} - C\int_{t/2}^t\int_0^1 \frac{\theta^\beta\theta_x^2}{\tau\theta^2}\mathrm{d}x\mathrm{d}s \geq C.
\end{aligned} \tag{2.29}$$

Inserting (2.29) into (2.25), for a large enough time T_0, when $t > T_0$, it follows that

$$\inf_{x\in\Omega}\tau(x,t) \geq C.$$

Step 3 (Upper bound for τ). By (2.17), (2.22)–(2.24), and Lemma 2.1, one obtains

$$\|\tau\|_\infty \leq C + C\int_0^t \mathrm{e}^{-c_2(t-s)}\|\tau\|_\infty\left(\int_0^1 \frac{\theta^\beta\theta_x^2}{\tau\theta^2}\mathrm{d}x + 1\right)\mathrm{d}s, \tag{2.30}$$

where we have used the results

$$\begin{cases}\|\theta\|_\infty \leq C + C\|\tau\|_\infty\int_0^1 \frac{\theta^\beta\theta_x^2}{\tau\theta^2}\mathrm{d}x & \text{when} \quad 0 < \beta \leq 1,\\ \|\theta\|_\infty \leq C + C\int_0^1 \frac{\theta^\beta\theta_x^2}{\tau\theta^2}\mathrm{d}x & \text{when} \quad 1 < \beta < \infty.\end{cases} \tag{2.31}$$

In fact, by Hölder's inequality, for $0 < \beta \leq 1$,

$$\begin{aligned}
&\left|\theta^{1/2}(x,t) - \theta^{1/2}(x_2(t),t)\right|\\
&\leq \int_0^1 \theta^{-\frac{1}{2}}\theta_x\mathrm{d}x\\
&\leq \|\tau\|_\infty^{1/2}\left(\int_0^1 \frac{\theta^\beta\theta_x^2}{\tau\theta^2}\mathrm{d}x\right)^{1/2}\left(\int_0^1 \theta^{1-\beta}\mathrm{d}x\right)^{1/2}\\
&\leq \|\tau\|_\infty^{1/2}\left(\int_0^1 \frac{\theta^\beta\theta_x^2}{\tau\theta^2}\mathrm{d}x\right)^{1/2}.
\end{aligned} \tag{2.32}$$

For $1 < \beta < \infty$,

$$\left|\theta^{\beta/2}(x,t) - \theta^{\beta/2}(x_2(t),t)\right| \le \int_0^1 \frac{\theta^{\beta/2}\theta_x}{\theta}\mathrm{d}x \le \left(\int_0^1 \frac{\theta^\beta\theta_x^2}{\tau\theta^2}\mathrm{d}x\right)^{1/2}\left(\int_0^1 \tau\mathrm{d}x\right)^{1/2}. \tag{2.33}$$

By means of (2.26) and (2.32)–(2.33), we can obtain (2.31).

Thus, the inequality (2.30) combined with Gronwall's inequality and Lemma 2.1 yields that for any $t \ge 0$,

$$\sup_{t\ge0} \|\tau(x,t)\|_\infty \le C.$$

□

However, we cannot get the time-space estimate of v_x in Lemma 2.1. To obtain this estimate, we need the following result.

Lemma 2.3. *Assume that the conditions listed in Lemma 2.1 hold. Then for any $p > 0$ and $T \ge 0$,*

$$\int_0^1 \theta^{1-p}\mathrm{d}x + \int_0^T\int_0^1 \left(\frac{\theta^\beta\theta_x^2}{\theta^{p+1}} + \frac{\theta^\alpha(u^2+u_x^2+w_x^2)}{\theta^p} + \frac{\theta^\alpha\tau}{\theta^p}\left(\frac{rv_x}{\tau} - \frac{v}{r}\right)^2\right)\mathrm{d}x\mathrm{d}s \le C. \tag{2.34}$$

Proof. By Lemma 2.1, the result of (2.34) has been established for $p = 1$. In the following steps, we do the estimate for $p > 0$ and $p \ne 1$. Multiplying (1.16) by θ^{-p}, integrating over $[0,1]$, and using integration by parts gives

$$\begin{aligned} &\frac{c_v}{p-1}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta^{1-p}\mathrm{d}x + p\int_0^1 \frac{\tilde\kappa r^2\theta^\beta\theta_x^2}{\tau\theta^{p+1}}\mathrm{d}x + \int_0^1 \frac{Q}{\theta^p}\mathrm{d}x \\ &= R\int_0^1 \frac{\theta^{1-p}}{\tau}(ru)_x\mathrm{d}x \\ &= R\int_0^1 \frac{\theta^{1-p}-E_0}{\tau}(ru)_x\mathrm{d}x + RE_0\int_0^1 \frac{(ru)_x}{\tau}\mathrm{d}x. \end{aligned} \tag{2.35}$$

Apparently, there exists constant $C(p)$ depending on p such that

$$\left|\theta^{1-p} - E_0\right| \le C(p)\left|\theta^{1/2} - E_0^{1/2}\right|\left(E_0^{1/2} + \theta^{\frac12-p}\right). \tag{2.36}$$

By means of (2.35), (2.36), Lemma 2.2, (1.13), and (1.12), we deduce

$$\begin{aligned} &\frac{c_v}{p-1}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta^{1-p}\mathrm{d}x + p\int_0^1 \frac{\tilde\kappa r^2\theta^\beta\theta_x^2}{\tau\theta^{p+1}}\mathrm{d}x + \int_0^1 \frac{Q}{\theta^p}\mathrm{d}x \\ &\le C(p)\|\theta^{1/2} - E_0^{1/2}\|_\infty \int_0^1 (E_0^{1/2} + \theta^{\frac12-p})(|u| + |u_x|)\mathrm{d}x + RE_0\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \ln\tau\mathrm{d}x \\ &\le C(p)\|\theta^{1/2} - E_0^{1/2}\|_\infty \left[\left(\int_0^1 \frac{u^2+u_x^2}{\theta}\mathrm{d}x\right)^{\frac12}\left(\int_0^1 \theta\mathrm{d}x\right)^{\frac12}\right. \\ &\quad \left. + \left(\int_0^1 \theta^{1-p}\mathrm{d}x\right)^{\frac12}\left(\int_0^1 \frac{u^2+u_x^2}{\tau\theta^p}\mathrm{d}x\right)^{\frac12}\right] + RE_0\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \ln\tau\mathrm{d}x \\ &\le C(p)\|\theta^{1/2} - E_0^{1/2}\|_\infty^2 + C(p)\int_0^1 \frac{u^2+u_x^2}{\theta}\mathrm{d}x + \delta\int_0^1 \frac{u^2+u_x^2}{\tau\theta^p}\mathrm{d}x \\ &\quad + C(\delta,p)\|\theta^{1/2} - E_0^{1/2}\|_\infty^2\int_0^1 \theta^{1-p}\mathrm{d}x + RE_0\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \ln\tau\mathrm{d}x. \end{aligned} \tag{2.37}$$

Thus, employing the truth of

$$\int_0^t \|\theta^{1/2} - E_0^{1/2}\|_\infty^2 \mathrm{d}s \le C, \tag{2.38}$$

we can conclude from (2.37), Grönwall's inequality, and Lemma 2.2 that (2.34) is correct. In fact,

$$\|\theta^{1/2} - E_0^{1/2}\|_\infty \le \|\theta^{1/2} - \bar{\theta}^{1/2}\|_\infty + \|\bar{\theta}^{1/2} - E_0^{1/2}\|_\infty. \tag{2.39}$$

By virtue of Lemmas 2.1–2.2 and (2.19), one has

$$\begin{aligned}
\left|\bar{\theta}^\zeta - E_0^\zeta\right| &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\eta}\left\{\left[\int_0^1 \left(\theta + \eta\frac{u^2+v^2+w^2}{2c_v}\right)\mathrm{d}x\right]^\zeta\right\}\mathrm{d}\eta\right| \\
&= \left|\zeta\int_0^1 \left[\int_0^1 \left(\theta + \eta\frac{u^2+v^2+w^2}{2c_v}\right)\mathrm{d}x\right]^{\zeta-1}\mathrm{d}\eta\int_0^1 \frac{(u,v,w)^2}{2}\mathrm{d}x\right| \\
&\le C\|(u,v,w)\|\|(u,v,w)\|_\infty \le C\int_0^1 \left|\left(u_x, \left(\frac{v}{r}\right)_x, w_x\right)\right|\mathrm{d}x \\
&\le C\left(\int_0^1 \left[\frac{u_x^2}{\theta} + \frac{1}{\theta}\left(\frac{rv_x}{\tau} - \frac{v}{r}\right)^2 + \frac{w_x^2}{\theta}\mathrm{d}x\right]\right)^{1/2}\left(\int_0^1 \theta\mathrm{d}x\right)^{1/2} \\
&\le C\left(\int_0^1 \left[\frac{u_x^2}{\theta} + \frac{1}{\theta}\left(\frac{rv_x}{\tau} - \frac{v}{r}\right)^2 + \frac{w_x^2}{\theta}\right]\mathrm{d}x\right)^{1/2},
\end{aligned} \tag{2.40}$$

where we have used the fact that

$$\left(\frac{v}{r}\right)_x = \frac{\tau}{r^2}\left(\frac{rv_x}{\tau} - \frac{v}{r}\right).$$

For $\beta < 1$, it follows from Lemma 2.1 and (2.19) that

$$\begin{aligned}
&\|\theta^{1/2} - \bar{\theta}^{1/2}\|_\infty \\
&\quad\le C\int_0^1 \theta^{-\frac{1}{2}}|\theta_x|\mathrm{d}x \\
&\quad\le C\left(\int_0^1 \frac{\theta^\beta\theta_x^2}{\theta^2}\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^1 \theta^{1-\beta}\mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad\le C\left(\int_0^1 \frac{\theta^\beta\theta_x^2}{\theta^2}\mathrm{d}x\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.41}$$

For $1 \le \beta < \infty$,

$$\|\theta^{\frac{1}{2}} - \bar{\theta}^{\frac{1}{2}}\|_\infty \le C\|\theta^{\frac{\beta}{2}} - \bar{\theta}^{\frac{\beta}{2}}\|_\infty \le C\int_0^1 \theta^{\frac{\beta}{2}-1}|\theta_x|\mathrm{d}x \le C\left(\int_0^1 \frac{\theta^\beta\theta_x^2}{\theta^2}\mathrm{d}x\right)^{\frac{1}{2}}. \tag{2.42}$$

Hence, by (2.39)–(2.42) and Lemmas 2.1–2.2, we can derive (2.38). The proof of Lemma 2.3 is thus complete. □

According to Lemmas 2.1–2.3, we can conclude that the following results have been established.

Corollary 2.1. *Assume that the conditions listed in Lemma 2.1 hold. Then for $-\infty < q < 1$, $0 < p < \infty$, and $T \geq 0$,*

$$C_1 \leq \tau \leq C, \quad C^{-1} \leq \int_0^1 \tau \mathrm{d}x \leq C, \quad C^{-1} \leq \int_0^1 \theta \mathrm{d}x \leq C,$$

$$\int_0^1 (|\ln \tau| + |\ln \theta| + \theta^q + u^2 + v^2 + w^2)\mathrm{d}x \leq C_3,$$

$$\int_0^T \int_0^1 \left[(u^2 + u_x^2 + v^2 + v_x^2 + w_x^2 + \tau_t^2)(1 + \theta^{-p}) + \frac{\theta^\beta \theta_x^2}{\theta^{1+p}}\right]\mathrm{d}x\mathrm{d}s \leq C. \tag{2.43}$$

Here, we have taken $p = \alpha$ in (2.34) to obtain the time-space estimates of v and v_x.

Using the result above, we establish the following estimate about τ_x.

Lemma 2.4. *Assume that the conditions listed in Lemma 2.1 hold. Then for $T \geq 0$,*

$$\int_0^1 \tau_x^2 \mathrm{d}x + \int_0^T \int_0^1 \tau_x^2 (1 + \theta)\mathrm{d}x\mathrm{d}s \leq C_2.$$

Proof. According to the chain rule, one has

$$\left(\frac{\lambda \tau_x}{\tau}\right)_t = \left(\frac{\lambda \tau_t}{\tau}\right)_x + \frac{\lambda_\theta}{\tau}(\tau_x \theta_t - \tau_t \theta_x). \tag{2.44}$$

By means of (1.12), (1.13), and (2.44), we have

$$\left(\frac{\lambda \tau_x}{\tau}\right)_t = \frac{u_t}{r} + P_x - \frac{v^2}{r^2} + \frac{2u\mu_x}{r} + \frac{\lambda_\theta}{\tau}(\tau_x \theta_t - \tau_t \theta_x). \tag{2.45}$$

Multiplying (2.45) by $\frac{\lambda \tau_x}{\tau}$, integrating over $[0, 1]$ about x, and using (1.12) and (2.44), we obtain

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 \left[\frac{1}{2}\left(\frac{\lambda \tau_x}{\tau}\right)^2 - \frac{\lambda u \tau_x}{r\tau}\right]\mathrm{d}x + \int_0^1 \frac{R\lambda\theta\tau_x^2}{\tau^3}\mathrm{d}x \\ &= \int_0^1 \left(\frac{u}{r}\right)_x \frac{\lambda \tau_t}{\tau}\mathrm{d}x + \int_0^1 \frac{R\lambda\tau_x\theta_x}{\tau^2}\mathrm{d}x + \int_0^1 \frac{\lambda\tau_x(u^2 - v^2)}{\tau r^2}\mathrm{d}x \\ &\quad + \int_0^1 \frac{2u\mu_x\lambda\tau_x}{r\tau}\mathrm{d}x + \int_0^1 \frac{\lambda_\theta}{\tau^2}(\lambda\tau_x - r^{-1}u\tau)(\tau_x\theta_t - \tau_t\theta_x)\mathrm{d}x \\ &:= \sum_{i=1}^5 I_i. \end{aligned} \tag{2.46}$$

By Hölder's inequality, (2.1), (1.12), and Corollary 2.1, one has

$$I_1 = \int_0^1 \left(\frac{u_x}{r} - \frac{\tau u}{r^3}\right)\frac{\lambda \tau_t}{\tau}\mathrm{d}x \leq C\|(u, u_x, \tau_t)\|^2 \leq C\|(u, u_x)\|^2. \tag{2.47}$$

Using Corollary 2.1 and taking $p = \beta$, one has

$$\int_0^T \int_0^1 \frac{\theta_x^2}{\theta}\mathrm{d}x\mathrm{d}s \leq C. \tag{2.48}$$

Hence, we argue the term I_2 as the following

$$I_2 \leq \delta \int_0^1 \frac{\tau_x^2\theta}{\tau^3}\mathrm{d}x + C(\delta)\int_0^1 \frac{\theta_x^2}{\theta}\mathrm{d}x. \tag{2.49}$$

By means of integration by parts, Corollary 2.1, and (2.1), one can derive

$$\begin{aligned} I_3 &= -\int_0^1 \log\tau\left(\frac{\lambda}{r^2}(u^2-v^2)\right)_x \mathrm{d}x \\ &\leq C\|\ln\tau\|_\infty \int_0^1 \left|\left(|\alpha|\theta_x u^2, |\alpha|\theta_x v^2, \theta^\alpha u^2, \theta^\alpha v^2, \theta^\alpha u_x^2, \theta^\alpha v_x^2\right)\right|\mathrm{d}x \\ &\leq C\int_0^1 (u^2, v^2, u_x^2, v_x^2)\mathrm{d}x. \end{aligned} \tag{2.50}$$

By virtue of (2.1), we derive

$$I_4 \leq C|\alpha| m_2^{-\frac{3}{2}} N\left(\int_0^1 u^2\mathrm{d}x\right)^{1/2}\left(\int_0^1 \frac{\tau_x^2\theta}{\tau^3}\mathrm{d}x\right)^{1/2} \leq \delta\int_0^1 \frac{\tau_x^2\theta}{\tau^3}\mathrm{d}x + C(\delta)\int_0^1 u^2\mathrm{d}x. \tag{2.51}$$

By means of (2.1), Corollary 2.1, and (1.16), one can deduce

$$\begin{aligned} I_5 &\leq C\int_0^1 |\alpha||\theta^{-1}|\left|\left(\tau_x^2\theta_t, u\tau_x\theta_t, \tau_x u\theta_x, u^2\theta_x, \tau_x u_x\theta_x, uu_x\theta_x\right)\right|\mathrm{d}x \\ &\leq C|\alpha| m_2^{-2} N\int_0^1 \frac{\tau_x^2\theta}{\tau^3}\mathrm{d}x + C|\alpha| m_2^{-\frac{3}{2}} N\left(\int_0^1 u^2+u_x^2\mathrm{d}x\right)^{1/2}\left(\int_0^1 \frac{\tau_x^2\theta}{\tau^2}\mathrm{d}x\right)^{1/2} \\ &\quad + C|\alpha| m_2^{-1} N\int_0^1 u^2+u_x^2\mathrm{d}x \\ &\leq \varepsilon\int_0^1 \frac{\tau_x^2\theta}{\tau^3}\mathrm{d}x + C(\varepsilon)\int_0^1 u^2+u_x^2\mathrm{d}x. \end{aligned} \tag{2.52}$$

Inserting (2.47) and (2.49)–(2.52) into (2.46), and choosing ε suitable small, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left[\frac{1}{2}\left(\frac{\lambda\tau_x}{\tau}\right)^2 - \frac{\lambda u\tau_x}{r\tau}\right]\mathrm{d}x + c\int_0^1 \theta\tau_x^2\mathrm{d}x \leq C\|(u, u_x, \theta_x/\sqrt{\theta}, v, v_x)\|^2. \tag{2.53}$$

Integrating (2.53) over $[0, t]$, using Cauchy-Schwarz's inequality, (2.48), and Corollary 2.1, for any $t \geq 0$, one has

$$\int_0^1 \tau_x^2\mathrm{d}x + \int_0^t\int_0^1 \tau_x^2\theta\mathrm{d}x\mathrm{d}s \leq C. \tag{2.54}$$

By virtue of (2.54), we have

$$\begin{aligned} \bar\theta\int_0^1 \tau_x^2\mathrm{d}x &= \int_0^1 \tau_x^2(\bar\theta-\theta)\mathrm{d}x + \int_0^1 \tau_x^2\theta\mathrm{d}x \\ &\leq \frac{\bar\theta}{2}\int_0^1 \tau_x^2\mathrm{d}x + \frac{1}{2\bar\theta}\|\theta-\bar\theta\|_\infty^2\int_0^1 \tau_x^2\mathrm{d}x + \int_0^1 \tau_x^2\theta\mathrm{d}x \\ &\leq \frac{\bar\theta}{2}\int_0^1 \tau_x^2\mathrm{d}x + C\|\theta-\bar\theta\|_\infty^2 + \int_0^1 \tau_x^2\theta\mathrm{d}x. \end{aligned} \tag{2.55}$$

It follows from (2.19) and (2.48) that

$$\int_0^T \|\theta - \bar{\theta}\|_\infty^2 \mathrm{d}s \le C \int_0^T \int_0^1 \frac{\theta_x^2}{\theta} \mathrm{d}x \int_0^1 \theta \mathrm{d}x\mathrm{d}t \le C. \tag{2.56}$$

Thus, it follows from (2.55)–(2.56) that

$$\int_0^T \int_0^1 \tau_x^2 \mathrm{d}x\mathrm{d}t \le C. \tag{2.57}$$

The proof of Lemma 2.4 has been completed by (2.54) and (2.57). □

Next, based on the estimate of τ_x, we are devoted to derive the estimates on the first-order derivatives of w_x.

Lemma 2.5. *Assume that the conditions listed in Lemma 2.1 hold. Then for $T \ge 0$,*

$$\int_0^1 w_x^2 \mathrm{d}x + \int_0^T \int_0^1 w_{xx}^2 \mathrm{d}x\mathrm{d}t \le C_3.$$

Proof. Multiplying (1.15) by w_{xx} and integrating over $[0, 1]$ about x, we find from (2.1) and Lemma 2.4 that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_x\|^2 + \int_0^1 \frac{\mu r^2 w_{xx}^2}{\tau} \mathrm{d}x \\
&\quad = -\int_0^1 r w_{xx} w_x \left(\frac{\mu r}{\tau}\right)_x \mathrm{d}x - \int_0^1 \mu w_{xx} w_x \mathrm{d}x \\
&\quad \le C \int_0^1 |w_{xx} w_x| (|\alpha| m_2^{-1} |\theta_x| + 1 + |\tau_x|) \mathrm{d}x \\
&\quad \le \varepsilon \|w_{xx}\|^2 + C(\varepsilon)\|w_x\|^2 + C(\varepsilon)\|\tau_x\|^2 \|w_x\|_\infty^2 \\
&\quad \le \varepsilon \|w_{xx}\|^2 + C(\varepsilon)\|w_x\|^2.
\end{aligned} \tag{2.58}$$

Taking ε suitably small in (2.58) finds

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w_x\|^2 + c \int_0^1 w_{xx}^2 \mathrm{d}x \le C\|w_x\|^2. \tag{2.59}$$

The proof of Lemma 2.5 is complete by integrating (2.59) over $(0, t)$ about time and choosing ε suitably small. □

Based on the above result, we have the following uniform first-order derivatives estimates on the velocity (u, v).

Lemma 2.6. *Assume that the conditions listed in Lemma 2.1 hold. Then for $T \ge 0$,*

$$\int_0^1 (u_x^2 + v_x^2 + \tau_t^2) \mathrm{d}x + \int_0^T \int_0^1 \left(u_{xx}^2 + v_{xx}^2 + \theta_x^2 + u_t^2 + v_t^2 + w_t^2 + \tau_{tx}^2\right) \mathrm{d}x\mathrm{d}t \le C_4.$$

Proof. Multiplying (1.13) and (1.14) by u_{xx} and v_{xx}, respectively, and integrating over Ω about x, by integration by parts, one has

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (u_x^2+v_x^2)\mathrm{d}x+\int_0^1 \frac{r^2}{\tau}(\lambda u_{xx}^2+\mu v_{xx}^2)\mathrm{d}x\\
&=\int_0^1 u_{xx} r P_x \mathrm{d}x+\int_0^1\left(v_{xx}\frac{uv}{r}-u_{xx}\frac{v^2}{r}\right)\mathrm{d}x\\
&\quad-\int_0^1 u_{xx} r\left[\left(\frac{\lambda(ru)_x}{\tau}\right)_x-\frac{\lambda r u_{xx}}{\tau}\right]\mathrm{d}x+2\int_0^1 u\mu_x u_{xx}\mathrm{d}x\\
&\quad-\int_0^1 v_{xx}\left[r v_x\left(\frac{\mu r}{\tau}\right)_x+2\mu v_x-(\mu v)_x-\frac{\mu\tau v}{r^2}\right]\mathrm{d}x\\
&:=\sum_{i=1}^5 II_i.
\end{aligned}\tag{2.60}$$

By Cauchy-Schwarz's inequality, one has

$$II_1\le\varepsilon\|u_{xx}\|^2+C(\varepsilon)\|(\theta_x,\tau_x)\|^2.\tag{2.61}$$

It follows from Sobolev's inequality, the boundary condition of v, and Corollary 2.1, that we have

$$II_2\le\varepsilon\|(u_{xx},v_{xx})\|^2+C(\varepsilon)\|v\|_\infty^2\|(u,v)\|^2\le\varepsilon\|(u_{xx},v_{xx})\|^2+C(\varepsilon)\|v_x\|^2.\tag{2.62}$$

Direct computation from (2.1) yields

$$\begin{aligned}
II_3&\le\varepsilon\|u_{xx}\|^2+C(\varepsilon)\int_0^1\left[\tau_x^2u_x^2+(1+|\alpha|m_2^{-2}\theta_x^2)|(u_x,u\tau_x,u)|^2\right]\mathrm{d}x\\
&\le\varepsilon\|u_{xx}\|^2+C(\varepsilon)\|(u_x,u)\|^2+C(\varepsilon)\|\tau_x\|^2\|(u_x,u)\|_\infty^2\\
&\le 2\varepsilon\|u_{xx}\|^2+C(\varepsilon)\|(u_x,u)\|^2, &(2.63)\\
II_4&\le\varepsilon\|u_{xx}\|^2+C(\varepsilon)|\alpha|N^2m_2^{-2}\|u\|^2\le\varepsilon\|u_{xx}\|^2+C(\varepsilon)\|u\|^2, &(2.64)
\end{aligned}$$

and

$$\begin{aligned}
II_5&\le\varepsilon\|v_{xx}\|^2+C(\varepsilon)\int_0^1\left[v_x^2(1+|\alpha|m_2^{-2}\theta_x^2+\tau_x^2)+v^2\right]\mathrm{d}x\\
&\le 2\varepsilon\|v_{xx}\|^2+C(\varepsilon)\|(v_x,v)\|^2.
\end{aligned}\tag{2.65}$$

Putting (2.61)–(2.65) into (2.60) and taking ε suitably small gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1(u_x^2+v_x^2)\mathrm{d}x+c\int_0^1(u_{xx}^2+v_{xx}^2)\mathrm{d}x\le C\|(\theta_x,\tau_x,v_x,u_x,u,v)\|^2.\tag{2.66}$$

Integrating (2.66) over $(0,T)$ about time, and using Lemma 2.4 and Corollary 2.1, we find

$$\int_0^1(u_x^2+v_x^2)\mathrm{d}x+\int_0^T\int_0^1(u_{xx}^2+v_{xx}^2)\mathrm{d}x\mathrm{d}t\le C+C\int_0^T\int_0^1\theta_x^2\mathrm{d}x\mathrm{d}t.\tag{2.67}$$

For $\beta > 1$, we take $p = \beta - 1$ in (2.43), and then

$$\int_0^T \int_0^1 \theta_x^2 \mathrm{d}x\mathrm{d}t \le C. \tag{2.68}$$

Substituting (2.68) into (2.67), it follows for $\beta > 1$ that

$$\int_0^1 (u_x^2 + v_x^2)\mathrm{d}x + \int_0^T \int_0^1 (u_{xx}^2 + v_{xx}^2 + \theta_x^2)\mathrm{d}x\mathrm{d}t \le C. \tag{2.69}$$

Next, we need to estimate the $L^2(\Omega \times (0, t))$-norm of θ_x for $0 < \beta \le 1$. We deduce from multiplying (1.16) by $\theta^{1-\frac{\beta}{2}}$ and integration by parts that

$$\begin{aligned}
&\frac{2c_v}{4-\beta}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta^{2-\frac{\beta}{2}}\mathrm{d}x + \frac{2-\beta}{2}\int_0^1 \frac{\tilde{\kappa} r^2 \theta^{\frac{\beta}{2}}\theta_x^2}{\tau}\mathrm{d}x \\
&= -R\int_0^1 \frac{\theta^{2-\frac{\beta}{2}}}{\tau}(ru)_x \mathrm{d}x + \int_0^1 \theta^{1-\frac{\beta}{2}} Q\mathrm{d}x \\
&= R\int_0^1 \frac{\bar{\theta}^{2-\frac{\beta}{2}} - \theta^{2-\frac{\beta}{2}}}{\tau}(ru)_x \mathrm{d}x - R\bar{\theta}^{2-\frac{\beta}{2}}\int_0^1 \frac{(ru)_x}{\tau}\mathrm{d}x + \int_0^1 \theta^{1-\frac{\beta}{2}} Q\mathrm{d}x \\
&\le C\int_0^1 \left|\bar{\theta}^{2-\frac{\beta}{2}} - \theta^{2-\frac{\beta}{2}}\right| |(u, u_x)|\mathrm{d}x - R\bar{\theta}^{2-\frac{\beta}{2}}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \ln\tau\mathrm{d}x + \int_0^1 \theta^{1-\frac{\beta}{2}} Q\mathrm{d}x.
\end{aligned} \tag{2.70}$$

Notice that

$$\begin{aligned}
&\int_0^1 \left|\bar{\theta}^{2-\frac{\beta}{2}} - \theta^{2-\frac{\beta}{2}}\right| |(u, u_x)|\mathrm{d}x \\
&\le C\|\bar{\theta}^{1-\frac{\beta}{4}} - \theta^{1-\frac{\beta}{4}}\|_\infty \left(\int_0^1 (1 + \theta^{2-\frac{\beta}{2}})\mathrm{d}x\right)^{1/2} \left(\int_0^1 (u^2 + u_x^2)\mathrm{d}x\right)^{1/2} \\
&\le C\left(\int_0^1 \theta^{-\frac{\beta}{4}}|\theta_x|\mathrm{d}x\right)^2 + C\int_0^1 (1 + \theta^{2-\frac{\beta}{2}})\mathrm{d}x \int_0^1 (u^2 + u_x^2)\mathrm{d}x \\
&\le C\int_0^1 \theta^{1-\frac{\beta}{2}}\mathrm{d}x \int_0^1 \frac{\theta_x^2}{\theta}\mathrm{d}x + C\int_0^1 (1 + \theta^{2-\frac{\beta}{2}})\mathrm{d}x \int_0^1 (u^2 + u_x^2)\mathrm{d}x \\
&\le C\int_0^1 \frac{\theta_x^2}{\theta}\mathrm{d}x + C\int_0^1 (1 + \theta^{2-\frac{\beta}{2}})\mathrm{d}x \int_0^1 (u^2 + u_x^2)\mathrm{d}x,
\end{aligned} \tag{2.71}$$

and

$$
\begin{aligned}
&\int_0^1 \theta^{1-\frac{\beta}{2}} Q \mathrm{d}x \\
&\le C\left(\|\bar{\theta}^{1-\frac{\beta}{2}} - \theta^{1-\frac{\beta}{2}}\|_\infty + 1\right)\int_0^1 (u^2 + u_x^2 + v^2 + v_x^2 + w_x^2)\mathrm{d}x \\
&\le C\int_0^1 \theta^{-\frac{\beta}{2}}|\theta_x|\mathrm{d}x \int_0^1 (u_x^2 + v_x^2 + w_x^2)\mathrm{d}x + C\int_0^1 (u_x^2 + v_x^2 + w_x^2)\mathrm{d}x \\
&\le C\int_0^1 (\theta^{-\frac{1}{2}} + \theta^{\frac{\beta}{4}})|\theta_x|\mathrm{d}x \int_0^1 (u_x^2 + v_x^2 + w_x^2)\mathrm{d}x + C\int_0^1 (u_x^2 + v_x^2 + w_x^2)\mathrm{d}x \\
&\le \varepsilon\int_0^1 \theta^{\frac{\beta}{2}}\theta_x^2\mathrm{d}x + C(\varepsilon)\left(\int_0^1 (u_x^2 + v_x^2 + w_x^2)\mathrm{d}x\right)^2 \\
&\quad + C\int_0^1 \left(\frac{\theta_x^2}{\theta} + u_x^2 + v_x^2 + w_x^2\right)\mathrm{d}x.
\end{aligned}
\tag{2.72}
$$

We can conclude from (2.70)–(2.72) that

$$
\int_0^1 \theta^{2-\frac{\beta}{2}}\mathrm{d}x + \int_0^T\int_0^1 \theta^{\beta/2}\theta_x^2\mathrm{d}x\mathrm{d}t \le C + C\int_0^T \left(\int_0^1 (u_x^2 + v_x^2 + w_x^2)\mathrm{d}x\right)^2 \mathrm{d}s,
$$

which combined with Young's inequality and Corollary 2.1 yields

$$
\begin{aligned}
&\int_0^T\int_0^1 \theta_x^2\mathrm{d}x\mathrm{d}t \\
&\le C\int_0^T\int_0^1 \frac{\theta^\beta\theta_x^2}{\theta^2}\mathrm{d}x\mathrm{d}s + C\int_0^T\int_0^1 \theta^{\beta/2}\theta_x^2\mathrm{d}x\mathrm{d}s \\
&\le C + C\int_0^T \left(\int_0^1 (u_x^2 + v_x^2 + w_x^2)\mathrm{d}x\right)^2 \mathrm{d}t.
\end{aligned}
\tag{2.73}
$$

By means of Lemma 2.5, (2.67), and (2.73), we find for $0 < \beta \le 1$,

$$
\int_0^1 (u_x^2 + v_x^2)\mathrm{d}x + \int_0^T\int_0^1 (u_{xx}^2 + v_{xx}^2 + \theta_x^2)\mathrm{d}x\mathrm{d}t \le C. \tag{2.74}
$$

By virtue of (1.12)–(1.16), (2.1), Corollary 2.1, Lemma 2.4, (2.69), and (2.74), it follows that

$$
\int_0^1 \tau_t^2\mathrm{d}x + \int_0^T\int_0^1 (u_t^2 + v_t^2 + w_t^2 + \tau_{tx}^2)\mathrm{d}x\mathrm{d}s \le C. \tag{2.75}
$$

□

To obtain the first-order derivative estimate of the temperature, we need to first establish the uniform upper and lower bounds of θ.

Lemma 2.7. *Assume that the conditions listed in Lemma 2.1 hold. Then for $T \ge 0$,*

$$
C_1^{-1} \le \theta \le C_1.
$$

Proof. First of all, multiplying (1.16) by θ , and integrating over $[0,1]$ about x, yields

$$\begin{aligned}
&\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta^2 \mathrm{d}x + \int_0^1 \frac{\tilde{\kappa} r^2 \theta^\beta \theta_x^2}{\tau}\mathrm{d}x \\
&= \int_0^1 \theta Q \mathrm{d}x - R\int_0^1 \frac{\theta^2 (ru)_x}{\tau}\mathrm{d}x \\
&\le C\|(u,u_x,v,v_x,w_x)\|_\infty^2 \int_0^1 \theta \mathrm{d}x + \|u_x\|_\infty^2 \int_0^1 \theta^2 \mathrm{d}x.
\end{aligned} \tag{2.76}$$

Applying Gronwall's inequality to (2.76), we can obtain

$$\int_0^1 \theta^2 \mathrm{d}x + \int_0^T \int_0^1 \theta^\beta \theta_x^2 \mathrm{d}x\mathrm{d}t \le C. \tag{2.77}$$

Based on the estimate above, we can get the bound of $\int_0^1 \theta^\beta \theta_x^2 \mathrm{d}x$ which will be used to obtain the upper bound of θ. Multiplying (1.16) by $\theta^\beta \theta_t$ and integrating over $(0,1)$ about x, it follows that

$$c_v \int_0^1 \theta^\beta \theta_t^2 \mathrm{d}x + R\int_0^1 \frac{\theta^{\beta+1}\theta_t (ru)_x}{\tau}\mathrm{d}x - \int_0^1 \theta^\beta \theta_t Q \mathrm{d}x = \int_0^1 \left(\frac{\tilde{\kappa} r^2 \theta^\beta \theta_x}{\tau}\right)_x \theta^\beta \theta_t \mathrm{d}x. \tag{2.78}$$

By integration by parts, one has

$$\begin{aligned}
&\int_0^1 \left(\frac{\tilde{\kappa} r^2 \theta^\beta \theta_x}{\tau}\right)_x \theta^\beta \theta_t \mathrm{d}x \\
&= -\int_0^1 \frac{\tilde{\kappa} r^2 \theta^\beta \theta_x}{\tau}\left(\theta^\beta \theta_x\right)_t \mathrm{d}x \\
&= -\frac{\tilde{\kappa}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{r^2}{\tau}(\theta^\beta \theta_x)^2 \mathrm{d}x + \frac{\tilde{\kappa}}{2}\int_0^1 \left(\frac{2ru}{\tau} - \frac{ru}{\tau} - \frac{r^3 u_x}{\tau^2}\right)(\theta^\beta \theta_x)^2 \mathrm{d}x.
\end{aligned} \tag{2.79}$$

Inserting (2.79) into (2.78), we can deduce that

$$\begin{aligned}
&\frac{\tilde{\kappa}}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \frac{r^2}{\tau}(\theta^\beta \theta_x)^2 \mathrm{d}x + c_v \int_0^1 \theta^\beta \theta_t^2 \mathrm{d}x \\
&= -R\int_0^1 \frac{\theta^{\beta+1}\theta_t (ru)_x}{\tau}\mathrm{d}x + \int_0^1 \theta^\beta \theta_t Q \mathrm{d}x + \frac{\tilde{\kappa}}{2}\int_0^1 \left(\frac{ru}{\tau} - \frac{r^3 u_x}{\tau^2}\right)(\theta^\beta \theta_x)^2 \mathrm{d}x \\
&\le \frac{c_v}{2}\int_0^1 \theta^\beta \theta_t^2 \mathrm{d}x + C\int_0^1 \theta^{\beta+2}(u^2 + u_x^2)\mathrm{d}x \\
&\quad + C\int_0^1 \theta^\beta \left(u^4 + u_x^4 + v^4 + v_x^4 + w_x^4\right)\mathrm{d}x + C\|(u,u_x)\|_\infty \int_0^1 (\theta^\beta \theta_x)^2 \mathrm{d}x \\
&\le \frac{c_v}{2}\int_0^1 \theta^\beta \theta_t^2 \mathrm{d}x + C\|(u^2,u_x^2,u^4,u_x^4,v^4,v_x^4,w_x^4)\|_\infty + C\left(\int_0^1 \theta^\beta \theta_t^2 \mathrm{d}x\right)^2.
\end{aligned} \tag{2.80}$$

By Sobolev's inequality, Corollary 2.1, and Lemmas 2.5–2.6, one can find that

$$\int_0^T \|(u^2,u_x^2,u^4,u_x^4,v^4,v_x^4,w_x^4)\|_\infty \mathrm{d}s \le C. \tag{2.81}$$

By virtue of (2.80), Grönwall's inequality, and (2.81), we can obtain

$$\int_0^1 (\theta^\beta \theta_x)^2 \mathrm{d}x + \int_0^T \int_0^1 \theta^\beta \theta_t^2 \mathrm{d}x\mathrm{d}s \leq C. \tag{2.82}$$

Thanks to (2.82), it follows that

$$\|\theta^{\beta+1} - \bar{\theta}^{\beta+1}\|_\infty \leq C\left(\int_0^1 (\theta^\beta \theta_x)^2 \mathrm{d}x\right)^{\frac{1}{2}} \leq C. \tag{2.83}$$

That is, for $t \geq 0$,

$$\|\theta\|_\infty \leq C. \tag{2.84}$$

Thanks to (2.77) and (2.84), one has

$$\begin{aligned}
&\int_0^T \int_0^1 (\theta^{\beta+1} - \bar{\theta}^{\beta+1})^2 \mathrm{d}x\mathrm{d}t \\
&\quad \leq \int_0^T \int_0^1 \theta^{2\beta} \theta_x^2 \mathrm{d}x\mathrm{d}t \\
&\quad \leq C \sup_{0\leq t\leq T} \|\theta\|_\infty^\beta \int_0^T \int_0^1 \theta^\beta \theta_x^2 \mathrm{d}x\mathrm{d}t \\
&\quad \leq C.
\end{aligned} \tag{2.85}$$

Combining (2.83) and (2.84), one has

$$\begin{aligned}
&\int_0^T \left| \frac{\mathrm{d}}{\mathrm{d}t} \int_0^1 (\theta^{\beta+1} - \bar{\theta}^{\beta+1})^2 \mathrm{d}x \right| \mathrm{d}t \\
&\quad \leq C \int_0^T \int_0^1 (\theta^{\beta+1} - \bar{\theta}^{\beta+1})^2 \mathrm{d}x\mathrm{d}t + C \int_0^T \|\theta^\beta \theta_t\|^2 \mathrm{d}t \\
&\quad \leq C \sup_{0\leq t\leq T} \|\theta\|_\infty^\beta \int_0^T \int_0^1 \theta^\beta \theta_x^2 \mathrm{d}x\mathrm{d}t \\
&\quad \leq C.
\end{aligned} \tag{2.86}$$

So, from (2.83), (2.85), and (2.86), one has

$$\lim_{t\to+\infty} \int_0^1 (\theta^{\beta+1} - \bar{\theta}^{\beta+1})^2 \mathrm{d}x = 0.$$

From (2.83), when $t \to +\infty$,

$$\|(\theta^{\beta+1} - \bar{\theta}^{\beta+1})\|_\infty^2 \leq C\|(\theta^{\beta+1} - \bar{\theta}^{\beta+1})\| \|\theta^\beta \theta_x\| \to 0,$$

and we can obtain that there exists some time $T_0 \gg 1$ such that when $t > T_0$,

$$\theta(x,t) \geq \frac{\gamma_1}{2}. \tag{2.87}$$

Fixing T_0 in (2.87), multiplying (1.16) by θ^{-p}, $p > 2$, and integrating over $[0, 1]$ about x yield

$$\begin{aligned}
&\frac{c_v}{p-1}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta^{-1}\|_{p-1}^{p-1} + p\int_0^1 \frac{\tilde{\kappa} r^2\theta^\beta\theta_x^2}{\tau\theta^{p+1}}\mathrm{d}x + \int_0^1 \frac{Q}{\theta^p}\mathrm{d}x \\
&= R\int_0^1 \frac{\theta}{\tau\theta^p}(ru)_x\mathrm{d}x \\
&\le \frac{1}{2}\int_0^1 \frac{u^2+u_x^2}{\tau\theta^p}\mathrm{d}x + C\|\theta^{-1}\|_{L^{p-1}}^{p-2}.
\end{aligned}$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta^{-1}\|_{L^{p-1}} \le C,$$

where C is a generic positive constant independent of p. Thus, integrating the above inequality over $(0, t)$ and letting $p \to \infty$, we arrive that

$$\theta^{-1}(x,t) \le C(T_0+1) \leftrightarrow \theta(x,t) \ge [C(T_0+1)]^{-1}, \forall (x,t) \in [0,1]\times[T_0,+\infty).$$

The proof of Lemma 2.7 is complete. □

Lemma 2.8. *Assume that the conditions listed in Lemma 2.1 hold. Then for $T \ge 0$,*

$$\int_0^1 \theta_x^2\mathrm{d}x + \int_0^T\int_0^1 (\theta_{xx}^2 + \theta_t^2)\mathrm{d}x\mathrm{d}s \le C_5.$$

Proof. Multiplying (1.16) by θ_{xx}, integrating over $[0, 1]$ on x, and by Hölder's, Poincaré's, and Cauchy-Schwarz's inequalities, Corollary 2.1, Lemma 2.4, and Lemma 2.7, we have

$$\begin{aligned}
&\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta_x^2\mathrm{d}x + \int_0^1 \frac{\kappa r^2\theta_{xx}^2}{\tau}\mathrm{d}x \\
&= \int_0^1 \theta_{xx}\left[\frac{R\theta}{\tau}(ru)_x - \theta_x\left(\frac{\kappa r^2}{\tau}\right)_x - Q\right]\mathrm{d}x \\
&\le \varepsilon\int_0^1 \theta_{xx}^2\mathrm{d}x + C(\varepsilon)\int_0^1\left[\theta^2(ru)_x^2 - \theta_x^2\left(\frac{\kappa r^2}{\tau}\right)_x^2 - Q^2\right]\mathrm{d}x \\
&\le \varepsilon\int_0^1 \theta_{xx}^2\mathrm{d}x + C(\varepsilon)\|u_x\|^2\|\theta\|_\infty^2 + C(\varepsilon)\|\theta_x\|^2 + C(\varepsilon)\|\theta_x\|_\infty^2\|\tau_x\|^2 \\
&\quad + C(\varepsilon)\int_0^1 (u^4+v^4+u_x^4+v_x^4+w_x^4)\mathrm{d}x \\
&\le \varepsilon\|\theta_{xx}\|^2 + C(\varepsilon)\Big(\|u_x\|^2 + \|\theta_x\|^2 + \|\theta_x\|\|\theta_{xx}\| + \|u\|^2\|u\|^2 + \|v\|_\infty^2\|v\|^2\Big) \\
&\quad + C(\varepsilon)\Big(\|u_x\|_\infty^2\|u_x\|^2 + \|v_x\|_\infty^2\|v_x\|^2 + \|w_x\|_\infty^2\|w_x\|^2\Big) \\
&\le \varepsilon\|\theta_{xx}\|^2 + C(\varepsilon)\|(u_x,v_x,w_x,u_{xx},v_{xx},w_{xx})\|^2 + C(\varepsilon)\|\theta_x\|^2.
\end{aligned} \tag{2.88}$$

Choosing ε suitably small in (2.88) gives

$$\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta_x^2\mathrm{d}x + c\int_0^1 \theta_{xx}^2\mathrm{d}x \le C\|(u_x,v_x,w_x)\|_1^2 + C\|\theta_x\|^2. \tag{2.89}$$

Integrating (2.89) and using Lemmas 2.5–2.6, one has

$$\|\theta_x(t)\|^2 + \int_0^T \|\theta_{xx}\|^2 \mathrm{d}s \leq C. \tag{2.90}$$

Hence, similar to (2.75), by means of (1.16), Corollary 2.1, Lemmas 2.4–2.7, and (2.90), one can deduce that

$$\int_0^T \int_0^1 \theta_t^2 \mathrm{d}x\mathrm{d}t \leq C.$$

□

Next, we derive the second-order derivatives estimates of (τ, u, v, w, θ).

Lemma 2.9. *Assume that the conditions listed in Lemma 2.1 hold. Then for $T \geq 0$,*

$$\begin{aligned}&\int_0^1 (u_t^2 + v_t^2 + w_t^2 + \theta_t^2 + u_{xx}^2 + v_{xx}^2 + w_{xx}^2 + \theta_{xx}^2 + \tau_{xt}^2)\mathrm{d}x \\ &\quad + \int_0^T \int_0^1 (u_{xt}^2 + \tau_{tt}^2 + v_{xt}^2 + w_{xt}^2 + \theta_{xt}^2)\mathrm{d}x\mathrm{d}s \leq C_6.\end{aligned}$$

Proof. Applying ∂_t to (1.13) and multiplying by u_t in L^2, one has

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 u_t^2 \mathrm{d}x + \int_0^1 \frac{\tilde{\lambda} r^2 \theta^\alpha u_{xt}^2}{\tau}\mathrm{d}x \\ &\quad = \int_0^1 r u_{xt}\left[P_t - \left(\frac{\lambda}{\tau}\right)_t (ru)_x - \frac{\lambda}{\tau}\Big((ru)_{xt} - ru_{xt}\Big)\right]\mathrm{d}x - \int_0^1 r_x u_t \left[\frac{\lambda(ru)_x}{\tau}\right]_t \mathrm{d}x \\ &\quad\quad + \int_0^1 u_t\left[\left(\frac{v^2}{r}\right)_t - r_t P_x + r_x P_t + r_t\left(\frac{\lambda(ru)_x}{\tau}\right)_x - 2(u\mu_x)_t\right]\mathrm{d}x \\ &\quad := \sum_{i=1}^{3} III_i.\end{aligned} \tag{2.91}$$

Applying ∂_t to (1.14) and multiplying by v_t in L^2, one has

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 v_t^2 \mathrm{d}x + \int_0^1 \frac{\tilde{\mu} r^2 \theta^\alpha v_{xt}^2}{\tau}\mathrm{d}x \\ &\quad = \int_0^1 \left\{v_t\left[2\Big(\mu v_x\Big)_t - r_x\left(\frac{\mu r v_x}{\tau}\right)_t - \Big(\mu v\Big)_{xt}\right] - r v_{xt} v_x \left[\frac{\mu r}{\tau}\right]_t\right\}\mathrm{d}x \\ &\quad\quad + \int_0^1 v_t\left[r_t\left(\frac{\mu r v_x}{\tau}\right)_x - \left(\frac{uv}{r}\right)_t - \left(\frac{\mu \tau v}{r^2}\right)_t\right]\mathrm{d}x \\ &\quad := \sum_{i=4}^{5} III_i.\end{aligned} \tag{2.92}$$

Applying ∂_t to (1.15) and multiplying by w_t in L^2, one has

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 w_t^2\mathrm{d}x + \int_0^1 \frac{\tilde{\mu}r^2\theta^\alpha w_{xt}^2}{\tau}\mathrm{d}x\\
&= \int_0^1 w_t r_t\left(\frac{\mu r w_x}{\tau}\right)_x\mathrm{d}x - \int_0^1 \left\{ r w_x w_{xt}\left(\frac{\mu r}{\tau}\right)_t + w_t\left[r_x\left(\frac{\mu r w_x}{\tau}\right)_t - \left(\mu w_x\right)_t\right]\right\}\mathrm{d}x\\
&:= \sum_{i=6}^{7} III_i.
\end{aligned}
\tag{2.93}
$$

Adding (2.91)–(2.93) together, we get

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 (u_t^2+v_t^2+w_t^2)\mathrm{d}x + \int_0^1 \frac{\tilde{\lambda}r^2\theta^\alpha u_{xt}^2 + \tilde{\mu}r^2\theta^\alpha v_{xt}^2 + \tilde{\mu}r^2\theta^\alpha w_{xt}^2}{\tau}\mathrm{d}x = \sum_{i=1}^{7} III_i. \tag{2.94}
$$

Before the computations of III_1 to III_7, we need to keep in mind the following facts:

$$
\begin{aligned}
&\|(u,v,w)\|_\infty \le C,\\
&a\le r\le b,\quad r_x = r^{-1}\tau,\quad r_t = u,\quad r_{tx} = u_x,\\
&C^{-1}\le\tau\le C,\quad C^{-1}\le\theta\le C,\\
&|(ru)_x| \le C|(u,u_x)|,\quad |(ru)_{xt} - ru_{xt}| \le C|(u^2,u_t,uu_x)|,\\
&|(ru)_{xt}| \le C|(u^2,u_t,uu_x,u_{xt})|.
\end{aligned}
$$

Then, by Hölder's, Sobolev's, and Cauchy-Schwarz's inequalities, one has

$$
\begin{aligned}
III_1 &\le C\int_0^1 |u_{xt}||(\theta_t,\tau_t,\theta_t u_x,\tau_t u_x,u,u_t,u_x)|\mathrm{d}x\\
&\le \varepsilon\|u_{xt}\|^2 + C(\varepsilon)\|(\theta_t,\tau_t,u,u_t)\|^2 + C(\varepsilon)\|(\theta_t,\tau_t)\|_\infty^2\|u_x\|^2\\
&\le \varepsilon\|u_{xt}\|^2 + \delta\|\theta_{xt}\|^2 + C(\varepsilon,\delta)\|(\theta_t,\tau_t,u,u_t,\tau_{xt})\|^2,
\end{aligned}
\tag{2.95}
$$

and

$$
\begin{aligned}
III_2 &\le C\int_0^1 |u_t||(\theta_t,\theta_t u_x,u^2,u_t,u_x,u_{xt},\tau_t,\tau_t^2)|\mathrm{d}x\\
&\le \varepsilon\|u_{xt}\|^2 + C(\varepsilon)\|(u_t,\theta_t,u,u_x,\tau_t)\|^2 + \varepsilon\|\theta_t\|_\infty^2\|u_x\|^2 + C(\varepsilon)\|\tau_t\|_\infty^2\|\tau_t\|^2\\
&\le \varepsilon\|u_{xt}\|^2 + \delta\|\theta_{xt}\|^2 + C(\varepsilon,\delta)\|(u_t,\theta_t,u,u_x,\tau_t,\tau_{xt})\|^2.
\end{aligned}
\tag{2.96}
$$

By virtue of (1.13), one has

$$
\left|\left(\frac{\lambda(ru)_x}{\tau}\right)_x\right| \le C\left|\left(u_t,v^2,\theta_x,\tau_x\right)\right|.
$$

Thus, it follows from Hölder's, Sobolev's, and Cauchy-Schwarz's inequalities that

$$
\begin{aligned}
III_3 &\le C\int_0^1 |u_t||(v_t,v^2,\theta_x,\tau_x,\tau_t,\theta_t,u_t,u_t\theta_x,\theta_x\theta_t,\theta_{xt})|\mathrm{d}x\\
&\le \varepsilon\|\theta_{xt}\|^2 + C(\varepsilon)\|(v_t,u_t,v,\theta_x,\theta_t,\theta_t,\tau_x,\tau_t)\|^2 + \varepsilon\|(u_t,\theta_t)\|_\infty^2\|\theta_x\|^2\\
&\le \varepsilon\|(u_{xt},\theta_{xt})\|^2 + C(\varepsilon)\|(v_t,u_t,\theta_t,\tau_t,v,\theta_x,\tau_x)\|^2,
\end{aligned}
\tag{2.97}
$$

and

$$\begin{aligned} III_4 &\le C\int_0^1 |v_t||(\theta_t v_x, v_x, v_{xt}, v_x\tau_t, \theta_x\theta_t, \theta_{xt}, \theta_x v_t)|\mathrm{d}x + C\int_0^1 |v_{xt}v_x||(\theta_t, v, \tau_t)|\mathrm{d}x \\ &\le \frac{\varepsilon}{2}\|(v_{xt}, \theta_{xt})\|^2 + C(\varepsilon)\|(v_t, v_x)\|^2 + C(\varepsilon)\|(\theta_t, \tau_t, v_t)\|_\infty^2\|(v_x, \theta_x)\|^2 \\ &\le \varepsilon\|(v_{xt}, \theta_{xt})\|^2 + C(\varepsilon)\|(v_t, v_x, \theta_t, \tau_t, \tau_{tx})\|^2. \end{aligned} \tag{2.98}$$

It follows from (1.14) that

$$\left|\left(\frac{\mu r v_x}{\tau}\right)_x\right| \le C|(v_t, v, v_x, \theta_x)|.$$

Then

$$III_5 \le C\int_0^1 |v_t||(v_t, v, \theta_x, v_x, u_t, \theta_t, \tau_t)|\mathrm{d}x \le C\|(v_t, v, \theta_x, v_x, u_t, \theta_t, \tau_t)\|^2. \tag{2.99}$$

By virtue of (1.15), we can obtain

$$III_6 \le C\int_0^1 |w_t||u|\left|\left(\frac{\mu r w_x}{\tau}\right)_x\right|\mathrm{d}x \le C\int_0^1 |w_t|\,|(w_t, w_x)|\,\mathrm{d}x \le C\|(w_t, w_x)\|^2, \tag{2.100}$$

and

$$\begin{aligned} III_7 &\le C\int_0^1 |w_{xt}|\Big|(w_x\theta_t, w_x\tau_t, w_x)\Big| + |w_t|\Big|(\theta_t w_x, \tau_t w_x, w_x, w_{xt})\Big|\mathrm{d}x \\ &\le \varepsilon\|w_{xt}\|^2 + C(\varepsilon)\|(w_x, w_t)\|^2 + C(\varepsilon)\|(\theta_t, \tau_t)\|_\infty^2\|w_x\|^2 \\ &\le \varepsilon\|(w_{xt}, \theta_{xt})\|^2 + C(\varepsilon)\|(w_x, w_t, \theta_t, \tau_t, \tau_{xt})\|^2. \end{aligned} \tag{2.101}$$

Putting (2.95)–(2.101) into (2.94) gives

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(u_t, v_t, w_t)\|^2 + c\|(u_{xt}, v_{xt}, w_{xt})\|^2 \\ &\quad\le \varepsilon\|(u_{xt}, v_{xt}, w_{xt}, \theta_{xt})\|^2 + C(\varepsilon)\|(u_t, v_t, w_t, \theta_t, w_x, \tau_x, \theta_x)\|^2 + C(\varepsilon)\|(\tau_t, u, v)\|_1^2. \end{aligned} \tag{2.102}$$

Applying ∂_t to (1.16) and multiplying by θ_t in L^2, it follows that

$$\begin{aligned} &\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta_t^2\mathrm{d}x + \int_0^1 \frac{\tilde{\kappa}\theta^\beta r^2\theta_{xt}^2}{\tau}\mathrm{d}x \\ &\quad= \int_0^1 \theta_t\left[Q_t - \left(P(ru)_x\right)_t\right] - \theta_x\theta_{xt}\left(\frac{\kappa r^2}{\tau}\right)_t\mathrm{d}x. \end{aligned} \tag{2.103}$$

First of all, by means of the definition of Q, one has

$$\begin{aligned} |\theta_t Q_t| \le& C|\theta_t||(u, \tau_t, u_t, u_x, u_{xt}, u_x\tau_t, u_x u_t, u_x^2, u_x u_{xt})| \\ &+ C|\theta_t||(\theta_t, \theta_t u_x^2, \tau_t u_x^2, \theta_t w_x^2, w_x^2, \tau_t w_x^2, w_x w_{xt}, \theta_t v_x^2, \tau_t v_x^2)| \\ &+ C|\theta_t||(v_x^2, v_{xt}, \tau_t v_x, v_t, v, v_x, v_x v_{xt}, v_x v_t)| \\ \le& C(\varepsilon)|(\theta_t, u, \tau_t, u_t, u_x, w_x, v_x, v_t, v, v_x)|^2 + \varepsilon|(u_{xt}, w_{xt}, v_{xt})|^2 \\ &+ C(\varepsilon)|(\tau_t, u_t, u_x)|^2|(u_x, w_x, v_x)|^2 + C(\varepsilon)|\theta_t|^2|(u_x, v_x, w_x)|^2. \end{aligned} \tag{2.104}$$

Using (2.104) and Sobolev's inequality, we can derive from (2.103) that

$$\begin{aligned}
&\frac{c_v}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \theta_t^2\mathrm{d}x + \int_0^1 \frac{\tilde{\kappa}\theta^\beta r^2\theta_{xt}^2}{\tau}\mathrm{d}x \\
&\le C\int_0^1 (|\theta_t||(Q_t,\theta_t,\tau_t,\theta_t u_x,\tau_t u_x,u,u_t,u_x,u_{xt})| + |\theta_x||(\theta_{xt}\theta_t,\theta_{xt},\theta_{xt}\tau_t)|)\,\mathrm{d}x \\
&\le C(\varepsilon)\|(\theta_t,u,\tau_t,u_t,u_x,\theta_x,w_x,v_x,v_t,v,v_x)\|^2 + \varepsilon\|(u_{xt},w_{xt},v_{xt},\theta_{xt})\|^2 \\
&\quad + C(\varepsilon)\|(\tau_t,\theta_t,u_t,u_x)\|_\infty^2\|(u_x,w_x,v_x,\theta_x)\|^2 + C(\varepsilon)\|\tau_t\|_\infty^2\|\theta_t\|^2 \\
&\le C(\varepsilon)\|(\theta_t,u,\tau_t,u_t,u_x,w_x,v_x,v_t,v,\tau_{tx},u_{xx})\|^2 + \varepsilon\|(u_{xt},w_{xt},v_{xt},\theta_{xt})\|^2 \\
&\quad + C(\varepsilon)\|(u_x,v_x,w_x)\|_1^2\|\theta_t\|^2 + C(\varepsilon)\|\tau_t\|_1^2\|\theta_t\|^2.
\end{aligned} \tag{2.105}$$

Adding (2.102) to (2.105) and choosing $\varepsilon > 0$ suitably small, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|(\sqrt{c_v}\theta_t,u_t,v_t,w_t)\|^2 + c\|(u_{xt},v_{xt},w_{xt},\theta_{xt})\|^2 \\
&\quad \le C\|(u_t,v_t,w_t,\theta_t,w_x,\tau_x,\theta_x)\|^2 + C\|(u_x,\tau_t,v)\|_1^2 \\
&\qquad + C\|(u_x,v_x,w_x)\|_1^2\|\theta_t\|^2 + C\|\tau_t\|_1^2\|\theta_t\|^2.
\end{aligned} \tag{2.106}$$

By means of (2.106) and Grönwall's inequality, we deduce

$$\|(u_t,v_t,w_t,\theta_t)\|^2 + \int_0^T \|(u_{xt},v_{xt},w_{xt},\theta_{xt})\|^2\mathrm{d}s \le C. \tag{2.107}$$

According to (1.13), one has

$$\frac{\lambda r^2 u_{xx}}{\tau} = u_t - \frac{v^2}{r} + rP_x + 2u\mu_x - r\left[\left(\frac{\lambda(ru)_x}{\tau}\right)_x - \frac{\lambda r u_{xx}}{\tau}\right],$$

which means that

$$|u_{xx}| \le C|(u_t,v,\theta_x,\tau_x,\theta_x u_x,\tau_x u_x,u,u_x)|.$$

Hence, by means of (2.107), we obtain

$$\|u_{xx}\|^2 \le C.$$

Similarly, use the equations (1.12)–(1.16), we also can derive

$$\|(v_{xx},w_{xx},\theta_{xx},\tau_{tx})\|^2 + \int_0^T \|\tau_{tt}\|^2\mathrm{d}s \le C_7. \tag{2.108}$$

Here, we omit the details of (2.108). The proof of Lemma 2.9 is complete. □

Lemma 2.10. *Assume that the conditions listed in Lemma 2.1 hold. Then for $T \ge 0$,*

$$\int_0^1 \tau_{xx}^2\mathrm{d}x + \int_0^T\int_0^1 (\tau_{xx}^2 + \tau_{xxt}^2 + u_{xxx}^2 + v_{xxx}^2 + w_{xxx}^2 + \theta_{xxx}^2)\mathrm{d}x\mathrm{d}s \le C_7.$$

Proof. Apply ∂_x to (2.45) and multiply by $(\lambda\tau_x/\tau)_x$ in L^2 to get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\frac{\lambda\tau_x}{\tau}\right)_x^2\mathrm{d}x+\int_0^1\frac{R\theta}{\lambda\tau}\left(\frac{\lambda\tau_x}{\tau}\right)_x^2\mathrm{d}x\\
&=\int_0^1\left(\frac{\lambda\tau_x}{\tau}\right)_x\left[\frac{\lambda_\theta}{\tau}(\tau_x\theta_t-\tau_t\theta_x)\right]_x\mathrm{d}x+\int_0^1\left(\frac{\lambda\tau_x}{\tau}\right)_x\left(\frac{u_t}{r}-\frac{v^2}{r^2}+\frac{2u\mu_x}{r}\right)_x\mathrm{d}x\\
&\quad-R\int_0^1\left(\frac{\lambda\tau_x}{\tau}\right)_x\left[\frac{2\theta_x\tau_x}{\tau^2}-\frac{\theta_{xx}}{\tau}-\frac{2\theta\tau_x^2}{\tau^3}-\frac{\theta\tau_x}{\lambda\tau}\left(\frac{\lambda}{\tau}\right)_x\right]\mathrm{d}x\\
&\le C(\varepsilon)\int_0^1\Big[|(\tau_x,\theta_x)|^2|(\tau_x\theta_t,\tau_t\theta_x)|^2+|(\tau_{xx}\theta_t,\tau_x\theta_{xt},\tau_{tx}\theta_x,\tau_t\theta_{xx})|^2\Big]\mathrm{d}x \qquad (2.109)\\
&\quad+C(\varepsilon)\int_0^1\Big[|(u_{xt},u_t,v_x,v,u_x\theta_x,\theta_{xx},\theta_x^2,\theta_x)|^2+|(\theta_{xx},\tau_x^2,\theta_x\tau_x)|^2\Big]\mathrm{d}x\\
&\quad+\varepsilon\int_0^1\left(\frac{\lambda\tau_x}{\tau}\right)_x^2\mathrm{d}x\\
&:=\sum_{i=8}^{9}III_i+\varepsilon\int_0^1\left(\frac{\lambda\tau_x}{\tau}\right)_x^2\mathrm{d}x,
\end{aligned}
$$

where the following fact has been used:

$$
\begin{aligned}
\left(\frac{\theta}{\tau}\right)_{xx}&=\frac{\theta_{xx}}{\tau}-2\frac{\theta_x\tau_x}{\tau^2}+2\frac{\theta\tau_x^2}{\tau^3}-\frac{\theta\tau_{xx}}{\tau^2}\\
&=\frac{\theta_{xx}}{\tau}-2\frac{\theta_x\tau_x}{\tau^2}+2\frac{\theta\tau_x^2}{\tau^3}-\frac{\theta}{\lambda\tau}\left[\left(\frac{\lambda\tau_x}{\tau}\right)_x-\frac{\lambda_x\tau_x}{\tau}+\frac{\lambda\tau_x^2}{\tau^2}\right].
\end{aligned}
$$

By Sobolev's inequality and Lemmas 2.6–2.9, we have

$$
\begin{aligned}
III_8\le&C(\varepsilon)\Big(\|(\tau_x,\theta_x)\|_\infty^4\|(\tau_t,\theta_t)\|^2+\|\theta_t\|_\infty^2\|\tau_{xx}\|^2\\
&+\|\tau_x\|_\infty^2\|\theta_{xt}\|^2+\|\tau_t\|_\infty^2\|\theta_{xx}\|^2+\|\theta_x\|_\infty^2\|\tau_{xt}\|^2\Big)\\
\le&C(\varepsilon)\Big(\|(\tau_x,\theta_x)\|^4+\|(\tau_x,\theta_x)\|^2\|(\tau_{xx},\theta_{xx})\|^2 \qquad (2.110)\\
&+\|\theta_t\|_1^2\|\tau_{xx}\|^2+\|\theta_{xt}\|^2\|\tau_x\|^2+\|\tau_t\|_1^2+\|\theta_x\|_1^2\Big)\\
\le&C(\varepsilon)\Big(\|(\tau_x,\theta_x,\tau_t)\|^2+\|\theta_t\|^2\|\tau_{xx}\|^2\Big),
\end{aligned}
$$

and

$$
\begin{aligned}
III_9&\le C(\varepsilon)\|(u_{xt},u_t,v_x,v,\theta_{xx},\theta_x)\|^2+C(\varepsilon)\|(u_x,\theta_x,\tau_x)\|_\infty^2\|(\theta_x,\tau_x)\|^2\\
&\le\varepsilon\|\tau_{xx}\|^2+C(\varepsilon)\|(u_{xt},u_t,v_x,v,\theta_{xx},\theta_x,\tau_x)\|^2.
\end{aligned}
\qquad (2.111)
$$

Noting that

$$
|\tau_{xx}|\le C\left|\left(\frac{\lambda\tau_x}{\tau}\right)_x\right|+C\left|\left(\theta_x\tau_x,\tau_x^2\right)\right|,
$$

we can derive from Sobolev's inequality and Lemma 2.4 that

$$
\begin{aligned}
\|\tau_{xx}\|^2&\le C\left\|\left(\frac{\lambda\tau_x}{\tau}\right)_x\right\|^2+C\|\theta_x\|_\infty^2\|\tau_x\|^2+C\|\tau_x\|_4^4\\
&\le C\left\|\left(\frac{\lambda\tau_x}{\tau}\right)_x\right\|^2+C\|\theta_x\|_1^2+C\|\tau_x\|^4+C\|\tau_x\|^3\|\tau_{xx}\|.
\end{aligned}
$$

So, it follows from Cauchy-Schwarz's inequality and Lemma 2.4 that

$$\|\tau_{xx}\|^2 \le C\|\theta_x\|_1^2 + C\|\tau_x\|^2 + C\left\|\left(\frac{\lambda\tau_x}{\tau}\right)_x\right\|^2. \tag{2.112}$$

Taking ε suitably small, putting (2.110)–(2.111) into (2.109), and using Lemmas 2.4, 2.8, and 2.9, we find

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\left(\frac{\lambda\tau_x}{\tau}\right)_x^2\mathrm{d}x + c\int_0^1\left(\frac{\lambda\tau_x}{\tau}\right)_x^2\mathrm{d}x\\ &\quad\le C\|(\tau_t,\theta_t,\theta_x,u_t,v)\|_1^2 + C\|\tau_x\|^2 + C\|\theta_t\|_1^2\left\|\left(\frac{\lambda\tau_x}{\tau}\right)_x\right\|^2.\end{aligned} \tag{2.113}$$

By (2.113), Grönwall's inequality, Corollary 2.1, and Lemmas 2.6 and 2.8–2.9, one obtains

$$\int_0^1\left(\frac{\lambda\tau_x}{\tau}\right)_x^2\mathrm{d}x + \int_0^T\int_0^1\left(\frac{\lambda\tau_x}{\tau}\right)_x^2\mathrm{d}x\mathrm{d}s \le C. \tag{2.114}$$

It follows from (2.112) and (2.114) that

$$\|\tau_{xx}\|^2 + \int_0^T\|\tau_{xx}\|^2\mathrm{d}s \le C. \tag{2.115}$$

Letting ∂_x act on (1.13) gives

$$\begin{aligned}&\frac{\tilde\lambda\theta^2r^2u_{xxx}}{\tau} + r_x\left(\frac{\lambda\tau_t}{\tau}\right)_x + r\left[\left(\frac{\lambda}{\tau}\right)_x\tau_t\right]_x + r\left(\frac{\lambda}{\tau}\right)_x(ru)_{xx}\\ &\quad= u_{xt} - \left(\frac{v^2}{r}\right)_x + (rP_x)_x + 2\Big(u\mu_x\Big)_x - \frac{r\lambda}{\tau}[(ru)_{xxx} - ru_{xxx}].\end{aligned} \tag{2.116}$$

It follows from (2.115) and (2.116) that

$$\begin{aligned}&\int_0^T\|u_{xxx}\|^2\mathrm{d}s\\ &\quad\le C\int_0^T\|(\theta_x\tau_t,\tau_{xt},\tau_t\tau_x)\|^2\mathrm{d}s + C\int_0^T\|(\theta_x^2\tau_t,\theta_{xx}\tau_t,\theta_x\tau_{tx},\theta_x\tau_t\tau_x)\|^2\mathrm{d}s\\ &\qquad+ C\int_0^T\|(\tau_{xx}\tau_t,\tau_x\tau_{tx},\tau_x^2\tau_t)\|^2\mathrm{d}s + C\int_0^T\|(\theta_x,\theta_x\tau_x,\theta_xu_x,\theta_xu_{xx})\|^2\mathrm{d}s\\ &\qquad+ C\int_0^T\|(u_{xt},v_x,v,\theta_x,\tau_x,\theta_{xx},\theta_x\tau_x,\tau_{xx})\|^2\mathrm{d}s + C\int_0^T\|(u_x\theta_x,\theta_x^2,\theta_{xx})\|^2\mathrm{d}s\\ &\qquad+ C\int_0^T\|(u,\tau_x,\tau_{xx},u_x,\tau_xu_x,u_{xx})\|^2\mathrm{d}s\\ &\quad\le C\int_0^T\|(\tau_t,\tau_{xx},\tau_{tx},\theta_x,u_x,u_{xt},v_x,v,\theta_{xx},u,\tau_x,u_{xx})\|^2\mathrm{d}s\\ &\quad\le C,\end{aligned}$$

where the following fact has been used:

$$\|(\theta_x, \tau_x, \theta_x^2, \tau_t, \theta_x\tau_t, \tau_x^2)\|_\infty \le C + C\|(\theta_x, \tau_t, \tau_x)\|_1^2 \le C.$$

Similarly, using (1.14)–(1.15), we also have

$$\int_0^T \|(v_{xxx}, w_{xxx})\|^2 \mathrm{d}s \le C.$$

Letting ∂_x act on (1.16) gives

$$\frac{\tilde{\kappa}\theta^\beta r^2 \theta_{xxx}}{\tau} = c_v\theta_{xt} + (P\tau_t)_x - \left(\frac{\kappa r^2}{\tau}\right)_{xx}\theta_x - 2\left(\frac{\kappa r^2}{\tau}\right)_x \theta_{xx} - Q_x. \tag{2.117}$$

It follows from (2.114) and (2.117) that

$$\begin{aligned}
&\int_0^T \|\theta_{xxx}\|^2 \mathrm{d}s \\
&\le C\int_0^T \|(\theta_{xt}, \theta_x\tau_t, \tau_x\tau_t, \tau_{tx})\|^2 \mathrm{d}s \\
&\quad + C\int_0^T \|(\theta_x^3, \theta_{xx}\theta_x, \theta_x^2, \theta_x^2\tau_x, \theta_x\tau_x, \theta_x\tau_x^2, \theta_x\tau_{xx})\|^2 \mathrm{d}s \\
&\quad + C\int_0^T \|(\theta_{xx}, \tau_x\theta_{xx})\|^2 + \|Q_x\|^2 \mathrm{d}s.
\end{aligned} \tag{2.118}$$

By the definition of Q, one has

$$\begin{aligned}
&\int_0^T \|Q_x\|^2 \mathrm{d}s \\
&\le C\int_0^T \|(\theta_x, \theta_x u_x, u, \tau_x, u_x, u_{xx}, u_x\tau_x, u_x^2, u_x u_{xx})\|^2 \mathrm{d}s \\
&\quad + C\int_0^T \|(\theta_x w_x^2, w_x^2, w_x w_{xx}, w_x^2\tau_x)\|^2 \mathrm{d}s \\
&\quad + C\int_0^T \|(\theta_x v_x^2, \tau_x v_x^2, v_x^2, v_x v_{xx}, v_x^2\tau_x, v_x, v_{xx}, v_x\tau_x, v)\|^2 \mathrm{d}s.
\end{aligned} \tag{2.119}$$

Since the following estimates have been obtained:

$$\|(\theta_x, \tau_x, u_x, w_x, v_x)\|_\infty \le C\|(\theta_x, \tau_x, u_x, w_x, v_x)\|_1 \le C,$$

putting (2.119) into (2.118) yields

$$\begin{aligned}
&\int_0^T \|\theta_{xxx}\|^2 \mathrm{d}s \\
&\le C\int_0^T \|(\theta_{xt}, \tau_t, \tau_{xt}, \theta_x, \theta_{xx}, \tau_x, \tau_{xx}, u_x, u, u_{xx}, w_x, w_{xx}, v_x, v_{xx}, v)\|^2 \mathrm{d}s \\
&\le C.
\end{aligned}$$

The proof of Lemma 2.10 is complete. □

3. The proof of Theorem 1.1

With all *a priori* estimates from Section 2 at hand, we can complete the proof of Theorem 1.1. For this purpose, it will be shown that the existence and uniqueness of local solutions to the initial-boundary value problem (1.12)–(1.19) can be obtained by using the Banach theorem and the contractivity of the operator defined by the linearization of the problem on a small time interval.

Lemma 3.1. *Letting* (1.20) *hold, then there exists* $T_0 = T_0(V_0, V_0, M_0) > 0$*, depending only on* β *,* V_0*, and* M_0*, such that the initial boundary value problem* (1.12)–(1.19) *has a unique solution* $(\tau, u, v, w, \theta) \in X(0, T_0; \frac{1}{2}V_0, \frac{1}{2}V_0, 2M_0)$.

Proof of Theorem 1.1: First, to prove Theorem 1.1, according to (1.20), one has

$$\begin{aligned} &\tau_0 \geq V_0, \theta_0 \geq V_0, \qquad \forall x \in \Omega,\\ &\|(\tau_0, u_0, v_0, w_0, \theta_0)\|_{H^2} \leq M_0. \end{aligned}$$

Combined with Lemma 3.1, there exists $t_1 = T_0(V_0, V_0, M_0)$ such that $(\tau, u, v, w, \theta) \in X(0, t_1; \frac{1}{2}V_0, \frac{1}{2}V_0, 2M_0)$.

We find the positive constant $|\alpha| \leq \alpha_1$, where α_1 satisfies

$$\left(\frac{1}{2}V_0\right)^{-|\alpha_1|} \leq 2, \qquad (2M_0)^{|\alpha_1|} \leq 2, \qquad |\alpha_1| H(\frac{1}{2}V_0, \frac{1}{2}V_0, 2M_0) \leq \epsilon_1, \tag{3.1}$$

where ϵ_1 is chosen in Lemma 2.1. That means that one can choose

$$|\alpha_1| := \min\left\{\frac{\ln 2}{|\ln 2 - \ln V_0|}, \frac{\ln 2}{|\ln 2 + \ln M_0|}, \epsilon_1 H^{-1}\left(\frac{1}{2}V_0, \frac{1}{2}V_0, 2M_0\right)\right\}. \tag{3.2}$$

One deduces from Lemmas 2.1–2.10 with $T = t_1$ that for each $t \in [0, t_1]$, the local solution (τ, u, v, w, θ) satisfies

$$C_0^{-1} \leq v(x,t) \leq C_0, \qquad C_1^{-1} \leq \theta(x,t) \leq C_1, \qquad x \in (0,1), \tag{3.3}$$

and

$$\sup_{0 \leq t \leq t_1} \|(\tau, u, v, w, \theta)\|_2^2 + \int_0^{t_1} \|\theta_t\|^2 \mathrm{d}t \leq C_8^2, \tag{3.4}$$

where $C_i (i = 2, \cdots, 7)$ is chosen in Section 2 and $C_8^2 := \sum_{i=2}^{7} C_i$. It follows from Lemma 2.9 and Lemma 2.10 that $(\tau, u, v, w, \theta) \in C([0, T); H^2)$. If one takes $(\tau, u, v, w, \theta)(\cdot, t_1)$ as the initial data and applies Lemma 3.1 again, the local solution (τ, u, v, w, θ) can be extended to the time interval $[t_1, t_1 + t_2]$ with $t_2(C_0, C_1, C_8)$ such that $(\tau, u, v, w, \theta) \in X(t_1, t_1 + t_2; \frac{1}{2}C_0, \frac{1}{2}C_1, \frac{1}{2}C_8)$. Moreover, for all $(x, t) \in [0, 1] \times [0, t_1 + t_2]$, one gets

$$\frac{1}{2}C_0 \leq v(x,t), \qquad \frac{1}{2}C_1 \leq \theta(x,t),$$

and

$$\sup_{t_1 \leq t \leq t_1 + t_2} \|(\tau, u, v, w, \theta)\|_2^2 + \int_{t_1}^{t_1 + t_2} \|\theta_t\|^2 \mathrm{d}t \leq 4C_8^2,$$

which combined with (3.3) and (3.4) implies that for all $t \in [0, t_1 + t_2]$,

$$\frac{1}{2}C_0 \leq v(x,t), \qquad \frac{1}{2}C_1 \leq \theta(x,t),$$

$$\sup_{0\leq t\leq t_1+t_2} \|(\tau,u,v,w,\theta)\|_2^2 + \int_0^{t_1+t_2} \|\theta_t\|^2 \mathrm{d}t \leq 5C_8^2.$$

Take $\alpha \leq \min\{\alpha_1, \alpha_2\}$, where $\alpha_i (i = 1, 2)$ are positive constants satisfying (3.1) and

$$\left(\frac{1}{2}C_0\right)^{-\alpha_2} \leq 2, \qquad (\sqrt{5}C_8)^{\alpha_2} \leq 2, \qquad \alpha_2 H(\frac{1}{2}C_0, \frac{1}{2}C_1, \sqrt{5}C_8) \leq \epsilon_1,$$

where the value of ϵ_1 is chosen in Lemma 2.1. That means that we can choose

$$|\alpha_2| := \min\left\{\frac{\ln 2}{|\ln 2 - \ln C_0|}, \frac{\ln 2}{|\ln \sqrt{5} + \ln C_8|}, \epsilon_1 H^{-1}\left(\frac{1}{2}C_0, \frac{1}{2}C_1, \sqrt{5}C_8\right)\right\}. \tag{3.5}$$

Then one can employ Lemmas 2.1–2.10 with $T = t_1 + t_2$ to infer the local solution (τ, u, v, w, θ) satisfying (3.3) and (3.4).

Choosing

$$\epsilon_0 = \min\{\alpha_1, \alpha_2\}, \tag{3.6}$$

and repeating the above procedure, one can extend the solution (τ, u, v, w, θ) step-by-step to a global one provided that $|\alpha| \leq \epsilon_0$. Furthermore,

$$\|(\tau,u,v,w,\theta)\|_{H^2}^2 + \int_0^{+\infty} \left[\|(u_x,v_x,w_x,\theta_x)\|^2 + \|\tau\|^2\right] \mathrm{d}t \leq C_9^2,$$

from which we derive that the solution $(\tau, u, v, w, \theta) \in X(0, +\infty; C_0, C_1, C_9)$. □

The large-time behavior (1.21) follows from Lemmas 2.4–2.10 by using a standard argument [21].

First, thanks to (1.15), (2.1), (2.43), (2.55), (2.62), (2.73), Corollary 2.1, and Lemmas 2.4–2.10, taking $\hat{\theta} = E_0$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \eta_{E_0}(\tau,u,v,w,\theta)\mathrm{d}x + c_1\|(u,v)\|_1^2 + c_1\|(w_x,\theta_x)\|^2 \leq 0, \tag{3.7}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left[\frac{1}{2}\left(\frac{\lambda\tau_x}{\tau}\right)^2 - \frac{\lambda u\tau_x}{r\tau}\right]\mathrm{d}x + c_2\|\tau_x\|^2 \leq C_{10}\|(u,u_x,\theta_x,v,v_x)\|^2, \tag{3.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|(u_x,v_x,w_x)\|^2 + c_3\|(u_{xx},v_{xx},w_{xx})\|^2 \leq C_{11}\|(\theta_x,\tau_x,v_x,u_x,u,v,w_x)\|^2, \tag{3.9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_x\|^2 + c_4\|\theta_{xx}\|^2 \leq C_{12}\|(u_x,v_x,w_x)\|_1^2 + C_{12}\|\theta_x\|^2. \tag{3.10}$$

By Cauchy-Schwarz's inequality, one has

$$\left|\frac{\lambda u\tau_x}{r\tau}\right| \leq \frac{1}{4}\left(\frac{\lambda\tau_x}{\tau}\right)^2 + C\|u\|^2. \tag{3.11}$$

Hence, by means of (3.11), Poincaré's inequalities, Corollary 2.1, and Lemma 2.7, one can deduce

$$c\|\tau_x\|^2 - C_{13}\|u\|^2 \leq \int_0^1 \left[\frac{1}{2}\left(\frac{\lambda\tau_x}{\tau}\right)^2 - \frac{\lambda u\tau_x}{r\tau}\right]\mathrm{d}x \leq C\|(\tau_x, u_x)\|^2.$$

Multiplying (3.7)–(3.10) by C_{14}, C_{15}, and C_{16}, respectively, and adding them together with (3.10), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{A} + c\|(u_x, v_x, w_x, \theta_x)\|_{H^1}^2 + c\|\tau_x\|^2 \leq 0, \tag{3.12}$$

where we have defined

$$\mathcal{A} := \int_0^1 C_{14}\eta_{E_0}(\tau, u, v, w, \theta) + C_{15}\left[\frac{1}{2}\left(\frac{\lambda\tau_x}{\tau}\right)^2 - \frac{\lambda u\tau_x}{r\tau}\right]\mathrm{d}x + C_{16}\|(u_x, v_x, w_x)\|^2 + \|\theta_x\|^2,$$

and chosen constants $C_{14} > C_{15} > C_{16} > 0$ suitably large such that

$$c_1C_{14} - C_{10}C_{15} - C_{11}C_{16} - C_{12} > 0,$$

$$c_2C_{15} - C_{11}C_{16} - C_{12} > 0,$$

$$c_3C_{16} - C_{12} > 0.$$

Taking $\frac{C_{14}}{2} > C_{13}$ and using Poincaré's inequality gives

$$c\|(\tau - \bar{\tau}, u, v, w, \theta - E_0)\|^2 \leq \mathcal{A} \leq C\|(u_x, v_x, w_x, \theta_x)\|_1^2 + C\|\tau_x\|^2, \tag{3.13}$$

where we have used the facts

$$\|\theta - E_0\|^2 \leq C\int_0^1 |\theta - \bar{\theta}|^2\mathrm{d}x + C\|(u, v, w)\|^2 \leq C\|(\theta_x, u_x, v_x, w_x)\|^2.$$

By means of (3.12) and (3.13), we can derive that

$$\|(\tau - \bar{\tau}, u, v, w, \theta - E_0)(t)\|_{H^1(\Omega)}^2 \leq C\mathrm{e}^{-ct}. \tag{3.14}$$

By means of $\bar{r}$, one has

$$r^2 - \bar{r}^2 = 2\int_0^x \tau - \bar{\tau}\mathrm{d}\xi. \tag{3.15}$$

By means of (3.14) and (3.15), we have

$$\|r - \bar{r}\|_2^2 \leq C\mathrm{e}^{-ct}.$$

The proof is thus complete. □

Author contributions

Dandan Song: Writing-original draft, Writing-review & editing, Supervision, Formal Analysis; Xiaokui Zhao: Writing-review & editing, Methodology, Supervision.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. Y. Cho, H. Kim, On classical solutions of the compressible Navier-Stokes equations with nonnegative initial densities, *Manuscripta Math.*, **120** (2006), 91–129. https://doi.org/10.1007/s00229-006-0637-y
2. H. J. Choe, H. Kim, Strong solutions of the Navier-Stokes equations for isentropic compressible fluids, *J. Differential Equations*, **190** (2003), 504–523. https://doi.org/10.1016/S0022-0396(03)00015-9
3. R. Danchin, Global existence in critical spaces for compressible Navier-Stokes equations, *Invent. Math.*, **141** (2000), 579–614. https://doi.org/10.1007/s002220000078
4. E. Feireisl, H. Petzeltová, Large-time behaviour of solutions to the Navier-Stokes equations of compressible flow, *Arch. Ration. Mech. Anal.*, **150** (1999), 77–96. https://doi.org/10.1007/s002050050181
5. D. Hoff, Global existence for 1D, compressible, isentropic Navier-Stokes equations with large initial data, *Trans. Amer. Math. Soc.*, **303** (1987), 169–181. https://doi.org/10.2307/2000785
6. X. D. Huang, Z. P. Xin, A blow-up criterion for classical solutions to the compressible Navier-Stokes equations, *Sci. China Math.*, **53** (2010), 671–686. https://doi.org/10.1007/s11425-010-0042-6
7. A. Matsumura, T. Nishida, The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids, *Proc. Japan Acad. Ser. A Math. Sci.*, **55** (1979), 337–342. https://doi.org/10.3792/pjaa.55.337
8. O. Rozanova, Blow-up of smooth highly decreasing at infinity solutions to the compressible Navier-Stokes equations, *J. Differential Equations*, **245** (2008), 1762–1774. https://doi.org/10.1016/j.jde.2008.07.007
9. P. L. Lions, *Mathematical topics in fluid mechanics. Vol. 2. Compressible models*, Oxford University Press, New York, 1998.

10. D. Hoff, Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data, *J. Differential Equations*, **120** (1995), 215–254. https://doi.org/10.1006/jdeq.1995.1111

11. D. Hoff, M. M. Santos, Lagrangean structure and propagation of singularities in multidimensional compressible flow, *Arch. Ration. Mech. Anal.*, **188** (2008), 509–543. https://doi.org/10.1007/s00205-007-0099-8

12. T. P. Liu, Z. P. Xin, T. Yang, Vacuum states for compressible flow, *Discrete Contin. Dynam. Systems*, **4** (1998), 1–32. https://doi.org/10.3934/dcds.1998.4.1

13. Z. P. Xin, Blowup of smooth solutions to the compressible Navier-Stokes equation with compact density, *Comm. Pure Appl. Math.*, **51** (1998), 229–240. https://doi.org/10.1002/(SICI)1097-0312(199803)51:3<229::AID-CPA1>3.0.CO;2-C

14. Y. B. Zel'dovich, Y. P. Raizer, *Physics of shock waves and high-temperature hydrodynamic phenomena. Vol. II*, Academic Press, New York, 1967.

15. R. H. Pan, W. Z. Zhang, Compressible Navier-Stokes equations with temperature dependent heat conductivity, *Commun. Math. Sci.*, **13** (2015), 401–425. https://doi.org/10.4310/CMS.2015.v13.n2.a7

16. B. Huang, X. D. Shi, Nonlinearly exponential stability of compressible Navier-Stokes system with degenerate heat-conductivity, *J. Differential Equations*, **268** (2020), 2464–2490. https://doi.org/10.1016/j.jde.2019.09.006

17. H. X. Liu, T. Yang, H. J. Zhao, Q. Zou, One-dimensional compressible Navier-Stokes equations with temperature dependent transport coefficients and large data, *SIAM J. Math. Anal.*, **46** (2014), 2185–2228. https://doi.org/10.1137/130920617

18. L. Wan, T. Wang, Symmetric flows for compressible heat-conducting fluids with temperature dependent viscosity coefficients, *J. Differential Equations*, **262** (2017), 5939–5977. https://doi.org/10.1016/j.jde.2017.02.022

19. T. Wang, H. J. Zhao, One-dimensional compressible heat-conducting gas with temperature-dependent viscosity, *Math. Models Methods Appl. Sci.*, **26** (2016), 2237–2275. https://doi.org/10.1142/S0218202516500524

20. Y. Sun, J. W. Zhang, X. K. Zhao, Nonlinearly exponential stability for the compressible Navier-Stokes equations with temperature-dependent transport coefficients, *J. Differential Equations*, **286** (2021), 676–709. https://doi.org/10.1016/j.jde.2021.03.044

21. M. Okada, S. Kawashima, On the equations of one-dimensional motion of compressible viscous fluids, *J. Math. Kyoto Univ.*, **23** (1983), 55–71. https://doi.org/10.1215/kjm/1250521610

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