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Research article

A critical Kirchhoff problem with a logarithmic type perturbation in high dimension

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Abstract: In this paper, the following critical Kirchhoff-type elliptic equation involving a logarithmic-type perturbation

$$-\Big(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x\Big)\Delta u = \lambda|u|^{q-2}u\ln|u|^2 + \mu|u|^2u$$

is considered in a bounded domain in $\mathbb{R}^4$. One of the main obstructions one encounters when looking for weak solutions to Kirchhoff problems in high dimensions is that the boundedness of the (PS) sequence is hard to obtain. By combining a result by Jeanjean [27] with the mountain pass lemma and Brézis–Lieb's lemma, it is proved that either the norm of the sequence of approximation solutions goes to infinity or the problem admits a nontrivial weak solution, under some appropriate assumptions on a, b, λ, and μ.

Keywords: critical; Kirchhoff equation; logarithmic perturbation; high dimension; mountain pass lemma
Mathematics Subject Classification: 35J60, 35B33

1. Introduction

In this paper, we consider the following Kirchhoff-type elliptic problem:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u = \lambda|u|^{q-2}u\ln|u|^2 + \mu|u|^2u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^4$ is a bounded domain with smooth boundary $\partial\Omega$, $2 < q < 4$, a, b, λ, and μ are positive parameters.

As a natural generalization of (1.1), we obtain the following Kirchhoff-type elliptic problem:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u = f(x, u), & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.2}$$

When the space dimension $N = 1$, the equation in (1.2) is closely related to the stationary version of the following wave equation:

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x\right)\frac{\partial^2 u}{\partial x^2} = f(x,u), \tag{1.3}$$

which was proposed by Kirchhoff [1] to describe the transversal oscillations of a stretched string. Here $\rho > 0$ is the mass per unit length, P_0 is the base tension, E is the Young modulus, h is the area of the cross section, and L is the initial length of the string. Such problems are widely applied in engineering, physics, and other applied sciences (see [2–4]). A remarkable feature of problem (1.2) is the presence of the nonlocal term, which brings essential difficulties when looking for weak solutions to it in the framework of variational methods since, in general, one cannot deduce from $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ the convergence $\int_\Omega |\nabla u_n|^2 \mathrm{d}x \to \int_\Omega |\nabla u|^2 \mathrm{d}x$.

In recent years, various techniques, such as the mountain pass lemma, the Nehari manifold approach, genus theory, Morse theory, etc., have been used to study the existence and multiplicity of weak solutions to Kirchhoff problems with different kinds of nonlinearities in the general dimension. We refer the interested reader to [5–10] and the references therein. In particular, when dealing with (1.2) by using variational methods, f is usually required to satisfy the following Aimbrosetti–Rabinowitz condition: i.e., for some $\nu > 4$ and $R > 0$, there holds

$$0 < \nu F(x,t) \le t f(x,t),\ \forall\ |t| > R,\ x \in \Omega,$$

which implies that f is 4-superlinear about t at infinity, that is,

$$\lim_{t\to+\infty}\frac{F(x,t)}{t^4} = +\infty, \tag{1.4}$$

where $F(x,t) = \int_0^t f(x,\tau)\mathrm{d}\tau$. This guarantees the boundedness of any (PS) sequence of the corresponding energy functional in $H_0^1(\Omega)$. In addition, assume that f satisfies the subcritical growth condition

$$|f(x,t)| \le C(|t|^{q-1} + 1),\ t \in \mathbb{R},\ x \in \Omega, \tag{1.5}$$

where $C > 0$, $2 < q < 2^* := \frac{2N}{N-2}$. Then it follows from [11, Lemma 1] that the functional satisfies the compactness condition. Combining (1.4) with (1.5), one has $q > 4$, which, together with $q < 2^*$, implies $N < 4$. Hence, problem (1.2) is usually studied in dimension three or less. For example, Chen et al. [12] studied problem (1.2) with $N \le 3$ and $f(x,u) = \lambda h(x)|u|^{q-2}u + g(x)|u|^{p-2}u$, where $a,\ b,\ \lambda > 0$, $1 < q < 2 < p < 2^*$, and $h, g \in C(\overline{\Omega})$ are sign-changing functions. They obtained the existence of multiple positive solutions with the help of Nehari manifolds and fibering maps. Silva [13] considered the existence and multiplicity of weak solutions to problem (1.2) in a bounded smooth domain $\Omega \subset \mathbb{R}^3$ with $f(x,u) = |u|^{\gamma-2}u$ and $\gamma \in (2.4)$ by using fibering maps and the mountain pass lemma. Later, the main results in [13] were extended to the parallel p-Kirchhoff problem in a previous work of ours [14].

For the critical problem (1.2) with $N \ge 4$, it was shown in [15] that when a and b satisfy appropriate constraints, the interaction between the Kirchhoff operator and the critical term makes some useful variational properties of the energy functional valid, such as the weak lower semi-continuity and the Palais–Smale properties. Naimen [16] considered problem (1.2) with $f(x,u) = \lambda u^q + \mu u^{2^*-1}(1 \le q < 2^* - 1)$ when $N = 4$. By applying the variational method and the concentration compactness argument, he proved the existence of solutions to problem (1.2). Later, Naimen and Shibata [17] considered the same problem with $N = 5$, and the existence of two solutions was obtained by using the variational method. Faraci and Silva [18] considered problem (1.2) with $f(x, u) = \lambda g(x, u) + |u|^{2^*-2}u$ in $\Omega \subset \mathbb{R}^N (N > 4)$. By using variational properties and the fiber maps of the energy functional associated with the problem, the existence, nonexistence, and multiplicity of weak solutions were obtained under some assumptions on a, b, λ, and $g(x, u)$. Li et al. [19] considered a Kirchhoff-type problem without a compactness condition in the whole space $\mathbb{R}^N (N \geq 3)$. By introducing an appropriate truncation on the nonlocal coefficient, they showed that the problem admits at least one positive solution.

On the other hand, equations with logarithmic nonlinearity have also been receiving increasing attention recently, due to their wide application in describing many phenomena in physics and other applied sciences such as viscoelastic mechanics and quantum mechanics theory ([20–23]). The logarithmic nonlinearity is sign-changing and satisfies neither the monotonicity condition nor the Ambrosetti–Rabinowitz condition, which makes the study of problems with logarithmic nonlinearity more interesting and challenging. Therefore, much effort has also been made in this direction during the past few years. For example, Shuai [24] considered problem (1.2) with $a > 0$, $b = 0$, and $f(x, u) = a(x)u \ln |u|$, where $a(x) \in C(\overline{\Omega})$ can be positive, negative, or sign-changing in $\overline{\Omega}$. Among many interesting results, he proved, by the use of the Nehari manifold, the symmetric mountain pass lemma and Clark's Theorem, that the problem possesses two sequences of solutions when $a(x)$ is sign-changing. Later, the first two authors of this paper and their co-author [25] investigated the following critical biharmonic elliptic problem:

$$\begin{cases} \Delta^2 u = \lambda u + \mu u \ln u^2 + |u|^{2^{**}-2}u, & x \in \Omega, \\ u = \dfrac{\partial u}{\partial \nu} = 0, & x \in \partial\Omega, \end{cases} \tag{1.6}$$

where $2^{**} := \frac{2N}{N-4}$ is the critical Sobolev exponent for the embedding $H_0^2(\Omega) \hookrightarrow L^{2^{**}}(\Omega)$. Both the cases $\mu > 0$ and $\mu < 0$ are considered in [25], and the existence of nontrivial solutions was derived by combing the variational methods with careful estimates on the logarithmic term. It is worth mentioning that the result with $\mu > 0$ implies that the logarithmic term plays a positive role in problem (1.6) to admit a nontrivial solution. Later, Zhang et al. [26] not only weakened the existence condition in [25], but also specified the types and the energy levels of solutions by using Brézis–Lieb's lemma and Ekeland's variational principle.

Inspired mainly by the above-mentioned literature, we consider a critical Kirchhoff problem with a logarithmic perturbation and investigate the combined effect of the nonlocal term, the critical term, and the logarithmic term on the existence of weak solutions to problem (1.1). We think that at least the following three essential difficulties make the study of such a problem far from trivial. The first one is that since the power corresponding to the critical term is equal to that corresponding to the nonlocal term in the energy functional $I(u)$ (see (2.2)), it is very difficult to obtain the boundedness of the (PS) sequences for $I(u)$. The second one is the lack of compactness of the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$, which prevents us from directly establishing the (PS) condition for $I(u)$. The third one is caused by the logarithmic nonlinearity, which we have mentioned above.

To overcome these difficulties and to investigate the existence of weak solutions to problem (1.1), we first consider a sequence of approximation problems (see problem (2.7) in Section 2) and obtain a bounded (PS) sequence for each approximation problem based on a result by Jeanjean [27]. Then, with the help of some delicate estimates on the truncated Talenti functions and Brézis–Lieb's lemma, we prove that the (PS) sequence has a strongly convergent subsequence. Then we obtain a solution u_n to problem (2.7) for almost every $\nu_n \in (\sigma, 1]$. Finally, we show that the original problem admits a mountain pass type solution if the sequence of the approximation solution $\{u_n\}$ is bounded by the mountain pass lemma.

The organization of this paper is as follows: In Section 2, some notations, definitions, and necessary lemmas are introduced. The main results of this paper are also stated in this section. In Section 3, we give detailed proof of the main results.

2. Preliminaries and main results

We start by introducing some notations and definitions that will be used throughout the paper. In what follows, we denote by $\|\cdot\|_p$ the $L^p(\Omega)$ norm for $1 \leq p \leq \infty$. The Sobolev space $H_0^1(\Omega)$ will be equipped with the norm $\|u\| := \|u\|_{H_0^1(\Omega)} = \|\nabla u\|_2$, which is equivalent to the full one due to Poincaré's inequality. The dual space of $H_0^1(\Omega)$ is denoted by $H^{-1}(\Omega)$. We use $\to$ and $\rightharpoonup$ to denote the strong and weak convergence in each Banach space, respectively, and use $C, C_1, C_2, \ldots$ to denote (possibly different) positive constants. $B_R(x_0)$ is a ball of radius R centered at x_0. We use ω_4 to denote the area of the unit sphere in $\mathbb{R}^4$. For all $t > 0$, $O(t)$ denotes the quantity satisfying $|\frac{O(t)}{t}| \leq C$, $O_1(t)$ means there exist two positive constants C_1 and C_2 such that $C_1 t \leq O_1(t) \leq C_2 t$, $o(t)$ means $|\frac{o(t)}{t}| \to 0$ as $t \to 0$, and $o_n(1)$ is an infinitesimal as $n \to \infty$. We use $S > 0$ to denote the best embedding constant from $H_0^1(\Omega)$ to $L^4(\Omega)$, i.e.,

$$\|u\|_4 \leq S^{-\frac{1}{2}}\|u\|, \ \forall\, u \in H_0^1(\Omega). \tag{2.1}$$

In this paper, we consider weak solutions to problem (1.1) in the following sense:

Definition 2.1. *(weak solution) A function $u \in H_0^1(\Omega)$ is called a weak solution to problem* (1.1) *if for every $\varphi \in H_0^1(\Omega)$, there holds*

$$a\int_\Omega \nabla u \nabla \varphi \mathrm{d}x + b\|u\|^2 \int_\Omega \nabla u \nabla \varphi \mathrm{d}x - \lambda \int_\Omega |u|^{q-2} u\varphi \ln u^2 \mathrm{d}x - \mu \int_\Omega |u|^2 u\varphi \mathrm{d}x = 0.$$

Define the energy functional associated with problem (1.1) by

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{2\lambda}{q^2}\|u\|_q^q - \frac{\lambda}{q}\int_\Omega |u|^q \ln u^2 \mathrm{d}x - \frac{\mu}{4}\|u\|_4^4, \ \forall\, u \in H_0^1(\Omega). \tag{2.2}$$

From $2 < q < 4$, one can see that $I(u)$ is well defined and is a C^1 functional in $H_0^1(\Omega)$ (see [18]). Moreover, each critical point of I is also a weak solution to problem (1.1).

We introduce a definition of a local compactness condition, usually called the $(PS)_c$ condition.

Definition 2.2. *($(PS)_c$ condition [28]) Assume that X is a real Banach space, $I : X \to \mathbb{R}$ is a C^1 functional, and $c \in \mathbb{R}$. Let $\{u_n\} \subset X$ be a $(PS)_c$ sequence of $I(u)$, i.e.,*

$$I(u_n) \to c \text{ and } I'(u_n) \to 0 \text{ in } X^{-1}(\Omega) \text{ as } n \to \infty,$$

where X^{-1} is the dual space of X. We say that I satisfies the $(PS)_c$ condition if any $(PS)_c$ sequence has a strongly convergent subsequence.

The following three lemmas will play crucial roles in proving our main results. The first one is the mountain pass lemma, the second one is Brézis–Lieb's lemma, and the third one will be used to deal with the logarithmic term.

Lemma 2.1. *(mountain pass lemma [28]) Assume that* $(X, \|\cdot\|_X)$ *is a real Banach space,* $I : X \to \mathbb{R}$ *is a* C^1 *functional, and there exist* $\beta > 0$ *and* $r > 0$ *such that* I *satisfies the following mountain pass geometry:*

(i) $I(u) \geq \beta > 0$ *if* $\|u\|_X = r$*;*

(ii) There exists a $\overline{u} \in X$ *such that* $\|\overline{u}\|_X > r$ *and* $I(\overline{u}) < 0$.

Then there exists a $(PS)_{c_*}$ *sequence such that*

$$c_* := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)) \geq \beta,$$

where

$$\Gamma = \{\gamma \in C([0, 1], X) : \gamma(0) = 0, \gamma(1) = \overline{u}\}.$$

c_* *is called the mountain level. Furthermore,* c_* *is a critical value of* I *if* I *satisfies the* $(PS)_{c_*}$ *condition.*

Lemma 2.2. *(Brézis–Lieb's lemma [29]) Let* $p \in (0, \infty)$*. Suppose that* $\{u_n\}$ *is a bounded sequence in* $L^p(\Omega)$ *and* $u_n \to u$ *a.e. on* Ω*. Then*

$$\lim_{n\to\infty}(\|u_n\|_p^p - \|u_n - u\|_p^p) = \|u\|_p^p.$$

Lemma 2.3. (1) *For all* $t \in (0, 1]$*, there holds*

$$|t \ln t| \leq \frac{1}{e}. \tag{2.3}$$

(2) *For any* $\alpha, \delta > 0$*, there exists a positive constant* $C_{\alpha,\delta}$ *such that*

$$|\ln t| \leq C_{\alpha,\delta}(t^\alpha + t^{-\delta}), \qquad \forall\, t > 0. \tag{2.4}$$

(3) *For any* $\delta > 0$*, there holds*

$$\frac{\ln t}{t^\delta} \leq \frac{1}{\delta e}, \qquad \forall\, t > 0. \tag{2.5}$$

(4) *For any* $q \in \mathbb{R}\backslash\{0\}$*, there holds*

$$2t^q - qt^q \ln t^2 \leq 2, \qquad \forall\, t > 0. \tag{2.6}$$

Proof. (1) Let $k_1(t) := t \ln t$, $t \in (0, 1]$. Then simple analysis shows that $k_1(t)$ is decreasing in $(0, \frac{1}{e})$, increasing in $(\frac{1}{e}, 1)$, and attaining its minimum at $t_{k_1} = \frac{1}{e}$ with $k_1(t_{k_1}) = -\frac{1}{e}$. Moreover, $k_1(t) \leq 0$ for all $t \in (0, 1]$. Consequently, $|k_1(t)| \leq \dfrac{1}{e}$, $t \in (0, 1]$.

(2) For any $\alpha,\ \delta > 0$, from

$$\lim_{t\to0^+} \frac{\ln t}{t^{-\delta}} = 0 \ \text{ and } \ \lim_{t\to+\infty} \frac{\ln t}{t^\alpha} = 0,$$

one sees that there exist constants $C_\delta, C_\alpha > 0$ and $0 < M_1 < M_2 < +\infty$ such that $|\ln t| \le C_\delta t^{-\delta}$ when $t \in (0, M_1)$ and $|\ln t| \le C_\alpha t^\alpha$ when $t \in (M_2, \infty)$. Moreover, it is obvious that there exists $C'_\alpha > 0$ such that $|\ln t| \le C'_\alpha t^\alpha$ when $t \in [M_1, M_2]$. Therefore,

$$|\ln t| \le (C'_\alpha + C_\alpha)t^\alpha + C_\delta t^{-\delta} \le C_{\alpha,\delta}(t^\alpha + t^{-\delta}), \quad \forall\, t > 0,$$

where $C_{\alpha,\delta} = \max\{C_\delta, C'_\alpha + C_\alpha\}$.

(3) For any $\delta > 0$, set $k_2(t) := \dfrac{\ln t}{t^\delta}$, $t > 0$. Then direct computation shows that $k'_2(t) > 0$ in $(0, e^{\frac{1}{\delta}})$, $k'_2(t) < 0$ in $(e^{\frac{1}{\delta}}, +\infty)$, and $k_2(t)$ attain their maximum at $t_{k_2} = e^{\frac{1}{\delta}}$. Therefore, $k_2(t) \le k_2(e^{\frac{1}{\delta}}) = \frac{1}{\delta e}$, $t > 0$.

(4) Let $k_3(t) := 2t^q - qt^q \ln t^2$, $t > 0$, where $q \in \mathbb{R}\backslash\{0\}$. From

$$k'_3(t) = -q^2 t^{q-1} \ln t^2, \ \ t > 0,$$

we know that $k_3(t)$ has a unique critical point, $t_{k_3} = 1$ in $(0, +\infty)$. Moreover, $k'_3(t) > 0$ in $(0, 1)$, $k'_3(t) < 0$ in $(1, +\infty)$, and $k_3(t)$ attain their maximum at t_{k_3}. Consequently, $k_3(t) \le k_3(1) = 2$. The proof is complete.

Following the ideas of [16], we consider the following approximation problem:

$$\begin{cases} -(a + b\int_\Omega |\nabla u|^2 \mathrm{d}x)\Delta u = \lambda |u|^{q-2}u \ln |u|^2 + \nu\mu|u|^2 u, & x \in \Omega, \\ u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{2.7}$$

where $\nu \in (\sigma, 1]$ for some $\sigma \in (\frac{1}{2}, 1)$. Associated functionals are defined by

$$I_\nu(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{2\lambda}{q^2}\|u\|_q^q - \frac{\lambda}{q}\int_\Omega |u|^q \ln u^2 \mathrm{d}x - \frac{1}{4}\nu\mu\|u\|_4^4, \ \forall\, u \in H_0^1(\Omega).$$

Noticing that $I_\nu(u) = I_\nu(|u|)$ and $I(u) = I(|u|)$, we may assume that $u \ge 0$ in the sequel.

To obtain the boundedness of the (PS) sequences for I_ν, we need the following result by Jeanjean: [27].

Lemma 2.4. *Assume that $(X, \|\cdot\|_X)$ is a real Banach space, and let $J \subset \mathbb{R}^+$ be an interval. We consider a family $(I_\nu)_{\nu\in J}$ of C^1-functionals on X of the form*

$$I_\nu(u) = A(u) - \nu B(u), \ \nu \in J,$$

where $B(u) \ge 0$ for all $u \in X$ and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to \infty$. Assume, in addition, that there are two points e_1, e_2 in X such that for all $\nu \in J$, there holds

$$c_\nu := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\nu(\gamma(t)) \ge \max\{I_\nu(e_1), I_\nu(e_2)\},$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = e_1, \gamma(1) = e_2\}.$$

Then, for almost every $\nu \in J$, there is a sequence $\{u_n\} \subset X$ such that

$$(i)\ \{u_n\} \text{ is bounded}, \ \ (ii)\ I_\nu(u_n) \to c_\nu, \text{ as } n \to \infty, \ \ (iii)\ I'_\nu(u_n) \to 0 \text{ in } X^{-1}(\Omega) \text{ as } n \to \infty,$$

where X^{-1} is the dual space of X.

At the end of this section, we state the main results of this paper, which can be summarized into the following theorem:

Theorem 2.1. *Let $b, \mu > 0$ satisfy $bS^2 < \mu < 2bS^2$, and take $\frac{1}{2} < \sigma < 1$ such that $\frac{bS^2}{\sigma} < \mu$. Assume one of the following (C_1), (C_2), or (C_3) holds.*

(C_1) $a > 0$, and $\lambda > 0$ is small enough.

(C_2) $\lambda > 0$, and $a > 0$ is large enough.

(C_3) $a > 0$, $\lambda > 0$, and $b < \frac{\mu}{S^2}$ is sufficiently close to $\frac{\mu}{S^2}$.

Then problem (2.7) has a solution for almost every $\nu \in (\sigma, 1]$. In addition, we can find an increasing sequence $\nu_n \in (\sigma, 1]$ such that $\nu_n \to 1$ as $n \to \infty$ and denote by u_n the corresponding solution to problem (2.7). Then either (i) or (ii) below holds.

(i) $\lim_{n\to\infty} \|u_n\|_{H_0^1(\Omega)} = \infty$;

(ii) $\{u_n\}$ is bounded in $H_0^1(\Omega)$ and consequently, problem (1.1) has a nontrivial weak solution.

In particular, if $\Omega \subset \mathbb{R}^4$ is strictly star-shaped, then problem (1.1) has a nontrivial weak solution.

3. Proofs of the main results

We are going to show that there exists a bounded (PS) sequence of the energy functional I_ν for almost every $\nu \in (\sigma, 1]$. For this, let us introduce the Talenti functions (see [16]). For any $\varepsilon > 0$, define

$$U_\varepsilon(x) = \frac{8^{\frac{1}{2}}\varepsilon}{\varepsilon^2 + |x|^2},\ x \in \mathbb{R}^4.$$

Then $U_\varepsilon(x)$ is a solution to the critical problem

$$-\Delta u = u^3, \quad x \in \mathbb{R}^4, \tag{3.1}$$

and it satisfies $\|U_\varepsilon\|^2 = \|U_\varepsilon\|_4^4 = S^2$, where $S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\|u\|_4^2} = \frac{\|U_\varepsilon\|^2}{\|U_\varepsilon\|_4^2}$ (an equivalent characterization of S defined in (2.1)).

Lemma 3.1. *Let $\tau \in C_0^\infty(\Omega)$ be a cut-off function such that $0 \le \tau(x) \le 1$ in Ω with $\tau(x) = 1$ if $|x| < R$ and $\tau(x) = 0$ if $|x| > 2R$, where $R > 0$ is a constant such that $B_{2R}(0) \subset \Omega$ (Here we assume, without loss of generality, that $0 \in \Omega$). Set $u_\varepsilon(x) = \tau(x)U_\varepsilon(x)$. Then we have*

$$\begin{aligned} \|u_\varepsilon\|^2 &= S^2 + O(\varepsilon^2), \\ \|u_\varepsilon\|_4^4 &= S^2 + O(\varepsilon^4), \\ \|u_\varepsilon\|_q^q &= O_1(\varepsilon^{4-q}) + O(\varepsilon^q), \end{aligned} \tag{3.2}$$

and

$$\int_\Omega u_\varepsilon^q \ln u_\varepsilon^2 \mathrm{d}x = O_1\left(\varepsilon^{4-q} \ln\left(\frac{1}{\varepsilon}\right)\right) + O(\varepsilon^q \ln \varepsilon) + O(\varepsilon^{4-q}), \tag{3.3}$$

where $q \in (2, 4)$.

Set $v_\varepsilon(x) = \dfrac{u_\varepsilon(x)}{\|u_\varepsilon\|_4}$. *Then, as* $\varepsilon \to 0$,

$$\|v_\varepsilon\|^2 = S + O(\varepsilon^2),$$
$$\|v_\varepsilon\|_4^4 = 1,$$
$$\|v_\varepsilon\|_q^q = O_1(\varepsilon^{4-q}) + O(\varepsilon^q),$$

and

$$\int_\Omega v_\varepsilon^q \ln v_\varepsilon^2 \mathrm{d}x = O_1\left(\varepsilon^{4-q}\ln\Big(\frac{1}{\varepsilon}\Big)\right) + O(\varepsilon^q \ln\varepsilon) + O(\varepsilon^{4-q}).$$

Proof. We only prove (3.2) and (3.3). The proof of other results is similar and can be referred to [30]. Using the properties of the cut-off function τ, one has

$$\begin{aligned}
&\int_\Omega u_\varepsilon^q \mathrm{d}x = C\int_{B_{2R}(0)} \tau^q \frac{\varepsilon^q}{(\varepsilon^2+|x|^2)^q}\mathrm{d}x\\
&= C\int_{B_R(0)} \frac{\varepsilon^q}{(\varepsilon^2+|x|^2)^q}\mathrm{d}x + C\int_{B_{2R}(0)\backslash B_R(0)} \frac{\tau^q\varepsilon^q}{(\varepsilon^2+|x|^2)^q}\mathrm{d}x\\
&:= J_1 + J_2.
\end{aligned}$$

By changing the variable and applying the polar coordinate transformation, we can estimate J_1 as follows:

$$\begin{aligned}
J_1 &= C\int_{B_{\frac{R}{\varepsilon}}(0)} \frac{\varepsilon^q\varepsilon^4}{(1+|y|^2)^q\varepsilon^{2q}}\mathrm{d}y = C\varepsilon^{4-q}\int_{B_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\mathrm{d}y\\
&= C\omega_4\varepsilon^{4-q}\int_0^{\frac{R}{\varepsilon}} \frac{r^3}{(1+r^2)^q}\mathrm{d}r\\
&= C_1\varepsilon^{4-q}\Big(\int_0^{+\infty} \frac{r^3}{(1+r^2)^q}\mathrm{d}r - \int_{\frac{R}{\varepsilon}}^{+\infty} \frac{r^3}{(1+r^2)^q}\mathrm{d}r\Big)\\
&= C_2\varepsilon^{4-q} + O(\varepsilon^q),
\end{aligned} \tag{3.4}$$

where we have used the fact that

$$\Big|\int_{\frac{R}{\varepsilon}}^{+\infty} \frac{r^3}{(1+r^2)^q}\mathrm{d}r\Big| \le \int_{\frac{R}{\varepsilon}}^{+\infty} r^{3-2q}\mathrm{d}r = O(\varepsilon^{2q-4}),$$

and

$$\int_0^{+\infty} \frac{r^3}{(1+r^2)^q}\mathrm{d}r \le C.$$

On the other hand,

$$|J_2| \le C\varepsilon^q \int_{B_{2R}(0)\backslash B_R(0)} \frac{1}{|x|^{2q}}\mathrm{d}x = O(\varepsilon^q). \tag{3.5}$$

Hence, (3.2) follows from (3.4) and (3.5).

Next, we shall prove (3.3). According to the definition of u_ε, we obtain

$$\begin{aligned}\int_\Omega u_\varepsilon^q \ln u_\varepsilon^2 \mathrm{d}x &= \int_\Omega \tau^q U_\varepsilon^q \ln(U_\varepsilon^2 \tau^2)\mathrm{d}x \\ &= \int_\Omega \tau^q U_\varepsilon^q \ln \tau^2 \mathrm{d}x + \int_\Omega \tau^q U_\varepsilon^q \ln U_\varepsilon^2 \mathrm{d}x \\ &:= J_3 + J_4.\end{aligned}$$

By direct computation, we have

$$|J_3| = \Big| \int_{B_{2R}(0)\setminus B_R(0)} \tau^q U_\varepsilon^q \ln \tau^2 \mathrm{d}x \Big| \le C \int_{B_{2R}(0)\setminus B_R(0)} U_\varepsilon^q \mathrm{d}x = O(\varepsilon^q).$$

Rewrite J_4 as follows:

$$J_4 = \int_{B_R(0)} U_\varepsilon^q \ln U_\varepsilon^2 \mathrm{d}x + \int_{B_{2R}(0)\setminus B_R(0)} \tau^q U_\varepsilon^q \ln U_\varepsilon^2 \mathrm{d}x := J_{41} + J_{42}.$$

By using inequality (2.4) with $\alpha_1 = \delta_1 < 4 - q$, one has

$$|J_{42}| \le C \int_{B_{2R}(0)\setminus B_R(0)} (U_\varepsilon^{q-\delta_1} + U_\varepsilon^{q+\delta_1})\mathrm{d}x = O(\varepsilon^{q-\delta_1}) + O(\varepsilon^{q+\delta_1}) = O(\varepsilon^{q-\delta_1}),$$

where

$$\begin{aligned}\int_{B_{2R}(0)\setminus B_R(0)} U_\varepsilon^{q-\delta_1} \mathrm{d}x &= C \int_{B_{2R}(0)\setminus B_R(0)} \frac{\varepsilon^{q-\delta_1}}{(\varepsilon^2 + |x|^2)^{q-\delta_1}} \mathrm{d}x \\ &\le C \int_{B_{2R}(0)\setminus B_R(0)} \frac{\varepsilon^{q-\delta_1}}{|x|^{2q-2\delta_1}} \mathrm{d}x \\ &= O(\varepsilon^{q-\delta_1}),\end{aligned}$$

and

$$\begin{aligned}\int_{B_{2R}(0)\setminus B_R(0)} U_\varepsilon^{q+\delta_1} \mathrm{d}x &= C \int_{B_{2R}(0)\setminus B_R(0)} \frac{\varepsilon^{q+\delta_1}}{(\varepsilon^2 + |x|^2)^{q+\delta_1}} \mathrm{d}x \\ &\le C \int_{B_{2R}(0)\setminus B_R(0)} \frac{\varepsilon^{q+\delta_1}}{|x|^{2q-2\delta_1}} \mathrm{d}x \\ &= O(\varepsilon^{q+\delta_1}).\end{aligned}$$

In addition,

$$\begin{aligned}J_{41} &= C \int_{B_R(0)} \frac{\varepsilon^q}{(\varepsilon^2 + |x|^2)^q} \ln\Big(C \frac{\varepsilon}{\varepsilon^2 + |x|^2}\Big)\mathrm{d}x \\ &= C\varepsilon^4 \int_{B_{\frac{R}{\varepsilon}}(0)} \frac{\varepsilon^q}{\varepsilon^{2q}(1 + |y|^2)^q} \ln\Big(C \frac{\varepsilon}{\varepsilon^2(1 + |y|^2)}\Big)\mathrm{d}y\end{aligned}$$

$$
\begin{aligned}
&= C\varepsilon^{4-q}\int_{B_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\ln\Big(C\frac{1}{\varepsilon(1+|y|^2)}\Big)\mathrm{d}y \\
&= C\varepsilon^{4-q}\ln\Big(\frac{1}{\varepsilon}\Big)\int_{B_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\mathrm{d}y + C\varepsilon^{4-q}\int_{B_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\ln\Big(\frac{C}{1+|y|^2}\Big)\mathrm{d}y \\
&= C\varepsilon^{4-q}\ln\Big(\frac{1}{\varepsilon}\Big)\Big(\int_{\mathbb{R}^4} \frac{1}{(1+|y|^2)^q}\mathrm{d}y - \int_{B^c_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\mathrm{d}y\Big) \\
&\quad + C\varepsilon^{4-q}\int_{B_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\ln\Big(\frac{C}{1+|y|^2}\Big)\mathrm{d}y \\
&= C_1\varepsilon^{4-q}\ln\Big(\frac{1}{\varepsilon}\Big) - C\varepsilon^{4-q}\ln\Big(\frac{1}{\varepsilon}\Big)\int_{B^c_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\mathrm{d}y \\
&\quad + C\varepsilon^{4-q}\int_{B_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\ln\Big(\frac{C}{1+|y|^2}\Big)\mathrm{d}y. \qquad (3.6)
\end{aligned}
$$

By direct computation, one obtains

$$
\Big|\int_{B^c_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\mathrm{d}y\Big| = \Big|\omega_4\int_{\frac{R}{\varepsilon}}^{+\infty} \frac{r^3}{(1+r^2)^q}\mathrm{d}r\Big| \le C\int_{\frac{R}{\varepsilon}}^{+\infty} r^{3-2q}\mathrm{d}r = O(\varepsilon^{2q-4}), \qquad (3.7)
$$

and

$$
\begin{aligned}
&\int_{B_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\ln\Big(\frac{C}{1+|y|^2}\Big)\mathrm{d}y \\
&= \int_{\mathbb{R}^4} \frac{1}{(1+|y|^2)^q}\ln\Big(\frac{C}{1+|y|^2}\Big)\mathrm{d}y - \int_{B^c_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\ln\Big(\frac{C}{1+|y|^2}\Big)\mathrm{d}y \\
&\le C + \Big|\int_{B^c_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\ln\Big(\frac{C}{1+|y|^2}\Big)\mathrm{d}y\Big| \\
&\le C + O(\varepsilon^{2q-4-2\delta_2}), \qquad (3.8)
\end{aligned}
$$

where we have used the fact that

$$
\begin{aligned}
&\Big|\int_{B^c_{\frac{R}{\varepsilon}}(0)} \frac{1}{(1+|y|^2)^q}\ln\Big(\frac{C}{1+|y|^2}\Big)\mathrm{d}y\Big| \\
&\le C\int_{B^c_{\frac{R}{\varepsilon}}(0)} \Big(\frac{1}{(1+|y|^2)^{q-\delta_2}} + \frac{1}{(1+|y|^2)^{q+\delta_2}}\Big)\mathrm{d}y \\
&= C\omega_4\int_{\frac{R}{\varepsilon}}^{+\infty} r^3\Big(\frac{1}{(1+r^2)^{q-\delta_2}} + \frac{1}{(1+r^2)^{q+\delta_2}}\Big)\mathrm{d}r \\
&\le C_1\int_{\frac{R}{\varepsilon}}^{+\infty} \Big(r^{3-2q+2\delta_2} + r^{3-2q-2\delta_2}\Big)\mathrm{d}r \\
&= O(\varepsilon^{2q-4-2\delta_2}),
\end{aligned}
$$

by recalling (2.4) with $\alpha_2 = \delta_2 < q-2$. Substituting (3.7) and (3.8) into (3.6), one arrives at

$$\begin{aligned} J_{41} &= C_1\varepsilon^{4-q}\ln\Big(\frac{1}{\varepsilon}\Big) - C\varepsilon^{4-q}\ln\Big(\frac{1}{\varepsilon}\Big)O(\varepsilon^{2q-4}) + C\varepsilon^{4-q}(C + O(\varepsilon^{2q-4-2\delta_2})) \\ &= C_1\varepsilon^{4-q}\ln\Big(\frac{1}{\varepsilon}\Big) + O(\varepsilon^q\ln\varepsilon) + O(\varepsilon^{4-q}) + O(\varepsilon^{q-2\delta_2}). \end{aligned}$$

Putting the estimates on J_3, J_{41}, and J_{42} together, one obtains

$$\begin{aligned} &\int_\Omega u_\varepsilon^q \ln u_\varepsilon^2 \mathrm{d}x \\ =&O(\varepsilon^q) + O(\varepsilon^{q-\delta_1}) + C_1\varepsilon^{4-q}\ln\Big(\frac{1}{\varepsilon}\Big) + O(\varepsilon^q\ln\varepsilon) + O(\varepsilon^{4-q}) + O(\varepsilon^{q-2\delta_2}). \end{aligned}$$

Therefore, (3.3) follows by taking δ_1 and δ_2 suitably small. The proof is complete.

With the help of Lemma 3.1, the existence of bounded (PS) sequences of I_ν can be derived.

Lemma 3.2. *Let $b>0$, $\mu>0$ satisfy $bS^2<\mu$ and take $\sigma<1$ such that $\frac{bS^2}{\sigma}<\mu$. Then there exists a bounded (PS) sequence of the energy functional I_ν for almost every $\nu \in (\sigma, 1]$.*

Proof. Applying (2.5) with $\delta < 4-q$, using the Sobolev embedding inequality, and noticing that $\nu \le 1$, one has

$$\begin{aligned} I_\nu(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{2}{q^2}\lambda\|u\|_q^q - \frac{\lambda}{q}\int_\Omega u^q\ln u^2\mathrm{d}x - \frac{1}{4}\nu\mu\|u\|_4^4 \\ &\ge \frac{a}{2}\|u\|^2 - \frac{\lambda}{q}\int_\Omega u^q\ln u^2\mathrm{d}x - \frac{1}{4}\mu\|u\|_4^4 \\ &\ge \frac{a}{2}\|u\|^2 - C\|u\|^{q+\delta} - \frac{1}{4}\mu S^{-2}\|u\|^4 \\ &= \|u\|^2\left(\frac{a}{2} - C\|u\|^{q+\delta-2} - \frac{1}{4}\mu S^{-2}\|u\|^4\right). \end{aligned}$$

Hence, there exist positive constants β and ρ such that

$$I_\nu(u) \ge \beta \text{ for all } \|u\| = \rho.$$

On the other hand, from (2.6), one has $t^q\ln t^2 \ge \frac{2}{q}(t^q-1)$ for $t>0$. Let v_ε be given in Lemma 3.1. Then, as $\varepsilon \to 0$, we have, for all $t>0$,

$$\begin{aligned} I_\nu(tv_\varepsilon) &= \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{b}{4}t^4\|v_\varepsilon\|^4 + \frac{2}{q^2}\lambda t^q\|v_\varepsilon\|_q^q - \frac{\lambda}{q}t^q\int_\Omega v_\varepsilon^q\ln(tv_\varepsilon)^2\mathrm{d}x - \frac{1}{4}\nu\mu t^4\|v_\varepsilon\|_4^4 \\ &\le \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{b}{4}t^4\|v_\varepsilon\|^4 + \frac{2}{q^2}\lambda t^q\|v_\varepsilon\|_q^q - \frac{2}{q^2}\lambda\int_\Omega((tv_\varepsilon)^q-1)\mathrm{d}x - \frac{1}{4}\nu\mu t^4\|v_\varepsilon\|_4^4 \\ &= \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{b}{4}t^4\|v_\varepsilon\|^4 + \frac{2}{q^2}\lambda|\Omega| - \frac{1}{4}\nu\mu t^4\|v_\varepsilon\|_4^4 \\ &= \frac{a}{2}t^2(S+O(\varepsilon^2)) + \frac{b}{4}t^4(S+O(\varepsilon^2))^2 + \frac{2}{q^2}\lambda|\Omega| - \frac{1}{4}\nu\mu t^4 \end{aligned}$$

$$= \frac{a}{2}St^2 - \frac{1}{4}(\nu\mu - bS^2)t^4 + \frac{2}{q^2}\lambda|\Omega| + O(\varepsilon^2),$$

which ensures that

$$I_\nu(tv_\varepsilon) \le \frac{a}{2}St^2 - \frac{1}{4}(\sigma\mu - bS^2)t^4 + \frac{2}{q^2}\lambda|\Omega| + 1,$$

for all sufficiently small $\varepsilon > 0$. Fix such an ε and denote it by ε_0. Then it follows from $\frac{bS^2}{\sigma} < \mu$ and the above inequality that

$$\lim_{t\to\infty} I_\nu(tv_{\varepsilon_0}) = -\infty \tag{3.9}$$

uniformly for $\nu \in (\sigma, 1]$, which implies that there exists a $t^* > 0$ such that $\|t^* v_{\varepsilon_0}\| > \rho$ and $I_\nu(t^* v_{\varepsilon_0}) < 0$ for all $\nu \in (\sigma, 1]$. Thus, I_ν satisfies the mountain pass geometry around 0, which is determined independently of $\nu \in (\sigma, 1]$. Now define

$$c_\nu := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\nu(\gamma(t)) \text{ and } \Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0, \gamma(1) = t^* v_{\varepsilon_0}\}.$$

Then, following the mountain pass lemma, we have

$$c_\nu \ge \beta > \max\{I_\nu(\gamma(0)), I_\nu(\gamma(1))\}, \tag{3.10}$$

for all $\nu \in (\sigma, 1]$. Set

$$I_\nu(u) = A(u) - \nu B(u), \ \forall\, u \in H_0^1(\Omega),$$

where $B(u) := \frac{1}{4}\mu\|u\|_4^4$ and $A(u) := \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{2}{q^2}\lambda\|u\|_q^q - \frac{\lambda}{q}\int_\Omega u^q \ln u^2 \mathrm{d}x$. By a simple analysis, one has

$$B(u) \ge 0 \text{ for all } u \in H_0^1(\Omega), \text{ and } A(u) \to +\infty, \ as\ \|u\| \to \infty. \tag{3.11}$$

In view of (3.10), (3.11), and according to Lemma 2.4, there exists a bounded $(PS)_{c_\nu}$ sequence of I_ν for almost every $\nu \in (\sigma, 1]$. The proof is complete.

Next, we prove the local compactness for $I_\nu(u)$, which will play a fundamental role in proving the main results.

Lemma 3.3. *Let $b > 0$, $\mu > 0$ satisfy $bS^2 < \mu < 2bS^2$ and take $\frac{1}{2} < \sigma < 1$ such that $\frac{bS^2}{\sigma} < \mu$. Suppose that one of the following (C_1), (C_2), or (C_3) holds.*
(C_1) $a > 0$, and $\lambda > 0$ is small enough.
(C_2) $\lambda > 0$, and $a > 0$ is large enough.
(C_3) $a > 0$, $\lambda > 0$, and $b < \frac{\mu}{S^2}$ is close enough to $\frac{\mu}{S_2}$.
Let $\{u_n\} \subset H_0^1(\Omega)$ be a bounded (PS) sequence for $I_\nu(u)$ with $\nu \in (\sigma, 1]$ at the level c with $c < c(K)$, where $c(K) := \frac{a^2S^2}{4(\nu\mu - bS^2)} = \frac{1}{2}aK + \frac{1}{4}bK^2 - \nu\mu\frac{K^2}{4S^2} > 0$ and $K := \frac{aS^2}{\mu\nu - bS^2}$, that is, $I_\nu(u_n) \to c$ and $I'_\nu(u_n) \to 0$ in $H^{-1}(\Omega)$ as $n \to \infty$. Then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$ itself) and a $u \in H_0^1(\Omega)$ such that $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$.

Proof. By the boundedness of $\{u_n\}$ in $H_0^1(\Omega)$ and the Sobolev embedding, one sees that there is a subsequence of $\{u_n\}$ (which we still denote by $\{u_n\}$) such that, as $n \to \infty$,

$$\begin{aligned} &u_n \rightharpoonup u \text{ in } H_0^1(\Omega),\\ &u_n \to u \text{ in } L^s(\Omega),\ 1 \le s < 4,\\ &u_n \to u \text{ a.e. in } \Omega. \end{aligned} \tag{3.12}$$

It follows from $u_n \to u$ a.e. in Ω as $n \to \infty$ that

$$u_n^q \ln u_n^2 \to u^q \ln u^2 \text{ a.e. in } \Omega \text{ as } n \to \infty. \tag{3.13}$$

Moreover, by virtue of (2.4) with $\alpha = \delta < 4 - q$, we get

$$\left|u_n^q \ln u_n^2\right| \le C_{\delta,\delta}(u_n^{q-\delta} + u_n^{q+\delta}) \to C_{\delta,\delta}(u^{q-\delta} + u^{q+\delta}) \text{ in } L^1(\Omega) \text{ as } n \to \infty. \tag{3.14}$$

With the help of (3.13), (3.14), and Lebesgue's dominating convergence theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega u_n^q \ln u_n^2 \mathrm{d}x = \int_\Omega u^q \ln u^2 \mathrm{d}x. \tag{3.15}$$

Similarly, we have

$$\lim_{n\to\infty} \int_\Omega u_n^{q-1} u \ln u_n^2 \mathrm{d}x = \int_\Omega u^q \ln u^2 \mathrm{d}x. \tag{3.16}$$

To prove $u_n \to u$ *in* $H_0^1(\Omega)$ as $n \to \infty$, set $w_n = u_n - u$. Then $\{w_n\}$ is also a bounded sequence in $H_0^1(\Omega)$. So there exists a subsequence of $\{w_n\}$ (which we still denote by $\{w_n\}$) such that $\lim\limits_{n\to\infty} \|w_n\|^2 = l \ge 0$. We claim that $l = 0$. Otherwise, according to (3.12) and Brézis–Lieb's lemma, we see that, as $n \to \infty$,

$$\begin{aligned} \|u_n\|^2 &= \|w_n\|^2 + \|u\|^2 + o_n(1),\\ \|u_n\|_4^4 &= \|w_n\|_4^4 + \|u\|_4^4 + o_n(1). \end{aligned} \tag{3.17}$$

It follows from (3.12), (3.16), and (3.17) that

$$\begin{aligned} o_n(1) =&\langle I_\nu'(u_n), u\rangle\\ =&a\|u\|^2 + b\|u_n\|^2\|u\|^2 - \lambda \int_\Omega u^q \ln u^2 \mathrm{d}x - \nu\mu\|u\|_4^4 + o_n(1)\\ =&a\|u\|^2 + b\|w_n\|^2\|u\|^2 + b\|u\|^4 - \lambda \int_\Omega u^q \ln u^2 \mathrm{d}x - \nu\mu\|u\|_4^4 + o_n(1),\ \text{as } n \to \infty. \end{aligned} \tag{3.18}$$

By (3.12), (3.15), (3.17), and (3.18), the boundedness of $\{u_n\}$ in $H_0^1(\Omega)$ and the Sobolev embedding, we obtain

$$\begin{aligned} o_n(1) =&\langle I_\nu'(u_n), u_n\rangle\\ =&a\|u\|^2 + a\|w_n\|^2 + 2b\|w_n\|^2\|u\|^2 + b\|w_n\|^4 + b\|u\|^4\\ &- \lambda \int_\Omega u^q \ln u^2 \mathrm{d}x - \nu\mu\|u\|_4^4 - \nu\mu\|w_n\|_4^4 + o_n(1) \end{aligned} \tag{3.19}$$

$$
\begin{aligned}
&=\langle I_\nu'(u_n), u\rangle + a\|w_n\|^2 + b\|w_n\|^4 + b\|w_n\|^2\|u\|^2 - \nu\mu\|w_n\|_4^4 + o_n(1)\\
&=a\|w_n\|^2 + b\|w_n\|^4 + b\|w_n\|^2\|u\|^2 - \nu\mu\|w_n\|_4^4 + o_n(1)\\
&\geq a\|w_n\|^2 + b\|w_n\|^4 + b\|w_n\|^2\|u\|^2 - \nu\mu S^{-2}\|w_n\|^4 + o_n(1)\\
&=[(a + b\|u\|^2) - (\nu\mu - bS^2)S^{-2}\|w_n\|^2]\|w_n\|^2 + o_n(1),\ as\ n \to \infty.
\end{aligned}
$$

Then, we have

$$
\lim_{n\to\infty} \|w_n\|^2 = l \geq \frac{(a + b\|u\|^2)S^2}{\nu\mu - bS^2}. \tag{3.20}
$$

It follows from $I_\nu(u_n) \to c$, (3.12), (3.15), and (3.17) that, as $n \to \infty$,

$$
\begin{aligned}
c + o_n(1) = I_\nu(u_n)&\\
&= \frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 + \frac{2}{q^2}\lambda\|u_n\|_q^q - \frac{\lambda}{q}\int_\Omega u_n^q \ln u_n^2 \mathrm{d}x - \frac{1}{4}\nu\mu\|u_n\|_4^4\\
&= \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{4}\|w_n\|^2\|u\|^2 - \frac{1}{4}\nu\mu\|w_n\|_4^4 + o_n(1)\\
&\quad + \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{b}{4}\|w_n\|^2\|u\|^2 + \frac{2}{q^2}\lambda\|u\|_q^q - \frac{\lambda}{q}\int_\Omega u^q \ln u^2 \mathrm{d}x - \frac{1}{4}\nu\mu\|u\|_4^4\\
&:= I_1 + I_2,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 &= \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{4}\|w_n\|^2\|u\|^2 - \frac{1}{4}\nu\mu\|w_n\|_4^4 + o_n(1),\\
I_2 &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{b}{4}\|w_n\|^2\|u\|^2 + \frac{2}{q^2}\lambda\|u\|_q^q - \frac{\lambda}{q}\int_\Omega u^q \ln u^2 \mathrm{d}x - \frac{1}{4}\nu\mu\|u\|_4^4 + o_n(1).
\end{aligned}
$$

By (3.19) and (3.20), we have, as $n \to \infty$,

$$
\begin{aligned}
I_1 + o_n(1) = I_1 - \frac{1}{4}\langle I_\nu'(u_n), u_n\rangle&\\
= \frac{a}{4}\|w_n\|^2 \geq \frac{a}{4}\frac{(a + b\|u\|^2)S^2}{\nu\mu - bS^2} &= \frac{a^2S^2}{4(\nu\mu - bS^2)} + \frac{abS^2}{4(\nu\mu - bS^2)}\|u\|^2.
\end{aligned} \tag{3.21}
$$

Applying (2.5) with $\delta < 4 - q$, from (3.20) and the Sobolev embedding, one has, as $n \to \infty$,

$$
\begin{aligned}
I_2 &\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{b}{4}\|w_n\|^2\|u\|^2 - \frac{\lambda}{q}C\|u\|^{q+\delta} - \frac{1}{4}\nu\mu S^{-2}\|u\|^4 + o_n(1)\\
&\geq \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 + \frac{b}{4}\frac{(a + b\|u\|^2)S^2}{\nu\mu - bS^2}\|u\|^2 - \frac{\lambda}{q}C\|u\|^{q+\delta} - \frac{1}{4}\nu\mu S^{-2}\|u\|^4 + o_n(1)\\
&= \Big(\frac{1}{2} + \frac{bS^2}{4(\nu\mu - bS^2)}\Big)a\|u\|^2 + \frac{1}{4}\Big(b + \frac{b^2S^2}{\nu\mu - bS^2} - \nu\mu S^{-2}\Big)\|u\|^4 - \frac{\lambda}{q}C\|u\|^{q+\delta} + o_n(1)\\
&= \frac{2\nu\mu - bS^2}{4(\nu\mu - bS^2)}a\|u\|^2 + \frac{\nu\mu(2bS^2 - \nu\mu)}{4S^2(\nu\mu - bS^2)}\|u\|^4 - \frac{\lambda}{q}C\|u\|^{q+\delta} + o_n(1).
\end{aligned} \tag{3.22}
$$

In view of (3.21), (3.22), and the assumption of $\nu \in (\sigma, 1]$, we obtain, as $n \to \infty$,

$$
c + o_n(1)
$$

$$
\begin{aligned}
&= I_\nu(u_n) = I_1 + I_2 \\
&\geq \frac{a^2S^2}{4(\nu\mu - bS^2)} + \frac{abS^2}{4(\nu\mu - bS^2)}\|u\|^2 + \frac{2\nu\mu - bS^2}{4(\nu\mu - bS^2)}a\|u\|^2 \\
&\quad + \frac{\nu\mu(2bS^2 - \nu\mu)}{4S^2(\nu\mu - bS^2)}\|u\|^4 - \frac{\lambda}{q}C\|u\|^{q+\delta} \\
&= \frac{a^2S^2}{4(\nu\mu - bS^2)} + \frac{\nu\mu a}{2(\nu\mu - bS^2)}\|u\|^2 + \frac{\nu\mu(2bS^2 - \nu\mu)}{4S^2(\nu\mu - bS^2)}\|u\|^4 - \frac{\lambda}{q}C\|u\|^{q+\delta} \\
&= \frac{a^2S^2}{4(\nu\mu - bS^2)} + \frac{\mu a}{2(\mu - bS^2\nu^{-1})}\|u\|^2 + \frac{\mu(2bS^2 - \nu\mu)}{4S^2(\mu - bS^2\nu^{-1})}\|u\|^4 - \frac{\lambda}{q}C\|u\|^{q+\delta} \\
&\geq \frac{a^2S^2}{4(\nu\mu - bS^2)} + \frac{\mu a}{2(\mu - bS^2)}\|u\|^2 + \frac{\mu(2bS^2 - \mu)}{4S^2(\mu - bS^2)}\|u\|^4 - \frac{\lambda}{q}C\|u\|^{q+\delta} \\
&:= \frac{a^2S^2}{4(\nu\mu - bS^2)} + h(\|u\|), \qquad (3.23)
\end{aligned}
$$

where

$$h(t) = \frac{\mu a}{2(\mu - bS^2)}t^2 + \frac{\mu(2bS^2 - \mu)}{4S^2(\mu - bS^2)}t^4 - \frac{1}{q}\lambda C t^{q+\delta}, \ for\ t > 0.$$

By a simple analysis, (C_1), (C_2), or (C_3) imply that

$$h(t) > 0, \ \text{for } t > 0. \qquad (3.24)$$

Indeed, if the parameters satisfy (C_1), then, for any $a > 0$, set

$$g_1(t) := \frac{\mu a q}{2C(\mu - bS^2)}t^{2-q-\delta} + \frac{\mu(2bS^2 - \mu)q}{4CS^2(\mu - bS^2)}t^{4-q-\delta}, \ t > 0.$$

From

$$g_1'(t) = \left(\frac{\mu q(2bS^2 - \mu)(4 - q - \delta)}{4CS^2(\mu - bS^2)}t^2 - \frac{\mu a q(q + \delta - 2)}{2C(\mu - bS^2)}\right)t^{1-q-\delta}, \ t > 0,$$

we know that $g_1(t)$ has a unique critical point

$$t_1 = \left(\frac{2aS^2(q + \delta - 2)}{(4 - q - \delta)(2bS^2 - \mu)}\right)^{\frac{1}{2}}$$

in $(0, +\infty)$. Moreover, $g_1'(t) < 0$ in $(0, t_1)$, $g_1'(t) > 0$ in $(t_1, +\infty)$ and $g_1(t)$ attain their minimum at t_1. Consequently,

$$
\begin{aligned}
g_1(t) \geq g_1(t_1) =& \frac{\mu a q}{2C(\mu - bS^2)}\left(\frac{2aS^2(q + \delta - 2)}{(4 - q - \delta)(2bS^2 - \mu)}\right)^{-\frac{q+\delta-2}{2}} \\
&+ \frac{\mu q(2bS^2 - \mu)}{4CS^2(\mu - bS^2)}\left(\frac{2aS^2(q + \delta - 2)}{(4 - q - \delta)(2bS^2 - \mu)}\right)^{\frac{4-q-\delta}{2}} > 0.
\end{aligned}
$$

Consequently, for $\lambda > 0$ small enough, we have $g_1(t) > \lambda$ for $t \in (0, +\infty)$. This implies (3.24). By applying a similar argument, one can show that (3.24) is also true if (C_2) or (C_3) hold. It then follows from (3.23) and (3.24) that $c \geq c(K) := \frac{a^2S^2}{4(\nu\mu - bS^2)}$, a contradiction. Thus $l = 0$, i.e., u_n converges strongly to u in $H_0^1(\Omega)$. The proof is complete.

With the help of the Talenti functions given in Lemma 3.1, we show that the mountain pass level of $I_\nu(u)$ around 0 is smaller than $c(K)$.

Lemma 3.4. *Let $b, \mu > 0$ satisfy $bS^2 < \mu$ and take $\sigma < 1$ such that $\frac{bS^2}{\sigma} < \mu$. Suppose that $\nu \in (\sigma, 1]$. Then there exists a $u^* > 0$ such that*

$$\sup_{t\geq 0} I_\nu(tu^*) < c(K), \tag{3.25}$$

where $c(K)$ is the positive constant given in Lemma 3.3.

Proof. Let v_ε be given in Lemma 3.1. According to the definition of I_ν, one sees that $\lim_{t\to 0} I_\nu(tv_\varepsilon) = 0$ and $\lim_{t\to+\infty} I_\nu(tv_\varepsilon) = -\infty$ uniformly for $\varepsilon \in (0, \varepsilon_1)$, where ε_1 is a sufficiently small but fixed number. Therefore, there exists $0 < t_1 < t_2 < +\infty$, independent of ε, such that

$$I_\nu(tv_\varepsilon) < c(K),\ \forall\, t \in (0, t_1] \cup [t_2, +\infty). \tag{3.26}$$

For $t \in [t_1, t_2]$, it follows from Lemma 3.1 that, as $n \to \infty$,

$$\begin{aligned}
I_\nu(tv_\varepsilon) &= \frac{a}{2}t^2\|v_\varepsilon\|^2 + \frac{b}{4}t^4\|v_\varepsilon\|^4 + \frac{2}{q^2}\lambda t^q\|v_\varepsilon\|_q^q - \frac{\lambda}{q}t^q\int_\Omega v_\varepsilon^q \ln(tv_\varepsilon)^2 dx - \frac{1}{4}\nu\mu t^4\|v_\varepsilon\|_4^4 \\
&= \frac{a}{2}t^2(S + O(\varepsilon^2)) + \frac{b}{4}t^4(S + O(\varepsilon^2))^2 + O_1(\varepsilon^{4-q}) + O(\varepsilon^q) \\
&\quad - \left[O_1\left(\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)\right) + O(\varepsilon^q \ln\varepsilon) + O(\varepsilon^{4-q})\right] - \frac{1}{4}\nu\mu t^4 \\
&= \frac{a}{2}St^2 - \frac{1}{4}(\nu\mu - bS^2)t^4 + O(\varepsilon^2) + O_1(\varepsilon^{4-q}) + O(\varepsilon^q) \\
&\quad - O_1\left(\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)\right) + O(\varepsilon^q\ln\varepsilon) + O(\varepsilon^{4-q}) \\
&:= g(t) - O_1\left(\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)\right) + O(\varepsilon^{4-q}),
\end{aligned}$$

where $g(t) := \frac{a}{2}St^2 - \frac{1}{4}(\nu\mu - bS^2)t^4$. According to the positivity of $g(t)$ for $t > 0$, suitably small, and the fact that $\lim_{t\to\infty} g(t) = -\infty$, there exists a $t_0 > 0$ such that $\max_{t>0} g(t) = g(t_0)$. So, one has $g'(t_0) = t_0[aS - (\nu\mu - bS^2)t_0^2] = 0$, that is,

$$t_0^2 = \frac{aS}{\nu\mu - bS^2}.$$

It follows from the definition of $g(t)$ that

$$\max_{t>0} g(t) = g(t_0) = \frac{a^2S^2}{4(\nu\mu - bS^2)}. \tag{3.27}$$

Then, for $t \in [t_1, t_2]$, one sees

$$I_\nu(tv_\varepsilon) \leq \max_{t\in[t_1,t_2]} g(t) - O_1\left(\varepsilon^{4-q}\ln\left(\frac{1}{\varepsilon}\right)\right) + O(\varepsilon^{4-q}) \tag{3.28}$$

$$\leq \max_{t>0} g(t) - O_1\left(\varepsilon^{4-q}\ln\Big(\frac{1}{\varepsilon}\Big)\right) + O(\varepsilon^{4-q}),\ as\ \varepsilon \to 0.$$

From

$$\lim_{\varepsilon\to 0} \frac{\varepsilon^{4-q}}{\varepsilon^{4-q}\ln\frac{1}{\varepsilon}} = 0,$$

we have

$$-O_1\left(\varepsilon^{4-q}\ln\Big(\frac{1}{\varepsilon}\Big)\right) + O(\varepsilon^{4-q}) < 0, \tag{3.29}$$

for suitably small ε. Fix such an $\varepsilon > 0$. It then follows from (3.27), (3.28), and (3.29) that

$$I_\nu(tv_\varepsilon) < c(K),\ t \in [t_1, t_2]. \tag{3.30}$$

Take $u^* \equiv v_\varepsilon$, and we obtain (3.25) by combining (3.26) with (3.30). The proof is complete.

On the basis of the above lemmas, we can now prove Theorem 2.1.

Proof of Theorem 2.1. From (3.10) and Lemma 3.4, we obtain

$$0 < \beta \leq c_\nu \leq \max_{t\in[0,1]} I_\nu(tt^* v_\varepsilon) \leq \sup_{t\geq 0} I_\nu(tv_\varepsilon) < c(K), \tag{3.31}$$

where c_ν is defined in Lemma 3.2. This, together with Lemmas 3.2 and 3.3, shows that there exists a bounded $(PS)_{c_\nu}$ sequence of the energy functional I_ν for almost every $\nu \in (\sigma, 1]$, which strongly converges to some nontrivial function in $H_0^1(\Omega)$ up to subsequences. Thus, the approximation problem (2.7) has a nontrivial weak solution for almost every $\nu \in (\sigma, 1]$. Take an increasing sequence $\nu_n \in (\sigma, 1]$ such that $\nu_n \to 1$ as $n \to \infty$, and denote the corresponding solution by u_n, which fulfills $I_\nu(u_n) = c_{\nu_n} \geq \beta$. It is obvious that either $\|u_n\| \to \infty$ as $n \to \infty$ or $\{u_n\} \subset H_0^1(\Omega)$ is bounded.

To show that problem (1.1) admits a mountain pass-type solution for the latter case, we first prove

$$c_{\nu_n} \to c_1 \text{ as } n \to \infty. \tag{3.32}$$

Assume by contradiction that

$$c_1 < \lim_{n\to\infty} c_{\nu_n},$$

where we have used the fact that c_ν is nonincreasing in ν since $B(u)$ is nonnegative for all $u \in H_0^1(\Omega)$. Let

$$\theta := \lim_{n\to\infty} c_{\nu_n} - c_1 > 0. \tag{3.33}$$

Following from the definition of c_1, there exists a $\gamma_1 \in \Gamma$ such that

$$\max_{t\in[0,1]} I(\gamma_1(t)) < c_1 + \frac{1}{4}\theta. \tag{3.34}$$

Then, in view of the fact that $I_{\nu_n}(\gamma_1(t)) = I(\gamma_1(t)) + \frac{1}{4}(1-\nu_n)\mu\|\gamma_1(t)\|_4^4$ and (3.34), we have

$$\max_{t\in[0,1]} I_{\nu_n}(\gamma_1(t)) < c_1 + \frac{1}{4}\theta + \frac{1}{4}(1-\nu_n)\mu \max_{t\in[0,1]} \|\gamma_1(t)\|_4^4. \tag{3.35}$$

Since $\mu\|\gamma_1(t)\|_4^4$ is continuous in t, we deduce $\mu \max_{t\in[0,1]} \|\gamma_1(t)\|_4^4 \le C$. From this and (3.35), one has

$$\lim_{n\to\infty} \max_{t\in[0,1]} I_{\nu_n}(\gamma_1(t)) \le c_1 + \frac{1}{2}\theta.$$

On the other hand, by virtue of the definition of c_{ν_n}, we have

$$\lim_{n\to\infty} \max_{t\in[0,1]} I_{\nu_n}(\gamma_1(t)) \ge \lim_{n\to\infty} c_{\nu_n}.$$

By using the above two inequalities, we obtain

$$\lim_{n\to\infty} c_{\nu_n} \le c_1 + \frac{1}{2}\theta,$$

a contradiction with (3.33).

Next, we claim that $\{u_n\}$ is a $(PS)_{c_1}$ sequence of $I(u)$. Indeed, by (3.32), one has

$$\begin{aligned}
& I(u_n) \\
&= \frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 + \frac{2}{q^2}\lambda\|u_n\|_q^q - \frac{\lambda}{q}\int_\Omega u_n^q \ln u_n^2 \mathrm{d}x - \frac{1}{4}\mu\|u_n\|_4^4 \\
&= \frac{a}{2}\|u_n\|^2 + \frac{b}{4}\|u_n\|^4 + \frac{2}{q^2}\lambda\|u_n\|_q^q - \frac{\lambda}{q}\int_\Omega u_n^q \ln u_n^2 \mathrm{d}x - \frac{1}{4}\nu_n\mu\|u_n\|_4^4 + \frac{1}{4}(\nu_n-1)\mu\|u_n\|_4^4 \\
&= I_{\nu_n}(u_n) + \frac{1}{4}(\nu_n-1)\mu\|u_n\|_4^4 \\
&= c_1 + o_n(1), \ as\ n\to\infty.
\end{aligned}$$

Similarly, for any $\varphi \in H_0^1(\Omega)$,

$$\begin{aligned}
& \langle I'(u_n), \varphi\rangle \\
&= a\int_\Omega \nabla u_n \nabla\varphi \mathrm{d}x + b\|u_n\|^2 \int_\Omega \nabla u_n\nabla\varphi \mathrm{d}x - \lambda\int_\Omega u_n^{q-1}\varphi \ln u_n^2 \mathrm{d}x - \mu\int_\Omega u_n^3\varphi \mathrm{d}x \\
&= a\int_\Omega \nabla u_n \nabla\varphi \mathrm{d}x + b\|u_n\|^2 \int_\Omega \nabla u_n\nabla\varphi \mathrm{d}x - \lambda\int_\Omega u_n^{q-1}\varphi \ln u_n^2 \mathrm{d}x \\
&\quad - \nu_n\mu\int_\Omega u_n^3\varphi \mathrm{d}x + (\nu_n-1)\mu\int_\Omega u_n^3\varphi \mathrm{d}x \\
&= \langle I'_{\nu_n}(u_n), \varphi\rangle + (\nu_n-1)\mu\int_\Omega u_n^3\varphi \mathrm{d}x \\
&= o_n(1), \ as\ n\to\infty.
\end{aligned}$$

Hence, $\{u_n\}$ is a bounded $(PS)_{c_1}$ sequence for I. It then follows from (3.31) and Lemma 3.3 that there exists a $u \in H_0^1(\Omega)$ such that $u_n \to u$ in $H_0^1(\Omega)$ as $n\to\infty$, and u is a mountain pass-type solution to problem (1.1).

Finally, we prove the last part of Theorem 2.1. Take a sequence $\{\nu_n\} \subset (\sigma, 1]$ such that $\nu_n \to 1$ as $n \to \infty$ and denote the corresponding solution to problem (2.7) by $\{u_n\}$. We first show that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Assume by contradiction that $\|u_n\| \to \infty$ as $n \to \infty$. Set $\widetilde{w}_n := \frac{u_n}{\|u_n\|} \geq 0$. Then $\|\widetilde{w}_n\| = 1$, and there is a subsequence of $\{\widetilde{w}_n\}$ (which we still denote by $\{\widetilde{w}_n\}$) such that $\widetilde{w}_n \rightharpoonup \widetilde{w}_0$ in $H_0^1(\Omega)$ as $n \to \infty$. Notice that, for all $\varphi \in H_0^1(\Omega)$, we have

$$
\begin{aligned}
0 &= \langle I'_{\nu_n}(u_n), \varphi\rangle \\
&= a\int_\Omega \nabla u_n \nabla\varphi \mathrm{d}x + b\|u_n\|^2 \int_\Omega \nabla u_n\nabla\varphi\mathrm{d}x - \lambda\int_\Omega u_n^{q-1}\varphi\ln u_n^2\mathrm{d}x - \nu_n\mu\int_\Omega u_n^3\varphi\mathrm{d}x \\
&= \|u_n\|^3\Bigg[\Big(\frac{a}{\|u_n\|^2} + b\Big)\int_\Omega \nabla\widetilde{w}_n\nabla\varphi\mathrm{d}x - \lambda\|u_n\|^{q-4}\int_\Omega \widetilde{w}_n^{q-1}\varphi\ln\widetilde{w}_n^2\mathrm{d}x \\
&\quad - \lambda\|u_n\|^{q-4}\ln\|u_n\|^2\int_\Omega \widetilde{w}_n^{q-1}\varphi\mathrm{d}x - \nu_n\mu\int_\Omega\widetilde{w}_n^3\varphi\mathrm{d}x\Bigg], \text{ as } n \to \infty.
\end{aligned}
$$

Letting $n \to \infty$ in the above equality and recalling the assumptions that $\nu_n \to 1$ and $\|u_n\| \to \infty$ as $n \to \infty$, one has

$$
b\int_\Omega \nabla\widetilde{w}_0\nabla\varphi\mathrm{d}x = \mu\int_\Omega \widetilde{w}_0^3\varphi\mathrm{d}x, \quad \forall\, \varphi \in H_0^1(\Omega), \tag{3.36}
$$

where we have used the fact that $\lim\limits_{x\to\infty} x^{q-4}\ln x = 0$ since $q < 4$. Since Ω is strictly star-shaped, we know from Pohozaev's identity that $\widetilde{w}_0 = 0$ (see [31]). Then, applying a similar argument to that of the proof of Theorem 1.6 in [16], one obtains $\mu = bS^2$, a contradiction with $bS^2 < \mu < 2bS^2$. Therefore, $\{u_n\}$ is bounded, and problem (1.1) has a nontrivial weak solution. The proof is complete.

Author contributions

Qi Li, Yuzhu Han and Bin Guo: Methodology; Qi Li: Writing-original draft; Yuzhu Han and Bin Guo: Writing-review & editing.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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