Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 16(3): 554–577.
DOI: 10.3934/cam.2024026
Received: 20 February 2024
Revised: 29 July 2024
Accepted: 29 July 2024
Published: 06 August 2024

Research article

On an anisotropic $\overrightarrow{p}(\cdot)$-Laplace equation with variable singular and sublinear nonlinearities

Mustafa Avci*

Faculty of Science and Technology, Applied Mathematics, Athabasca University, AB, Canada

* **Correspondence:** Email: mavci@athabascau.ca, avcixmustafa@gmail.com

Abstract: In the present paper, we study an anisotropic $\overrightarrow{p}(\cdot)$-Laplace equation with combined effects of variable singular and sublinear nonlinearities. Using the Ekeland's variational principle and a constrained minimization, we show the existence of a positive solution for the case where the variable singularity $\beta(x)$ assumes its values in the interval $(1, \infty)$.

Keywords: anisotropic singular $\overrightarrow{p}(\cdot)$-Laplacian; variable strong singularity; anisotropic variable Sobolev space; Ekeland's variational principle
Mathematics Subject Classification: 35A15, 35A21, 35J75, 58E30

1. Introduction

In this article, we study the following anisotropic singular $\overrightarrow{p}(\cdot)$-Laplace equation

$$\begin{cases} -\sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u\right) &= f(x)u^{-\beta(x)} + g(x)u^{q(x)} \text{ in } \Omega, \\ u &> 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$; $f \in L^1(\Omega)$ is a positive function; $g \in L^\infty(\Omega)$ is a nonnegative function; $\beta \in C(\overline{\Omega})$ such that $1 < \beta(x) < \infty$ for any $x \in \overline{\Omega}$; $q \in C(\overline{\Omega})$ such that $0 < q(x) < 1$ for any $x \in \overline{\Omega}$; $p_i \in C(\overline{\Omega})$ such that $2 \leq p_i(x) < N$ for any $x \in \overline{\Omega}$, $i \in \{1, ..., N\}$.
The differential operator

$$\sum_{i=1}^{N} \partial_{x_i}\left(|\partial_{x_i} u|^{p_i(x)-2}\partial_{x_i} u\right),$$

that appears in problem (1.1) is an anisotropic variable exponent $\overrightarrow{p}(\cdot)$-Laplace operator, which represents an extension of the $p(\cdot)$-Laplace operator

$$\sum_{i=1}^{N} \partial_{x_i} \left(|\partial_{x_i} u|^{p(x)-2} \partial_{x_i} u \right),$$

obtained in the case for each $i \in \{1, ..., N\}$, $p_i(\cdot) = p(\cdot)$.

In the variable exponent case, $p(\cdot)$, the integrability condition changes with each point in the domain. This makes variable exponent Sobolev spaces very useful in modeling materials with spatially varying properties and in studying partial differential equations with non-standard growth conditions [1–8].

Anisotropy, on the other hand, adds another layer of complexity, providing a robust mathematical framework for modeling and solving problems that involve complex materials and phenomena exhibiting non-uniform and direction-dependent properties. This is represented mathematically by having different exponents for different partial derivatives. We refer to the papers [9–21] and references for further reading.

The progress in researching anisotropic singular problems with $\overrightarrow{p}(\cdot)$-growth, however, has been relatively slow. There are only a limited number of studies available on this topic in academic literature. We could only refer to the papers [22–24] that were published recently. In [22], the author studied an anisotropic singular problems with constant case $p(\cdot) = p$ but with a variable singularity, where existence and regularity of positive solutions was obtained via the approximation methods. In [23], the author obtained the existence and regularity results of positive solutions by using the regularity theory and approximation methods. In [24], the authors showed the existence of positive solutions using the regularity theory and maximum principle. However, none of these papers studied combined effects of variable singular and sublinear nonlinearities.

We would also like to mention that the singular problems of the type

$$\begin{cases} -\Delta u &= f(x)u^{-\beta} \text{ in } \Omega, \\ u &> 0 \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{cases} \tag{1.2}$$

have been intensively studied because of their wide applications to physical models in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogenous catalysts, glacial advance, etc. (see, e.g., [25–30]).

These studies, however, have mainly focused on the case $0 < \beta < 1$, i.e., the weak singularity (see, e.g. [31–36]), and in this case, the corresponding energy functional is continuous.
When $\beta > 1$ (the strong singularity), on the other hand, the situation changes dramatically, and numerous challenges emerge in the analysis of differential equations of the type (1.2), where the primary challenge encountered is due to the lack of integrability of $u^{-\beta}$ for $u \in H_0^1(\Omega)$ [37–41].

To overcome these challenges, as an alternative approach, the so-called "compatibility relation" between $f(x)$ and β has been introduced in the recent studies [37, 40, 42]. This method, used along with a constrained minimization and the Ekeland's variational principle [43], suggests a practical approach to obtain solutions to the problems of the type (1.2). In the present paper, we generalize these results to nonstandard $p(\cdot)$-growth.

The paper is organized as follows. In Section 2, we provide some fundamental information for the theory of variable Sobolev spaces since it is our work space. In Section 3, first we obtain the auxiliary results. Then, we present our main result and obtain a positive solution to problem (1.1). In Section 4, we provide an example to illustrate our results in a concrete way.

2. Preliminaries

We start with some basic concepts of variable Lebesgue-Sobolev spaces. For more details, and the proof of the following propositions, we refer the reader to [1, 2, 44, 45].

$$C_+\left(\overline{\Omega}\right) = \left\{p; p \in C\left(\overline{\Omega}\right),\ \inf p(x) > 1,\ \text{for all } x \in \overline{\Omega}\right\}.$$

For $p \in C_+(\overline{\Omega})$ denote

$$p^- := \inf_{x\in\overline{\Omega}} p(x) \leq p(x) \leq p^+ := \sup_{x\in\overline{\Omega}} p(x) < \infty.$$

For any $p \in C_+\left(\overline{\Omega}\right)$, we define the variable exponent Lebesgue space by

$$L^{p(\cdot)}(\Omega) = \left\{u \mid u : \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega |u(x)|^{p(x)} dx < \infty\right\},$$

then, $L^{p(\cdot)}(\Omega)$ endowed with the norm

$$|u|_{p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

becomes a Banach space.

Proposition 2.1. *For any $u \in L^{p(\cdot)}(\Omega)$ and $v \in L^{p'(\cdot)}(\Omega)$, we have*

$$\int_\Omega |uv| dx \leq C(p^-, (p^-)')|u|_{p(\cdot)}|v|_{p'(\cdot)}$$

where $L^{p'(x)}(\Omega)$ is the conjugate space of $L^{p(\cdot)}(\Omega)$ such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

The convex functional $\Lambda : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\Lambda(u) = \int_\Omega |u(x)|^{p(x)} dx,$$

is called modular on $L^{p(\cdot)}(\Omega)$.

Proposition 2.2. *If $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, ...)$, we have*

(i) $|u|_{p(\cdot)} < 1 (= 1; > 1) \Leftrightarrow \Lambda(u) < 1 (= 1; > 1)$;
(ii) $|u|_{p(\cdot)} > 1 \implies |u|^{p^-}_{p(\cdot)} \leq \Lambda(u) \leq |u|^{p^+}_{p(\cdot)}$;

(iii) $|u|_{p(\cdot)} \le 1 \implies |u|_{p(\cdot)}^{p^+} \le \Lambda(u) \le |u|_{p(\cdot)}^{p^-}$;
(iv) $\lim_{n\to\infty} |u_n|_{p(\cdot)} = 0 \Leftrightarrow \lim_{n\to\infty} \Lambda(u_n) = 0$; $\lim_{n\to\infty} |u_n|_{p(\cdot)} = \infty \Leftrightarrow \lim_{n\to\infty} \Lambda(u_n) = \infty$.

Proposition 2.3. *If $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, ...)$, then the following statements are equivalent:*

(i) $\lim_{n\to\infty} |u_n - u|_{p(\cdot)} = 0$;
(ii) $\lim_{n\to\infty} \Lambda(u_n - u) = 0$;
(iii) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \Lambda(u_n) = \Lambda(u)$.

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\},$$

with the norm

$$\|u\|_{1,p(\cdot)} = |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)},$$

or equivalently

$$\|u\|_{1,p(\cdot)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} + \left|\frac{u(x)}{\lambda}\right|^{p(x)}\right) dx, \le 1\right\}$$

for all $u \in W^{1,p(\cdot)}(\Omega)$.

As shown in [46], the smooth functions are in general not dense in $W^{1,p(\cdot)}(\Omega)$, but if the variable exponent $p \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous, that is

$$|p(x) - p(y)| \le -\frac{M}{\log(|x-y|)}, \text{ for all } x, y \in \Omega \text{ such that } \quad |x-y| \le \frac{1}{2}, \tag{2.1}$$

then the smooth functions are dense in $W^{1,p(\cdot)}(\Omega)$ and so the Sobolev space with zero boundary values, denoted by $W_0^{1,p(\cdot)}(\Omega)$, as the closure of $C_0^\infty(\Omega)$ does make sense. Therefore, the space $W_0^{1,p(\cdot)}(\Omega)$ can be defined as $\overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,p(\cdot)}} = W_0^{1,p(\cdot)}(\Omega)$, and hence, $u \in W_0^{1,p(\cdot)}(\Omega)$ iff there exists a sequence (u_n) of $C_0^\infty(\Omega)$ such that $\|u_n - u\|_{1,p(\cdot)} \to 0$.

As a consequence of Poincaré inequality, $\|u\|_{1,p(\cdot)}$ and $|\nabla u|_{p(\cdot)}$ are equivalent norms on $W_0^{1,p(\cdot)}(\Omega)$ when $p \in C_+(\overline{\Omega})$ is logarithmic Hölder continuous. Therefore, for any $u \in W_0^{1,p(\cdot)}(\Omega)$, we can define an equivalent norm $\|u\|$ such that

$$\|u\| = |\nabla u|_{p(\cdot)}.$$

Proposition 2.4. *If $1 < p^- \le p^+ < \infty$, then the spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.5. *Let $q \in C(\overline{\Omega})$. If $1 \le q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$ is compact and continuous, where*

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, & \textit{if } p(x) < N, \\ +\infty, & \textit{if } p(x) \ge N. \end{cases}$$

Finally, we introduce the anisotropic variable exponent Sobolev spaces.
Let us denote by $\vec{p}:\overline{\Omega}\to\mathbb{R}^N$ the vectorial function $\vec{p}(\cdot)=(p_1(\cdot),...,p_N(\cdot))$ with $p_i\in C_+(\overline{\Omega})$, $i\in\{1,...,N\}$. We will use the following notations.
Define $\overrightarrow{P}_+,\overrightarrow{P}_-\in\mathbb{R}^N$ as

$$\overrightarrow{P}_+=(p_1^+,...,p_N^+),\quad \overrightarrow{P}_-=(p_1^-,...,p_N^-),$$

and $P_+^+,P_-^+,P_-^-\in\mathbb{R}^+$ as

$$P_+^+=\max\{p_1^+,...,p_N^+\},\ P_-^+=\max\{p_1^-,...,p_N^-\},\ P_-^-=\min\{p_1^-,...,p_N^-\},$$

Below, we use the definitions of the anisotropic variable exponent Sobolev spaces as given in [12] and assume that the domain $\Omega\subset\mathbb{R}^N$ satisfies all the necessary assumptions given in there.

The anisotropic variable exponent Sobolev space is defined by

$$W^{1,\vec{p}(\cdot)}(\Omega)=\{u\in L^{P_+^+}(\Omega):\partial_{x_i}u\in L^{p_i(\cdot)}(\Omega),\quad i\in\{1,...,N\}\},$$

which is associated with the norm

$$\|u\|_{W^{1,\vec{p}(\cdot)}(\Omega)}=|u|_{P_+^+(\cdot)}+\sum_{i=1}^{N}|\partial_{x_i}u|_{p_i(\cdot)}.$$

$W^{1,\vec{p}(\cdot)}(\Omega)$ is a reflexive Banach space under this norm.
The subspace $W_0^{1,\vec{p}(\cdot)}(\Omega)\subset W^{1,\vec{p}(\cdot)}(\Omega)$ consists of the functions that are vanishing on the boundary, that is,

$$W_0^{1,\vec{p}(\cdot)}(\Omega)=\{u\in W^{1,\vec{p}(\cdot)}(\Omega):u=0\ \text{on}\ \partial\Omega\},$$

We can define the following equivalent norm on $W_0^{1,\vec{p}(\cdot)}(\Omega)$

$$\|u\|_{\vec{p}(\cdot)}=\sum_{i=1}^{N}|\partial_{x_i}u|_{p_i(\cdot)}.$$

since the smooth functions are dense in $W_0^{1,\vec{p}(\cdot)}(\Omega)$, as the variable exponent $p_i\in C_+(\overline{\Omega})$, $i\in\{1,...,N\}$ is logarithmic Hölder continuous.

The space $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is also a reflexive Banach space (for the theory of the anisotropic Sobolev spaces see, e.g., the monographs [2,47,48] and the papers [12,15]).

Throughout this article, we assume that

$$\sum_{i=1}^{N}\frac{1}{p_i^-}>1, \tag{2.2}$$

and define $P_-^*\in\mathbb{R}^+$ and $P_{-,\infty}\in\mathbb{R}^+$ by

$$P_-^*=\frac{N}{\sum_{i=1}^{N}\frac{1}{p_i^-}-1},\quad P_{-,\infty}=\max\{P_-^+,P_-^*\}.$$

Proposition 2.6. [*[15], Theorem 1*] *Suppose that* $\Omega \subset \mathbb{R}^N (N \geq 3)$ *is a bounded domain with smooth boundary and relation (2.2) is fulfilled. For any* $q \in C\left(\overline{\Omega}\right)$ *verifying*

$$1 < q(x) < P_{-,\infty} \text{ for all } x \in \overline{\Omega},$$

the embedding

$$W_0^{1,\vec{p}(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega),$$

is continuous and compact.

3. The main results

We define the singular energy functional $\mathcal{J} : W_0^{1,\vec{p}(\cdot)}(\Omega) \to \mathbb{R}$ corresponding to equation (1.1) by

$$\mathcal{J}(u) = \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u|^{p_i(x)}}{p_i(x)} dx - \int_\Omega \frac{g(x)|u|^{q(x)+1}}{q(x)+1} dx + \int_\Omega \frac{f(x)|u|^{1-\beta(x)}}{\beta(x)-1} dx.$$

Definition 3.1. A function u is called a weak solution to problem (1.1) if $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $u > 0$ in Ω and

$$\int_\Omega \left[\sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)-2} \partial_{x_i} u \cdot \partial_{x_i} \varphi - [g(x)u^{q(x)} + f(x)u^{-\beta(x)}]\varphi \right] dx = 0, \tag{3.1}$$

for all $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Definition 3.2. Due to the singularity of $\mathcal{J}$ on $W_0^{1,\vec{p}(\cdot)}(\Omega)$, we apply a constrained minimization for problem (1.1). As such, we introduce the following constrains:

$$\mathcal{N}_1 = \left\{ u \in W_0^{1,\vec{p}(\cdot)}(\Omega) : \int_\Omega \left[\sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} - g(x)|u|^{q(x)+1} - f(x)|u|^{1-\beta(x)} \right] dx \geq 0 \right\},$$

and

$$\mathcal{N}_2 = \left\{ u \in W_0^{1,\vec{p}(\cdot)}(\Omega) : \int_\Omega \left[\sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)} - g(x)|u|^{q(x)+1} - f(x)|u|^{1-\beta(x)} \right] dx = 0 \right\}.$$

Remark 1. $\mathcal{N}_2$ can be considered as a Nehari manifold, even though in general it may not be a manifold. Therefore, if we set

$$c_0 := \inf_{u \in \mathcal{N}_2} \mathcal{J}(u),$$

then one might expect that c_0 is attained at some $u \in \mathcal{N}_2$ (i.e., $\mathcal{N}_2 \neq \emptyset$) and that u is a critical point of $\mathcal{J}$.

Throughout the paper, we assume that the following conditions hold:

(A_1) $\beta : \overline{\Omega} \to (1, \infty)$ is a continuous function such that $1 < \beta^- \leq \beta(x) \leq \beta^+ < \infty$.
(A_2) $q : \overline{\Omega} \to (0, 1)$ is a continuous function such that $0 < q^- \leq q(x) \leq q^+ < 1$ and $q^+ + 1 \leq \beta^-$.

(A_3) $2 \le P_-^- \le P_+^+ < P_-^*$ for almost all $x \in \overline{\Omega}$.
(A_4) $f \in L^1(\Omega)$ is a positive function, that is, $f(x) > 0$ a.e. in Ω.
(A_5) $g \in L^\infty(\Omega)$ is a nonnegative function.

Lemma 3.3. *For any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ satisfying $\int_\Omega f(x)|u|^{1-\beta(x)}dx < \infty$, the functional $\mathcal{J}$ is well-defined and coercive on $W_0^{1,\vec{p}(\cdot)}(\Omega)$.*

Proof. Denote by $\mathcal{I}_1, \mathcal{I}_2$ the indices sets $\mathcal{I}_1 = \{i \in \{1,2,...,N\} : |\partial_{x_i}u|_{p_i(\cdot)} \le 1\}$ and $\mathcal{I}_2 = \{i \in \{1,2,...,N\} : |\partial_{x_i}u|_{p_i(\cdot)} > 1\}$. Using Proposition 2.2, it follows

$$
\begin{aligned}
|\mathcal{J}(u)| &\le \frac{1}{P_-^-}\sum_{i=1}^N \int_\Omega |\partial_{x_i}u|^{p_i(x)}dx - \frac{|g|_\infty}{q^+ + 1}\int_\Omega |u|^{q(x)+1}dx + \frac{1}{\beta^- - 1}\int_\Omega f(x)|u|^{1-\beta(x)}dx \\
&\le \frac{1}{P_-^-}\left(\sum_{i\in\mathcal{I}_1} |\partial_{x_i}u|_{p_i(\cdot)}^{P_-^-} + \sum_{i\in\mathcal{I}_2} |\partial_{x_i}u|_{p_i(\cdot)}^{P_+^+}\right) - \frac{|g|_\infty}{q^+ + 1}\min\{|u|_{q(x)+1}^{q^+ +1}, |u|_{q(x)+1}^{q^- +1}\} \\
&+ \frac{1}{\beta^- - 1}\int_\Omega f(x)|u|^{1-\beta(x)}dx \\
&\le \frac{1}{P_-^-}\left(\sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}^{P_+^+} + \sum_{i\in\mathcal{I}_1} |\partial_{x_i}u|_{p_i(\cdot)}^{P_-^-}\right) - \frac{|g|_\infty}{q^+ + 1}\min\{|u|_{q(x)+1}^{q^+ +1}, |u|_{q(x)+1}^{q^- +1}\} \\
&+ \frac{1}{\beta^- - 1}\int_\Omega f(x)|u|^{1-\beta(x)}dx \\
&\le \frac{1}{P_-^-}\left(\sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}^{P_+^+} + N\right) - \frac{|g|_\infty}{q^+ + 1}\min\{|u|_{q(x)+1}^{q^+ +1}, |u|_{q(x)+1}^{q^- +1}\} \\
&+ \frac{1}{\beta^- - 1}\int_\Omega f(x)|u|^{1-\beta(x)}dx
\end{aligned} \tag{3.2}
$$

which shows that $\mathcal{J}$ is well-defined on $W_0^{1,\vec{p}(\cdot)}(\Omega)$.
Applying similar steps and using the generalized mean inequality for $\sum_{i=1}^N |\partial_{x_i}u|_{p_i(\cdot)}^{P_-^-}$ gives

$$
\begin{aligned}
\mathcal{J}(u) &\ge \frac{1}{P_+^+}\sum_{i=1}^N \int_\Omega |\partial_{x_i}u|^{p_i(x)}dx - \frac{|g|_\infty}{q^- + 1}\int_\Omega |u|^{q(x)+1}dx + \frac{1}{\beta^+ - 1}\int_\Omega f(x)|u|^{1-\beta(x)}dx \\
&\ge \frac{1}{P_+^+}\left(\sum_{i\in\mathcal{I}_1} |\partial_{x_i}u|_{p_i(\cdot)}^{P_+^+} + \sum_{i\in\mathcal{I}_2} |\partial_{x_i}u|_{p_i(\cdot)}^{P_-^-}\right) - \frac{|g|_\infty}{q^- + 1}\int_\Omega |u|^{q(x)+1}dx \\
&+ \frac{1}{\beta^+ - 1}\int_\Omega f(x)|u|^{1-\beta(x)}dx \\
&\ge \frac{N}{P_+^+}\left(\frac{\|u\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^-}} - 1\right) - \frac{|g|_\infty}{q^- + 1}\|u\|_{\vec{p}(\cdot)}^{q^+ +1} + \frac{1}{\beta^+ - 1}\int_\Omega f(x)|u|^{1-\beta(x)}dx
\end{aligned} \tag{3.3}
$$

That is, $\mathcal{J}$ is coercive (i.e., $\mathcal{J}(u) \to \infty$ as $\|u\|_{\vec{p}(\cdot)} \to \infty$), and bounded below on $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Next, we provide a-priori estimate.

Lemma 3.4. *Assume that* $(u_n) \subset \mathcal{N}_1$ *is a nonnegative minimizing sequence for the minimization problem* $\lim_{n\to\infty} \mathcal{J}(u_n) = \inf_{\mathcal{N}_1} \mathcal{J}$. *Then, there are positive real numbers* δ_1, δ_2 *such that*

$$\delta_1 \leq \|u_n\|_{\vec{p}(\cdot)} \leq \delta_2$$

Proof. We assume by contradiction that there exists a subsequence (u_n) (not relabelled) such that $u_n \to 0$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Thus, we can assume that $\|u_n\|_{\vec{p}(\cdot)} < 1$ for n large enough, and therefore, $|\partial_{x_i} u_n|_{L^{p_i(\cdot)}} < 1$. Then, using Proposition 2.2, we have

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx \leq \sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(\cdot)}^{p_i^-} \leq \sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-} \tag{3.4}$$

We recall the following elementary inequality: for all $r, s > 0$ and $m > 0$ it holds

$$r^m + s^m \leq K(r+s)^m \tag{3.5}$$

where $K := \max\{1, 2^{1-m}\}$. If we let $r = |\partial_{x_1} u_n|_{L^{p_1(\cdot)}}^{P_-^-}$, $s = |\partial_{x_2} u_n|_{L^{p_2(\cdot)}}^{P_-^-}$ and $m = P_-^-$ in (3.5), it reads

$$|\partial_{x_1} u_n|_{L^{p_1(\cdot)}}^{P_-^-} + |\partial_{x_2} u_n|_{L^{p_2(\cdot)}}^{P_-^-} \leq K(|\partial_{x_1} u_n|_{L^{p_1(\cdot)}} + |\partial_{x_2} u_n|_{L^{p_2(\cdot)}})^{P_-^-} \tag{3.6}$$

where $K = \max\{1, 2^{1-P_-^-}\} = 1$. Applying this argument to the following terms in the sum $\sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-}$ consecutively leads to

$$\begin{aligned} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx &\leq \sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(\cdot)}^{p_i^-} \leq \sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(\cdot)}^{P_-^-} \\ &\leq \left(\sum_{i=1}^N |\partial_{x_i} u_n|_{p_i(\cdot)} \right)^{P_-^-} \leq \|u_n\|_{\vec{p}(\cdot)}^{P_-^-} \end{aligned} \tag{3.7}$$

Now, using (3.7) and the reversed Hölder's inequality, we have

$$\left(\int_\Omega f(x)^{1/\beta^-} dx \right)^{\beta^-} \left(\int_\Omega |u_n| dx \right)^{1-\beta^-} \leq \int_\Omega f(x)|u_n|^{1-\beta^-} dx \leq \int_\Omega f(x)|u_n|^{1-\beta(x)} dx \tag{3.8}$$

By the assumption, $(u_n) \subset \mathcal{N}_1$. Thus, using (3.8) and Proposition 2.2 leads to

$$\begin{aligned} \left(\int_\Omega f(x)^{1/\beta^-} dx \right)^{\beta^-} \left(\int_\Omega |u_n| dx \right)^{1-\beta^-} &\leq \int_\Omega f(x)|u_n|^{1-\beta^-} dx \\ &\leq \|u_n\|_{\vec{p}(\cdot)}^{P_-^-} - \frac{|g|_\infty}{q^- + 1} \|u_n\|^{q^+ + 1} \to 0 \end{aligned} \tag{3.9}$$

Considering the assumption (A_2), this can only happen if $\int_\Omega |u_n| dx \to \infty$, which is not possible. Therefore, there exists a positive real number δ_1 such that $\|u_n\|_{\vec{p}(\cdot)} \geq \delta_1$.
Now, let's assume, on the contrary, that $\|u_n\|_{\vec{p}(\cdot)} > 1$ for any n. We know, by the coerciveness of $\mathcal{J}$, that

the infimum of $\mathcal{J}$ is attained, that is, $\infty < m := \inf_{u\in W_0^{1,\vec{p}(\cdot)}(\Omega)} \mathcal{J}(u)$. Moreover, due to the assumption $\lim_{n\to\infty} \mathcal{J}(u_n) = \inf_{\mathcal{N}_1} \mathcal{J}$, $(\mathcal{J}(u_n))$ is bounded. Then, applying the same steps as in (3.3), it follows

$$
\begin{aligned}
&C\|u_n\|_{\vec{p}(\cdot)} + \mathcal{J}(u_n)\\
&\geq \frac{N}{P_+^+}\left(\frac{\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^-}} - 1\right) - \frac{|g|_\infty}{q^- + 1}\|u_n\|_{\vec{p}(\cdot)}^{q^+ + 1} + \frac{1}{\beta^+ - 1}\int_\Omega f(x)|u_n|^{1-\beta(x)}dx
\end{aligned}
$$

for some constant $C > 0$. If we drop the nonnegative terms, we obtain

$$
C\|u_n\|_{\vec{p}(\cdot)} + \mathcal{J}(u_n) \geq \frac{1}{P_+^+}\left(\frac{\|u_n\|_{\vec{p}(\cdot)}^{P_-^-}}{N^{P_-^- - 1}} - N\right) - \frac{|g|_\infty}{q^- + 1}\|u\|_{\vec{p}(\cdot)}^{q^+ + 1}
$$

Dividing the both sides of the above inequality by $\|u_n\|_{\vec{p}(\cdot)}^{q^+ + 1}$ and passing to the limit as $n \to \infty$ leads to a contradiction since we have $q^- + 1 < P_-^-$. Therefore, there exists a positive real number δ_2 such that $\|u_n\|_{\vec{p}(\cdot)} \leq \delta_2$.

Lemma 3.5. *$\mathcal{N}_1$ is closed in $W_0^{1,\vec{p}(\cdot)}(\Omega)$.*

Proof. Assume that $(u_n) \subset \mathcal{N}_1$ such that $u_n \to \hat{u}$ (*strongly*) in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Thus, $u_n(x) \to \hat{u}(x)$ a.e. in Ω, and $\partial_{x_i} u_n \to \partial_{x_i}\hat{u}$ in $L^{p_i(\cdot)}(\Omega)$ for $i = 1, 2, ..., N$. Then, using Fatou's lemma, it reads

$$
\begin{aligned}
&\int_\Omega \left[\sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} - g(x)|u_n|^{q(x)+1} - f(x)|u_n|^{1-\beta(x)}\right] dx \geq 0\\
&\liminf_{n\to\infty}\left[\int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx\right] - \int_\Omega g(x)|\hat{u}|^{q(x)+1}dx \geq \liminf_{n\to\infty}\left[\int_\Omega f(x)|u_n|^{1-\beta(x)}dx\right]
\end{aligned}
$$

and hence,

$$
\int_\Omega \left[\sum_{i=1}^N |\partial_{x_i}\hat{u}|^{p_i(x)} - g(x)|\hat{u}|^{q(x)+1} - f(x)|\hat{u}|^{1-\beta(x)}\right] dx \quad \geq \quad 0
$$

which means $\hat{u} \in \mathcal{N}_1$. $\mathcal{N}_1$ is closed in $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Lemma 3.6. *For any $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ satisfying $\int_\Omega f(x)|u|^{1-\beta(x)}dx < \infty$, there exists a unique continuous scaling function $u \in W_0^{1,\vec{p}(\cdot)}(\Omega) \to (0,\infty) : u \longmapsto t(u)$ such that $t(u)u \in \mathcal{N}_2$, and $t(u)u$ is the minimizer of the functional $\mathcal{J}$ along the ray $\{tu : t > 0\}$, that is, $\inf_{t>0} \mathcal{J}(tu) = \mathcal{J}(t(u)u)$.*

Proof. Fix $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $\int_\Omega f(x)|u|^{1-\beta(x)}dx < \infty$. For any $t > 0$, the scaled functional, $\mathcal{J}(tu)$, determines a curve that can be characterized by

$$
\Phi(t) := \mathcal{J}(tu), \quad t \in [0,\infty). \tag{3.10}
$$

Then, for a $t \in [0, \infty)$, $tu \in \mathcal{N}_2$ if and only if

$$\Phi'(t) = \frac{d}{dt}\Phi(t)\Big|_{t=t(u)} = 0. \tag{3.11}$$

First, we show that $\Phi(t)$ attains its minimum on $[0, \infty)$ at some point $t = t(u)$.
Considering the fact $0 < \int_\Omega f(x)|u|^{1-\beta(x)}dx < \infty$, we will examine two cases for t.
For $0 < t < 1$:

$$\Phi(t) = \mathcal{J}(tu) \geq \frac{t^{P_+^+}}{P_+^+}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u|^{p_i(x)}dx - \frac{t^{q^-+1}}{q^-+1}\int_\Omega g(x)|u|^{q(x)+1}dx$$
$$+ \frac{t^{1-\beta^-}}{\beta^+-1}\int_\Omega f(x)|u|^{1-\beta(x)}dx := \Psi_0(t)$$

Then, $\Psi_0 : (0, 1) \to \mathbb{R}$ is continuous. Taking the derivative of Ψ_0 gives

$$\Psi_0'(t) = t^{P_+^+-1}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u|^{p_i(x)}dx - t^{q^-}\int_\Omega g(x)|u|^{q(x)+1}dx$$
$$+\left(\frac{1-\beta^-}{\beta^+-1}\right)t^{-\beta^-}\int_\Omega f(x)|u|^{1-\beta(x)}dx \tag{3.12}$$

It is easy to see from (3.12) that $\Psi_0'(t) < 0$ when $t > 0$ is small enough. Therefore, $\Psi_0(t)$ is decreasing when $t > 0$ is small enough. In the same way,

$$\Phi(t) = \mathcal{J}(tu) \leq \frac{t^{P_-^-}}{P_-^-}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u|^{p_i(x)}dx - \frac{t^{q^++1}}{q^++1}\int_\Omega g(x)|u|^{q(x)+1}dx$$
$$+ \frac{t^{1-\beta^+}}{\beta^--1}\int_\Omega f(x)|u|^{1-\beta(x)}dx := \Psi_1(t)$$

Then, $\Psi_1 : (0, 1) \to \mathbb{R}$ is continuous. Taking the derivative of Ψ_1 gives

$$\Psi_1'(t) = t^{P_-^--1}\sum_{i=1}^{N}\int_\Omega |\partial_{x_i}u|^{p_i(x)}dx - t^{q^+}\int_\Omega g(x)|u|^{q(x)+1}dx$$
$$+\left(\frac{1-\beta^+}{\beta^+-1}\right)t^{-\beta^+}\int_\Omega f(x)|u|^{1-\beta(x)}dx \tag{3.13}$$

But (3.13) also suggests that $\Psi_1'(t) < 0$ when $t > 0$ is small enough. Thus, $\Psi_1(t)$ is decreasing when $t > 0$ is small enough. Therefore, since $\Psi_0(t) \leq \Phi(t) \leq \Psi_1(t)$ for $0 < t < 1$, $\Phi(t)$ is decreasing when $t > 0$ is small enough.
For $t > 1$: Following the same arguments shows that $\Psi_0'(t) > 0$ and $\Psi_1'(t) > 0$ when $t > 1$ is large enough, and therefore, both $\Psi_0(t)$ and $\Psi_1(t)$ are increasing. Thus, $\Phi(t)$ is increasing when $t > 1$ is large enough. In conclusion, since $\Phi(0) = 0$, $\Phi(t)$ attains its minimum on $[0, \infty)$ at some point, say $t = t(u)$. That is, $\frac{d}{dt}\Phi(t)|_{t=t(u)} = 0$. Then, $t(u)u \in \mathcal{N}_2$ and $\inf_{t>0}\mathcal{J}(tu) = \mathcal{J}(t(u)u)$.
Next, we show that scaling function $t(u)$ is continuous on $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

Let $u_n \to u$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)\backslash\{0\}$, and $t_n = t(u_n)$. Then, by the definition, $t_n u_n \in \mathcal{N}_2$. Defined in this way, the sequence t_n is bounded. Assume on the contrary that $t_n \to \infty$ (up to a subsequence). Then, using the fact $t_n u_n \in \mathcal{N}_2$ it follows

$$\begin{aligned}\int_\Omega \sum_{i=1}^N |\partial_{x_i} t_n u_n|^{p_i(x)} dx - \int_\Omega g(x)|t_n u_n|^{q(x)+1} dx &= \int_\Omega f(x)|t_n u_n|^{1-\beta(x)} dx \\ t_n^{P_-^-} \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx - t_n^{q^-+1} \int_\Omega g(x)|u_n|^{q(x)+1} dx &\leq t_n^{1-\beta^-} \int_\Omega f(x)|u_n|^{1-\beta(x)} dx\end{aligned}$$

which suggests a contradiction when $t_n \to \infty$. Hence, sequence t_n is bounded. Therefore, there exists a subsequence t_n (not relabelled) such that $t_n \to t_0$, $t_0 \geq 0$. On the other hand, from Lemma 3.4, $\|t_n u_n\|_{\vec{p}(\cdot)} \geq \delta_1 > 0$. Thus, $t_0 > 0$ and $t_0 u \in \mathcal{N}_2$. By the uniqueness of the map $t(u)$, $t_0 = t(u)$, which concludes the continuity of $t(u)$. In conclusion, for any $\in W_0^{1,\vec{p}(\cdot)}(\Omega)$ satisfying $\int_\Omega f(x)|u|^{1-\beta(x)} dx < \infty$, the function $t(u)$ scales $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ continuously to a point such that $t(u)u \in \mathcal{N}_2$.

Lemma 3.7. *Assume that $(u_n) \subset \mathcal{N}_1$ is the nonnegative minimizing sequence for the minimization problem $\lim_{n\to\infty} \mathcal{J}(u_n) = \inf_{\mathcal{N}_1} \mathcal{J}$. Then, there exists a subsequence (u_n) (not relabelled) such that $u_n \to u^*$ (strongly) in $W_0^{1,\vec{p}(\cdot)}(\Omega)$.*

Proof. Since (u_n) is bounded in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ and $W_0^{1,\vec{p}(\cdot)}(\Omega)$ is reflexive, there exists a subsequence (u_n), not relabelled, and $u^* \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that

- $u_n \rightharpoonup u^*$ (weakly) in $W_0^{1,\vec{p}(\cdot)}(\Omega)$,
- $u_n \to u^*$ in $L^{s(\cdot)}(\Omega)$, $1 < s(x) < P_{-,\infty}$, for all $x \in \overline{\Omega}$,
- $u_n(x) \to u^*(x)$ a.e. in Ω.

Since the norm $\|\cdot\|_{\vec{p}(\cdot)}$ is a continuous convex functional, it is weakly lower semicontinuous. Using this fact along with the Fatou's lemma, and Lemma 3.4, it reads

$$\begin{aligned}\inf_{\mathcal{N}_1} \mathcal{J} &= \lim_{n\to\infty} \mathcal{J}(u_n) \\ &\geq \liminf_{n\to\infty} \left[\int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} dx \right] - \int_\Omega \frac{g(x)|u^*|^{q(x)+1}}{q(x)+1} dx \\ &+ \liminf_{n\to\infty} \left[\int_\Omega \frac{f(x)|u_n|^{1-\beta(x)}}{\beta(x)-1} dx \right] \\ &\geq \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u^*|^{p_i(x)}}{p_i(x)} dx - \int_\Omega \frac{g(x)|u^*|^{q(x)+1}}{q(x)+1} dx + \int_\Omega \frac{f(x)|u^*|^{1-\beta(x)}}{\beta(x)-1} dx \\ &= \mathcal{J}(u^*) \geq \mathcal{J}(t(u^*)u^*) \geq \inf_{\mathcal{N}_2} \mathcal{J} \geq \inf_{\mathcal{N}_1} \mathcal{J} \end{aligned} \tag{3.14}$$

The above result implies, up to subsequences, that

$$\lim_{n\to\infty} \|u_n\|_{\vec{p}(\cdot)} = \|u^*\|_{\vec{p}(\cdot)}. \tag{3.15}$$

Thus, (3.15) along with $u_n \rightharpoonup u^*$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$ show that $u_n \to u^*$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$.

The following is the main result of the present paper.

Theorem 3.8. *Assume that the conditions* $(A_1)-(A_5)$ *hold. Then, problem (1.1) has at least one positive* $W_0^{1,\vec{p}(\cdot)}(\Omega)$*-solution if and only if there exists* $\overline{u} \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ *satisfying* $\int_\Omega f(x)|\overline{u}|^{1-\beta(x)}dx < \infty$.

Proof. $(\Rightarrow)$: Assume that the function $u \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ is a weak solution to problem (1.1). Then, letting $u = \varphi$ in Definition (3.1) gives

$$\begin{aligned}\int_\Omega f(x)|u|^{1-\beta(x)}dx &= \int_\Omega \sum_{i=1}^N |\partial_{x_i} u|^{p_i(x)}dx - \int_\Omega g(x)|u|^{q(x)+1}dx \\ &\leq \|u\|_{\vec{p}(\cdot)}^{P_M} - |g|_\infty |u|_{q(x)+1}^{q_M} \\ &\leq \|u\|_{\vec{p}(\cdot)}^{P_M} < \infty,\end{aligned}$$

where $P_M := \max\{P_-^-, P_+^+\}$ and $q_M := \max\{q^-, q^+\}$, changing according to the base.
$(\Leftarrow)$: Assume that there exists $\overline{u} \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ such that $\int_\Omega f(x)|\overline{u}|^{1-\beta(x)}dx < \infty$. Then, by Lemma 3.6, there exists a unique number $t(\overline{u}) > 0$ such that $t(\overline{u})\overline{u} \in \mathcal{N}_2$.
The information we have had about $\mathcal{J}$ so far and the closeness of $\mathcal{N}_1$ allow us to apply Ekeland's variational principle to the problem $\inf_{\mathcal{N}_1} \mathcal{J}$. That is, it suggests the existence of a corresponding minimizing sequence $(u_n) \subset \mathcal{N}_1$ satisfying the following:

(E_1) $\mathcal{J}(u_n) - \inf_{\mathcal{N}_1} \mathcal{J} \leq \frac{1}{n}$,
(E_2) $\mathcal{J}(u_n) - \mathcal{J}(v) \leq \frac{1}{n}\|u_n - v\|_{\vec{p}(\cdot)}$, $\forall v \in \mathcal{N}_1$.

Due to the fact $\mathcal{J}(|u_n|) = \mathcal{J}(u_n)$, it is not wrong to assume that $u_n \geq 0$ a.e. in Ω. Additionally, considering that $(u_n) \subset \mathcal{N}_1$ and following the same approach as it is done in the $(\Rightarrow)$ part, we can obtain that $\int_\Omega f(x)|u_n|^{1-\beta(x)}dx < \infty$. If all this information and the assumptions (A_1), (A_2) are taken into consideration, it follows that $u_n(x) > 0$ a.e. in Ω.
The rest of the proof is split into two cases.
Case I: $(u_n) \subset \mathcal{N}_1 \setminus \mathcal{N}_2$ for n large.
For a function $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ with $\varphi \geq 0$, and $t > 0$, we have

$$0 < (u_n(x) + t\varphi(x))^{1-\beta(x)} \leq u_n(x)^{1-\beta(x)} \text{ a.e. in } \Omega.$$

Therefore, using (A_1),(A_2) gives

$$\begin{aligned}\int_\Omega f(x)(u_n + t\varphi)^{1-\beta(x)}dx &\leq \int_\Omega f(x)u_n^{1-\beta(x)}dx \\ &\leq \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)}dx - \int_\Omega g(x)u_n^{q(x)+1}dx < \infty \qquad (3.16)\end{aligned}$$

Then, when $t > 0$ is small enough in (3.16), we obtain

$$\int_\Omega f(x)(u_n + t\varphi)^{1-\beta(x)}dx \leq \int_\Omega \sum_{i=1}^N |\partial_{x_i}(u_n + t\varphi)|^{p_i(x)}dx - \int_\Omega g(x)(u_n + t\varphi)^{q(x)+1}dx \qquad (3.17)$$

which means that $v := u_n + t\varphi \in \mathcal{N}_1$. Now, using (E_2), it reads

$$\begin{aligned}
\frac{1}{n}\|t\varphi\|_{\vec{p}(\cdot)} &\geq \mathcal{J}(u_n) - \mathcal{J}(v) \\
&= \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} dx - \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i}(u_n + t\varphi)|^{p_i(x)}}{p_i(x)} dx \\
&- \int_\Omega \frac{g(x) u_n^{q(x)+1}}{q(x)+1} dx + \int_\Omega \frac{g(x)(u_n + t\varphi)^{q(x)+1}}{q(x)+1} dx \\
&+ \int_\Omega \frac{f(x) u_n^{1-\beta(x)}}{\beta(x)-1} dx - \int_\Omega \frac{f(x)(u_n + t\varphi)^{1-\beta(x)}}{\beta(x)-1} dx
\end{aligned}$$

Dividing the above inequality by t and passing to the infimum limit as $t \to 0$ gives

$$\begin{aligned}
\liminf_{t\to 0} \frac{\|\varphi\|_{\vec{p}(\cdot)}}{n} &+ \underbrace{\liminf_{t\to 0}\left[\int_\Omega \sum_{i=1}^N \frac{\left[|\partial_{x_i}(u_n + t\varphi)|^{p_i(x)} - |\partial_{x_i} u_n|^{p_i(x)}\right]}{t p_i(x)} dx\right]}_{:=I_1} \\
&- \underbrace{\liminf_{t\to 0}\left[\int_\Omega g(x) \frac{\left[(u_n + t\varphi)^{q(x)+1} - u_n^{q(x)+1}\right]}{t(q(x)+1)} dx\right]}_{:=I_2} \\
&\geq \underbrace{\liminf_{t\to 0}\left[\int_\Omega f(x) \frac{\left[(u_n + t\varphi)^{1-\beta(x)} - u_n^{1-\beta(x)}\right]}{t(1-\beta(x))} dx\right]}_{:=I_3}
\end{aligned}$$

Calculation of I_1, I_2 gives

$$I_1 = \frac{d}{dt}\left(\int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i}(u_n + t\varphi)|^{p_i(x)}}{p_i(x)} dx\right)\Bigg|_{t=0} = \int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)-2} \partial_{x_i} u_n \cdot \partial_{x_i}\varphi dx \tag{3.18}$$

and

$$I_2 = \frac{d}{dt}\left(\int_\Omega g(x) \frac{(u_n + t\varphi)^{q(x)+1}}{q(x)+1} dx\right)\Bigg|_{t=0} = \int_\Omega g(x) u_n^{q(x)} \varphi dx. \tag{3.19}$$

For I_3: Since for $t > 0$ it holds

$$u_n^{1-\beta(x)}(x) - (u_n(x) + t\varphi(x))^{1-\beta(x)} \geq 0, \text{ a.e. in } \Omega$$

we can apply Fatou's lemma, that is,

$$\begin{aligned}
I_2 &= \liminf_{t\to 0} \int_\Omega f(x) \frac{\left[(u_n + t\varphi)^{1-\beta(x)} - u_n^{1-\beta(x)}\right]}{t(1-\beta(x))} dx \\
&\geq \int_\Omega \liminf_{t\to 0} f(x) \frac{\left[(u_n + t\varphi)^{1-\beta(x)} - u_n^{1-\beta(x)}\right]}{t(1-\beta(x))} dx
\end{aligned}$$

$$\geq \int_{\Omega} f(x)u_n^{-\beta(x)}\varphi dx \tag{3.20}$$

Now, substituting I_1, I_2, I_3 gives

$$\frac{||\varphi||_{\vec{p}(\cdot)}}{n} + \int_{\Omega}\sum_{i=1}^{N}|\partial_{x_i}u_n|^{p_i(x)-2}\partial_{x_i}u_n\cdot\partial_{x_i}\varphi dx - \int_{\Omega} g(x)u_n^{q(x)}\varphi dx \geq \int_{\Omega} f(x)u_n^{-\beta(x)}\varphi dx$$

From Lemma 3.7, we know that $u_n \to u^*$ in $W_0^{1,\vec{p}(\cdot)}(\Omega)$. Thus, also considering Fatou's lemma, we obtain

$$\int_{\Omega}\sum_{i=1}^{N}|\partial_{x_i}u^*|^{p_i(x)-2}\partial_{x_i}u^*\cdot\partial_{x_i}\varphi dx - \int_{\Omega} g(x)(u^*)^{q(x)}\varphi dx - \int_{\Omega} f(x)(u^*)^{-\beta(x)}\varphi dx \geq 0, \tag{3.21}$$

for any $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ with $\varphi \geq 0$. Letting $\varphi = u^*$ in (3.21) shows clearly that $u^* \in \mathcal{N}_1$.
Lastly, from Lemma 3.7, we can conclude that

$$\lim_{n\to\infty}\mathcal{J}(u_n) = \mathcal{J}(u^*) = \inf_{\mathcal{N}_2}\mathcal{J},$$

which means

$$u^* \in \mathcal{N}_2, \quad (\text{with } t(u^*) = 1) \tag{3.22}$$

Case II: There exists a subsequence of (u_n) (not relabelled) contained in $\mathcal{N}_2$.
For a function $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \geq 0$, $t > 0$, and $u_n \in \mathcal{N}_2$, we have

$$\begin{aligned}\int_{\Omega} f(x)(u_n+t\varphi)^{1-\beta(x)}dx &\leq \int_{\Omega} f(x)u_n^{1-\beta(x)}dx \\ &= \int_{\Omega}\sum_{i=1}^{N}|\partial_{x_i}u|^{p_i(x)}dx - \int_{\Omega} g(x)u_n^{q(x)+1}dx < \infty,\end{aligned} \tag{3.23}$$

and hence, there exists a unique continuous scaling function, denoted by $\theta_n(t) := t(u_n + t\varphi) > 0$, corresponding to $(u_n + t\varphi)$ so that $\theta_n(t)(u_n + t\varphi) \in \mathcal{N}_2$ for $n = 1, 2, ...$ Obviously, $\theta_n(0) = 1$. Since $\theta_n(t)(u_n + t\varphi) \in \mathcal{N}_2$, we have

$$\begin{aligned}0 &= \int_{\Omega}\sum_{i=1}^{N}|\partial_{x_i}\theta_n(t)(u_n+t\varphi)|^{p_i(x)}dx - \int_{\Omega} g(x)(\theta_n(t)(u_n+t\varphi))^{q(x)+1}dx \\ &- \int_{\Omega} f(x)(\theta_n(t)(u_n+t\varphi))^{1-\beta(x)}dx \\ &\geq \int_{\Omega}\sum_{i=1}^{N}|\partial_{x_i}\theta_n(t)(u_n+t\varphi)|^{p_i(x)}dx - \theta_n^{q_M+1}(t)\int_{\Omega} g(x)(u_n+t\varphi)^{q(x)+1}dx \\ &- \theta_n^{1-\beta_m}(t)\int_{\Omega} f(x)(u_n+t\varphi)^{1-\beta(x)}dx,\end{aligned} \tag{3.24}$$

and

$$0=\int_\Omega \sum_{i=1}^N |\partial_{x_i}u_n|^{p_i(x)}dx-\int_\Omega g(x)u_n^{q(x)+1}dx-\int_\Omega f(x)u_n^{1-\beta(x)}dx. \tag{3.25}$$

where $\beta_m := \min\{\beta^-, \beta^+\}$. Then, using (3.24) and (3.25) together gives

$$\begin{aligned}0\geq&\Bigg[-(q^++1)[\theta_n(0)+\tau_1(\theta_n(t)-\theta_n(0))]^{q_m}\int_\Omega g(x)(u_n+t\varphi)^{q(x)+1}dx\\&-(1-\beta_m)[\theta_n(0)+\tau_2(\theta_n(t)-\theta_n(0))]^{-\beta_m}\int_\Omega f(x)(u_n+t\varphi)^{1-\beta(x)}dx\Bigg](\theta_n(t)-\theta_n(0))\\&+\int_\Omega\sum_{i=1}^N|\partial_{x_i}\theta_n(t)(u_n+t\varphi)|^{p_i(x)}dx-\int_\Omega\sum_{i=1}^N|\partial_{x_i}(u_n+t\varphi)|^{p_i(x)}dx\\&+\int_\Omega\sum_{i=1}^N|\partial_{x_i}(u_n+t\varphi)|^{p_i(x)}dx-\int_\Omega\sum_{i=1}^N|\partial_{x_i}u_n|^{p_i(x)}dx\\&-\left[\int_\Omega g(x)(u_n+t\varphi)^{q(x)+1}dx-\int_\Omega g(x)u_n^{q(x)+1}dx\right]\\&-\left[\int_\Omega f(x)(u_n+t\varphi)^{1-\beta(x)}dx-\int_\Omega f(x)u_n^{1-\beta(x)}dx\right]\end{aligned}\tag{3.26}$$

for some constants $\tau_1, \tau_2 \in (0,1)$. To proceed, we assume that $\theta_n'(0) = \frac{d}{dt}\theta_n(t)|_{t=0} \in [-\infty, \infty]$. In case this limit does not exist, we can consider a subsequence $t_k > 0$ of t such that $t_k \to 0$ as $k \to \infty$.
Next, we show that $\theta_n'(0) \neq \infty$.
Dividing the both sides of (3.26) by t and passing to the limit as $t \to 0$ leads to

$$\begin{aligned}0\geq&\Bigg[P_-^-\int_\Omega\sum_{i=1}^N|\partial_{x_i}u_n|^{p_i(x)}dx+(\beta_m-1)\int_\Omega f(x)u_n^{1-\beta(x)}dx\\&-(q^++1)\int_\Omega g(x)u_n^{q(x)+1}dx\Bigg]\theta_n'(0)\\&+P_-^-\int_\Omega\sum_{i=1}^N|\partial_{x_i}u_n|^{p_i(x)-2}\partial_{x_i}u_n\cdot\partial_{x_i}\varphi dx-(q^++1)\int_\Omega g(x)u_n^{q(x)}\varphi dx\\&+(\beta_m-1)\int_\Omega f(x)u_n^{-\beta(x)}\varphi dx\end{aligned}\tag{3.27}$$

or

$$\begin{aligned}0\geq&\left[(P_-^--q^+-1)\int_\Omega\sum_{i=1}^N|\partial_{x_i}u_n|^{p_i(x)}dx+(\beta_m+q^+)\int_\Omega f(x)u_n^{1-\beta(x)}dx\right]\theta_n'(0)\\&+P_-^-\int_\Omega\sum_{i=1}^N|\partial_{x_i}u_n|^{p_i(x)-2}\partial_{x_i}u_n\cdot\partial_{x_i}\varphi dx-(q^++1)\int_\Omega g(x)u_n^{q(x)}\varphi dx\\&+(\beta_m-1)\int_\Omega f(x)u_n^{-\beta(x)}\varphi dx\end{aligned}\tag{3.28}$$

which, along with Lemma 3.4, concludes that $-\infty \leq \theta_n'(0) < \infty$, and hence, $\theta_n'(0) \leq \bar{c}$, uniformly in all large n.

Next, we show that $\theta_n'(0) \neq -\infty$.

First, we apply Ekeland's variational principle to the minimizing sequence $(u_n) \subset \mathcal{N}_2(\subset \mathcal{N}_1)$. Thus, letting $v := \theta_n(t)(u_n + t\varphi)$ in (E_2) gives

$$\begin{aligned}
&\frac{1}{n}\left[|\theta_n(t) - 1|\|u_n\|_{\vec{p}(\cdot)} + t\theta_n(t)\|\varphi\|_{\vec{p}(\cdot)}\right] \geq \mathcal{J}(u_n) - \mathcal{J}(\theta_n(t)(u_n + t\varphi))\\
&= \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} dx - \int_\Omega \frac{g(x)u_n^{q(x)+1}}{q(x)+1} dx + \int_\Omega \frac{f(x)u_n^{1-\beta(x)}}{\beta(x)-1} dx\\
&- \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i}\theta_n(t)(u_n + t\varphi)|^{p_i(x)}}{p_i(x)} dx + \int_\Omega \frac{g(x)[\theta_n(t)(u_n + t\varphi)]^{q(x)+1}}{q(x)+1} dx\\
&- \int_\Omega \frac{f(x)[\theta_n(t)(u_n + t\varphi)]^{1-\beta(x)}}{\beta(x)-1} dx\\
&\geq \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i} u_n|^{p_i(x)}}{p_i(x)} dx - \int_\Omega \sum_{i=1}^N \frac{|\partial_{x_i}\theta_n(t)(u_n + t\varphi)|^{p_i(x)}}{p_i(x)} dx\\
&- \int_\Omega \frac{g(x)u_n^{q(x)+1}}{q(x)+1} dx + \int_\Omega \frac{g(x)[\theta_n(t)(u_n + t\varphi)]^{q(x)+1}}{q(x)+1} dx\\
&- \frac{1}{\beta^- - 1} \int_\Omega \sum_{i=1}^N |\partial_{x_i}\theta_n(t)(u_n + t\varphi)|^{p_i(x)} dx
\end{aligned} \tag{3.29}$$

If we use Lemma 3.4 to manipulate the norm $\|u + t\varphi\|_{\vec{p}(\cdot)}$, the integral in the last line of (3.29) can be written as follows

$$\begin{aligned}
\frac{1}{\beta^- - 1} \int_\Omega \sum_{i=1}^N |\partial_{x_i}\theta_n(t)(u_n + t\varphi)|^{p_i(x)} dx &\leq \frac{\theta_n^{P_M}(t)}{\beta^- - 1} \int_\Omega \sum_{i=1}^N |\partial_{x_i}(u_n + t\varphi)|^{p_i(x)} dx\\
&\leq \frac{\theta_n^{P_M}(t)}{\beta^- - 1} \|u_n + t\varphi\|_{\vec{p}(\cdot)}^{P_M}\\
&\leq \frac{2^{P_+^+ - 1}\theta_n^{P_M}(t) C^{P_M}(\delta_2)\|\varphi\|_{\vec{p}(\cdot)}^{P_M}}{\beta^- - 1} t
\end{aligned} \tag{3.30}$$

Then,

$$\begin{aligned}
&\frac{1}{n}\left[|\theta_n(t) - 1|\|u_n\|_{\vec{p}(\cdot)} + t\theta_n(t)\|\varphi\|_{\vec{p}(\cdot)}\right]\\
&+ \int_\Omega \sum_{i=1}^N \frac{\left[|\partial_{x_i}(u_n + t\varphi)|^{p_i(x)} - |\partial_{x_i} u_n|^{p_i(x)}\right]}{p_i(x)} dx + \frac{2^{P_+^+ - 1}\theta_n^{P_M}(t) C^{P_M}(\delta_2)\|\varphi\|_{\vec{p}(\cdot)}^{P_M}}{\beta^- - 1} t\\
&\geq \left[\left(\frac{1}{q^- + 1}\right)[\theta_n(0) + \tau_1(\theta_n(t) - \theta_n(0))]^{q_m} \int_\Omega g(x)(u_n + t\varphi)^{q(x)+1} dx\right](\theta_n(t) - \theta_n(0))\\
&\geq -\int_\Omega \sum_{i=1}^N \frac{\left[|\partial_{x_i}\theta_n(t)(u_n + t\varphi)|^{p_i(x)} - |\partial_{x_i}(u_n + t\varphi)|^{p_i(x)}\right]}{p_i(x)} dx
\end{aligned}$$

$$+\frac{1}{q^-+1}\int_\Omega g(x)\left[(u_n+t\varphi)^{q(x)+1}-u_n^{q(x)+1}\right]dx \tag{3.31}$$

Dividing by t and passing to the limit as $t\to 0$ gives

$$\begin{aligned}
&\frac{1}{n}\|\varphi\|_{\vec{p}(\cdot)}+\frac{2^{P_+^+-1}\theta_n^{P_M}(t)C^{P_M}(\delta_2)\|\varphi\|_{\vec{p}(\cdot)}^{P_M}}{\beta^- -1}\\
&\geq\left[\left(-1+\frac{1}{q^-+1}\right)\int_\Omega\sum_{i=1}^N|\partial_{x_i}u_n|^{p_i(x)}dx-\frac{1}{q^-+1}\int_\Omega f(x)u_n^{1-\beta(x)}dx\right.\\
&\left.-\frac{\|u_n\|_{\vec{p}(\cdot)}}{n}\mathrm{sgn}[\theta_n(t)-1]\right]\theta_n'(0)\\
&-\int_\Omega\sum_{i=1}^N|\partial_{x_i}u_n|^{p_i(x)-2}\partial_{x_i}u_n\cdot\partial_{x_i}\varphi dx+\int_\Omega g(x)u_n^{q(x)}dx
\end{aligned} \tag{3.32}$$

which concludes that $\theta_n'(0)\neq-\infty$. Thus, $\theta_n'(0)\geq\underline{c}$ uniformly in large n.
In conclusion, there exists a constant, $C_0>0$ such that $|\theta_n'(0)|\leq C_0$ when $n\geq N_0,\ N_0\in\mathbb{N}$.

Next, we show that $u^*\in\mathcal{N}_2$.
Using (E_2) again, we have

$$\begin{aligned}
&\frac{1}{n}\left[|\theta_n(t)-1|\|u_n\|_{\vec{p}(\cdot)}+t\theta_n(t)\|\varphi\|_{\vec{p}(\cdot)}\right]\geq\mathcal{J}(u_n)-\mathcal{J}(\theta_n(t)(u_n+t\varphi))\\
&=\int_\Omega\sum_{i=1}^N\frac{|\partial_{x_i}u_n|^{p_i(x)}}{p_i(x)}dx-\int_\Omega\frac{g(x)u_n^{q(x)+1}}{q(x)+1}dx+\int_\Omega\frac{f(x)u_n^{1-\beta(x)}}{\beta(x)-1}dx\\
&-\int_\Omega\sum_{i=1}^N\frac{|\partial_{x_i}\theta_n(t)(u_n+t\varphi)|^{p_i(x)}}{p_i(x)}dx+\int_\Omega\frac{g(x)[\theta_n(t)(u_n+t\varphi)]^{q(x)+1}}{q(x)+1}dx\\
&-\int_\Omega\frac{f(x)[\theta_n(t)(u_n+t\varphi)]^{1-\beta(x)}}{\beta(x)-1}dx\\
&=-\int_\Omega\sum_{i=1}^N\frac{|\partial_{x_i}(u_n+t\varphi)|^{p_i(x)}}{p_i(x)}dx+\int_\Omega\sum_{i=1}^N\frac{|\partial_{x_i}u_n|^{p_i(x)}}{p_i(x)}dx\\
&-\int_\Omega\frac{f(x)(u_n+t\varphi)^{1-\beta(x)}}{\beta(x)-1}dx+\int_\Omega\frac{f(x)u_n^{1-\beta(x)}}{\beta(x)-1}dx\\
&-\int_\Omega\sum_{i=1}^N\frac{|\partial_{x_i}\theta_n(t)(u_n+t\varphi)|^{p_i(x)}}{p_i(x)}dx+\int_\Omega\sum_{i=1}^N\frac{|\partial_{x_i}(u_n+t\varphi)|^{p_i(x)}}{p_i(x)}dx\\
&-\int_\Omega\frac{f(x)[\theta_n(t)(u_n+t\varphi)]^{1-\beta(x)}}{\beta(x)-1}dx+\int_\Omega\frac{f(x)(u_n+t\varphi)^{1-\beta(x)}}{\beta(x)-1}dx\\
&\int_\Omega\frac{g(x)[\theta_n(t)(u_n+t\varphi)]^{q(x)+1}}{q(x)+1}dx-\int_\Omega\frac{g(x)(u_n+t\varphi)^{q(x)+1}}{q(x)+1}dx\\
&-\int_\Omega\frac{g(x)u_n^{q(x)+1}}{q(x)+1}dx+\int_\Omega\frac{g(x)(u_n+t\varphi)^{q(x)+1}}{q(x)+1}dx
\end{aligned} \tag{3.33}$$

Dividing by t and passing to the limit as $t \to 0$ gives

$$\begin{aligned}
&\frac{1}{n}\left[|\theta_n'(0)|\|u_n\|_{\vec{p}(\cdot)} + \|\varphi\|_{\vec{p}(\cdot)}\right] \\
&\geq -\int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)-2}\partial_{x_i} u_n \cdot \partial_{x_i}\varphi dx + \int_\Omega f(x)u_n^{-\beta(x)}\varphi dx + \int_\Omega g(x)u_n^{q(x)}\varphi dx \\
&\left[-\int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)} dx + \int_\Omega g(x)u_n^{q(x)+1} dx + \int_\Omega f(x)u_n^{1-\beta(x)} dx\right]\theta_n'(0) \\
&= -\int_\Omega \sum_{i=1}^N |\partial_{x_i} u_n|^{p_i(x)-2}\partial_{x_i} u_n \cdot \partial_{x_i}\varphi dx + \int_\Omega g(x)u_n^{q(x)}\varphi dx + \int_\Omega f(x)u_n^{-\beta(x)}\varphi dx \qquad (3.34)
\end{aligned}$$

If we consider that $|\theta_n'(0)| \leq C_0$ uniformly in n, we obtain that $\int_\Omega f(x)u_n^{-\beta(x)}dx < \infty$. Therefore, for $n \to \infty$ it reads

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i} u^*|^{p_i(x)-2}\partial_{x_i} u^* \cdot \partial_{x_i}\varphi dx - \int_\Omega g(x)(u^*)^{q(x)}\varphi dx - \int_\Omega f(x)(u^*)^{-\beta(x)}\varphi dx \geq 0 \qquad (3.35)$$

for all $\varphi \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, $\varphi \geq 0$. Letting $\varphi = u^*$ in (3.35) shows clearly that $u^* \in \mathcal{N}_1$.
This means, as with the Case I, that we have

$$u^* \in \mathcal{N}_2 \qquad (3.36)$$

By taking into consideration the results (3.21), (3.22), (3.35), and (3.36), we infer that $u^* \in \mathcal{N}_2$ and (3.35) holds, in the weak sense, for both cases. Additionally, since $u^* \geq 0$ and $u^* \neq 0$, by the strong maximum principle for weak solutions, we must have $u^*(x) > 0$ almost everywhere in Ω.
Next, we show that $u^* \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ is a weak solution to problem (1.1).
For a random function $\phi \in W_0^{1,\vec{p}(\cdot)}(\Omega)$, and $\varepsilon > 0$, let $\varphi = (u^* + \varepsilon\phi)^+ = \max\{0, u^* + \varepsilon\phi\}$. We split Ω into two sets as follows:

$$\Omega_\geq = \{x \in \Omega : u^*(x) + \varepsilon\phi(x) \geq 0\}, \qquad (3.37)$$

and

$$\Omega_< = \{x \in \Omega : u^*(x) + \varepsilon\phi(x) < 0\}. \qquad (3.38)$$

If we replace φ with $(u^* + \varepsilon\phi)$ in (3.35), it follows

$$\begin{aligned}
0 &\leq \int_\Omega \sum_{i=1}^N |\partial_{x_i} u^*|^{p_i(x)-2}\partial_{x_i} u^* \cdot \partial_{x_i}\varphi dx - \int_\Omega [g(x)(u^*)^{q(x)} + f(x)(u^*)^{-\beta(x)}]\varphi dx \\
&= \int_{\Omega_\geq} \sum_{i=1}^N |\partial_{x_i} u^*|^{p_i(x)-2}\partial_{x_i} u^* \cdot \partial_{x_i}(u^* + \varepsilon\phi) dx \\
&- \int_{\Omega_\geq} [g(x)(u^*)^{q(x)}(u)^* + f(x)(u^*)^{-\beta(x)}](u^* + \varepsilon\phi) dx \\
&= \int_\Omega - \int_{\Omega_<} \left[\sum_{i=1}^N |\partial_{x_i} u^*|^{p_i(x)-2}\partial_{x_i} u^* \cdot \partial_{x_i}(u^* + \varepsilon\phi)\right.
\end{aligned}$$

$$- [g(x)(u^*)^{q(x)} + f(x)(u^*)^{-\beta(x)}](u^* + \varepsilon\phi)\Big] dx$$

$$= \int_\Omega \sum_{i=1}^N |\partial_{x_i} u^*|^{p_i(x)} dx + \varepsilon \int_\Omega \sum_{i=1}^N |\partial_{x_i} u^*|^{p_i(x)-2} \partial_{x_i} u^* \cdot \partial_{x_i}\phi dx$$

$$- \int_\Omega f(x)(u^*)^{1-\beta(x)} dx - \varepsilon \int_\Omega f(x)(u^*)^{-\beta(x)}\phi dx$$

$$- \int_\Omega g(x)(u^*)^{q(x)+1} dx - \varepsilon \int_\Omega g(x)(u^*)^{q(x)}\phi dx$$

$$- \int_{\Omega_<} \left[\sum_{i=1}^N |\partial_{x_i} u^*|^{p_i(x)-2} \partial_{x_i} u^* \cdot \partial_{x_i}(u^* + \varepsilon\phi) \right.$$

$$\left. -[g(x)(u^*)^{q(x)} + f(x)(u^*)^{-\beta(x)}](u^* + \varepsilon\phi) \right] dx \tag{3.39}$$

Since $u^* \in \mathcal{N}_2$, we have

$$0 \le \varepsilon \left[\int_\Omega \sum_{i=1}^N |\partial_{x_i} u^*|^{p_i(x)-2} \partial_{x_i} u^* \cdot \partial_{x_i}\phi - [g(x)(u^*)^{q(x)} + f(x)(u^*)^{-\beta(x)}]\phi \right] dx$$

$$- \varepsilon \int_{\Omega_<} \sum_{i=1}^N |\partial_{x_i} u^*|^{p_i(x)-2} \partial_{x_i} u^* \cdot \partial_{x_i}\phi dx + \varepsilon \int_{\Omega_<} g(x)(u^*)^{q(x)}\phi dx$$

$$+ \varepsilon \int_{\Omega_<} f(x)(u^*)^{-\beta(x)}\phi dx \tag{3.40}$$

Dividing by ε and passing to the limit as $\varepsilon \to 0$, and considering that $|\Omega_<| \to 0$ as $\varepsilon \to 0$ gives

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i} u^*|^{p_i(x)-2} \partial_{x_i} u^* \cdot \partial_{x_i}\phi dx - \int_\Omega g(x)(u^*)^{q(x)}\phi dx$$

$$\ge \int_\Omega f(x)(u^*)^{-\beta(x)}\phi dx, \ \forall \phi \in W_0^{1,\vec{p}(\cdot)}(\Omega) \tag{3.41}$$

However, since the function $\phi \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ is chosen randomly, it follows that

$$\int_\Omega \sum_{i=1}^N |\partial_{x_i} u^*|^{p_i(x)-2} \partial_{x_i} u^* \cdot \partial_{x_i}\phi dx - \int_\Omega g(x)(u^*)^{q(x)}\phi dx$$

$$= \int_\Omega f(x)(u^*)^{-\beta(x)}\phi dx \tag{3.42}$$

which concludes that $u^* \in W_0^{1,\vec{p}(\cdot)}(\Omega)$ is a weak solution to problem (1.1).

4. Example

Suppose that

$$\begin{cases} g(x) &= e^{k\cos(|x|)}, \\ \text{and} & \\ f(x) &= \frac{(1-|x|)^k}{\beta(x)}, \ x \in B_1(0) \subset \mathbb{R}^N, \ k > 0. \end{cases}$$

Then equation (1.1) becomes

$$\begin{cases} -\sum_{i=1}^{N}\partial_{x_i}\left(|\partial_{x_i}u|^{p_i(x)-2}\partial_{x_i}u\right) & = & \frac{(1-|x|)^k}{\beta(x)}u^{-\beta(x)} + e^{k\cos(|x|)}u^{q(x)} \text{ in } B_1(0), \\ u & > & 0 \text{ in } B_1(0), \\ u & = & 0 \text{ on } \partial B_1(0). \end{cases} \tag{4.1}$$

Theorem 4.1. *Assume that the conditions* $(A_1)-(A_3)$ *hold. If* $1<\beta^+<1+\frac{k+1}{\alpha}$ *and* $\alpha>1/2$*, then, problem (4.1) has at least one positive* $W_0^{1,\vec{p}(\cdot)}(B_1(0))$*-solution.*

Proof. Function $f(x)=\frac{(1-|x|)^k}{\beta(x)}\leq\frac{(1-|x|)^k}{\beta^-}$ is clearly non-negative and bounded above within the unit ball $B_1(0)$ since $|x|<1$. Hence, $f(x)\in L^1(B_1(0))$.
Now, let's choose $\overline{u}=(1-|x|)^{\alpha}$. Since $\overline{u}$ is also non-negative and bounded within $B(0,1)$, it is in $\overline{u}\in L^{P_+^+}(B(0,1))$. Indeed,

$$\sum_{i=1}^{N}\int_{B_1(0)}((1-|x|)^{\alpha})^{p_i(x)}dx$$
$$\leq N\left[\int_{B_1(0)}((1-|x|)^{\alpha})^{P_-^-}dx+\int_{B_1(0)}((1-|x|)^{\alpha})^{P_+^+}dx\right]<\infty.$$

Next, we show that $\partial_{x_i}\overline{u}\in L^{p_i(\cdot)}(B_1(0))$ for $i\in\{1,...,N\}$. Fix $i\in\{1,...,N\}$. Then

$$\partial_{x_i}(1-|x|)^{\alpha}=\alpha(1-|x|)^{\alpha-1}\frac{-x_i}{|x|}$$

Considering that $x\in B_1(0)$, we obtain

$$\int_{B_1(0)}|\partial_{x_i}(1-|x|)^{\alpha}|^{p_i(x)}dx\leq\alpha^{P_M}\int_{B_1(0)}(1-|x|)^{(\alpha-1)P_-^-}dx$$

Therefore,

$$\sum_{i=1}^{N}\int_{B_1(0)}|\partial_{x_i}(1-|x|)^{\alpha}|^{p_i(x)}dx\leq N\alpha^{P_M}\sum_{i=1}^{N}\int_{B(0,1)}(1-|x|)^{(\alpha-1)P_-^-}dx<\infty$$

if $\alpha>\frac{P_-^--1}{P_-^-}$. Thus, $\partial_{x_i}\overline{u}\in L^{p_i(\cdot)}(B_1(0))$ for $i\in\{1,...,N\}$, and as a result, $\overline{u}\in W_0^{1,\vec{p}(\cdot)}(B_1(0))$.
Finally, we show that $\int_{B(0,1)}\frac{(1-|x|)^k(1-|x|)^{\alpha(1-\beta(x))}}{\beta(x)}dx<\infty$. Then,

$$\int_{B_1(0)}\frac{(1-|x|)^k(1-|x|)^{\alpha(1-\beta(x))}}{\beta(x)}dx\leq\frac{1}{\beta^-}\int_{B_1(0)}(1-|x|)^{k+\alpha(1-\beta^+)}dx<\infty.$$

Thus, by Theorem 3.8, problem (4.1) has at least one positive $W_0^{1,\vec{p}(\cdot)}(B_1(0))$-solution.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by Athabasca University Research Incentive Account [140111 RIA].

Conflict of interest

The author declares there is no conflict of interest.

References

1. D. V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue spaces: Foundations and harmonic analysis*, Birkhäuser Basel, 2013. https://doi.org/10.1007/978-3-0348-0548-3
2. L. Diening, P. Harjulehto, P. Hästö, M. Ruzicka, *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer Berlin, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-18363-8
3. M. Ruzicka, *Electrorheological fluids: modeling and mathematical theory*, Springer Berlin, Heidelberg, 2007. https://doi.org/10.1007/BFb0104029
4. B. Cekic, A. Kalinin, R. Mashiyev, M. Avci, $L^{p(x)}(\Omega)$-estimates of vector fields and some applications to magnetostatics problems, *J. Math. Anal. Appl.*, **389** (2012), 838–851. https://doi.org/10.1016/j.jmaa.2011.12.029
5. Y. Chen, S. Levine, M. Rao, Variable exponent, linear growth functionals in image restoration, *SIAM. J. Appl. Math.*, **66** (2006), 1383–1406. https://doi.org/10.1137/0506245
6. M. Avci, Ni-serrin type equations arising from capillarity phenomena with non-standard growth, *Bound. Value. Probl.*, **55** (2013), 1–13. https://doi.org/10.1186/1687-2770-2013-55
7. S. Heidarkhani, S. Moradi, M. Avci, Critical points approaches to a nonlocal elliptic problem driven by a $p(x)$-biharmonic operator, *Georgian. Math. J.*, **29** (2022), 55–69. https://doi.org/10.1515/gmj-2021-2115
8. Z. Yucedag, Existence of solutions for $p(x)$-laplacian equations without Ambrosetti–Rabinowitz type condition, *B. Malays. Math. Sci. So.*, **38** (2015), 1023–1033. https://doi.org/10.1007/s40840-014-0057-1
9. G. A. Afrouzi, N. T. Chung, S. Mahdavi, Existence and multiplicity of solutions for anisotropic elliptic systems with non-standard growth conditions, *Electron. J. Differ. Eq.*, **32** (2012), 1–15.
10. M. M. Boureanu, Multiple solutions for two general classes of anisotropic systems with variable exponents, *J. Anal. Math.*, **150** (2023), 1–51. https://doi.org/10.1007/s11854-023-0287-y
11. M. S. B. Elemine Vall, A. Ahmed, Multiplicity of solutions for a class of Neumann elliptic systems in anisotropic Sobolev spaces with variable exponent, *Adv. Oper. Theory*, **4** (2019), 497–513. https://doi.org/10.15352/aot.1808-1409
12. X. Fan, Anisotropic variable exponent sobolev spaces and-laplacian equations, *Complex Var Elliptic*, **56** (2011), 623–642. https://doi.org/10.1080/17476931003728412
13. J. Henríquez-Amador, A. Vélez-Santiago, Generalized anisotropic neumann problems of ambrosetti–prodi type with nonstandard growth conditions, *J. Math. Anal. Appl.*, **494** (2021), 124668. https://doi.org/10.1016/j.jmaa.2020.124668

14. B. Kone, S. Ouaro, S. Traore, Weak solutions for anisotropic nonlinear elliptic equations with variable exponents, *Electron. J. Diffe. Eq.*, **144** (2009), 1–11.

15. M. Mihăilescu, P. Pucci, V. Rădulescu, Eigenvalue problems for anisotropic quasilinear elliptic equations with variable exponent, *J. Math. Anal. Appl.*, **340** (2008), 687–698. https://doi.org/10.1016/j.jmaa.2007.09.015

16. I. Fragalà, F. Gazzola, B. Kawohl, Existence and nonexistence results for anisotropic quasilinear elliptic equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **21** (2004), 715–734. https://doi.org/10.1016/j.anihpc.2003.12.001

17. L. Kozhevnikova, On solutions of anisotropic elliptic equations with variable exponent and measure data, *Complex Var Elliptic*, **66** (2020), 333–367. https://doi.org/10.1080/17476933.2019.1579206

18. N. Mokhtar, F. Mokhtari, Anisotropic nonlinear elliptic systems with variable exponents and degenerate coercivity, *Appl. Anal.*, **100** (2021), 2347–2367. https://doi.org/10.1080/00036811.2023.2240333

19. A. R. Leggat, S.EH Miri, An existence result for a singular-regular anisotropic system, *Rend. Circ. Mat. Palermo, II*, **72** (2023), 977—996. https://doi.org/10.1007/s12215-022-00718-x

20. N. S. Papageorgiou, C. Vetro, F. Vetro, Singular Anisotropic Problems with Competition Phenomena, *J Geom Anal*, **33** (2023), 173. https://doi.org/10.1007/s12220-023-01227-8

21. M. Bohner, G. Caristi, A. Ghobadi, S. Heidarkhani Three solutions for discrete anisotropic Kirchhoff-type problems, *Dem. Math*, **56** (2023), 20220209. https://doi.org/10.1515/dema-2022-0209

22. S. EH Miri, On an anisotropic problem with singular nonlinearity having variable exponent, *Ric Mat*, **66** (2017), 415–424. https://doi.org/10.1007/s11587-016-0309-5

23. M. Naceri, Singular anisotropic elliptic problems with variable exponents, *Mem. Differ. Equ. Math.*, **85** (2022), 119–132.

24. N. S. Papageorgiou, A. Scapellato, Positive solutions for anisotropic singular dirichlet problems, *B. Malays. Math. Sci. So.*, **45** (2022), 1141–1168. https://doi.org/10.1007/s40840-022-01249-5

25. D. D. Repovš, K. Saoudi, The nehari manifold approach for singular equations involving the $p(x)$-laplace operator, *Complex Var Elliptic*, **68** (2006), 135–149. https://doi.org/10.1080/17476933.2021.1980878

26. K. Kefi, V. D. Rădulescu, On a $p(x)$-biharmonic problem with singular weights, *Z. Angew. Math. Phys.*, **68** (2017), 1–13. https://doi.org/10.1007/s00033-017-0827-3

27. Q. Zhang, Existence of solutions for $p(x)$-laplacian equations with singular coefficients in $\mathbb{R}^n$, *J. Math. Anal. Appl.*, **348** (2008), 38–50. https://doi.org/10.1016/j.jmaa.2008.06.026

28. A. Mokhtari, K. Saoudi, N. Chung, A fractional $p(x,\cdot)$-laplacian problem involving a singular term, *Indian J. Pure Ap Mat*, **53** (2021), 100–111. https://doi.org/10.1007/s13226-021-00037-4

29. C. Luning, W. Perry, An iterative method for solution of a boundary value problem in non-newtonian fluid flow, *J Non-Newton Fluid*, **15** (1984), 145–154. https://doi.org/10.1016/0377-0257(84)80002-6

30. J. Hernandez, F. J. Mancebo, J. M. Vega, Positive solutions for singular nonlinear elliptic equations, *P. Roy. Soc. Edinb. A.*, **137** (2007), 41–62. https://doi.org/10.1017/S030821050500065X

31. C. O. Alves, D. P. Covei, Existence of solution for a class of nonlocal elliptic problem via sub–supersolution method, *Nonlinear Anal-Real*, **23** (2015), 1–8. https://doi.org/10.1016/j.nonrwa.2014.11.003

32. M. Avci, Positive ground state solutions to a nonlocal singular elliptic problem, *Canad. J. Appl. Math.*, **1** (2019), 1–14.

33. J. Diaz, J. Hernández, J. Rakotoson, On very weak positive solutions to some semilinear elliptic problems with simultaneous singular nonlinear and spatial dependence terms, *Milan J Math*, **79** (2011), 233–245. https://doi.org/10.1007/s00032-011-0151-x

34. J. Shi, M. Yao, On a singular nonlinear semilinear elliptic problem, *P. Roy. Soc. Edinb. A.*, **128** (1998), 1389—1401. https://doi.org/10.1017/S0308210500027384

35. Y. Haitao, Multiplicity and asymptotic behavior of positive solutions for a singular semilinear elliptic problem, *J. Differ. Equations.*, **189** (2003), 487–512. https://doi.org/10.1016/S0022-0396(02)00098-0

36. S. Yijing, W. Shaoping, L. Yiming, Combined effects of singular and superlinear nonlinearities in some singular boundary value problems, *J. Differ. Equations.*, **176** (2001), 511–531. https://doi.org/10.1006/jdeq.2000.3973

37. S. Yijing, Z. Duanzhi, The role of the power 3 for elliptic equations with negative exponents, *Calc. Var. Partial. Dif.*, **49** (2014), 909–922. https://doi.org/10.1007/s00526-013-0604-x

38. S. Yijing, Compatibility phenomena in singular problems, *P. Roy. Soc. Edinb. A.*, **143** (2013), 1321–1330. https://doi.org/10.1017/S030821051100117X

39. A. C. Lazer, P. J. McKenna, On a singular nonlinear elliptic boundary-value problem, *P. Am. Math. Soc.*, **111** (1991), 721–730. https://doi.org/10.2307/2048410

40. Z. Zhang, J. Cheng, Existence and optimal estimates of solutions for singular nonlinear dirichlet problems, *Nonlinear Anal-Theor*, **57** (2004), 473–484. https://doi.org/10.1016/j.na.2004.02.025

41. L. Boccardo, L. Orsina, Semilinear elliptic equations with singular nonlinearities, *Calc. Var. Partial. Dif.*, **37** (2010), 363–380. https://doi.org/10.1007/s00526-009-0266-x

42. G. Tarantello, On nonhomogeneous elliptic equations involving critical sobolev exponent, *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **9** (1992), 281–304. https://doi.org/10.1016/S0294-1449(16)30238-4

43. I. Ekeland, On the variational principle, *J. Math. Anal. Appl.*, **47** (1974), 324–353. https://doi.org/10.1016/0022-247X(74)90025-0

44. X. Fan, D. Zhao, On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$, *J. Math. Anal. Appl.*, **263** (2001), 424–446. https://doi.org/10.1006/jmaa.2000.7617

45. V. D. Radulescu, D. D. Repovs, *Partial differential equations with variable exponents: variational methods and qualitative analysis*, CRC press, 2015. https://doi.org/10.1201/b18601

46. V. Zhikov, *Lavrentiev phenomenon and homogenization for some variational problems*, Composite media and homogenization theory. Proceedings of the second workshop, 1995, 273–288.

47. J. Rákosník, Some remarks to anisotropic sobolev spaces I, *Beiträgezur Analysis*, **13** (1979), 55–88.

48. J. Rákosník, Some remarks to anisotropic sobolev spaces II, *Beiträgezur Analysis*, **15** (1981), 127–140.

AIMS Press