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Research article

On a singular parabolic p-Laplacian equation with logarithmic nonlinearity

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Abstract: In this paper, we considered a singular parabolic p-Laplacian equation with logarithmic nonlinearity in a bounded domain with homogeneous Dirichlet boundary conditions. We established the local solvability by the technique of cut-off combining with the method of Faedo-Galerkin approximation. Based on the potential well method and Hardy-Sobolev inequality, the global existence of solutions was derived. In addition, we obtained the results of the decay. The blow-up phenomenon of solutions with different indicator ranges was also given. Moreover, we discussed the blow-up of solutions with arbitrary initial energy and the conditions of extinction.

Keywords: Non-Newton filtration equation; singular potential; logarithmic nonlinearity; global existence; decay; blow-up
Mathematics Subject Classification: 35A02, 35B44, 35L67

1. Introduction

In this paper, we are concerned with the following initial-boundary problem:

$$\begin{cases} |x|^{-s}u_t - \Delta_p u = |u|^{q-2}u\ln|u|, & x\in\Omega,\ t>0; \\ u(x,t)=0, & x\in\partial\Omega,\ t>0; \\ u(x,0)=u_0(x), & x\in\Omega, \end{cases} \tag{1.1}$$

where $\Omega\subset\mathbb{R}^N(N\geq N_\Omega)$ is a bounded domain with smooth boundary $\partial\Omega$, $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ with $u_0\in W_0^{1,p}(\Omega)$, $x=(x_1,x_2,\ldots,x_N)\in\mathbb{R}^N$ with $|x|=\sqrt{x_1^2+x_2^2+\cdots+x_N^2}$, and the parameters satisfy

$$0\leq s\leq 2,\quad \max\left\{\frac{2N}{N+2},1\right\}<p\leq q<p\left(1+\frac{2}{N}\right).$$

As is well known, according to the law of conservation, many diffusion processes with reactions can be described by the following equation (see [1]):

$$u_t - \nabla \cdot (D\nabla u) = f(x, t, u, \nabla u), \tag{1.2}$$

where $u(x, t)$ stands for the mass concentration in chemical reaction processes or temperature in heat conduction, at position x in the diffusion medium and time t. The function D is called the diffusion coefficient or the thermal diffusivity, the term $\nabla \cdot (D\nabla u)$ represents the rate of change due to diffusion, and $f(x, t, u, \nabla u)$ is the rate of change due to reaction.

In the past few years, many researchers had focused on Equation (1.2). For more details, one can refer to [2–6]. For the source $f(x, t, u, \nabla u) = u^q$, there has already been much discussion. For example, for $D = |x|^2$, in 2004, Tan [7] considered the existence and asymptotic estimates of global solutions as well as finite time blow-up of local solutions based on the classical Hardy inequality [8]. Han [9] considered the blow-up properties of solutions to the following non-Newton filtration equation with special a medium void:

$$|x|^{-2} u_t - \triangle_p u = u^q. \tag{1.3}$$

A new criterion for the solutions to blow up in finite time was established by using the Hardy inequality. Moreover, the upper and lower bounds for the blow-up time were also estimated. The results solved an open problem proposed by Liu [10] in 2016.

When the source $f(u)$ is a logarithmic nonlinearity, Deng and Zhou [11] investigated the following semilinear heat equation with singular potential and logarithmic nonlinearity

$$|x|^{-s} u_t - \triangle u = u \ln |u|, \tag{1.4}$$

under an appropriate initial-boundary value condition. They did make full use of the logarithmic Sobolev inequality in [12, 13] to handle the difficulty caused by the logarithmic nonlinear term $u \ln |u|$. Taking the combination of a family of potential wells, the existence of global solutions and infinite time blow-up solutions were obtained.

Liu and Fang [14] considered a fourth-order singular parabolic equation involving logarithmic nonlinearity and p-biharmonic operator

$$|x|^{-s} u_t + \triangle(|\triangle u|^{p-2} \triangle u) = |u|^{q-2} u \log |u|, \tag{1.5}$$

and they established the local solvability by the technique of cut-off combining with the methods of Faedo-Galerkin approximation and multiplier. Meantime, by virtue of the family of potential wells, they used the technique of modified differential inequality and the improved logarithmic Sobolev inequality to obtain the global solvability and the infinite and finite time blow-up phenomena, and derived the upper bound of blow-up time as well as the estimate of the blow-up rate. Furthermore, the results of blow-up with arbitrary initial energy and extinction phenomena were presented.

Motivated by these works, in this paper, we consider the Problem (1.1) with the presence of nonlinear diffusion $\triangle_p u := \text{div}(|\nabla u|^{p-2} \nabla u)$ and logarithmic nonlinearity $|u|^{q-2} u \ln |u|$. To the best our knowledge, this is the first work in the literature that takes into account a singular parabolic p-Laplacian equation with logarithmic nonlinearity.

The rest of this paper is organized as follows. In Section 2, we introduce some symbols and definitions. In Section 3, we prove the local existence and uniqueness theorem. In Section 4, we prove the global existence and asymptotic behavior theorems of solutions. In Section 5, the blow-up phenomena of solutions are discussed. Finally, the extinction phenomenon of the solution is given in Section 6.

2. Preliminaries

In this section, we introduce some notations and lemmas that will be used throughout the paper. In what follows, we denote by $\|\cdot\|_r$ $(r \geq 1)$ the norm in $L^r(\Omega)$ and by $(\cdot,\cdot)$ the $L^2(\Omega)$ inner product. When $p > 1,\ p \neq 2$, we use $W_0^{1,p}(\Omega)$ to denote the Sobolev space such that both u and ∇u belong to $L^p(\Omega)$ for any $u \in W_0^{1,p}(\Omega)$, denote by $W^{-1,p'}(\Omega)$ its dual space, and by $\langle\cdot,\cdot\rangle$ the duality pairing between them. We will equip $W_0^{1,p}(\Omega)$ with the norm $\|u\|_{W_0^{1,p}} = \|\nabla u\|_p$, which is equivalent to the full one due to the Poincarés inequality. We use $\lambda_1 > 0$ to denote the first eigenvalue of $-\triangle$ in Ω under the homogeneous Dirichlet boundary condition. We also use notation X_0 to denote $W_0^{1,p}(\Omega)\backslash\{0\}$.

Due to the presence of the inverse coefficient $|x|^{-s}$, it is worth emphasizing the difference between the two cases when $0 \in \Omega$ and $0 \notin \Omega$.

If $0 \in \Omega$, then $|x|^{-s}$ develops a singularity. This necessitates the use of the Hardy-Sobolev inequality, which is valid for $N_\Omega \geq 3$, in the proofs of our main results.

On the other hand, if $0 \notin \Omega$, then there is no singularity and (1.1) can be regarded as a slight extension. In this case, our results are valid for all $N \in \{1, 2, 3, \ldots\}$. To deal with these two cases simultaneously, we employ the notation

$$N_\Omega = \begin{cases} 3, & if\ 0 \in \Omega, \\ 1, & if\ 0 \notin \Omega. \end{cases}$$

First, for Problem (1.1), we introduce the potential energy functional

$$J(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \ln|u| dx + \frac{1}{q^2}\|u\|_q^q, \tag{2.1}$$

and the Nehari functional

$$I(u) = \|\nabla u\|_p^p - \int_\Omega |u|^q \ln|u| dx. \tag{2.2}$$

By a direct computation,

$$J(u) = \frac{1}{q}I(u) + \left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q. \tag{2.3}$$

By $I(u)$ and $J(u)$, we define the potential well:

$$W_1 = \{u \in X_0 : J(u) < d\},\ W_2 = \{u \in X_0 : J(u) = d\},\ W = W_1 \cup W_2,$$

$$W_1^+ = \{u \in W_1, I(u) > 0\},\ W_2^+ = \{u \in W_2, I(u) > 0\},\ W^+ = W_1^+ \cup W_2^+,$$

$$W_1^- = \{u \in W_1, I(u) < 0\},\ W_2^- = \{u \in W_2, I(u) < 0\},\ W^- = W_1^- \cup W_2^-,$$

and the Nehari manifold

$$N = \{u \in X_0,\ I(u) = 0\}.$$

The depth of the potential well is defined as

$$d = \inf_{u \in N} J(u).$$

The solution $u(x,t)$ to Problem (1.1) is considered in weak sense as follows. Sometimes $u(x,t)$ will be simply written as $u(t)$ if no confusion arises.

Lemma 2.1. [15] *Let μ be a positive number. Then we have the following inequalities:*

$$s^p \ln s \le (e\mu)^{-1} s^{p+\mu}, \quad for\ all\ s \ge 1,$$

$$|s^p \ln s| \le (ep)^{-1}, \quad for\ all\ 0 < s < 1.$$

Lemma 2.2. [15, 16] *Assume that $q < \frac{Np}{N-p}$, i.e., $q < \infty$ for $N \le p$ and $r \le q < \frac{Np}{N-p}$ for $N > p$ and $r \ge 1$. Then for any $u \in W_0^{1,p}(\Omega)$, it holds that*

$$\|u\|_q \le C_G \|\nabla u\|_p^{\theta} \|u\|_r^{1-\theta},$$

where $\theta \in (0,1)$ is determined by $\theta = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{N} - \frac{1}{p} + \frac{1}{r}\right)^{-1}$ and the constant $C_G > 0$ depends on N, p, q, and r.

Remark 2.1. *From $p > \frac{2N}{N+2}$, we deduce*

$$p\left(1 + \frac{2}{N}\right) < \begin{cases} \dfrac{Np}{N-p}, & if\ N > p, \\ +\infty, & if\ N \le p. \end{cases}$$

Then by the Sobolev inequality, we have $W_0^{1,p}(\Omega) \hookrightarrow L^{q+a}(\Omega)$ for $p > 1$ and $\forall\ a \ge 0$.

Lemma 2.3. *Let $u(t) \in X_0$ and p, q satisfy $\max\left\{\frac{2N}{N+2}, 1\right\} < p \le q < p\left(1 + \frac{2}{N}\right)$. We have the following statements:*
(i) If $0 < \|u\|_p \le r$, then $I(u) \ge 0$;
(ii) If $I(u) < 0$, then $\|u\|_p > r$;
(iii) If $I(u) = 0$, then $\|u\|_p = 0$ or $\|u\|_p \ge r$,
where

$$r = \left(\frac{1}{B_*^{q+\alpha}}\right)^{\frac{1}{q+\alpha-p}}.$$

Proof. (i) A direct computation yields

$$\ln |u(x)| < \frac{|u(x)|^{\alpha}}{\alpha}, a.e. x \in \Omega, \forall \alpha > 0. \tag{2.4}$$

Then, by the definition of $I(u)$, we have

$$\begin{aligned} I(u) &= \|\nabla u\|_p^p - \int_{\Omega} |u|^q \ln |u|\, dx \\ &= \|\nabla u\|_p^p - \|u\|_{q+\alpha}^{q+\alpha} \\ &\ge \left(1 - B_*^{q+\alpha} \|\nabla u\|_p^{q+\alpha-p}\right) \|\nabla u\|_p^p, \end{aligned} \tag{2.5}$$

where B_* is the imbedding constant for $W_0^{1,p}(\Omega) \hookrightarrow L^{q+\alpha}(\Omega)$. If $0 < \|\nabla u\|_p \le r$, this implies that $\|\nabla u\|_p^{q+\alpha-p} \le \frac{1}{B_*^{q+\alpha}}$. Therefore, we gain $I(u) \ge 0$ by (2.5).

(ii) From (2.5) and $I(u) < 0$, we can see that

$$\left(1 - B_*^{q+\alpha} \|\nabla u\|_p^{q+\alpha-p}\right) \|\nabla u\|_p^p < 0,$$

which means that

$$\|\nabla u\|_p > \left(\frac{1}{B_*^{q+\alpha}}\right)^{\frac{1}{q+\alpha-p}} = r.$$

(iii) If $I(u) = 0$, then from (2.5) we attain

$$\|\nabla u\|_p \geq \left(\frac{1}{B_*^{q+\alpha}}\right)^{\frac{1}{q+\alpha-p}} \quad or \quad \|\nabla u\|_p = 0.$$

The prove is complete.

Next, in Lemma 2.4, we describe some basic properties of the fiber mapping $J(\lambda u)$ that can be verified directly.

Lemma 2.4. [17] *Assume that* $u \in X_0$*, then*
(i) $\lim_{\lambda \to 0^+} J(\lambda u) = 0$, $\lim_{\lambda \to +\infty} J(\lambda u) = -\infty$.
(ii) *There exists a unique* $\lambda^* = \lambda^*(u) > 0$ *such that* $\frac{d}{d\lambda} J(\lambda u)\big|_{\lambda=\lambda^*} = 0$.
(iii) $J(\lambda u)$ *is increasing on* $0 < \lambda < \lambda^*$*, decreasing on* $\lambda^* < \lambda < +\infty$*, and attains the maximum at* $\lambda = \lambda^*$.
(iv) $I(\lambda u) > 0$ *for* $0 < \lambda < \lambda^*$*,* $I(\lambda u) < 0$ *for* $\lambda^* < \lambda < +\infty$*, and* $I(\lambda^* u) = 0$.

Lemma 2.5. [15, 18] (Logarithmic Sobolev Inequality). Let $q > 1$, $\mu > 0$, and $u \in W_0^{1,q}\left(\mathbb{R}^N\right) \setminus \{0\}$. Then we have

$$q \int_{R^N} |u(x)|^q \ln\left(\frac{|u(x)|}{\|u\|_{L^q(R^N)}}\right) dx + \frac{N}{q} \ln\left(\frac{q\mu e}{N\vartheta_q}\right) \int_{R^N} |u(x)|^q dx \leq \mu \int_{R^N} |\nabla u(x)|^q dx,$$

where

$$\vartheta_q = \frac{q}{N}\left(\frac{q-1}{e}\right)^{q-1} \pi^{-\frac{q}{2}} \left[\frac{\Gamma\left(\frac{N}{2}+1\right)}{\Gamma\left(N\frac{q-1}{q}+1\right)}\right]^{\frac{q}{N}}.$$

Remark 2.2. *If* $u \in W_0^{1,q}(\Omega) \setminus \{0\}$*, then by defining* $u(x) = 0$ *for* $x \in \mathbb{R}^N \setminus \Omega$*, we derive*

$$q \int_{\Omega} |u(x)|^q \ln\left(\frac{|u(x)|}{\|u\|_{L^q(\Omega)}}\right) dx + \frac{N}{q} \ln\left(\frac{q\mu e}{N\vartheta_q}\right) \int_{\Omega} |u(x)|^q dx \leq \mu \int_{\Omega} |\nabla u(x)|^q dx, \tag{2.6}$$

for any real number $\mu > 0$.

Lemma 2.6. [14, 19] (Hardy-Sobolev inequality). *Let* $R^N = R^k \times R^{N-k}$, $2 \leq k \leq N$ *and* $x = (y, z) \in R^N = R^k \times R^{N-k}$*. For given* n, β *satisfying* $1 < p < N$, $0 \leq \beta \leq p$*, and* $\beta < k$*, let* $m(\beta, N, p) = \frac{p(N-\beta)}{(N-p)}$*. Then there exists a positive constant* C_H *depending on* β, N, p*, and* k *such that for any* $u \in W_0^{1,p}(R^N)$*, it holds that*

$$\int_{\mathbb{R}^N} |u(x)|^m |y|^{-\beta} dx \leq C_H \left(\int_{\mathbb{R}^N} |\nabla u|^p dx\right)^{\frac{N-\beta}{N-p}}.$$

Remark 2.3. *(i) When $m = p = \beta$, this inequality is the classical Hardy inequality.*
(ii) If $m = 2, \beta = s$ in Lemma 2.4, we have $p = \frac{2N}{N-s+2} > 2$, and then Lemma 2.6 becomes

$$\int_\Omega |u(x)|^2|x|^{-s}dx \le C_H\Big(\int_\Omega |\nabla u|^{\frac{2N}{N-s+2}}dx\Big)^{\frac{N-s+2}{N}}.$$

Lemma 2.7. [15,20] *Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function and σ be a positive constant such that*

$$\int_t^{+\infty} f^{1+\sigma}(s)ds \le \frac{1}{\omega} f^{\sigma}(0)f(t), \quad \forall\ t \ge 0.$$

Then we have
(i) $f(t) \le f(0)e^{1-\omega t}$, for all $t \ge 0$, whenever $\sigma = 0$.
(i) $f(t) \le f(0)\Big(\frac{1+\sigma}{1+\omega\sigma t}\Big)^{\frac{1}{\sigma}}$, for all $t \ge 0$, whenever $\sigma > 0$.

The following is the concavity lemma.

Lemma 2.8. [21–23] *Suppose that a positive, twice-differentiable function $\Psi(t)$ satisfies the inequality*

$$\Psi''(t)\Psi(t) - (1+\theta)(\Psi'(t))^2 \ge 0,$$

where $\theta > 0$. If $\Psi(0) > 0$ and $\Psi'(0) > 0$, then $\Psi(t) \to \infty$ as

$$t \to t_* \le t^* = \frac{\Psi(0)}{\theta\Psi'(0)}.$$

Lemma 2.9. [24] *Suppose that $0 < l < r \le 1$ and $\epsilon_1, \epsilon_2 \ge 0$ are positive constants. If nonnegative and absolutely continuous function $h(t)$ satisfies*

$$h'(t) + \epsilon_1 h^l(t) \le \epsilon_2 h^r(t), t \ge 0,$$

$$h(0) > 0, \epsilon_2 h^{r-l}(0) < \epsilon_1,$$

then we have

$$h(t) \le \left[-\epsilon_0(1-l)t + h^{1-l}(0)\right]^{\frac{1}{1-l}}, 0 < t < T_0,$$

and

$$h(t) \equiv 0, t \ge T_0,$$

where $\epsilon_0 = \epsilon_1 - \epsilon_2 h^{r-l}(0)$ and $T_0 = \frac{h^{1-l}(0)}{\epsilon_0(1-l)}$.

Definition 2.1. (Weak Solution). *A function $u := u(x,t) \in L^\infty(0,T;X_0)$ with $|x|^{-\frac{s}{2}} u_t \in L^2(0,T;L^2(\Omega))$ is called a weak solution of Problem (1.1) on $\Omega \times [0,T)$ if $u(x,0) = u_0(x)$ in X_0 and*

$$\langle |x|^{-s} u_t, v\rangle + \left\langle |\nabla u|^{p-2}\nabla u, \nabla v\right\rangle = \left\langle |u|^{q-2}u\ln|u|, v\right\rangle, \quad a.e.\ t \in (0,T),$$

for any $v \in W_0^{1,p}(\Omega)$. Moreover,

$$\int_0^t \left\||x|^{-\frac{s}{2}} u_\tau\right\|_2^2 d\tau + J(u(x,t)) = J(u_0), \quad a.e.\ \ t \in (0,T).$$

Definition 2.2. (Maximal Existence Time). *Let $u(x,t)$ be a weak solution of Problem* (1.1)*, we define the maximal existence time T_{max} as follows:*

$$T_{\max} = \sup\{T > 0;\ u(x,t) \text{ exists on } [0,T]\}.$$

(*i*) *If $T_{\max} = +\infty$, we say that the solution $u(t)$ is global;*
(*ii*) *If $T_{\max} < +\infty$, we say that the solution $u(t)$ blows up in finite time and $T_{\max}$ is the blow-up time.*

Definition 2.3. (Finite Time Blow-Up). *Let $u(x,t)$ be a weak solution of Problem* (1.1)*, then $u(x,t)$ is called the finite time blow-up if the maximal existence time $T_{max} < +\infty$ and*

$$\lim_{t\to T_{max}^-}\int_0^t \left\||x|^{-\frac{s}{2}}\, u(\tau)\right\|_2^2 d\tau = +\infty.$$

Definition 2.4. (Infinite Time Blow-Up). *Let $u(x,t)$ be a weak solution of Problem* (1.1)*, then $u(x,t)$ is called the infinite blow-up if*

$$\lim_{t\to+\infty}\left\||x|^{-\frac{s}{2}}\, u(t)\right\|_2^2 = +\infty.$$

3. Local existence

In this section, we state the local existence and uniqueness of weak solutions to Problem (1.1).

Theorem 3.1. *Let $u_0 \in X_0$, and p, q satisfy* $\max\left\{\frac{2N}{N+2}, 1\right\} < p \le q < p\left(1+\frac{2}{N}\right)$. *Then there exist a $T > 0$ and a unique weak solution $u(x,t) \in L^\infty(0,T;X_0)$ of Problem* (1.1) *with $|x|^{-\frac{s}{2}}\, u_t \in L^2(0,T;L^2(\Omega))$ satisfying $u(0) = u_0$. Moreover, $u(x,t)$ satisfies the energy equality*

$$\int_0^t \left\||x|^{-\frac{s}{2}}\, u_t\right\|_2^2 dt + J(u) = J(u_0), \quad 0 \le t \le T.$$

Proof. We divide the proof of Theorem 3.1 into 5 steps.

Step 1. Approximate problem

In order to deal with the singular potential, we introduce the cut-off function

$$\rho_n(x) = \min\{|x|^{-s}, n\}, \forall\, n \in N^+.$$

We denote the solutions corresponding to ρ_n of Problem (1.1) as u_n,

$$\begin{cases} \rho_n(x)\, u_{nt} - \Delta_p u_n = |u_n|^{q-2}\, u_n \ln|u_n|, & x \in \Omega, t > 0, \\ u_n(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u_n(x,0) = u_{n0}, & x \in \Omega. \end{cases} \tag{3.1}$$

We noticed that $u_{n0} \in C_0^\infty(\Omega)$, and then $u_{n0} \to u_0(x)$ *in* $W_0^{1,p}(\Omega)$. Let $\left\{\omega_j\right\}_{j=1}^\infty$ be a system of basis in $W_0^{1,p}(\Omega)$ which is normalized orthogonal in $L^2(\Omega)$ and construct the approximate solution

$$u_n^k(x,t) = \sum_{j=1}^k a_{nj}^k(t)\omega_j(x) \quad for\ k = 1, 2, \cdots,\ j = 1, 2, \cdots, k.$$

We solve the problem

$$\left\langle \rho_n(x)u_{nt}^k, \omega_j \right\rangle + \left\langle \left|\nabla u_n^k\right|^{p-2}\nabla u_n^k, \nabla \omega_j \right\rangle + = \left\langle \left|u_n^k\right|^{q-2} u_n^k \ln\left|u_n^k\right|, \omega_j \right\rangle, \tag{3.2}$$

and

$$u_n^k(x,0) = \sum_{j=1}^{k} b_{nj}^k \omega_j(x) = u_{n0}^k \to u_{n0}(x) \;\; in \; W_0^{1,p}(\Omega) \tag{3.3}$$

as $k \to +\infty, n \to +\infty$. Hence $\left\{a_{nj}^k\right\}_{j=1}^k$ is determined by the following Cauchy problem:

$$\begin{cases} \sum_{j=1}^{k} \left(\int_\Omega \rho_n(x)\omega_j(x)\omega_j dx \right) \left[a_{nj}^k(t)\right]_t = G_{nj}^k(t), \\ a_{nj}^k(0) = b_{nj}^k, \end{cases}$$

where

$$\begin{aligned} G_{nj}^k(t) = & \int_\Omega \left| \sum_{j=1}^{k} a_{nj}^k(t)\omega_j(x) \right|^{q-2} \sum_{j=1}^{k} a_{nj}^k(t)\omega_j(x) \ln \left| \sum_{j=1}^{k} a_{nj}^k(t)\omega_j(x) \right| \omega_j dx \\ & - \int_\Omega \left| \sum_{j=1}^{k} a_{nj}^k(t)\nabla\omega_j(x) \right|^{p-2} \sum_{j=1}^{k} a_{nj}^k(t)\nabla\omega_j(x)\nabla\omega_j dx. \end{aligned}$$

Therefore, the standard theory of ordinary differential equations yields that there exists a $T > 0$ such that $a_{nj}^k(t) \in C^1([0,T])$. As a consequence, $u_n^k \in C^1([0,T], W_0^{1,p}(\Omega))$.

Step 2: Priori estimates

We discuss the following two cases:

Case 1: $\max\left\{\frac{2N}{N+2}, 1\right\} < p \leq q$ and $2 \leq q < p\left(1 + \frac{2}{N}\right)$

Multiply (3.2) by $a_{nj}^k(t)$, sum for $j = 1, \cdots, k$, and recall $u_n^k(x,t)$ to find

$$\left\langle \rho_n(x)u_{nt}^k, u_n^k \right\rangle + \left\langle \left|\nabla u_n^k\right|^{p-2}\nabla u_n^k, \nabla u_n^k \right\rangle = \left\langle \left|u_n^k\right|^{q-2} u_n^k \ln\left|u_n^k\right|, u_n^k \right\rangle. \tag{3.4}$$

Integrating over $(0,t)$ on both sides of (3.4), we get,

$$\frac{1}{2}\left\|(\rho_n(x))^{\frac{1}{2}} u_n^k(t)\right\|_2^2 + \int_0^t \left\|\nabla u_n^k(s)\right\|_p^p ds = \int_0^t \int_\Omega \left|u_n^k(s)\right|^q \ln\left|u_n^k(s)\right| dxds + \frac{1}{2}\left\|(\rho_n(x))^{\frac{1}{2}} u_n^k(0)\right\|_2^2.$$

Set

$$S_n^k(t) = \frac{1}{2}\left\|(\rho_n(x))^{\frac{1}{2}} u_n^k(t)\right\|_2^2 + \int_0^t \left\|\nabla u_n^k(s)\right\|_p^p ds. \tag{3.5}$$

Combining the above equalities, and we have

$$S_n^k(t) \leq S_n^k(0) + \int_0^t \int_\Omega \left|u_n^k(s)\right|^q \ln\left|u_n^k(s)\right| dxds. \tag{3.6}$$

From Lemma 2.1, we get

$$\begin{aligned}\int_{\Omega}\left|u_n^k(t)\right|^q\ln\left|u_n^k(t)\right|dx &= \int_{\Omega_1=\left\{x\in\Omega;\left|u_n^k(x)\right|\geq 1\right\}}\left|u_n^k(t)\right|^q\ln\left|u_n^k(t)\right|dx\\ &\quad+\int_{\Omega_2=\left\{x\in\Omega;\left|u_n^k(x)\right|<1\right\}}\left|u_n^k(t)\right|^q\ln\left|u_n^k(t)\right|dx\\ &\leq (e\mu)^{-1}\int_{\Omega_1=\left\{x\in\Omega;\left|u_n^k(x)\right|\geq 1\right\}}\left|u_n^k(t)\right|^{q+\mu}dx\\ &\leq (e\mu)^{-1}\left\|u_n^k(t)\right\|_{q+\mu}^{q+\mu}.\end{aligned}\tag{3.7}$$

Then, by Lemma 2.2 and Young's inequality, (3.7) becomes

$$\begin{aligned}\int_{\Omega}\left|u_n^k(t)\right|^q\ln\left|u_n^k(t)\right|dx &\leq (e\mu)^{-1}\left\|u_n^k(t)\right\|_{q+\mu}^{q+\mu}\\ &\leq (e\mu)^{-1}C_G\left\|\nabla u_n^k(t)\right\|_p^{\theta(q+\mu)}\left\|u_n^k(t)\right\|_2^{(1-\theta)(q+\mu)}\\ &\leq (e\mu)^{-1}C_G\varepsilon\left\|\nabla u_n^k(t)\right\|_p^p+(e\mu)^{-1}C_GC(\varepsilon)\left\|u_n^k(t)\right\|_2^{\frac{p(1-\theta)(q+\mu)}{p-\theta(q+\mu)}},\end{aligned}\tag{3.8}$$

where $\varepsilon\in(0,1)$, and $\theta=\left(\frac{1}{2}-\frac{1}{q+\mu}\right)\left(\frac{1}{N}-\frac{1}{p}+\frac{1}{2}\right)^{-1}=\frac{(q+\mu-2)Np}{(q+\mu)(2p-2N+Np)}$. We note that since $0<\mu<p(1+\frac{2}{N})-q$, $\theta(q+\mu)<p$ holds. Let

$$\alpha=\frac{p(1-\theta)(q+\mu)}{2\left[p-\theta(q+\mu)\right]}=\frac{p(N+q+\mu)-N(q+\mu)}{p(N+2)-N(q+\mu)},$$

then $\alpha>1$ since $\max\{1,\frac{2N}{N+2}\}<p\leq q$, $2\leq q<p\left(1+\frac{2}{N}\right)$. Besides, since Ω is a bounded domain in $\mathbb{R}^N$, it leads to

$$\int_{\Omega}\left|u_n^k(t)\right|^2dx=\int_{\Omega}(\rho_n(x))^{-1}\rho_n(x)\left|u_n^k(t)\right|^2dx\leq C(\Omega)\left\|(\rho_n(x))^{\frac{1}{2}}u_n^k(t)\right\|_2^2,\tag{3.9}$$

where $C(\Omega)$ is related to Ω.

Thus, from (3.5), (3.6), (3.8), and (3.9), we get

$$\begin{aligned}S_n^k(t)&\leq S_n^k(0)+\int_0^t(e\mu)^{-1}C_G\varepsilon\left\|\nabla u_n^k(t)\right\|_p^p ds+\int_0^t(e\mu)^{-1}C_GC(\varepsilon)\left\|u_n^k(t)\right\|_2^{2\alpha}ds\\ &\leq S_n^k(0)+(e\mu)^{-1}C_G\varepsilon S_n^k(t)+(e\mu)^{-1}C_GC(\varepsilon)C(\Omega)\int_0^t\left(S_n^k(t)\right)^{\alpha}ds,\end{aligned}$$

and

$$S_n^k(t)\leq\frac{S_n^k(0)}{1-(e\mu)^{-1}C_G\varepsilon}+\frac{(e\mu)^{-1}C_GC(\varepsilon)C(\Omega)}{1-(e\mu)^{-1}C_G\varepsilon}\int_0^t\left(S_n^k(t)\right)^{\alpha}ds.$$

Therefore,

$$S_n^k(t)\leq C_1+C_2\int_0^t\left(S_n^k(t)\right)^{\alpha}ds,\tag{3.10}$$

where $1-(e\mu)^{-1}C_G\varepsilon>0$, $C_1=\frac{S_n^k(0)}{1-(e\mu)^{-1}C_G\varepsilon}$, and $C_2=\frac{(e\mu)^{-1}C_GC(\varepsilon)C(\Omega)}{1-(e\mu)^{-1}C_G\varepsilon}$. From the Gronwall-Bellman-Bihari inequality, we obtain

$$S_n^k(t)\leq C_3,$$

and

$$\frac{1}{2}\left\|(\rho_n(x))^{\frac{1}{2}}u_n^k(t)\right\|_2^2+\int_0^t\left\|\nabla u_n^k(s)\right\|_p^p ds\leq C_3,\quad \forall\, n,k\in N^+, \tag{3.11}$$

where C_3 is a constant which is dependent on T.

Multiplying (3.2) by $\left[a_{nj}^k(t)\right]_t$, summing on $j=1,2,\cdots,k$, and then integrating on $(0,t)$, we know that

$$\int_0^t\left\|(\rho_n(x))^{\frac{1}{2}}u_{nt}^k(s)\right\|_2^2 ds+J\left(u_n^k(t)\right)=J\left(u_{n0}^k\right),\quad 0\leq t\leq T. \tag{3.12}$$

By the continuity of the functional J and (3.3), there exists a constant $C>0$ satisfying

$$J(u_{n0}^k)\leq C,\ \ \text{for any positive integer } n \text{ and } k. \tag{3.13}$$

Applying (2.1), (3.5), (3.8), (3.11), (3.12), and (3.13), we obtain

$$\int_0^t\left\|(\rho_n(x))^{\frac{1}{2}}u_{nt}^k(s)\right\|_2^2 ds+\left(\frac{1}{p}-\frac{C_G\varepsilon}{e\mu q}\right)\left\|\nabla u_n^k(t)\right\|_p^p+\frac{1}{q^2}\left\|u_n^k(t)\right\|_q^q-C_4\leq J\left(u_{n0}^k(t)\right)\leq C, \tag{3.14}$$

where $C_4=\frac{2C_GC(\varepsilon)C(\Omega)}{e\mu}(C_3)^{\alpha}$, for all $n,k\in N^+$.

Case 2: $\max\{1,\frac{2N}{N+2}\}<p\leq q<2$

Combining $\ln|u(x)|<\frac{|u(x)|^a}{a}$ a.e. $x\in\Omega,\forall\, a>0$ and (3.5), and taking $a=2-q$, we obtain

$$S_n^k(t)\leq S_n^k(0)+\frac{1}{2-q}\int_0^t\left\|u_n^k(s)\right\|_2^2 ds.$$

Together with (3.9), it can become

$$S_n^k(t)\leq S_n^k(0)+\frac{2}{(2-q)\,C(\Omega)}\int_0^t S_n^k(s)\,ds.$$

Then by means of Gronwall's inequality, we have

$$S_n^k(t)\leq C_5,$$

and

$$\frac{1}{2}\left\|(\rho_n(x))^{\frac{1}{2}}u_n^k(t)\right\|_2^2+\int_0^t\left\|\nabla u_n^k(s)\right\|_p^p ds\leq C_5, \tag{3.15}$$

where $C_5=S_n^k(0)e^{\frac{2T}{(2-q)C(\Omega)}}$.

From (2.1), (3.12), (3.13), and (3.15), we have

$$\begin{aligned}&\int_0^t\left\|(\rho_n(x))^{\frac{1}{2}}u_{nt}^k(s)\right\|_2^2 ds+\frac{1}{p}\left\|\nabla u_n^k(t)\right\|_p^p+\frac{1}{q^2}\left\|u_n^k(t)\right\|_q^q\\ &\leq C+\frac{1}{q}\int_\Omega\left|u_n^k(t)\right|^q\ln\left|u_n^k(t)\right|dx\\ &\leq C+\frac{2}{q\,(2-q)\,C(\Omega)}\left\|(\rho_n(x))^{\frac{1}{2}}u_n^k(t)\right\|_2^2\\ &\leq C+\frac{2C_5}{q\,(2-q)\,C(\Omega)},\end{aligned} \tag{3.16}$$

for all $k, n \in N^+$.

Therefore, we can derive

$$\left\|u_n^k(t)\right\|_{L^\infty\left(0,T;W_0^{1,p}(\Omega)\right)} \leq C, \quad \text{for any positive integer } n \text{ and } k, \tag{3.17}$$

$$\left\|u_n^k(t)\right\|_{L^\infty(0,T;L^q(\Omega))} \leq C, \quad \text{for any positive integer } n \text{ and } k, \tag{3.18}$$

$$\left\|(\rho_n(x))^{\frac{1}{2}} u_{nt}^k(t)\right\|_{L^2\left(0,T;L^2(\Omega)\right)} \leq C, \quad \text{for any positive integer } n \text{ and } k. \tag{3.19}$$

Combining (3.9) and (3.16), we have

$$\left\|u_{nt}^k(t)\right\|_{L^2\left(0,T;L^2(\Omega)\right)} \leq C, \quad \text{for any positive integer } n \text{ and } k. \tag{3.20}$$

Step 3: Pass to the limit

By (3.17), (3.19), and the Aubin-Lions-Simon Lemma (see [25], Corollary 4), we get

$$u_n^k \to u \ \ in \ \ C(0,T;L^2(\Omega)), \tag{3.21}$$

as $k, n \to +\infty$. Thus, $u_n^k(x,0) \to u(x,0)$ in $L^2(\Omega)$. Combining (3.3) with $u_{n0} \to u_0(x)$ in $W_0^{1,p}(\Omega)$, we observe that $u(x,0) = u_0$ in $W_0^{1,p}(\Omega)$.

From (3.21), we have $u_n^k \to u$ a.e.$(x,t) \in \Omega \times (0,T)$. This implies

$$\left|u_n^k\right|^{q-2} u_n^k \ln\left|u_n^k\right| \to |u|^{q-2} u \ln|u| \quad a.e.(x,t) \in \Omega \times (0,T).$$

It follows from (3.14) and the Hölder inequality that

$$\begin{aligned}
\left\|\left|\nabla u_n^k(t)\right|^{p-2}\nabla u_n^k(t)\right\|_{W^{-1,p'}(\Omega)} &= \sup_{\varphi\in W_0^{1,p}(\Omega)\backslash\{0\}} \frac{\int_\Omega \left|\nabla u_n^k(t)\right|^{p-2}\nabla u_n^k(t)\cdot\varphi dx}{\|\varphi\|_{W_0^{1,p}(\Omega)}} \\
&\leq \frac{\left(\int_\Omega \left|\left|\nabla u_n^k(t)\right|^{p-2}\nabla u_n^k(t)\right|^{\frac{p}{p-1}} dx\right)^{\frac{p-1}{p}} \left(\int_\Omega |\varphi|^p dx\right)^{\frac{1}{p}}}{\|\varphi\|_{W_0^{1,p}(\Omega)}} \\
&\leq \left\|\nabla u_n^k(t)\right\|_p^{p-1} < C.
\end{aligned}$$

That means

$$\left\|\left|\nabla u_n^k(t)\right|^{p-2}\nabla u_n^k(t)\right\|_{L^\infty\left(0,T;W^{-1,p'}(\Omega)\right)} \leq C, \quad \text{for any positive integer } n \text{ and } k. \tag{3.22}$$

On the other hand, there is

$$\begin{aligned}
\int_\Omega \left|\left|u_n^k\right|^{q-2}u_n^k \ln\left|u_n^k\right|\right|^{\frac{p}{p-1}} dx &= \int_{\Omega_1=\left\{x\in\Omega;\left|u_n^k(x)\right|\geq 1\right\}} \left|\left|u_n^k\right|^{q-2}u_n^k \ln\left|u_n^k\right|\right|^{\frac{p}{p-1}} dx \\
&+ \int_{\Omega_1=\left\{x\in\Omega;\left|u_n^k(x)\right|<1\right\}} \left|\left|u_n^k\right|^{q-2}u_n^k \ln\left|u_n^k\right|\right|^{\frac{p}{p-1}} dx.
\end{aligned}$$

From Lemma 2.1 and Lemma 2.2, we have

$$\begin{aligned}
&\int_{\Omega}\left|\left|u_n^k(t)\right|^{q-2}u_n^k(t)\ln\left|u_n^k(t)\right|\right|^{\frac{p}{p-1}}dx\\
=&\int_{\Omega_1}\left|\left|u_n^k(t)\right|^{q-2}u_n^k(t)\ln\left|u_n^k(t)\right|\right|^{\frac{p}{p-1}}dx+\int_{\Omega_2}\left|\left|u_n^k(t)\right|^{q-2}u_n^k(t)\ln\left|u_n^k(t)\right|\right|^{\frac{p}{p-1}}dx\\
\le&\int_{\Omega_1}\left|\left|u_n^k(t)\right|^{-\mu}\ln\left|u_n^k(t)\right|\cdot\left|u_n^k(t)\right|^{q-1+\mu}\right|^{\frac{p}{p-1}}dx+\int_{\Omega_2}\left|\left|u_n^k(t)\right|^{q-1}\ln\left|u_n^k(t)\right|\right|^{\frac{p}{p-1}}dx\\
\le&(e\mu)^{-\frac{p}{p-1}}\left\|u_n^k(t)\right\|_{\frac{p}{p-1}(q-1+\mu)}^{\frac{p}{p-1}(q-1+\mu)}+\left[e\left(q-1\right)\right]^{-\frac{p}{p-1}}|\Omega|\\
\le&(e\mu)^{-\frac{p}{p-1}}B_1\left\|\nabla u_n^k(t)\right\|_p^{\frac{p}{p-1}(q-1+\mu)}+\left[e\left(q-1\right)\right]^{-\frac{p}{p-1}}|\Omega|<C,
\end{aligned}$$

where B_1 is the best constant of the Sobolev embedding $W_0^{1,p}(\Omega)\hookrightarrow L^{\frac{p}{p-1}(q-1+\mu)}(\Omega)$. Here we choose $0<\mu\le p\left(1+\frac{p-1}{N-p}\right)-q$, $q<p\left(1+\frac{p-1}{N-p}\right)$, and we know that

$$\left\|\left|u_n^k(t)\right|^{q-2}u_n^k(t)\ln\left|u_n^k(t)\right|\right\|_{L^\infty\left(0,T;L^{\frac{p}{p-1}}(\Omega)\right)}\le C,\qquad \text{for any positive integer } n \text{ and } k. \tag{3.23}$$

By (3.17)–(3.19), (3.22), (3.23), there exist functions u,χ and a subsequence of $\left\{u_n^k\right\}_{n,k=1}^{\infty}$ which we still denote by $\left\{u_n^k\right\}_{n,k=1}^{\infty}$ such that

$$u_n^k\to u\quad \textit{weakly}\quad \textit{star}\quad \textit{in}\quad L^\infty(0,T;W_0^{1,p}(\Omega)) \tag{3.24}$$

$$(\rho_n(x))^{\frac{1}{2}}u_{nt}^k\to|x|^{-\frac{s}{2}}u_t\quad \textit{weakly}\quad \textit{in}\quad L^2(0,T;L^2(\Omega)) \tag{3.25}$$

$$\left|\nabla u_n^k\right|^{p-2}\nabla u_n^k\to\chi\quad \textit{weakly}\quad \textit{star}\quad \textit{in}\quad L^\infty(0,T;W^{-1,p'}(\Omega)) \tag{3.26}$$

$$\left|u_n^k\right|^{q-2}u_n^k\ln\left|u_n^k\right|\to|u|^{q-2}u\ln|u|\quad \textit{weakly}\quad \textit{star}\quad \textit{in}\quad L^\infty(0,T;L^{\frac{p}{p-1}}(\Omega)). \tag{3.27}$$

Next, by the method of Browder and Minty in the theory of monotone operators, we obtain $\chi=|\nabla u|^{p-2}\nabla u$.

By (3.24)–(3.27), passing to the limit in (3.2) as $n,k\to+\infty$, it follows that u satisfies the initial condition $u(0)=u_0$,

$$\left\langle|x|^{-s}u_t,\omega\right\rangle+\left\langle|\nabla u|^{p-2}\nabla u,\nabla\omega\right\rangle=\left\langle|u|^{q-2}u\ln|u|\,,\omega\right\rangle, \tag{3.28}$$

for all $\omega\in W_0^{1,p}(\Omega)$, and for a.e. $t\in[0,T]$.

Step 4. Uniqueness

Suppose there are two solutions u_1 and u_2 to Problem (1.1), and we have

$$\left\langle|x|^{-s}u_{1t},v\right\rangle+\left\langle|\nabla u_1|^{p-2}\nabla u_1,\nabla v\right\rangle=\left\langle|u_1|^{q-2}u_1\ln|u_1|\,,v\right\rangle, \tag{3.29}$$

and

$$\left\langle|x|^{-s}u_{2t},v\right\rangle+\left\langle|\nabla u_2|^{p-2}\nabla u_2,\nabla v\right\rangle=\left\langle|u_2|^{q-2}u_2\ln|u_2|\,,v\right\rangle. \tag{3.30}$$

Let $w=u_1-u_2$ and $w(0)=0$, then by subtracting (3.29) and (3.30), we can derive

$$\int_\Omega|x|^{-s}w_tv\,dx+\int_\Omega\left(|\nabla u_1|^{p-2}\nabla u_1-|\nabla u_2|^{p-2}\nabla u_2\right)\nabla v\,dx=\int_\Omega\left(|u_1|^{q-2}u_1\ln|u_1|-|u_2|^{q-2}u_2\ln|u_2|\right)v\,dx.$$

Let $v = w$, and we recall the following elementary vector inequalities that are used frequently: for all $a, b \in \mathbb{R}^N$, we have $0 \le (p-1)\frac{|a-b|^2}{(|a|+|b|)^{2-p}} \le \left(|a|^{p-2}a - |b|^{p-2}b\right)\cdot(a-b)$, if $1 < p < 2$. So, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\||x|^{-\frac{s}{2}}w\right\|_2^2 &\le \int_\Omega \frac{|u_1|^{q-2}u_1\ln|u_1| - |u_2|^{q-2}u_2\ln|u_2|}{w}w^2dx\\ &\le \int_\Omega \frac{f(u_1)-f(u_2)}{w}w^2dx.\end{aligned}$$

Integrating it on $[0, t]$, we obtain

$$\left\||x|^{-\frac{s}{2}}w\right\|_2^2 \le 2\int_0^t\int_\Omega \frac{f(u_1)-f(u_2)}{w}w^2dxdt, \tag{3.31}$$

where $F(s) = |s|^{q-2}\, s\ln|s|$. Combining with (3.9), we get

$$\|w\|_2^2 \le 2M_T\int_0^t \|w\|_2^2\,dt.$$

By the locally Lipschitz continuity of $F : R^N \to R$, the uniqueness follows from Gronwall's inequality.

Step 5: Energy equality

We multiply (1.1) with u_t and integrate over $\Omega\times(0,t)$ to obtain the equality

$$\int_0^t \left\||x|^{-\frac{s}{2}}u_t(s)\right\|_2^2ds + J(u(t)) = J(u_0), \quad 0 \le t \le T. \tag{3.32}$$

The proof of Theorem 3.1 is complete.

4. Global existence and decay rate

4.1. Global existence

In this section, we are concerned with the existence of a global weak solution to Problem (1.1).

Theorem 4.1. *Assume that* $u_0 \in W^+$, $\max\left\{\frac{2N}{N+2}, 1\right\} < p \le q < p(1+\frac{2}{N})$*, and then Problem* (1.1) *admits a global solution* $u \in L^\infty(0,\infty; X_0)$ *with* $|x|^{-\frac{s}{2}}u_t \in L^2(0,\infty; L^2(\Omega))$ *and* $u(t) \in W^+$ *for* $0 \le t \le \infty$.

Proof. Now, we prove Theorem 4.1. In order to prove the existence of global weak solutions, we consider the following two steps:

Step 1. The initial data $u_0 \in W_1^+$

From (3.32), we know that

$$\int_0^t \left\||x|^{-\frac{s}{2}}u_t(s)\right\|_2^2ds + J(u(t)) = J(u_0) < d, \quad 0 \le t \le T_{max}, \tag{4.1}$$

where T_{max} is the maximal existence time of solution $u(t)$. We shall prove that $T_{max} = +\infty$. Next, we will show that

$$u(t) \in W_1^+ \ for\ all\ \ 0 \le t \le T_{max}. \tag{4.2}$$

Indeed, assume that (4.2) does not hold and let t_* be the smallest time for which $u(t_*) \notin W_1^+$. Then, by the continuity of $u(t)$, one has $u(t_*) \in \partial W_1^+$. Hence, it follows that

$$J(u(t_*)) = d, \tag{4.3}$$

or

$$I(u(t_*)) = 0. \tag{4.4}$$

Nevertheless, it is clear that (4.3) could not occur by (4.1) while if (4.4) holds then, by the definition of d, we have

$$J(u(t_*)) \geq \inf_{u \in N} J(u) = d,$$

which also contradicts with (4.1). Hence, (4.2) is valid.

Next, it is discussed in two cases.

Case 1: $p < q$

As a consequence, it follows from this fact and the definition of functional $J(u(t))$ that

$$\int_0^t \left\| |x|^{-\frac{s}{2}} u_t(s) \right\|_2^2 ds + \frac{1}{q} I(u(t)) + \left(\frac{1}{p} - \frac{1}{q} \right) \|\nabla u(t)\|_p^p + \frac{1}{q^2} \|u(t)\|_q^q < d, \tag{4.5}$$

and

$$\int_0^t \left\| |x|^{-\frac{s}{2}} u_t(s) \right\|_2^2 ds + \left(\frac{1}{p} - \frac{1}{q} \right) \|\nabla u(t)\|_p^p + \frac{1}{q^2} \|u(t)\|_q^q < d. \tag{4.6}$$

This estimate allows us to take $T_{max} = +\infty$. So, we can conclude that there is a unique global weak solution $u(t) \in W_1^+$ of Problem (1.1) which satisfies that

$$\int_0^t \left\| |x|^{-\frac{s}{2}} u_t(s) \right\|_2^2 ds + J(u(t)) = J(u_0), \qquad 0 \leq t \leq +\infty.$$

Case 2: $p = q$

Similar to Case 1, we can derive

$$\int_0^t \left\| |x|^{-\frac{s}{2}} u_t(s) \right\|_2^2 ds + \frac{1}{p^2} \|u(t)\|_p^p < d.$$

By Lemma 2.5, we have

$$\int_\Omega |u(t)|^q \ln |u(t)| \, dx \leq \left[\ln \|u(t)\|_{L^q(\Omega)} - \frac{n}{q^2} \ln \left(\frac{q\mu e}{N\vartheta_q} \right) \right] \int_\Omega |u(t)|^q dx + \frac{\mu}{q} \int_\Omega |\nabla u(t)|^q dx.$$

From (2.2), (4.2), and the above inequality, we know that

$$\begin{aligned}
\|\nabla u(t)\|_p^p &= I(u(t)) + \int_\Omega |u(t)|^p \ln |u(t)| \, dx \\
&= 2I(u(t)) + 2 \int_\Omega |u(t)|^p \ln |u(t)| \, dx - \|\nabla u(t)\|_p^p \\
&\leq 2I(u(t)) + 2 \left[\ln \|u(t)\|_{L^p(\Omega)} - \frac{n}{p^2} \ln \left(\frac{p\mu e}{N\vartheta_p} \right) \right] \int_\Omega |u(t)|^p dx + \left(\frac{2\mu}{p} - 1 \right) \|\nabla u(t)\|_p^p \\
&\leq 2pJ(u(t)) + 2 \left[\frac{1}{p^2} + \ln \|u(t)\|_{L^p(\Omega)} - \frac{n}{p^2} \ln \left(\frac{p\mu e}{N\vartheta_p} \right) \right] \|u(t)\|_p^p \\
&\leq C_d.
\end{aligned} \tag{4.7}$$

Combining the two cases above, we know that the estimate allows us to take $T_{max} = +\infty$. It means that there is a unique global weak solution $u(t) \in W_1^+$ of Problem (1.1).

Step 2. The initial data $u_0 \in W_2^+$

First, we choose a sequence $\{\theta_m\}_{m=1}^{\infty} \subset (0,1)$ such that $\lim_{m\to\infty} \theta_m = 1$. Then we consider the following problem

$$\begin{cases} |x|^{-s}\, u_t - \Delta_p u = |u|^{q-2} u \ln |u|\,, & (x,t) \in \Omega \times R^+, \\ u = 0, & (x,t) \in \partial\Omega \times R^+, \\ u(x,0) = u_{0m}(x), & x \in \Omega, \end{cases} \tag{4.8}$$

where $u_{0m} = \theta_m u_0$. First of all, we claim that $u_{0m} \in W_1^+$, and then $J(u_{0m}) < d$ and $I(u_{0m}) > 0$. In fact, from $u_0 \in W_1^+$, $\lim_{m\to\infty} \theta_m = 1$ and $I(u_0) > 0$, and we can see

$$\begin{aligned} I(u_{0m}) &= \theta_m^p \, \|\nabla u_0\|_p^p - \theta_m^q \ln |\theta_m| \, \|u_0\|_q^q - \theta_m^q \int_\Omega |u_0|^q \ln |u_0| dx \\ &\geq \theta_m^p \left(\|\nabla u_0\|_p^p - \theta_m^{q-p} \int_\Omega |u_0|^q \ln |u_0| dx \right) \\ &\geq \theta_m^p I(u_0) > 0. \end{aligned}$$

On the other hand, by direct calculations, we obtain

$$\frac{d}{d(\theta_m)} J(\theta_m u_0) = \frac{1}{\theta_m} \left(\theta_m^p \, \|\nabla u_0\|_p^p - \theta_m^q \ln |\theta_m| \, \|u_0\|_q^q - \theta_m^q \int_\Omega |u_0|^q \ln |u_0| dx \right) = \frac{1}{\theta_m} I(u_{0m}) > 0,$$

which implies that $J(\theta_m u_0)$ is strictly increasing with respect to θ_m and

$$J(u_{0m}) = J(\theta_m u_0) < J(u_0) = d.$$

Since $u_{0m} \to u_0$ as $m \to +\infty$, our result can be derived by the same processes as the proof of Step 1.

Theorem 4.1 is complete.

4.2. Decay estimates

Theorem 4.2. *Let $u(t)$ be the solution of Problem* (1.1) *and p, q satisfy*

$$2 < p < q < p\left(1 + \frac{2}{N}\right).$$

If $u_0 \in W_1^+$, then there exist positive constants c_2 such that

$$\|\nabla u\,(t)\|_p^2 \leq \|\nabla u_0\|_p^2 \left(\frac{p-1}{1 + c_2(p-2)t} \right)^{\frac{1}{p-2}}, \quad t \geq 0.$$

Especially, if $p = 2$, then there exist positive constants c_4 such that

$$\|\nabla u\,(t)\|_2 \leq \|\nabla u_0\|_2 \, e^{\frac{1}{2}(1 - c_4 t)}, \quad t \geq 0.$$

Proof. We are now in a position to prove the algebraic decay results. Thanks to $u_0 \in W_1^+$, we get $u(t) \in W_1^+$. From (2.3), we have

$$\left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u\|_p^p + \frac{1}{q^2} \|u\|_q^q \leq J(u(t)) \leq J(u_0) < d. \tag{4.9}$$

By (4.9), through a direct calculation, we arrive at

$$\lambda_0 \left\{ \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u\|_p^p + \frac{1}{q^2} \|u\|_q^q \right\} \geq J\left(\lambda^* u(t)\right) \geq d,$$

where $\lambda_0 = \max\{(\lambda^*)^p, (\lambda^*)^q\}$. Combining with (4.9), we get

$$\lambda_0 \geq \max\left\{ \left(\frac{d}{J(u_0)}\right)^{\frac{1}{p}}, \left(\frac{d}{J(u_0)}\right)^{\frac{1}{q}} \right\} > 1, \tag{4.10}$$

which means that $\lambda^* > 1$, $\lambda^* \geq \left(\frac{d}{J(u_0)}\right)^{\frac{1}{p}}$.

From (2.2), we have

$$\begin{aligned} 0 = I(\lambda^* u) &= (\lambda^*)^p \|\nabla u\|_p^p - (\lambda^*)^q \int_\Omega |u|^q \ln|u| \, dx - (\lambda^*)^q \ln(\lambda^*) \|u\|_q^q \\ &= (\lambda^*)^q I(u) - ((\lambda^*)^q - (\lambda^*)^p) \|\nabla u(t)\|_p^p - (\lambda^*)^q \ln(\lambda^*) \|u\|_q^q. \end{aligned} \tag{4.11}$$

Namely,

$$I(u(t)) = \|u\|_q^q \ln \lambda^* + \left[1 - (\lambda^*)^{p-q}\right] \|\nabla u\|_p^p \geq c_1 \|\nabla u(t)\|_p^p, \tag{4.12}$$

where $c_1 = 1 - \left(\frac{d}{J(u_0)}\right)^{1-\frac{q}{p}}$, $p < q$.

According to Lemma 2.6, and (2.2), we obtain

$$\begin{aligned} \int_t^T I(u) \, ds &= \int_t^T \left(\|\nabla u\|_p^p - \int_\Omega |u|^q \ln|u| \, dx \right) ds \\ &= -\frac{1}{2} \int_t^T \frac{d}{dt} \left\| |x|^{-\frac{s}{2}} u \right\|_2^2 ds \\ &= \frac{1}{2} \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 - \frac{1}{2} \left\| |x|^{-\frac{s}{2}} u(T) \right\|_2^2 \\ &\leq \frac{1}{2} \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 \\ &\leq \frac{1}{2} C_H \|\nabla u(t)\|_p^2. \end{aligned} \tag{4.13}$$

By (4.12) and (4.13), we get

$$\int_t^T \|\nabla u(t)\|_p^p \, ds \leq \frac{C_H B_2}{2c_1} \|\nabla u(t)\|_p^2 = \frac{1}{c_2} \|\nabla u(t)\|_p^2. \tag{4.14}$$

Let $T \to +\infty$ in (4.14), and by the virtue of Lemma 2.7, it follows that

$$\|\nabla u(t)\|_p^2 \leq \|\nabla u_0\|_p^2 \left(\frac{p-1}{1 + c_2(p-2)t} \right)^{\frac{1}{p-2}}, \quad t \geq 0.$$

Theorem 4.2 is complete.

5. Blow-up phenomena of weak solutions

In this section, we present the blow-up phenomena of the solutions to (1.1) including infinite and finite time blow-up, and give some bounders of the blow-up. For simplicity, we shall write $L(t) = \frac{1}{2}\left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2$ in the sequel.

5.1. Infinite blow-up

This subsection is devoted to infinite blow-up for Problem (1.1).

Theorem 5.1. (Infinite Blow-Up). *Let $u_0 \in W^-$ and p, q satisfy $1 < p \le q < 2$. Then $u(t)$ blows up in infinite time.*

Proof. We divide the proof into 2 steps.
Step 1: $u_0 \in W_1^-$

We claim that $u(t) \in W_1^-$ for all $t \in [0, T_{max})$ provided that $u_0 \in W_1^-$. Let $u(t)$ be the weak solution of Problem (1.1) with $u_0 \in W_1^-$, which means that $u_0 \neq 0$ and $J(u_0) < d,\ I(u_0) < 0$.

If $J(u_0) < 0$. By the energy equality, we arrive at

$$\frac{1}{q}I(u) + \left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q = J(u(t)) \le J(u_0) < 0 < d. \tag{5.1}$$

It means that $u(t) \neq 0$, $J(u(t)) < d$, and $I(u(t)) < 0$, which implies that $u(t) \in W_1^-$.

If $0 < J(u_0) < d$. From the energy equality, we obtain

$$0 < \int_0^t \left\||x|^{-\frac{s}{2}}u_t(s)\right\|_2^2 ds + J(u(t)) = J(u_0) < d, \tag{5.2}$$

which means that $u(x,t) \neq 0$. Next, we will show that $I(u(t)) < 0$ for all $t \in [0, T_{\max})$. Otherwise, by the continuity of $I(u)$, there would exist a $t_* \in (0, T_{\max})$ such that $I(u(t)) < 0$, $t \in [0, t_*)$ and $I(u(t_*)) = 0$. It means that $u(t_*) \in N$. Then, from the definition of d, it holds that $J(u(t_*)) \ge d$ which contradicts (5.2). Then $u(t) \in W_1^-$ for all $t \in [0, T_{\max})$.

From Lemma 2.4(iv), as $I(u(t)) < 0$, there is a $\lambda^* < 1$ such that $I(\lambda^* u) = 0$. Then

$$\begin{aligned} d \le J(\lambda^* u) &= \frac{1}{q}I(\lambda^* u) + (\lambda^*)^p\left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla u\|_p^p + \frac{(\lambda^*)^q}{q^2}\|u\|_q^q \\ &< \left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q. \end{aligned} \tag{5.3}$$

Then, by taking the derivative of $L(t)$, we obtain

$$\begin{aligned} \frac{d}{dt}L(t) &= \int_\Omega u \cdot \frac{u_t}{|x|^s}dx = -I(u) \\ &= -qJ(u) + \left(\frac{q}{p} - 1\right)\|\nabla u\|_p^p + \frac{1}{q}\|u\|_q^q \\ &\ge q(d - J(u(t))) = q(d - J(u_0)) = C_0 > 0, \quad t \in [0, T_{max}]. \end{aligned} \tag{5.4}$$

Combining (5.4) and $L(t) - L(0) = \int_0^t L'(t)dt$, we can derive

$$\left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2 \geq \left\||x|^{-\frac{s}{2}}u_0\right\|_2^2 + C_0 t > 0, \quad t \in [0, T_{max}]. \tag{5.5}$$

Now, we prove that $u(t)$ cannot blow up in finite time. Arguing by contradiction, we assume that $u(t)$ blows up in finite time, which implies that

$$\lim_{t\to T_{\max}^-} \left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2 = +\infty. \tag{5.6}$$

Meantime, by (5.4), we have

$$L(t)\ln L(t) - L'(t) = \frac{1}{2}\left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2 \ln\left(\frac{1}{2}\left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2\right) + I(u(t)). \tag{5.7}$$

Next, combining $\ln|u(x)| < \frac{|u(x)|^\delta}{\delta}$ a.e. $x \in \Omega$, $\forall\, \delta > 0$ and (2.2), and taking $\delta = 2 - q$, we obtain

$$I(u(t)) \geq \|\nabla u(t)\|_p^p - \frac{1}{2-q}\|u(t)\|_2^2. \tag{5.8}$$

On the other hand, from (5.4) and (5.6), we can see that there exists a $t_1 \in (0, T_{max})$ such that

$$\frac{1}{2}\left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2 > \frac{1}{2}\left\||x|^{-\frac{s}{2}}u(t_1)\right\|_2^2 = \exp\left\{\frac{2L^s}{2-q}\right\}, \tag{5.9}$$

where $|x| < L$. Then by combining (5.7), (5.8), and (5.9) we can derive

$$\begin{aligned}
&L(t)\ln L(t) - L'(t)\\
&\geq \frac{1}{2}\left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2 \ln\left(\frac{1}{2}\left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2\right) + \|\nabla u(t)\|_p^p - \frac{1}{2-q}\|u(t)\|_2^2\\
&\geq \left(\frac{1}{2}L^{-s}\ln\left(\frac{1}{2}\left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2\right) - \frac{1}{2-q}\right)\|u(t)\|_2^2 > 0,
\end{aligned} \tag{5.10}$$

which means that

$$L(t)\ln L(t) - L'(t) > 0.$$

Through a direct calculation, we have

$$\frac{d}{dt}\ln(L(t)) = \frac{L'(t)}{L(t)} < \ln(L(t)), \quad t \in [t_1, T_{max}]. \tag{5.11}$$

Then by virtue of Gronwall's inequality, we get

$$\ln(L(t)) < \exp\{t - t_1\}\ln(L(t_1)), \quad t \in [t_1, T_{\max}],$$

which implies that

$$\left\||x|^{-\frac{s}{2}}u(t)\right\|_2^2 < \left\||x|^{-\frac{s}{2}}u(t_1)\right\|_2^{2\exp\{t-t_1\}}, \quad t \in [t_1, T_{\max}].$$

That contradicts with (5.6). Therefore, $T_{max} = +\infty$ and $u(t)$ blows up in infinite time.

Step 2: $u_0 \in W_2^-$

First of all, $u_0 \in W_2^-$ means that $I(u(t)) < 0, J(u(t)) = d$, $\forall t \in [0, T_{max}]$. We claim that $u(t) \in W_2^-$ for all $t \in [0, T_{max})$ provided that $u_0 \in W_2^-$. Otherwise, by continuity, there would exist a $t_1 \in [0, T_{max})$ such that $I(u(t)) < 0$ for $t \in [0, t_1)$ and $I(u(t_1)) = 0$. Recalling the definition of d, it is clear that $J(u(t_1)) \geq d$. On the other hand, from $\int_\Omega \left| u \cdot \frac{u_t}{|x|^s} \right| dx = -I(u(t)) > 0,\ t \in [t_1, T_{\max}]$, we know that $u_t \neq 0$ and $\int_0^{t_1} \left\| |x|^{-\frac{s}{2}} u_\tau(\tau) \right\|_2^2 d\tau > 0$, $t_1 \in [0, T_{max}]$. Meanwhile, it follows from the energy equality that

$$J(u(t_1)) = J(u_0) - \int_0^{t_0} \left\| |x|^{-\frac{s}{2}} u_\tau(\tau) \right\|_2^2 d\tau < J(u_0) = d,$$

which contradicts with $J(u(t_1)) \geq d$. Therefore, there exists a $t_2 \in [0, T_{max}]$ such that $I(u(t_2)) < 0$ and $J(u(t_2)) < d$. If we take t_2 as the initial time, then similar to Step 1, we can obtain that the weak solution $u(t)$ of Problem (1.1) blows up in infinite time.

5.2. Finite time blow-up

Theorem 5.2. (Finite Blow-Up). *Let $u_0 \in W^-$ and p, q satisfy $2 < p \leq q < p\left(\frac{2}{N} + 1\right)$. Then $u(t)$ blows up in finite time. Moreover,*

$$T^* \leq \frac{\left\| |x|^{-\frac{s}{2}} u(0) \right\|_2^2}{(p-2)\, pJ(u_0)}.$$

Proof. We shall apply the first-order differential inequality technique to show the finite time blow-up result for Problem (1.1) with negative initial energy. For this, set $K(t) = -J(u(t))$. Then $L(0) > 0, K(0) > 0$. From Problem (1.1), it follows that

$$\frac{d}{dt}K(t) = -\frac{d}{dt}J(u(t)) = \left\| |x|^{-\frac{s}{2}} u_t(t) \right\|_2^2 \geq 0,$$

which means that $K(t) \geq K(0) = -J(u_0) > 0$ for all $t \in [0, T^*)$. Recalling (2.2) and (2.3), we obtain, for any $t \in [0, T^*)$, that

$$\begin{aligned} \frac{d}{dt}L(t) &= \int_\Omega u \cdot \frac{u_t}{|x|^s} dx = -I(u) \\ &= -qJ(u) + \left(\frac{q}{p} - 1\right) \|\nabla u\|_p^p + \frac{1}{q} \|u\|_q^q \\ &\geq -qJ(u) = qK(t) > 0. \end{aligned} \tag{5.12}$$

Making use of the Holder inequality and Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned} L(t) K'(t) &= \frac{1}{2} \left(\left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 \right) \left(\left\| |x|^{-\frac{s}{2}} u_t(t) \right\|_2^2 \right) \\ &\geq \frac{1}{2} (L'(t))^2 \geq \frac{q}{2} L'(t) K(t), \end{aligned} \tag{5.13}$$

which then implies

$$\left[K(t) L^{-\frac{q}{2}}(t) \right]' = L^{-\frac{q}{2}-1}(t) \left(L(t) K'(t) - \frac{q}{2} L'(t) K(t) \right) \geq 0. \tag{5.14}$$

Therefore,

$$0 < k = K(0)L^{-\frac{q}{2}}(0) \le K(t)L^{-\frac{q}{2}}(t) \le \frac{1}{q}L'(t)L^{-\frac{q}{2}}(t) = \frac{2}{(2-q)p}\left[L^{\frac{2-q}{2}}(t)\right]'. \qquad (5.15)$$

Integrating (5.15) over $[0,t]$ for any $t \in (0,T^*)$ and noticing that $q > 2$, one has

$$kt \le \frac{2}{(2-q)q}\left[L^{\frac{2-q}{2}}(t) - L^{\frac{2-q}{2}}(0)\right],$$

or equivalently

$$0 \le L^{\frac{2-q}{2}}(t) \le L^{\frac{2-q}{2}}(0) - \frac{(q-2)q}{2}kt, \quad t \in (0,T^*]. \qquad (5.16)$$

It is obvious that (5.16) cannot hold for all $t > 0$. Therefore, $T^* < +\infty$. Moreover, it can be inferred from (5.16) that

$$T^* \le \frac{2L(0)}{(q-2)qK(0)} = \frac{\left\||x|^{-\frac{s}{2}}u(0)\right\|_2^2}{(q-2)qJ(u_0)}.$$

The proof is complete.

For the case of $J(u_0) \ge 0$, we obtain blow-up results when the initial energy is 'subcritical' and when the initial Nehari functional is negative which means that $u_0 \in W^-$. More precisely, we have the following theorem.

Theorem 5.3. *Assume that* $2 < p \le q < p\left(\frac{2}{N}+1\right)$, $u_0 \in W^-$. *Then the weak solution* $u(t)$ *to Problem* (1.1) *blows up in finite time. Furthermore, if* $u_0 \in W_1^-$, *then* T_{max} *can be estimated from above as follows:*

$$T_{\max} \le \frac{\beta b^2}{(q-2)\beta b - \left\||x|^{-\frac{s}{2}}u(0)\right\|_2^2},$$

where β, b *are constants that will be determined in the proof.*

Proof. We will divide the proof into two cases.
Case 1: $u_0 \in W_1^-$

We claim that $u(t) \in W_1^-$ for all $t \in [0,T_{max})$ provided that $u_0 \in W_1^-$. Otherwise, by continuity, there would exist a $t_0 \in [0,T_{max})$ such that $I(u(t)) > 0$ for $t \in [0,t_0)$ and $I(u(t_0)) = 0$. Recalling the definition of d, it is clear that $J(u(t_0)) \ge d$, which contradicts with $J(u(t)) \le J(u_0) < d$.

From Lemma 2.4(iv), as $I(u(t)) < 0$, there is a $\lambda^* < 1$ such that $I(\lambda^* u) = 0$. Then

$$\begin{aligned} d \le J(\lambda^* u) &= \frac{1}{q}I(\lambda^* u) + (\lambda^*)^p\left(\frac{1}{p}-\frac{1}{q}\right)\|\nabla u\|_p^p + \frac{(\lambda^*)^q}{q^2}\|u\|_q^q \\ &< \left(\frac{1}{p}-\frac{1}{q}\right)\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q. \end{aligned} \qquad (5.17)$$

We show that $T_{max} < +\infty$. For any $T \in [0,T_{max})$, define the positive function

$$F(t) = \int_0^t L(t)\,dt + (T-t)L(0) + \frac{\beta}{2}(t+b)^2, \qquad (5.18)$$

where $\beta > 0, b > 0$. By direct computations,

$$\begin{aligned} F'(t) &= L(t) - L(0) + \beta(t+b) = \int_0^t \frac{d}{dt} L(t)\,dt + \beta(t+b) \\ &= \int_0^t \int_\Omega u \cdot \frac{u_t}{|x|^s} dxdt + \beta(t+b), \end{aligned} \tag{5.19}$$

$$\begin{aligned} F''(t) &= L'(t) + \beta = -I(u) + \beta \\ &= -qJ(u) + \left(\frac{q}{p} - 1\right) \|\nabla u\|_p^p + \frac{1}{q} \|u\|_q^q + \beta \\ &= \left(\frac{q}{p} - 1\right) \|\nabla u\|_p^p + \frac{1}{q} \|u\|_q^q + \beta - q\left[J(u_0) - \int_0^t \left\| |x|^{-\frac{s}{2}} u_t(t) \right\|_2^2 dt\right]. \end{aligned} \tag{5.20}$$

Applying the Cauchy-Schwarz inequality, Young inequality, and Holder's inequality to yields

$$\begin{aligned} f(t) &= \left[\int_0^t \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 dt + \beta(t+b)^2\right] \cdot \left[\int_0^t \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 + \beta\right] \\ &-\left[\int_0^t \int_\Omega u \cdot \frac{u_t}{|x|^s} dxdt + \beta(t+b)\right]^2 \geq 0. \end{aligned} \tag{5.21}$$

Therefore, by recalling (5.19) and (5.20), and noticing the nonnegativity of $f(t)$, we arrive at

$$\begin{aligned} & F(t) F''(t) - (1+\theta)[F'(t)]^2 \\ = {} & F(t) F''(t) + (1+\theta)\left[f(t) - [2F(t) - 2(T-t)L(0)]\left[\int_0^t \left\| |x|^{-\frac{s}{2}} u_t(t) \right\|_2^2 dt + \beta\right]\right] \\ \geq {} & F(t) F''(t) - 2(1+\theta) F(t)\left[\int_0^t \left\| |x|^{-\frac{s}{2}} u_t(t) \right\|_2^2 dt + \beta\right] \\ \geq {} & F(t)\left\{F''(t) - 2(1+\theta)\int_0^t \left\| |x|^{-\frac{s}{2}} u_t(t) \right\|_2^2 dt - 2(1+\theta)\beta\right\}. \end{aligned} \tag{5.22}$$

Choosing $\theta = \dfrac{q-2}{2}$ and recalling (5.17) lets us obtain

$$F(t) F''(t) - \frac{q}{2}[F'(t)]^2 \geq F(t)[q(d - J(u_0)) - (q-1)\beta]. \tag{5.23}$$

In view of (5.18) and (5.23), we get, for any $t \in (0, T_{max})$ and $\beta \in \left(0, \frac{q(d-J(u_0))}{q-1}\right]$, that

$$F(t) F''(t) - \left(1 + \frac{q-2}{2}\right)[F'(t)]^2 \geq 0.$$

Therefore, Lemma 2.8 guarantees that $F(0) > 0$ and $F'(0) = \beta b > 0$, and then $\exists\, T_1 \,:\, 0 < T_1 < \frac{2F(0)}{(q-2)F'(0)}$, such that $F(t) \to \infty,\ t \to T_1$

$$T_{\max} \leq \frac{\beta b^2}{(q-2)\beta b - \left\| |x|^{-\frac{s}{2}} u(0) \right\|_2^2},$$

where $b > \max\left\{0, \frac{\left\| |x|^{-\frac{s}{2}} u(0) \right\|_2^2}{(q-2)\beta}\right\}$.

Case 2: $u_0 \in W_2^-$

By similar arguments as those in the proof of case 1, when $u_0 \in W_2^-$, by continuity, we see that there exists a $t_2 > 0$ such that $I(u(t_2)) < 0$ and $\left\| |x|^{-\frac{s}{2}} u_t \right\|_2^2 > 0$ for all $t \in [0, t_2)$. From the energy equality, we get

$$J(u(t_2)) = J(u_0) - \int_0^{t_2} \left\| |x|^{-\frac{s}{2}} u_\tau(\tau) \right\|_2^2 d\tau < J(u_0) = d.$$

The remainder of the proof is the same as in Case 1.

In the following, we shall derive a lower bound for the blow-up time T_*.

Theorem 5.4. *Assume* $2 < p \leq q < p\left(\frac{2}{N} + 1\right)$. *Let* $u(t)$ *be a weak solution to Problem* (1.1) *that blows up at* T_*. *Then*

$$T_* \geq \frac{L^{1-\alpha}(0)}{C_L(\alpha - 1)},$$

where $C_L > 0, \alpha > 1$ *are two constants that will be determined in the proof.*

Proof. Combining (2.2) and (3.8), we have

$$\begin{aligned} \frac{d}{dt} L(t) &= \int_\Omega u \cdot \frac{u_t}{|x|^s} dx = -I(u) = -\|\nabla u\|_p^p + \int_\Omega |u|^q \ln |u| \, dx \\ &\leq \left((e\mu)^{-1} C_G \varepsilon - 1\right) \|\nabla u\|_p^p + (e\mu)^{-1} C_G C(\varepsilon) \|u\|_2^{2\alpha}, \end{aligned} \tag{5.24}$$

where $\alpha > 1$. As $(e\mu)^{-1} C_G \varepsilon - 1 < 0$, recalling the definition of $L(t)$, we get

$$\frac{d}{dt} L(t) \leq (e\mu)^{-1} C_G C(\varepsilon) C(\Omega) \left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^{2\alpha} \leq C_L L^\alpha(t), \tag{5.25}$$

where $C_L = (e\mu)^{-1} C_G C(\varepsilon) C(\Omega)$. Integrating (5.25) over $[0, t)$, we get

$$\frac{1}{1-\alpha} \left[L^{1-\alpha}(t) - L^{1-\alpha}(0) \right] \leq C_L t.$$

Since $\alpha > 1$, letting $t \to T_*$ in the above inequality and recalling that $\lim_{t \to T_*} L(t) = +\infty$, we obtain

$$T_* \geq \frac{L^{1-\alpha}(0)}{C_L(\alpha - 1)}.$$

The proof is complete.

6. Extinction phenomenon

In this section, we present the result of extinction for Problem (1.1).

Theorem 6.1. (Extinction). *Assuming* $\frac{2N}{N+2} < p < q < 2$ *and*

$$0 < \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2 < 2 \left(\frac{2}{C_p} \right)^{\frac{p}{q+\delta-p}} \left[\frac{1}{\delta} |\Omega|^{1-\frac{q+\delta}{2}} (2C(\Omega))^{\frac{q+\delta}{2}} \right]^{\frac{-2}{q+\delta-p}},$$

then the weak solution of Problem (1.1) *becomes extinct in finite time. Furthermore, we have the following estimates*

$$\left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 \le 2\left[\left(\frac{1}{2} \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2 \right)^{1-\frac{p}{2}} - \epsilon_0 \left(1 - \frac{p}{2}\right) t \right]^{\frac{2}{2-p}}, \quad 0 < t < T_*,$$

and

$$\left\| |x|^{-\frac{s}{2}} u(t) \right\|_2^2 \equiv 0, \quad t \ge T_*.$$

The extinction time is

$$T_* = \frac{2}{2-p} \cdot \frac{\left(\frac{1}{2} \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2 \right)^{1-\frac{p}{2}}}{\epsilon_0},$$

where $\epsilon_0,\ \delta,\ C_p$ *are given in the following.*

Proof. Multiplying (1.1) by $u(t)$ and integrating over Ω, we have

$$L'(t) + \|\nabla u(t)\|_p^p = \int_\Omega |u|^q \ln |u| \, dx. \tag{6.1}$$

Meanwhile, by $\frac{2N}{N+2} < p < 2$, we have $0 \le s \le 2 < (N+2) - \frac{2N}{p}$, which implies that $1 < \frac{2N}{N-s+2} < p$ in Lemma 2.6, and we can see that there exists a constant $C_p > 0$ such that

$$\int_\Omega \frac{|u(x)|^2}{|x|^s} dx \le C_H \|\nabla u\|^2_{\frac{2N}{N-s+2}} \le C_p \|\nabla u\|_p^2. \tag{6.2}$$

Combining (2.4), (3.9), (6.1), (6.2), and Hölder's inequality, we deduce that there exists a $0 < \delta \le 2 - q$ such that

$$\begin{aligned} L'(t) + \left(\frac{2}{C_p}\right)^{\frac{p}{2}} L^{\frac{p}{2}}(t) &\le \frac{1}{\delta} \|u\|_{q+\delta}^{q+\delta} \\ &\le \frac{1}{\delta} |\Omega|^{1-\frac{q+\delta}{2}} \|u\|_2^{q+\delta} \\ &\le \frac{1}{\delta} |\Omega|^{1-\frac{q+\delta}{2}} (2C(\Omega))^{\frac{q+\delta}{2}} L^{\frac{q+\delta}{2}}(t). \end{aligned} \tag{6.3}$$

Then by Lemma 2.9, we know that

$$\epsilon_1 = \left(\frac{2}{C_p}\right)^{\frac{p}{2}}, \quad \epsilon_2 = \frac{1}{\delta} |\Omega|^{1-\frac{q+\delta}{2}} (2C(\Omega))^{\frac{q+\delta}{2}}, \quad 0 < l = \frac{p}{2} < r = \frac{q+\delta}{2} \le 1.$$

We assume that

$$0 < \left\| |x|^{-\frac{s}{2}} u_0 \right\|_2^2 < 2\left(\frac{\epsilon_1}{\epsilon_2}\right)^{\frac{2}{q+\delta-p}} = 2\left(\frac{2}{C_p}\right)^{\frac{p}{q+\delta-p}} \left[\frac{1}{\delta} |\Omega|^{1-\frac{q+\delta}{2}} (2C(\Omega))^{\frac{q+\delta}{2}} \right]^{\frac{-2}{q+\delta-p}},$$

and then we can see that

$$L(t) \le \left[-\epsilon_0 \left(1 - \frac{p}{2}\right) t + L^{1-\frac{p}{2}}(0) \right]^{\frac{2}{2-p}}, 0 < t < T_*,$$

and

$$L(t) \equiv 0, t \geq T_*,$$

where

$$\epsilon_0 = \epsilon_1 - \epsilon_2 L^{r-l}(0) = \left(\frac{2}{C_p}\right)^{\frac{p}{2}} - \left[\frac{1}{\delta}|\Omega|^{1-\frac{q+\delta}{2}} (2C(\Omega))^{\frac{q+\delta}{2}}\right] \left\||x|^{-\frac{s}{2}} u_0\right\|_2^{q+\delta-p},$$

and

$$T_* = \frac{2}{2-p} \cdot \frac{\left(\frac{1}{2}\left\||x|^{-\frac{s}{2}} u_0\right\|_2^2\right)^{1-\frac{p}{2}}}{\epsilon_0}.$$

The proof is complete.

Author contributions

Xiulan Wu: Methodology, Writing-original draft, Writing-review & editing; Yanxin Zhao and Xiaoxin Yang: Methodology, Writing-original draft.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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