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Research article

Existence and multiplicity results for a kind of double phase problems with mixed boundary value conditions

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Abstract: In this article, we study a double phase variable exponents problem with mixed boundary value conditions of the form

$$\begin{cases} D(u) + |u|^{p(x)-2}u + b(x)|u|^{q(x)-2}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Lambda_1, \\ \left(|\nabla u|^{p(x)-2}u + b(x)|\nabla u|^{q(x)-2}u\right) \cdot \nu = g(x,u) & \text{on } \Lambda_2. \end{cases}$$

First of all, using the mountain pass theorem, we establish that this problem admits at least one nontrivial weak solution without assuming the Ambrosetti–Rabinowitz condition. In addition, we give a result on the existence of an unbounded sequence of nontrivial weak solutions by employing the Fountain theorem with the Cerami condition.

Keywords: double phase problems; variational methods; critical point theory; Cerami condition
Mathematics Subject Classification: 35A01, 35J20, 35J60, 35J66

1. Introduction

The study of solutions to superlinear problems driven by the double phase operator is a new and important topic, since it sheds light on a range of applications in the field of mathematical physics such as elasticity theory, strongly anisotropic materials, Lavrentiev's phenomenon, etc. (see [1–3]).

In the present paper, we study the existence and multiplicity of solutions for the following double–phase problems with mixed boundary conditions:

$$\begin{cases} D(u) + |u|^{p(x)-2}u + b(x)|u|^{q(x)-2}u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \Lambda_1, \\ \left(|\nabla u|^{p(x)-2}u + b(x)|\nabla u|^{q(x)-2}u\right) \cdot \nu = g(x,u) & \text{on } \Lambda_2, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with Lipschitz boundary $\partial\Omega$, Λ_1, Λ_2 are disjoint open subsets of $\partial\Omega$ such that $\partial\Omega = \overline{\Lambda_1} \cup \overline{\Lambda_2}$ and $\Lambda_1 \neq \emptyset$, $1 < p(x) < q(x) < N$ for all $x \in \overline{\Omega}$, $b : \overline{\Omega} \mapsto [0, +\infty)$ is Lipschitz continuous, ν denotes the outer unit normal of Ω at the point $x \in \Lambda_2$, $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Lambda_2 \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, and D is the double phase variable exponents operator given by

$$D(u) := \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u + b(x)|\nabla u|^{q(x)-2}\nabla u), \quad \text{for } u \in W^{1,\mathcal{D}}(\Omega). \tag{1.2}$$

Note that the differential operator defined above in (1.2) is called the double phase operator with variable exponents, which is a natural generalization of the classical double phase operator when p and q are constant functions

$$u \mapsto \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + b(x)|\nabla u|^{q-2}\nabla u\right).$$

From the physical point of view, while studying the behavior of strongly anisotropic materials, V.V. Zhikov [2] in 1986 discovered that their hardening properties changed radically point by point, what is known as the Lavrentiev phenomenon [3]. To describe this phenomenon, he initially introduced the functional

$$v \longmapsto \int_\Omega (|\nabla v|^p + b(x)|\nabla v|^q)\, dx, \tag{1.3}$$

where the integrand changes its ellipticity and growth properties according to the point in the domain. In the framework of mathematics, the functional (1.3) has been investigated by many authors with respect to regularity and nonstandard growth. For instance, we refer to the papers of P. Baroni et al. [4, 5], P. Baroni et al. [6], G. Cupini [7], and the references therein.

Multiple authors have recently concentrated on the study of double phase problems in the case when the exponents p and q are constants, and a plethora of results have been obtained; see, for example, W. Liu and G. Dai [8], M. El Ahmadi et al. [9], L. Gasiński and P. Winkert [10], N. Cui and H.R. Sun [11], Y. Yang et al. [12], and the references therein. For example, N. Cui and H.R. Sun [11] considered the following problem in the particular case: $p(x) = p$, $q(x) = q$, and $\lambda = 1$

$$\begin{cases} D(u) + |u|^{p-2}u + b(x)|u|^{q-2}u = f(x, u) & \text{in } \Omega, \\ \left(|\nabla u|^{p-2}u + b(x)|\nabla u|^{q-2}u\right) \cdot \nu = g(x, u) & \text{on } \partial\Omega, \end{cases}$$

where $D(u) := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u + b(x)|\nabla u|^{q-2}\nabla u\right)$. The authors have proved the existence and multiplicity of nontrivial weak solutions for the above problem with superlinear nonlinearity. Their approach was based on critical point theory with Cerami condition.

Very recently, Y. Yang et al. [12] considered the problem (1.1) in the particular case of $p(x) = p$ and $q(x) = q$. Based on the maximum principle and homological local linking, they proved the existence of at least two bounded nontrivial weak solutions.

The main novelty of the current paper is the combination of the double phase variable exponents operator with mixed boundary conditions, that is, the Dirichlet condition on Λ_1 and the Steklov condition on Λ_2, which is different from [13]. To the best of our knowledge, there are only a few results related to the study of such problems.

To state our results, we make the subsequent hypotheses on f and g:

(H_0) There exist $C_1, C_2 > 0$, $s_1 \in C_+(\Omega)$, and $s_2 \in C_+(\Lambda_2)$ such that

(i) $|f(x, t)| \leq C_1(1 + |t|^{s_1(x)-1})$ for all $(x, t) \in \Omega \times \mathbb{R}$,

(ii) $|g(x,t)| \leq C_2(1+|t|^{s_2(x)-1})$ for all $(x,t) \in \Lambda_2 \times \mathbb{R}$.

(H_1) (i) $1 < p^+ \leq q^+ < s_1^- \leq s_1^+ < p^*(x)$ for all $x \in \Omega$,

(ii) $1 < p^+ \leq q^+ < s_2^- \leq s_2^+ < p_*(x)$ for all $x \in \Lambda_2$,

where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N, \end{cases}$$
$$p_*(x) := \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases} .$$

(H_2) (i) $\liminf\limits_{|t|\to\infty} \frac{F(x,t)}{|t|^{q^+}} = +\infty$ uniformly a.e. $x \in \Omega$,

(ii) $\liminf\limits_{|t|\to\infty} \frac{G(x,t)}{|t|^{q^+}} = +\infty$ uniformly a.e. $x \in \Lambda_2$,

where $F(x,t) = \int_0^t f(x,s)ds$ and $G(x,t) = \int_0^t g(x,s)ds$.

(H_3) (i) There exist $c_1, r_1 \geq 0$ and $l_1 \in L^\infty(\Omega)$ with $l_1(x) > \frac{N}{p^-}$ such as

$$|F(x,t)|^{l(x)} \leq c_1 |t|^{l(x)p^-} \mathcal{F}(x,t),$$

for all $(x,t) \in \Omega \times \mathbb{R}$, $|t| \geq r_1$ and $\mathcal{F}(x,t) := \frac{1}{q^+} f(x,t)t - F(x,t) \geq 0$.

(ii) There exist $c_2, r_2 \geq 0$ and $l_2 \in L^\infty(\Lambda_2)$ with $l_2(x) > \frac{N-1}{p^- - 1}$ such as

$$|G(x,t)|^{l(x)} \leq c_2 |t|^{l(x)p^-} \mathcal{G}(x,t),$$

for all $(x,t) \in \Lambda_2 \times \mathbb{R}$, $|t| \geq r_2$ and $\mathcal{G}(x,t) := \frac{1}{q^+} g(x,t)t - G(x,t) \geq 0$.

(H_4) (i) $f(x,t) = \circ(|t|^{p^+-1})$ as $t \to 0$ uniformly for a.e. $x \in \Omega$,

(ii) $g(x,t) = \circ(|t|^{p^+-1})$ as $t \to 0$ uniformly for a.e. $x \in \Lambda_2$.

(H_5) (i) $f(x,-t) = -f(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$,

(ii) $g(x,-t) = -g(x,t)$ for all $(x,t) \in \Lambda_2 \times \mathbb{R}$.

Let us consider $\phi : X_0 \to \mathbb{R}$ the Euler functional corresponding to problem (1.1), which is defined as follows:

$$\phi(u) = I(u) - \varphi(u),$$

where

$$I(u) = \int_\Omega \left(\frac{1}{p(x)} |\nabla u|^{p(x)} + \frac{b(x)}{q(x)} |\nabla u|^{q(x)} \right) dx + \int_\Omega \left(\frac{1}{p(x)} |u|^{p(x)} + \frac{b(x)}{q(x)} |u|^{q(x)} \right) dx,$$

and

$$\varphi(u) = \int_\Omega F(x,u)dx + \int_{\Lambda_2} G(x,u)d\sigma,$$

with X_0 will be defined in preliminaries and $d\sigma$ is the measure on the boundary.
Then, it follows from the hypothesis (H_0) that the functional $\phi \in C^1(X_0, \mathbb{R})$, and its Fréchet derivative is

$$\begin{aligned}\langle \phi'(u), v\rangle &= \int_\Omega \left(|\nabla u|^{p(x)-2} + b(x)|\nabla u|^{q(x)-2}\right) \nabla u.\nabla v \, dx \\ &+ \int_\Omega \left(|u|^{p(x)-2} + b(x)|u|^{q(x)-2}\right) u.v \, dx - \int_\Omega f(x,u)v dx - \int_{\Lambda_2} g(x,u)v d\sigma,\end{aligned}$$

for any $u, v \in X_0$. It is clear that any critical point of ϕ is a weak solution to the problem (1.1).

Now, we present the main results of this paper.

Theorem 1. Suppose that $(H_0), (H_1), (H_2), (H_3)$ and (H_4) hold. Then problem (1.1) has at least one nontrivial weak solution.

Theorem 2. Suppose that $(H_0), (H_1), (H_2), (H_3)$ and (H_5) hold. Then problem (1.1) possesses a sequence of weak solutions (u_n) such that $\phi(u_n) \to +\infty$ as $n \to +\infty$.

2. Preliminaries

To study double phase problems, we need some definitions and basic properties of $W^{1,\mathcal{D}}(\Omega)$, which are called Musielak–Orlicz–Sobolev spaces. For more details, see [14–19] and references therein.

First, we recall the definition of variable exponent Lebesgue space. For $p \in C_+(\overline{\Omega}) := \{p \in C(\overline{\Omega}) : p^- := \inf_{x\in\overline{\Omega}} p(x) > 1\}$, we designate the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega |u(x)|^{p(x)} dx < +\infty\right\},$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}.$$

Proposition 1. [20]

1. The Sobolev space $(L^{p(x)}(\Omega), |.|_{p(x)})$ is defined as the dual space $L^{q(x)}(\Omega)$, where $q(x)$ is conjugate to $p(x)$, i.e., $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$\left|\int_\Omega uv \, dx\right| \le \left(\frac{1}{p^-} + \frac{1}{q^-}\right)|u|_{p(x)}|v|_{q(x)} \le 2|u|_{p(x)}|v|_{q(x)}.$$

2. If $p_1, p_2 \in C_+(\overline{\Omega})$, $p_1(x) \le p_2(x)$, for all $x \in \overline{\Omega}$, then $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ and the embedding is continuous.

Let $p \in C_+(\partial\Omega) := \{p \in C(\partial\Omega) : p^- := \inf_{x\in\partial\Omega} p(x) > 1\}$ and denote by $d\sigma$ the Lebesgue measure on the boundary. We define

$$L^{p(x)}(\partial\Omega) = \left\{u : \partial\Omega \to \mathbb{R} \text{ is measurable and } \int_{\partial\Omega} |u|^{p(x)} d\sigma < +\infty\right\},$$

with the norm

$$|u|_{p(x),\partial\Omega} = \inf\Big\{\alpha > 0 : \int_{\partial\Omega} \Big|\frac{u(x)}{\alpha}\Big|^{q(x)} d\sigma \leq 1\Big\}.$$

Now, we give the main properties of the Musielak–Orlicz–Sobolev functional space that we will use in the rest of this paper. Denote by $N(\Omega)$ the set of all generalized N-functions. Let us denote by

$$\mathcal{D} : \Omega \times [0, +\infty) \to [0, +\infty)$$

the function defined as

$$\mathcal{D}(x, t) = t^{p(x)} + b(x)t^{q(x)}, \quad \text{for all } (x, t) \in \Omega \times [0, +\infty),$$

where the weight function $b(.)$ and the variable exponents $p(.), q(.) \in C_+(\overline{\Omega})$ satisfies the following hypothesis:

$$p(x) < q(x) < N, \frac{Nq(x)}{N + q(x) - 1} < p(x) \text{ for all } x \in \overline{\Omega} \text{ and } 0 \leq b(.) \in L^\infty(\Omega). \tag{2.1}$$

Note that the role of assuming the inequality $\frac{Nq(x)}{N+q(x)-1} < p(x)$ is to ensure that $q(x) < p_*(x)$ and $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, where $p^*(x) = \frac{Np(x)}{N-p(x)}$ and $p_*(x) = \frac{(N-1)p(x)}{N-p(x)}$.
It is clear that $\mathcal{D}$ is a generalized N-function, locally integrable, and

$$\mathcal{D}(x, 2t) \leq 2^{q^+}\mathcal{D}(x, t), \quad \text{for all } (x, t) \in \Omega \times [0, +\infty),$$

which is called condition (Δ_2).

We designate the Musielak–Orlicz space by

$$L^{\mathcal{D}}(\Omega) = \Big\{u : \Omega \to \mathbb{R} \text{ is measurable and } \int_\Omega \mathcal{D}(x, |u|)dx < +\infty\Big\},$$

equipped with the so-called Luxemburg norm

$$|u|_{\mathcal{D}} = \inf\Big\{\mu > 0 : \int_\Omega \mathcal{D}(x, |\frac{u}{\mu}|)dx \leq 1\Big\}.$$

The Musielak–Orlicz–Sobolev space $W^{1,\mathcal{D}}(\Omega)$ is defined as

$$W^{1,\mathcal{D}}(\Omega) = \Big\{u \in L^{\mathcal{D}}(\Omega) : |\nabla u| \in L^{\mathcal{D}}(\Omega)\Big\},$$

endowed with the norm

$$\| u \|_{1,\mathcal{D}} = |u|_{\mathcal{D}} + |\nabla u|_{\mathcal{D}}.$$

With such norms, $L^{\mathcal{D}}(\Omega)$ and $W^{1,\mathcal{D}}(\Omega)$ are separable, uniformly convex, and reflexive Banach spaces.

On $L^{\mathcal{D}}(\Omega)$, we consider the function $\rho : L^{\mathcal{D}}(\Omega) \to \mathbb{R}$ defined by

$$\rho(u) = \int_\Omega \Big(|u|^{p(x)} + b(x)|u|^{q(x)}\Big)dx.$$

The relationship between ρ and $|.|_{\mathcal{D}}$ is established by the next result.

Proposition 2. (see [16]) For $u \in L^{\mathcal{D}}(\Omega)$, $(u_n) \subset L^{\mathcal{D}}(\Omega)$, and $\mu > 0$, we have

1. For $u \neq 0$, $|u|_{\mathcal{D}} = \mu \Longleftrightarrow \rho(\frac{u}{\mu}) = 1$;
2. $|u|_{\mathcal{D}} < 1\,(= 1, > 1) \Longleftrightarrow \rho(u) < 1\,(= 1, > 1)$;
3. $|u|_{\mathcal{D}} > 1 \Longrightarrow |u|_{\mathcal{D}}^{p^-} \leq \rho(u) \leq |u|_{\mathcal{D}}^{q^+}$;
4. $|u|_{\mathcal{D}} < 1 \Longrightarrow |u|_{\mathcal{D}}^{q^+} \leq \rho(u) \leq |u|_{\mathcal{D}}^{p^-}$;
5. $\lim\limits_{n\to+\infty} |u_n|_{\mathcal{D}} = 0 \Leftrightarrow \lim\limits_{n\to+\infty} \rho(u_n) = 0$ and $\lim\limits_{n\to+\infty} |u_n|_{\mathcal{D}} = +\infty \Leftrightarrow \lim\limits_{n\to+\infty} \rho(u_n) = +\infty$.

On $W^{1,\mathcal{D}}(\Omega)$, we introduce the equivalent norm by

$$\|u\| := \inf\left\{\mu > 0 : \int_\Omega \left[\left|\frac{\nabla u}{\lambda}\right|^{p(x)} + b(x)\left|\frac{\nabla u}{\lambda}\right|^{q(x)} + \left|\frac{u}{\lambda}\right|^{p(x)} + b(x)\left|\frac{u}{\mu}\right|^{q(x)}\right] \mathrm{d}x \leq 1\right\}. \tag{2.2}$$

Similar to Proposition (2), we have

Proposition 3. (see [16]) Let

$$\hat{\rho}(u) = \int_\Omega \left(|\nabla u|^{p(x)} + b(x)|\nabla u|^{q(x)}\right) dx + \int_\Omega \left(|u|^{p(x)} + b(x)|u|^{q(x)}\right) dx.$$

For $u \in W^{1,\mathcal{D}}(\Omega)$, $(u_n) \subset W^{1,\mathcal{D}}(\Omega)$, and $\mu > 0$, we have

1. For $u \neq 0$, $\|u\| = \mu \Longleftrightarrow \hat{\rho}(\frac{u}{\mu}) = 1$;
2. $\|u\| < 1\,(= 1, > 1) \Longleftrightarrow \hat{\rho}(u) < 1\,(= 1, > 1)$;
3. $\|u\| > 1 \Longrightarrow \|u\|^{p^-} \leq \hat{\rho}(u) \leq \|u\|^{q^+}$;
4. $\|u\| < 1 \Longrightarrow \|u\|^{q^+} \leq \hat{\rho}(u) \leq \|u\|^{p^-}$;
5. $\lim\limits_{n\to+\infty} \|u_n\| = 0 \Leftrightarrow \lim\limits_{n\to+\infty} \hat{\rho}(u_n) = 0$ and $\lim\limits_{n\to+\infty} \|u_n\| = +\infty \Leftrightarrow \lim\limits_{n\to+\infty} \hat{\rho}(u_n) = +\infty$.

We recall that problem (1.1) has a mixed boundary condition. For this, our Banach space workspace is given by

$$X_0 := \left\{u \in W^{1,\mathcal{D}}(\Omega) : u = 0 \ \text{ on } \ \Lambda_2\right\},$$

endowed with the equivalent norm (2.2). Obviously, since X_0 is a closed subspace of $W^{1,\mathcal{D}}(\Omega)$, then $(X_0, \|.\|)$ is a reflexive Banach space.

Proposition 4. (see [16]) Let hypothesis (2.1) be satisfied. Then the following embeddings hold:

1. There is a continuous embedding $L^{\mathcal{D}}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ for $r \in C(\overline{\Omega})$ with $1 \leq r(x) \leq p(x)$ for all $x \in \overline{\Omega}$.
2. There is a compact embedding $W^{1,\mathcal{D}}(\Omega) \hookrightarrow L^{r(x)}(\Omega)$ for $r \in C(\overline{\Omega})$ with $1 \leq r(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

3. If $p \in C_+(\overline{\Omega}) \cap W^{1,\gamma}(\Omega)$ for some $\gamma \geq N$. Then, there is a continuous embedding $W^{1,\mathcal{D}}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ for $r \in C(\partial\Omega)$ with $1 \leq r(x) \leq p_*(x)$ for all $x \in \partial\Omega$.

4. There is a compact embedding $W^{1,\mathcal{D}}(\Omega) \hookrightarrow L^{r(x)}(\partial\Omega)$ for $r \in C(\partial\Omega)$ with $1 \leq r(x) < p_*(x)$ for all $x \in \partial\Omega$.

It is important to note that when we replace $W^{1,\mathcal{D}}(\Omega)$ by X_0 in Proposition 4, the embeddings 2 and 4 remain valid.

Let $A : W^{1,\mathcal{D}}(\Omega) \to \left(W^{1,\mathcal{D}}(\Omega)\right)^*$ be defined by

$$\langle A(u), v\rangle = \int_\Omega \left(|\nabla u|^{p(x)-2} + b(x)|\nabla u|^{q(x)-2}\right)\nabla u.\nabla v dx + \int_\Omega \left(|u|^{p(x)-2} + b(x)|u|^{q(x)-2}\right) u.v dx,$$

for all $u, v \in W^{1,\mathcal{D}}(\Omega)$, where $\left(W^{1,\mathcal{D}}(\Omega)\right)^*$ denotes the dual space of $W^{1,\mathcal{D}}(\Omega)$ and $\langle .,.\rangle$ stands for the duality pairing between $W^{1,\mathcal{D}}(\Omega)$ and $\left(W^{1,\mathcal{D}}(\Omega)\right)^*$.

Proposition 5. (see [16, Proposition 3.4]) Let hypothesis (2.1) be satisfied.

1. $A : W^{1,\mathcal{D}}(\Omega) \to \left(W^{1,\mathcal{D}}(\Omega)\right)^*$ is a continuous, bounded, and strictly monotone operator.

2. $A : W^{1,\mathcal{D}}(\Omega) \to \left(W^{1,\mathcal{D}}(\Omega)\right)^*$ satisfies the (S_+)-property, i.e., if $u_n \rightharpoonup u$ in $W^{1,\mathcal{D}}(\Omega)$ and $\overline{\lim_{n\to+\infty}}\langle A(u_n) - A(u), u_n - u\rangle \leq 0$, then $u_n \to u$ in $W^{1,\mathcal{D}}(\Omega)$.

Definition 1. Let $u \in X_0$. We say that u is a **weak solution** to the problem (1.1) if

$$\begin{aligned}\int_\Omega \left(|\nabla u|^{p(x)-2} + b(x)|\nabla u|^{q(x)-2}\right)\nabla u.\nabla v\, dx + \int_\Omega \left(|u|^{p(x)-2} + b(x)|u|^{q(x)-2}\right) u.v\, dx \\ - \int_\Omega f(x,u)v dx - \int_{\Lambda_2} g(x,u)v d\sigma, = 0,\end{aligned}$$

for all $v \in X_0$.

Now, we give the definition of the Cerami condition that was first introduced by G. Cerami in [21].

Definition 2. Let $(X, \|.\|)$ be a real Banach space and $\phi \in C^1(X,\mathbb{R})$. We say that ϕ satisfies the Cerami condition (we denote (C)−condition) in X, if any sequence $(u_n) \subset X$ such that $(\phi(u_n))$ is bounded and $\|\phi'(u_n)\|(1 + \|u_n\|) \to 0$ as $n \to +\infty$ has a strong convergent subsequence in X.

Remark 1. 1. It is clear from the above definition that if ϕ satisfies the (PS)-condition, then it satisfies the (C)-condition. However, there are functionals that satisfy the (C)-condition but do not satisfy the (PS)-condition (see [21]). Consequently, the (PS)-condition implies the (C)-condition.

2. The (C)-condition and the (PS)-condition are equivalent if ϕ is bounded below (see [22]).

Next, we present the following theorems, which will play a fundamental role in the proof of the main theorems.

Theorem 3. (see [23]) Let $(X, \|.\|)$ be a real Banach space; $\phi \in C^1(X, \mathbb{R})$ satisfies the (C)-condition; $\phi(0) = 0$, and the following conditions hold:

1. There exist positive constants ρ and α such that $\phi(u) \geq \alpha$ for any $u \in X$ with $\|u\| = \rho$.
2. There exists a function $e \in X$ such that $\|e\| > \rho$ and $\phi(e) \leq 0$.

Then, the functional ϕ has a critical value $c \geq \alpha$, that is, there exists $u \in X$ such that $\phi(u) = c$ and $\phi'(u) = 0$ in X^*.

Let X be a real, reflexive, and separable Banach space. Then there exist $\{e_j\}_{j\in\mathbb{N}} \subset X$ and $\{e_j^*\}_{j\in\mathbb{N}} \subset X^*$ such that

$$X = \overline{span\{e_j : j = 1, 2, ...\}}, \quad X^* = \overline{span\{e_j^* : j = 1, 2, ...\}},$$

and $\left\langle e_i^*, e_j \right\rangle = 1$ if $i = j$, $\left\langle e_i^*, e_j \right\rangle = 0$ if $i \neq j$.
We denote $X_j = span\{e_j\}$,$Y_k = \bigoplus_{j=1}^{k} X_j$, and $Z_k = \overline{\bigoplus_{j=k}^{+\infty} X_j}$.

Theorem 4. (see [24]) Assume that X is a Banach space, and let $\phi : X \to \mathbb{R}$ be an even functional of class $C^1(X, \mathbb{R})$ that satisfies the (C)−condition . For every $k \in \mathbb{N}$, there exists $\gamma_k > \eta_k > 0$ such that
(A_1) $b_k := \inf\{\phi(u) : u \in Z_k, \|u\| = \eta_k\} \to +\infty$ as $k \to +\infty$;
(A_2) $c_k := \max\{\phi(u) : u \in Y_k, \|u\| = \gamma_k\} \leq 0$.
Then, ϕ has a sequence of critical values tending to $+\infty$.

3. Proofs of Theorems 1-2

First of all, we are going to show that the functional ϕ fulfills the (C)-condition.

Lemma 3.1. If assumptions $(H_0), (H_1), (H_2)$ and (H_3) hold, then the functional ϕ satisfies the (C)-condition.

Proof. Let $(u_n) \subset X_0$ be a Cerami sequence for ϕ, namely,

$$(\phi(u_n)) \text{ is bounded and } \quad \|\phi'(u_n)\|_{X_0^*}(1 + \|u_n\|) \to 0, \tag{3.1}$$

which implies that

$$\sup |\phi(u_n)| \leq M \text{ and } \quad \langle \phi'(u_n), u_n \rangle = \circ_n(1), \tag{3.2}$$

where $\lim\limits_{n\to+\infty} \circ_n(1) = 0$ and $M > 0$.
We need to prove the boundedness of the sequence (u_n) in X_0. To this end, assume to the contrary, that the sequence (u_n) is unbounded in X_0. Without loss of generality, we can assume that $\|u_n\| > 1$. By virtue of (H_3), for n large enough, we have

$$\begin{aligned}
M + 1 &\geq \phi(u_n) - \frac{1}{q^+}\langle \phi'(u_n), u_n \rangle \\
&= \int_\Omega \left(\frac{1}{p(x)} |\nabla u_n|^{p(x)} + \frac{b(x)}{q(x)} |\nabla u_n|^{q(x)} \right) dx + \int_\Omega \left(\frac{1}{p(x)} |u_n|^{p(x)} + \frac{b(x)}{q(x)} |u_n|^{q(x)} \right) dx \\
&\quad - \int_\Omega F(x, u_n) dx - \int_{\Lambda_2} G(x, u_n) d\sigma + \frac{1}{q^+} \int_\Omega f(x, u_n) u_n dx + \frac{1}{q^+} \int_{\Lambda_2} g(x, u_n) u_n d\sigma
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{q^+}\int_\Omega\left(|\nabla u_n|^{p(x)}+b(x)|\nabla u_n|^{q(x)}\right)dx-\frac{1}{q^+}\int_\Omega\left(|u_n|^{p(x)}+b(x)|u_n|^{q(x)}\right)dx\\
&\geq\frac{1}{q^+}\int_\Omega\left(|\nabla u_n|^{p(x)}+b(x)|\nabla u_n|^{q(x)}\right)dx+\frac{1}{q^+}\int_\Omega\left(|u_n|^{p(x)}+b(x)|u_n|^{q(x)}\right)dx\\
&-\frac{1}{q^+}\int_\Omega\left(|\nabla u_n|^{p(x)}+b(x)|\nabla u_n|^{q(x)}\right)dx-\frac{1}{q^+}\int_\Omega\left(|u_n|^{p(x)}+b(x)|u_n|^{q(x)}\right)dx\\
&+\int_\Omega\mathcal{F}(x,u_n)dx+\int_{\Lambda_2}\mathcal{G}(x,u_n)d\sigma,
\end{aligned}
$$

where $\mathcal{F}(x,u_n):=\frac{1}{q^+}f(x,u_n)u_n-F(x,u_n)\geq 0$ and $\mathcal{G}(x,u_n)=\frac{1}{q^+}g(x,u_n)u_n-G(x,u_n)\geq 0$. Then, we obtain

$$
M+1\geq\int_\Omega\mathcal{F}(x,u_n)dx+\int_{\Lambda_2}\mathcal{G}(x,u_n)d\sigma,
$$

which implies

$$
M+1\geq\int_\Omega\mathcal{F}(x,u_n)dx, \tag{3.3}
$$

and

$$
M+1\geq\int_{\Lambda_2}\mathcal{G}(x,u_n)d\sigma. \tag{3.4}
$$

On the other hand, by Proposition 3, we have

$$
\begin{aligned}
M&\geq\phi(u_n)\\
&=\int_\Omega\left(\frac{1}{p(x)}|\nabla u_n|^{p(x)}+\frac{b(x)}{q(x)}|\nabla u_n|^{q(x)}\right)dx+\int_\Omega\left(\frac{1}{p(x)}|u_n|^{p(x)}+\frac{b(x)}{q(x)}|u_n|^{q(x)}\right)dx\\
&-\int_\Omega F(x,u_n)dx-\int_{\Lambda_2}G(x,u_n)d\sigma\\
&\geq\frac{1}{q^+}\left(\int_\Omega(|\nabla u_n|^{p(x)}+b(x)|\nabla u_n|^{q(x)})dx+\int_\Omega\left(|u_n|^{p(x)}+b(x)|u_n|^{q(x)}\right)dx\right)\\
&-\int_\Omega F(x,u_n)dx-\int_{\Lambda_2}G(x,u_n)d\sigma\\
&\geq\frac{1}{q^+}\hat{\rho}(u_n)-\int_\Omega F(x,u_n)dx-\int_{\Lambda_2}G(x,u_n)d\sigma.
\end{aligned}
$$

Because $\|u_n\|>1$, we can obtain

$$
M\geq\frac{1}{q^+}\|u_n\|^{p^-}-\int_\Omega F(x,u_n)dx-\int_{\Lambda_2}G(x,u_n)d\sigma. \tag{3.5}
$$

Since $\|u_n\|\to+\infty$ as $n\to+\infty$, we deduce that

$$
\int_\Omega F(x,u_n)dx+\int_{\Lambda_2}G(x,u_n)d\sigma\geq\frac{1}{q^+}\|u_n\|^{p^-}-M\to+\infty\quad\text{as } n\to+\infty. \tag{3.6}
$$

Furthermore, using Proposition 3, we have

$$
\phi(u_n)=\int_\Omega\left(\frac{1}{p(x)}|\nabla u_n|^{p(x)}+\frac{b(x)}{q(x)}|\nabla u_n|^{q(x)}\right)dx+\int_\Omega\left(\frac{1}{p(x)}|u_n|^{p(x)}+\frac{b(x)}{q(x)}|u_n|^{q(x)}\right)dx
$$

$$
\begin{aligned}
&- \int_\Omega F(x,u_n)dx - \int_{\Lambda_2} G(x,u_n)d\sigma \\
&\le \frac{1}{p^-}\|u_n\|^{q^+} - \int_\Omega F(x,u_n)dx - \int_{\Lambda_2} G(x,u_n)d\sigma.
\end{aligned}
$$

Then, we obtain

$$
\phi(u_n) + \int_\Omega F(x,u_n)dx + \int_{\Lambda_2} G(x,u_n)d\sigma \le \frac{1}{p^-}\|u_n\|^{q^+}. \tag{3.7}
$$

In view of condition (H_2), there exist $T_1, T_2 > 0$ such that

$$
\begin{aligned}
F(x,t) &> |t|^{q^+} \quad \text{for all } x \in \Omega \text{ and } |t| > T_1,\\
G(x,t) &> |t|^{q^+} \quad \text{for all } x \in \Lambda_2 \text{ and } |t| > T_2.
\end{aligned}
$$

Since $F(x,.)$ and $G(x,.)$ are continuous functions on $[-T_1,T_1]$ and $[-T_2,T_2]$, respectively, there exist $C_0, C_0^* > 0$ such that

$$
\begin{aligned}
|F(x,t)| &\le C_0 \quad \text{for all } (x,t) \in \Omega \times [-T_1,T_1],\\
|G(x,t)| &\le C_0^* \quad \text{for all } (x,t) \in \Lambda_2 \times [-T_2,T_2].
\end{aligned}
$$

Then, there exist two real numbers K and K', such that

$$
\begin{aligned}
F(x,t) &\ge K \ \text{ for all } (x,t) \in \Omega \times \mathbb{R},\\
G(x,t) &\ge K' \ \text{ for all } (x,t) \in \Lambda_2 \times \mathbb{R}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{F(x,u_n) - K}{\frac{1}{p^-}\|u_n\|^{q^+}} &\ge 0,\\
\frac{G(x,u_n) - K'}{\frac{1}{p^-}\|u_n\|^{q^+}} &\ge 0,
\end{aligned} \tag{3.8}
$$

for all $(x,n) \in \overline{\Omega} \times \mathbb{N}$.

Put $\beta_n = \frac{u_n}{\|u_n\|}$, so $\|\beta_n\| = 1$. Up to subsequences, for some $\beta \in X_0$, we have

$$
\begin{aligned}
\beta_n &\rightharpoonup \beta && \text{in } X_0,\\
\beta_n &\to \beta && \text{in } L^{s(x)}(\Omega),\\
\beta_n &\to \beta && \text{in } L^{r(x)}(\Lambda_2),\\
\beta_n(x) &\to \beta(x) && \text{a.e., in } \Omega,\\
\beta_n(x) &\to \beta(x) && \text{a.e., in } \Lambda_2,
\end{aligned} \tag{3.9}
$$

for $s(x) < p^*(x)$ and $r(x) < p_*(x)$.
Define the sets $\Omega_0 = \{x \in \Omega : \beta(x) \neq 0\}$ and $\Gamma = \{x \in \Lambda_2 : \beta(x) \neq 0\}$.
Obviously, since $\|u_n\| \to +\infty$ as $n \to +\infty$, we have

$$
|u_n(x)| = |\beta_n(x)|\|u_n\| \to +\infty,
$$

for any $x \in \Omega_0 \cup \Gamma$.
Therefore, due to (H_2), for all $x \in \Omega_0 \cup \Gamma$, we deduce

$$\begin{aligned} \frac{F(x,u_n)}{\frac{1}{p^-}\|u_n\|^{q^+}} &= p^- \frac{F(x,u_n)}{|u_n(x)|^{q^+}}|\beta_n(x)|^{q^+} \to +\infty, \\ \frac{G(x,u_n)}{\frac{1}{p^-}\|u_n\|^{q^+}} &= p^- \frac{G(x,u_n)}{|u_n(x)|^{q^+}}|\beta_n(x)|^{q^+} \to +\infty. \end{aligned} \tag{3.10}$$

Thus, $|\Omega_0| = 0$ and $|\Gamma| = 0$. In fact, suppose by contradiction that $|\Omega_0| \neq 0$ or $|\Gamma| \neq 0$. Using (3.6), (3.7), (3.10), and Fatou's lemma, we get

$$\begin{aligned} 1 &= \liminf_{n\to+\infty} \frac{\int_\Omega F(x,u_n)dx + \int_{\Lambda_2} G(x,u_n)d\sigma}{\phi(u_n) + \int_\Omega F(x,u_n)dx + \int_{\Lambda_2} G(x,u_n)d\sigma} \\ &\geq \liminf_{n\to+\infty} \frac{\int_\Omega F(x,u_n)dx + \int_{\Lambda_2} G(x,u_n)d\sigma}{\frac{1}{p^-}\|u_n\|^{q^+}} \\ &\geq \liminf_{n\to+\infty} \Big[\int_{\Omega_0} \frac{F(x,u_n)}{\frac{1}{p^-}\|u_n\|^{q^+}}dx + \int_\Gamma \frac{G(x,u_n)}{\frac{1}{p^-}\|u_n\|^{q^+}}d\sigma \Big] - \limsup_{n\to+\infty} \int_{\Omega_0} \frac{K}{\frac{1}{p^-}\|u_n\|^{q^+}}dx \\ &\quad - \limsup_{n\to+\infty} \int_\Gamma \frac{K'}{\frac{1}{p^-}\|u_n\|^{q^+}}d\sigma \\ &\geq \liminf_{n\to+\infty} \int_{\Omega_0} \frac{F(x,u_n)}{\frac{1}{p^-}\|u_n\|^{q^+}}dx + \liminf_{n\to+\infty} \int_\Gamma \frac{G(x,u_n)}{\frac{1}{p^-}\|u_n\|^{q^+}}d\sigma - \limsup_{n\to+\infty} \int_{\Omega_0} \frac{K}{\frac{1}{p^-}\|u_n\|^{q^+}}dx \\ &\quad - \limsup_{n\to+\infty} \int_\Gamma \frac{K'}{\frac{1}{p^-}\|u_n\|^{q^+}}d\sigma \\ &= \liminf_{n\to+\infty} \int_{\Omega_0} \frac{F(x,u_n)-K}{\frac{1}{p^-}\|u_n\|^{q^+}}dx + \liminf_{n\to+\infty} \int_\Gamma \frac{g(x,u_n)-K'}{\frac{1}{p^-}\|u_n\|^{q^+}}d\sigma \\ &\geq \int_{\Omega_0} \liminf_{n\to+\infty} \frac{F(x,u_n)-K}{\frac{1}{p^-}\|u_n\|^{q^+}}dx + \int_\Gamma \liminf_{n\to+\infty} \frac{G(x,u_n)-K'}{\frac{1}{p^-}\|u_n\|^{q^+}}d\sigma \\ &\geq \int_{\Omega_0} \liminf_{n\to+\infty} \frac{F(x,u_n)}{\frac{1}{p^-}\|u_n\|^{q^+}}dx + \int_\Gamma \liminf_{n\to+\infty} \frac{G(x,u_n)}{\frac{1}{p^-}\|u_n\|^{q^+}}d\sigma \\ &\quad - \int_{\Omega_0} \limsup_{n\to+\infty} \frac{K}{\frac{1}{p^-}\|u_n\|^{q^+}}dx - \int_\Gamma \limsup_{n\to+\infty} \frac{K'}{\frac{1}{p^-}\|u_n\|^{q^+}}dx \\ &= +\infty, \end{aligned}$$

which is a contradiction. Therefore, $\beta(x) = 0$ for a.e. $x \in \Omega$ and for a.e. $x \in \Lambda_2$.
From (3.5) and (3.9), respectively, we can deduce that

$$\begin{aligned} &\beta_n \to 0 \quad \text{in } L^{s(x)}(\Omega), \quad \beta_n \to 0 \quad \text{in } L^{r(x)}(\Lambda_2), \\ &\beta_n(x) \to 0 \quad \text{a.e. in } \Omega, \quad \beta_n(x) \to 0 \quad \text{a.e. in } \Lambda_2, \end{aligned} \tag{3.11}$$

for $s(x) < p^*(x)$, $r(x) < p_*(x)$, and

$$0 < \frac{1}{q^+} \le \limsup_{n\to+\infty}\Big[\int_\Omega \frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx + \int_{\Lambda_2} \frac{|G(x,u_n)|}{\|u_n\|^{p^-}}d\sigma\Big]$$
$$\le \limsup_{n\to+\infty}\int_\Omega \frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx + \limsup_{n\to+\infty}\int_{\Lambda_2} \frac{|G(x,u_n)|}{\|u_n\|^{p^-}}d\sigma. \tag{3.12}$$

Using (H_0) and (H_1), we obtain

$$\begin{aligned}
\int_{\{0\le|u_n(x)|\le r_1\}} \frac{|F(x,u_n)|}{\|u_n\|^{p^-}}\,dx &\le C_1 \int_{\{0\le|u_n(x)|\le r_1\}} \frac{|u_n| + \frac{1}{s_1(x)}|u_n|^{s(x)}}{\|u_n\|^{p^-}}\,dx \\
&\le C_1\frac{|u_n|_1}{\|u_n\|^{p^-}} + \frac{C_1}{s_1^-}\int_{\{0\le|u_n(x)|\le r_1\}} |u_n|^{s_1(x)-p^-}|\beta_n|^{p^-}dx \\
&\le C_1C_3\frac{\|u_n\|}{\|u_n\|^{p^-}} + \frac{Cr_1^{s-p^-}}{s^-}|\beta_n|_{p^-}^{p^-} \\
&\le \frac{CC_3}{\|u_n\|^{p^--1}} + \frac{C_1C_4r_1^{s-p^-}}{s_1^-}\|\beta_n\|^{p^-} \\
&\to 0, \quad \text{ as } n\to+\infty,
\end{aligned} \tag{3.13}$$

where $C_3, C_4 > 0$, s is either s_1^+ or s_1^- and r_1 comes from (H_3).
Put $l_1'(x) = \frac{l(x)}{l(x)-1}$. Since $l_1 \in L^\infty(\Omega)$ with $l_1(x) > \frac{N}{p^-}$, it follows that $l_1'(x)p^- < p^*(x)$.
On the other hand, by virtue of hypothesis $(H_3)(i)$, (3.3), and (3.11), we deduce

$$\begin{aligned}
\int_{\{|u_n(x)|\ge r_1\}} \frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx &\le 2\Big[\int_{\{|u_n(x)|\ge r_1\}}\Big(\frac{|F(x,u_n)|}{|u_n|^{p^-}}\Big)^{l_1(x)}dx\Big]^{\frac{1}{l_1(x)}}\Big[\int_{\{|u_n(x)|\ge r_1\}}|\beta_n|^{l_1'(x)p^-}dx\Big]^{\frac{1}{l_1'(x)}} \\
&\le 2c_1^{\frac{1}{l_1(x)}}\Big[\int_{\{|u_n(x)|\ge r_1\}}\mathcal{F}(x,u_n)dx\Big]^{\frac{1}{l_1(x)}}\Big[\int_{\{|u_n(x)|\ge r_1\}}|\beta_n|^{l_1'(x)p^-}dx\Big]^{\frac{1}{l_1'(x)}} \\
&\le 2c_1^{\frac{1}{l_1(x)}}\Big[\int_\Omega\mathcal{F}(x,u_n)dx\Big]^{\frac{1}{l_1(x)}}\Big[\int_\Omega|\beta_n|^{l_1'(x)p^-}dx\Big]^{\frac{1}{l_1'(x)}} \\
&\le 2c_1^{\frac{1}{l_1(x)}}(M+1)^{\frac{1}{l(x)}}\Big[\int_\Omega|\beta_n|^{l'(x)p^-}\,dx\Big]^{\frac{1}{l'(x)}} \\
&\to 0 \quad \text{ as } n\to+\infty.
\end{aligned}$$

Combining this with (3.13), we obtain

$$\begin{aligned}
\int_\Omega \frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx &= \int_{\{0\le|u_n(x)|\le r_1\}} \frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx + \int_{\{|u_n(x)|\ge r_1\}} \frac{|F(x,u_n)|}{\|u_n\|^{p^-}}dx \\
&\longrightarrow 0, \text{ as } n\to+\infty.
\end{aligned} \tag{3.14}$$

Similarly, let $l_2'(x) = \frac{l_2(x)}{l_2(x)-1}$. Since $l_2 \in L^\infty(\Omega)$ with $l_2(x) > \frac{N-1}{p^--1}$, it follows that $l_2'(x)p^- < p_*(x)$. Then, by $(H_3)(ii)$, (3.4), and (3.11), we can prove in a similar way that

$$\int_{\Lambda_2} \frac{|G(x,u_n)|}{\|u_n\|^{p^-}}d\sigma \longrightarrow 0, \quad \text{ as } n\to+\infty. \tag{3.15}$$

Consequently, combining (3.14) with (3.15), we obtain

$$\int_{\Omega} \frac{|F(x, u_n)|}{\|u_n\|^{p^-}} dx + \int_{\Lambda_2} \frac{|G(x, u_n)|}{\|u_n\|^{p^-}} d\sigma \longrightarrow 0, \quad \text{as } n \to +\infty,$$

which is a contradiction to (3.12). Thus, (u_n) is bounded in X_0.

Finally, we need to prove that any (C)-sequence has a convergent subsequence. Let $(u_n) \subset X_0$ be a (C)-sequence. Then, (u_n) is bounded in X_0. Passing to the limit, if necessary, to a subsequence, from Proposition 4, we have

$$\begin{aligned} u_n \rightharpoonup u \quad &\text{in } X_0, \ u_n \to u \ \text{ in } L^{s_1(x)}(\Omega), \ u_n \to u \ \text{ in } L^{s_2(x)}(\Lambda_2), \\ u_n(x) &\to u(x) \ \text{ a.e. } x \in \Omega, \ u_n(x) \to u(x) \ \text{ a.e. } x \in \Lambda_2, \end{aligned} \tag{3.16}$$

for $1 \leq s_1(x) < p^*(x)$ and $1 \leq s_2(x) < p_*(x)$. It is easy to check from (H_0), (3.16) and Hölder's inequality that

$$\begin{aligned} \Big| \int_{\Omega} f(x, u_n)(u_n - u) dx \Big| &\leq C_1 |1 + |u_n|^{s_1(x)-1}|_{s_1'(x)} |u_n - u|_{s_1(x)} \\ &\longrightarrow 0 \quad \text{as } n \to +\infty, \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \Big| \int_{\Lambda_2} g(x, u_n)(u_n - u) d\sigma \Big| &\leq C_2 |1 + |u_n|^{s_2(x)-1}|_{s_2'(x)} |u_n - u|_{s_2(x)} \\ &\longrightarrow 0 \quad \text{as } n \to +\infty, \end{aligned} \tag{3.18}$$

where $\frac{1}{s_1(x)} + \frac{1}{s_1'(x)} = 1$ and $\frac{1}{s_2(x)} + \frac{1}{s_2'(x)} = 1$.
Next, since $u_n \rightharpoonup u$, from (3.1), we have

$$\langle \phi'(u_n), u_n - u \rangle \longrightarrow 0, \quad \text{as } n \to +\infty. \tag{3.19}$$

Then

$$\begin{aligned} \langle \phi'(u_n), u_n - u \rangle &= \langle A(u_n), u_n - u \rangle - \int_{\Omega} f(x, u_n)(u_n - u) dx - \int_{\Lambda_2} g(x, u_n)(u_n - u) d\sigma \\ &\longrightarrow 0 \quad \text{as } n \to +\infty, \end{aligned}$$

where A is given in Proposition 5.
Finally, the combination of (3.17), (3.18), and (3.19) implies

$$\langle A(u_n), u_n - u \rangle \longrightarrow 0 \quad \text{as } n \to +\infty.$$

Since the operator A satisfies the (S_+) property in view of Proposition 5, we can obtain that $u_n \to u$ in X_0. The proof is complete.

Proof of Theorem 1

Let us check that the functional ϕ satisfies the geometric conditions of the mountain pass in Theorem 3. By Lemma 3.1, ϕ satisfies the $(C)-$condition. According to the definition of ϕ, we have $\phi(0) = 0$. Then, to apply Theorem 3, it remains to prove that

(i) There exist positive constants ρ and α such that $\phi(u) \geq \alpha$ for any $u \in X_0$ with $\|u\| = \rho$.

(ii) There exists a function $e \in X_0$ such that $\|e\| > \rho$ and $\phi(e) \leq 0$.

For (i), let $\|u\| < 1$. Then, by Proposition 3, we have

$$\begin{aligned} \phi(u) &\geq \frac{1}{p^+}\hat{\rho}(u) - \int_\Omega F(x,u)dx - \int_{\Lambda_2} G(x,u)d\sigma \\ &\geq \frac{1}{p^+}\|u\|^{q^+} - \int_\Omega F(x,u)dx - \int_{\Lambda_2} G(x,u)d\sigma. \end{aligned} \tag{3.20}$$

Using (H_0) and (H_4), for $\varepsilon > 0$ be small enough, there exist $C_1(\varepsilon), C_2(\varepsilon) > 0$ such that

$$\begin{aligned} F(x,t) &\leq \varepsilon|t|^{p^+} + C_1(\varepsilon)|t|^{s_1(x)}, \quad \forall (x,t) \in \Omega \times \mathbb{R}, \\ G(x,t) &\leq \varepsilon|t|^{p^+} + C_2(\varepsilon)|t|^{s_2(x)}, \quad \forall (x,t) \in \Lambda_2 \times \mathbb{R}. \end{aligned} \tag{3.21}$$

Since $p^+ < s_1(x) < p^*(x)$ and $p^+ < s_2(x) < p_*(x)$ for all $x \in \overline{\Omega}$ and for all $x \in \overline{\Lambda_2}$ in view of condition (H_1), we have from Proposition 4 that

$$\begin{aligned} &X_0 \hookrightarrow L^{p^+}(\Omega), \quad X_0 \hookrightarrow L^{s_1(x)}(\Omega), \\ &X_0 \hookrightarrow L^{p^+}(\Lambda_2), \quad X_0 \hookrightarrow L^{s_2(x)}(\Lambda_2). \end{aligned}$$

So, there exist $c_i > 0 (i = 3, ...6)$ such that

$$\begin{aligned} &|u|_{p^+} \leq c_3\|u\|, \qquad |u|_{s_1(x)} \leq c_4\|u\|, \ \ \forall u \in X_0 \\ &|u|_{p^+,\Lambda_2} \leq c_6\|u\|, \quad |u|_{s(x)} \leq c_7\|u\|, \ \ \forall u \in X_0. \end{aligned}$$

Therefore, by (3.20) and (3.21), for $\|u\| < 1$ sufficiently small, we obtain

$$\begin{aligned} \phi(u) &\geqslant \frac{1}{q^+}\|u\|^{q^+} - \varepsilon \int_\Omega |u|^{p^+}dx - C_1(\varepsilon)\int_\Omega |u|^{s_1(x)}dx - \varepsilon \int_{\Lambda_2} |u|^{p^+}d\sigma - C_2(\varepsilon)\int_{\Lambda_2} |u|^{s_2(x)}d\sigma \\ &\geqslant \frac{1}{q^+}\|u\|^{q^+} - \varepsilon c_3^{p^+}\|u\|^{p^+} - C_1(\varepsilon)c_4^{s_1^-}\|u\|^{s_1^-} - \varepsilon c_5^{p^+}\|u\|^{p^+} - C_2(\varepsilon)c_6^{s_2^-}\|u\|^{s_2^-}. \end{aligned}$$

Since $s_1^- > p^+$ in view of condition (H_1) and $\|u\| < 1$, then $\|u\|^{s_1^-} < \|u\|^{p^+}$. Thus, we obtain

$$\begin{aligned} \phi(u) &\geqslant \frac{1}{q^+}\|u\|^{q^+} - \varepsilon c_3^{p^+}\|u\|^{p^+} - C_1(\varepsilon)c_4^{s_1^-}\|u\|^{p^+} - \varepsilon c_5^{p^+}\|u\|^{p^+} - C_2(\varepsilon)c_6^{s_2^-}\|u\|^{s_2^-} \\ &\geq \frac{1}{q^+}\|u\|^{q^+} - \left(\varepsilon c_3^{p^+} + C_1(\varepsilon)c_4^{s_1^-} + \varepsilon c_5^{p^+}\right)\|u\|^{p^+} - C_2(\varepsilon)c_6^{s_2^-}\|u\|^{s_2^-}. \end{aligned}$$

Since $s_2^- > q^+ \geq p^+$, then by the standard argument, there exist positive constants ρ and α such that $\phi(u) \geq \alpha$ for any $u \in X_0$ with $\|u\| = \rho$.

Next, we affirm that there exists $e \in X_0$ with $\|u\| > \rho$ such that

$$\phi(e) < 0. \tag{3.22}$$

In fact, from (H_2), it follows that for every $k>0$, there exist constants T_k and T_k^* such that

$$F(x,t)>k|t|^{q^+} \quad \text{for all } x\in\Omega \text{ and } |t|>T_k,$$
$$G(x,t)>k|t|^{q^+} \quad \text{for all } x\in\Lambda_2 \text{ and } |t|>T_k^*.$$

Since $F(x,.)$ and $G(x,.)$ are continuous functions on $[-T_k,T_k]$ and $\left[-T_k^*,T_k^*\right]$, respectively, there exist constants $C_0, C_0^*>0$ such that

$$|F(x,t)|\le C_0 \quad \text{for all } (x,t)\in\Omega\times[-T_k,T_k],$$
$$|G(x,t)|\le C_0^* \quad \text{for all } (x,t)\in\Lambda_2\times[-T_k^*,T_k^*].$$

Thus,

$$\begin{aligned} F(x,t)&\ge k|t|^{q^+}-C_0, \quad \text{for all } (x,t)\in\Omega\times\mathbb{R},\\ G(x,t)&\ge k|t|^{q^+}-C_0^*, \quad \text{for all } (x,t)\in\Lambda_2\times\mathbb{R}. \end{aligned} \tag{3.23}$$

Let $w\in X_0\setminus\{0\}$ such that $\|w\|=1$ and $l>1$ be large enough. Using the above inequality, we obtain

$$\begin{aligned} \phi(lw)&=\int_\Omega\left(\frac{1}{p(x)}|\nabla lw|^{p(x)}+\frac{b(x)}{q(x)}|\nabla lw|^{q(x)}\right)dx+\int_\Omega\left(\frac{1}{p(x)}|lw|^{p(x)}+\frac{b(x)}{q(x)}|lw|^{q(x)}\right)dx\\ &\quad-\int_\Omega F(x,lw)dx-\int_{\Lambda_2}G(x,lw)d\sigma\\ &\le\frac{l^{q^+}}{p^-}\hat{\rho}(w)-kl^{q^+}\int_\Omega|w|^{q^+}dx-kl^{q^+}\int_{\Lambda_2}|w|^{q^+}d\sigma+C_0|\Omega|+C_0^*|\Lambda_2|\\ &\le\frac{l^{q^+}}{p^-}-kl^{q^+}\int_\Omega|w|^{q^+}dx-kl^{q^+}\int_{\Lambda_2}|w|^{q^+}d\sigma+C_0|\Omega|+C_0^*|\Lambda_2|\\ &=l^{q^+}\left(\frac{1}{p^-}-k\int_\Omega|w|^{q^+}dx-k\int_{\Lambda_2}|w|^{q^+}d\sigma\right)+C_0|\Omega|+C_0^*|\Lambda_2|. \end{aligned}$$

As

$$\frac{1}{p^-}-k\int_\Omega|w|^{q^+}dx-k\int_{\Lambda_2}|w|^{q^+}d\sigma<0,$$

for k large enough, we deduce

$$\phi(lw)\to-\infty, \quad \text{as } l\to+\infty.$$

Thus, there exist $t_0>1$ and $e=t_0w\in X_0\setminus\overline{B_\rho(0)}$ such that $\phi(e)<0$.

Proof of Theorem 2

To prove Theorem 2, we need the following auxiliary lemmas:

Lemma 3.2. (see [25, 26]) For $s\in C_+(\overline{\Omega})$ and $r\in C_+(\overline{\Lambda_2})$ such that $s(x)<p^*(x)$ for all $x\in\overline{\Omega}$ and $r(x)<p_*(x)$ for all $x\in\overline{\Lambda_2}$. Let

$$\delta_k=\sup\{|u|_{s(x)}:\|u\|=1,u\in Z_k\},$$
$$\delta_k'=\sup\{|u|_{s(x),\Lambda_2}:\|u\|=1,u\in Z_k\}.$$

Then, $\lim\limits_{k\to+\infty}\delta_k=\lim\limits_{k\to+\infty}\delta_k'=0.$

Lemma 3.3. (see [27]) For all $s \in C_+(\overline{\Omega})$ ($r \in C_+(\overline{\Lambda_2})$) and $u \in L^{s(x)}(\Omega)$ ($v \in L^{r(x)}(\Lambda_2)$), there exists $y \in \Omega$ ($z \in \Lambda_2$) such that

$$\begin{aligned} &\int_\Omega |u|^{s(x)}dx = |u|_{s(x)}^{s(y)}, \\ &\int_{\Lambda_2} |u|^{r(x)}d\sigma = |u|_{r(x),\Lambda_2}^{r(z)}. \end{aligned} \tag{3.24}$$

Now, we return to the proof of Theorem 2. To this end, based on Fountain Theorem 4, we will show that the problem (1.1) possesses infinitely many weak solutions with unbounded energy. Evidently, according to (H_5), ϕ is an even functional. By Lemma 3.1, we know that ϕ satisfies the (C)-condition. Then, to prove Theorem 2, it only remains to verify the following assertions:
(A_1) $b_k := \inf\{\phi(u) : u \in Z_k, \|u\| = \eta_k\} \to +\infty$ as $k \to +\infty$,
(A_2) $c_k := \max\{\phi(u) : u \in Y_k, \|u\| = \gamma_k\} \le 0$.

(A_1) For any $u \in Z_k$ such that $\|u\| = \eta_k > 1$. It follows from (H_0), Proposition 3, and Lemma 3.3 that

$$\begin{aligned} \phi(u) &= \int_\Omega \left(\frac{1}{p(x)}|\nabla u|^{p(x)} + \frac{b(x)}{q(x)}|\nabla u|^{q(x)}\right)dx + \int_\Omega \left(\frac{1}{p(x)}|u|^{p(x)} + \frac{b(x)}{q(x)}|u|^{q(x)}\right)dx \\ &\quad - \int_\Omega F(x,u)dx - \int_{\Lambda_2} G(x,u)d\sigma \\ &\ge \frac{1}{q^+}\|u\|^{p^-} - C_1\int_\Omega |u|dx - C_1\int_\Omega \frac{|u|^{s_1(x)}}{s_1(x)}dx - C_2\int_{\Lambda_2}|u|d\sigma - C_2\int_{\Lambda_2}\frac{|u|^{s_2(x)}}{s_2(x)}d\sigma \\ &\ge \frac{1}{q^+}\|u\|^{p^-} - C_1c_1\|u\| - \frac{C_1}{s_1^-}|u|_{s_1(x)}^{s_1(y)} - C_2c_2\|u\| - \frac{C_2}{s_2^-}|u|_{s_2(x),\lambda_2}^{s_2(z)} \\ &\ge \frac{1}{q^+}\|u\|^{p^-} - c_3\|u\| - \frac{C_1}{s_1^-}|u|_{s_1(x)}^{s_1(y)} - \frac{C_2}{s_2^-}|u|_{s_2(x),\lambda_2}^{s_2(z)}, \end{aligned}$$

where $c_3 = \max\{C_1c_1, C_2c_2\}$.
Then, it follows that

$$\begin{aligned} \phi(u) &\ge \begin{cases} \frac{1}{q^+}\|u\|^{p^-} - c_3\|u\| - \frac{C_1}{s_1^-} - \frac{C_2}{s_2^-} & \text{if } |u|_{s_1(x)} \le 1, |u|_{s_2(x)} \le 1 \\ \frac{1}{q^+}\|u\|^{p^-} - c_3\|u\| - \frac{C_1}{s_1^-}(\delta_k\|u\|)^{s_1^+} - \frac{C_2}{s_2^-}(\delta'_k\|u\|)^{s_2^+} & \text{if } |u|_{s_1(x)} > 1, |u|_{s_2(x)} > 1 \\ \frac{1}{q^+}\|u\|^{p^-} - c_3\|u\| - \frac{C_1}{s_1^-} - \frac{C_2}{s_2^-}(\delta'_k\|u\|)^{s_2^+} & \text{if } |u|_{s_1(x)} \le 1, |u|_{s_2(x)} > 1 \\ \frac{1}{q^+}\|u\|^{p^-} - c_3\|u\| - \frac{C_1}{s_1^-}(\delta_k\|u\|)^{s_1^+} - \frac{C_2}{s_2^-} & \text{if } |u|_{s_1(x)} > 1, |u|_{s_2(x)} \le 1 \end{cases} \\ &\ge \frac{1}{q^+}\|u\|^{p^-} - c_3\|u\| - \frac{2C_1}{s_1^-}(\delta_k\|u\|)^{s_1^+} - \frac{2C_2}{s_2^-}(\delta'_k\|u\|)^{s_2^+} - \frac{2C_1}{s_1^-} - \frac{2C_2}{s_2^-} \\ &\ge \frac{1}{2q^+}\|u\|^{p^-} + \left(\frac{1}{4q^+}\|u\|^{p^-} - \frac{2C_1}{s_1^-}\delta_k^{s_1^+}\|u\|^{s_1^+}\right) + \left(\frac{1}{4q^+}\|u\|^{p^-} - \frac{2C_1}{s_2^-}(\delta'_k)^{s_2^+}\|u\|^{s_2^+}\right) \\ &\quad - c_3\|u\| - \frac{2C_1}{s_1^-} - \frac{2C_2}{s_2^-}. \end{aligned} \tag{3.25}$$

Let us consider the following equations:

$$\frac{1}{4q^+}t^{p^-} - \frac{2C_1}{s_1^-}\delta_k^{s_1^+}t^{s_1^+} = 0, \tag{3.26}$$

and

$$\frac{1}{4q^+}t^{p^-} - \frac{2C_1}{s_2^-}(\delta_k')^{s_2^+}t^{s_2^+} = 0. \tag{3.27}$$

Let a_k and d_k be the two non-zero solutions of (3.26) and (3.27), respectively. Then, we obtain

$$a_k = \left(\frac{4q^+2C_1}{s_1^-}\delta_k^{s_1^+}\right)^{\frac{1}{p^- - s_1^+}} \to +\infty \quad \text{and} \quad d_k = \left(\frac{4q^+2C_2}{s_2^-}(\delta_k')^{s_2^+}\right)^{\frac{1}{p^- - s_2^+}} \to +\infty \quad \text{as } k \to +\infty.$$

We fix η_k as follows

$$\eta_k = \min\{a_k, d_k\}.$$

Then, by Lemma 3.2, (3.25) and $s_1^+, s_2^+ > q^+ > p^-$, we obtain

$$\phi(u) \geq \frac{1}{2q^+}\eta_k^{p^-} - c_3\eta_k - C_6 \to +\infty \quad \text{as} \quad k \to +\infty,$$

where $C_6 > 0$. Hence, (A_1) holds.

(A_2) In view of Proposition 3 and (3.23), for $u \in Y_k$ with $||u| > 1$, we have

$$\begin{aligned}
\phi(u) &\leq \frac{1}{p^-}||u||^{q^+} - \int_\Omega F(x,u)dx - \int_{\Lambda_2} G(x,u)d\sigma \\
&\leq \frac{1}{p^-}||u||^{q^+} - k\int_\Omega |u|^{q^+}dx - k\int_{\Lambda_2} |u|^{q^+}d\sigma + C_0|\Omega| + C_0^*|\Lambda_2| \\
&\leq \frac{1}{p^-}||u||^{q^+} - k\left(|u|_{q^+}^{q^+} + |u|_{q^+,\Lambda_2}^{q^+}\right) + C_0|\Omega| + C_0^*|\Lambda_2|.
\end{aligned}$$

Since $dimY_k < \infty$, then all norms are equivalent in Y_k. Therefore, as $\frac{1}{p^-} < 1$, for k large enough, we obtain

$$\phi(u) \to -\infty \quad \text{as} \quad ||u|| \to +\infty.$$

Finally, the assertion (A_2) is also valid.
This completes the proof.

Author contributions

Mahmoud El Ahmadi: Writing-original draft, Writing-review & editing; Mohammed Barghouthe: Formal Analysis, Methodology; Anass Lamaizi: Formal Analysis; Mohammed Berrajaa: Supervision, Validation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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