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Research article

Multiple positive solutions for the logarithmic Schrödinger equation with a Coulomb potential

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Abstract: In this article, we mainly study the global existence of multiple positive solutions for the logarithmic Schrödinger equation with a Coulomb type potential

$$-\Delta u + V(\epsilon x)u = \lambda(I_\alpha * |u|^p)|u|^{p-1} + u \log u^2 \text{ in } \mathbb{R}^3,$$

where $u \in H^1(\mathbb{R}^3)$, $\epsilon > 0$, V is a continuous function with a global minimum, and Coulomb type energies with $0 < \alpha < 3$ and $p \geq 1$. We explore the existence of local positive solutions without the functional having to be a combination of a C^1 functional and a convex semicontinuous functional, as is required in the global case.

Keywords: variational method; logarithmic Schrödinger equation; multiple solutions; Coulomb potential
Mathematics Subject Classification: 35Q55, 35A15, 35J60, 35B09

1. Introduction

Recently, some studies have focused on the nonlinear Schrödinger equation

$$i\epsilon\partial_t\Psi = -\epsilon^2\Delta\Psi + (V(x) + w)\Psi - \lambda(I_\alpha * |\Psi|^p)|\Psi|^{p-1} - \Psi \log |\Psi|^2, \tag{1.1}$$

where $\Psi : [0, \infty) \times \mathbb{R}^N \to \mathbb{C}, N \geq 3$, $\alpha \in (0, N)$, $p > 1$, λ is a physical constant and I_α is the Riesz potential, defined for $x \in \mathbb{R}^N \setminus \{0\}$ as

$$I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}.$$

The problem described in equation (1.1) has various practical applications in fields such as quantum mechanics, quantum optics, nuclear physics, transport and diffusion phenomena, open quantum systems, effective quantum gravity, theory of superfluidity, and Bose-Einstein condensation. Notably, periodic potentials V can play a significant role in crystals and artificial crystals formed by light beams. While the logarithmic Schrödinger equation has been excluded as a fundamental quantum wave equation based on precise neutron diffraction experiments, there is ongoing discussion regarding its suitability as a simplified model for certain physical phenomena. The existence and uniqueness of solutions for the associated Cauchy problem have been investigated in an appropriate functional framework [1–3], and orbital stability of the ground state solution with respect to radial perturbations has also been studied [4–6]. The results regarding the wave equation can be referred to in [7–10].

In the Schrödinger equation, the convolution term involve the Coulomb interaction between electrons or interactions between other particles. In Schrödinger equations with convolution terms, this term typically represents the potential energy arising from interactions between particles. Physically, it implies that particles are influenced not only by external potential fields but also by the potential fields created by other particles. These interactions could involve electromagnetic forces, gravitational forces, or other types of interactions depending on the nature of the system. The introduction of the convolution term adds complexity to the Schrödinger equation because particle interactions are often non-local, extending across the entire spatial domain [11]. Overall, Schrödinger equations with convolution terms provide a more realistic description of interactions in multi-particle systems, enabling a more accurate understanding and prediction of the behavior of microscopic particles under mutual influences.

Understanding the solutions of the elliptic equation

$$-\Delta u + V(\epsilon x)u = \lambda(I_\alpha * |u|^p)|u|^{p-1} + u\log u^2 \text{ in } \mathbb{R}^N \tag{1.2}$$

holds significant significance in the examination of standing wave solutions for equation (1.1). These standing wave solutions, characterized by the form $\Phi(t,x) = e^{iwt/\epsilon}u(x)$, play a crucial role in various contexts and provide valuable insights into the behavior and properties of the equation.

In 2018, C. O. Alves and Daniel C. de Morais Filho [12] focus on investigating the existence and concentration of positive solutions for a logarithmic elliptic equation

$$\begin{cases} -\epsilon^2\Delta u + V(x)u = u\log u^2, \text{ in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{cases}$$

where $\epsilon > 0$, $N \geq 3$ and V is a continuous function with a global minimum. To study the problem, the authors utilize a variational method developed by Szulkin for functionals that are a sum of a C^1 functional with a convex lower semicontinuous functional.

In 2020, Alves and Ji [13] investigated the existence of multiple positive solutions for a logarithmic Schrödinger equation

$$\begin{cases} -\epsilon^2\Delta u + V(x)u = u\log u^2, & \text{in } \mathbb{R}^N, \\ u \in H^1\left(\mathbb{R}^N\right), \end{cases}$$

where $\epsilon > 0$, $N \geq 1$ and V is a continuous function with a global minimum. By employing the variational method, the study demonstrates that when the parameter ϵ is sufficiently small, the number of nontrivial solutions is influenced by the "shape" of the graph of the function V.

In recent years, many authors have studied the nonlinear Schrödinger equation with the potential V. In 2022, Guo et al. [14] utilized fractional logarithmic Sobolev techniques and the linking theorem to elucidate existence theorems for equations with logarithmic nonlinearity. Further, a recent study [15] delineates conditions for a singular nonnegative solution in bounded $\mathbb{R}^n$ domains ($n \geq 2$), providing comprehensive insights into its behavior.

Inspired by the outcomes observed in the aforementioned papers, in this paper we aim to investigate the existence of multiple positive solutions for the problem (1.2) when $N = 3$, $\lambda > 0$ and $1 \leq p \leq 2^*$. It is noteworthy that the introduction of a convolution term presents a notable aspect. The difficulty arises in analyzing the unique existence of solutions to the energy functional when both the convolution term and the logarithmic term operate concurrently. Addressing this challenge involves employing specialized analytical techniques, setting it apart from the methods utilized in [13], marking a novel approach.

In this paper, we shall prove the existence of solution for (1.2) in $H^1(\mathbb{R}^3)$. The associated energy functional of (1.2) will be defined as $J_\epsilon : H^1(\mathbb{R}^3) \to (-\infty, +\infty)$,

$$J_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + (V(\varepsilon x) + 1)u^2)dx - \frac{\lambda}{2p}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{3-\alpha}}dxdy - \int_{\mathbb{R}^3} H(u)dx, \tag{1.3}$$

where

$$\int_{\mathbb{R}^3} H(u)dx = \int_{\mathbb{R}^3} -\frac{u^2}{2}dx + \frac{u^2\log u^2}{2}dx, \quad \forall u \in \mathbb{R}^3,$$

with

$$H(u) = \int_0^u s\log s^2 ds = -\frac{u^2}{2} + \frac{u^2\log u^2}{2},$$

and

$$L(u) = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)|^p|u(y)|^p}{|x-y|^{3-\alpha}}dxdy.$$

Given the infinite character and lack of C^1 smoothness of the functional J_ε, a new approach is required to find weak solutions since traditional methods are not effective here. In this scenario, the fundamental element of our approach lies in harnessing the groundbreaking minimax method introduced by Szulkin [16]. Furthermore, we will employ the Gagliardo-Nirenberg inequality [17, 18], the Brezis-Lieb lemma [19], and other specifically techniques for handling the nonlinear Coulomb potential, culminating in a robust result of strong convergence.

In our research, the potential V is based on the following assumptions [13]:

1°. $V : \mathbb{R}^3 \to \mathbb{R}$ is a continuous function such that

$$\lim_{|x|\to+\infty} V(x) = V_\infty.$$

with $0 < V(x) < V_\infty$ for any $x \in \mathbb{R}^3$.

2°. There are l points $z_1, \cdots, z_l$ in $\mathbb{R}^3$ with $z_1 = 0$ such that

$$1 = V(z_i) = \min_{x\in\mathbb{R}^3} V(x), \quad \text{for } 1 \leq i \leq l.$$

By employing the variational method, we can establish the existence of non-trivial solutions for the logarithmic Schrödinger equation with a Coulomb-type potential when ϵ is sufficiently small ($\epsilon > 0$). This outcome is contingent upon the distinctive characteristics of the graph of the function V.

A positive solution of problem (1.2) means that there exists a positive function $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfy $u^2\log u^2 < +\infty$ and

$$\int_{\mathbb{R}^3} \nabla u \cdot \nabla v + V(\varepsilon x)u \cdot v dx = \lambda \int_{\mathbb{R}^3} (I_\alpha * |u|^p)|u|^{p-1} v dx + \int_{\mathbb{R}^3} uv\log u^2, \text{ for all } v \in C_0^\infty\left(\mathbb{R}^3\right).$$

The main result is as follows.

Suppose that V satisfies $1°$ and $2°$. There exists $\varepsilon^* > 0$ such that problem (1.2) has l positive soutions in $H^1(\mathbb{R}^3)$ for $\varepsilon \in (0, \varepsilon^*)$.

The paper is organized as follows. In Section 2. we present several preliminary results that will be employed in the proofs of our main theorems. In Section 3. we prove the main result which are in the local case. In Section 4. we generalize the local results to the global space.

Notation: Henceforth, in this paper, unless otherwise specified, we adopt the following notations:

- $B_R(u)$ denotes an open ball centered at u with a radius of $R > 0$.
- If g is a measurable function, the integral $\int_{\mathbb{R}^N} g(x)dx$ will be denoted by $\int g(x)dx$.
- C, C_1, C_2 etc. will denote positive constants of negligible importance with respect to their exact values.
- $L_R(u)$ denotes the function $L(u)$ within the ball $B_R(0)$.
- $\|\cdot\|_p$ denotes the usual norm of the Lebesgue space $L^p(\mathbb{R}^3)$, for $p \in [1, +\infty)$.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.
- The expression $\iint \cdot\, dxdy$ denotes $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \cdot\, dxdy$.
- $2^* = \frac{2N}{N-2}$.

2. Preliminaries

In this section, we give some results and technical tools used for the main results.

First, we define the effective domain of J,

$$D(J_\epsilon) := \{u \in H^1(\mathbb{R}^3) : J_\epsilon(u) < +\infty\}.$$

Considering the problem

$$-\Delta u + V(0)u = \lambda(I_\alpha * |u|^p)|u|^{p-1} + u \log u^2 \quad \text{in } \mathbb{R}^3, \tag{2.1}$$

the corresponding energy functional associated to (2.1) is

$$J_0(u) = \frac{1}{2}\int (|\nabla u|^2 + (V(0)+1)u^2)dx - \frac{\lambda}{2p}\iint \frac{|u(x)|^p|u(y)|^p}{|x-y|^{3-\alpha}}dxdy - \frac{1}{2}\int u^2 \log u^2 dx.$$

And define the Nehari manifold

$$\Sigma_0 = \{u \in D(J_0) \backslash (0) : J_0'(u)u = 0\},$$

where

$$D(J_0) = \{u \in H^1(\mathbb{R}^3) : J_0(u) < +\infty\}.$$

The problem (2.1) has a positive solution attained at the infimum,

$$c_0 := \inf_{u \in \Sigma_0} J_0(u),$$

which will be proved in the Lemma 3. We shall additionally utilize the energy level

$$c_\infty := \inf_{u \in \Sigma_\infty} J_\infty(u),$$

through replacing $V(0)$ by V_∞, and

$$\Sigma_\infty = \{u \in D(J_\infty) \backslash (0) : J'_\infty(u)u = 0\},$$

it is clear that

$$c_0 < c_\infty.$$

Regarding to the values of c_0 and c_∞, it should be noted that they correspond to the critical levels of the functionals J_0 and J_∞, commonly referred to as the Mountain Pass levels.

Based on the approach discussed in previous studies [12, 20, 21], we address the issue of J_0 and J_∞ lacking smoothness by decomposing them into a sum of a differentiable C^1 functional and a convex lower semicontinuous functional, respectively. Following by [13], to facilitate this decomposition, for $\delta > 0$, we define the following functions:

$$F_1(s) = \begin{cases} 0, & s = 0, \\ -\frac{1}{2}s^2 \log s^2, & 0 < |s| < \delta, \\ -\frac{1}{2}s^2\left(\log \delta^2 + 3\right) + 2\delta|s| - \frac{1}{2}\delta^2, & |s| \geq \delta, \end{cases}$$

and

$$F_2(s) = \begin{cases} 0, & |s| < \delta, \\ \frac{1}{2}s^2 \log\left(s^2/\delta^2\right) + 2\delta|s| - \frac{3}{2}s^2 - \frac{1}{2}\delta^2, & |s| \geq \delta. \end{cases}$$

Therefore

$$F_2(s) - F_1(s) = \frac{1}{2}s^2 \log s^2, \quad \forall s \in \mathbb{R}. \tag{2.2}$$

The functionals $J_0, J_\infty : H^1(\mathbb{R}^3) \to (-\infty, +\infty]$ can be reformulated as an alternative form denoted by

$$J_0(u) = \Phi_0(u) + \Psi(u) \quad \text{and} \quad J_\infty(u) = \Phi_\infty(u) + \Psi(u), \quad u \in H^1(\mathbb{R}^3) \tag{2.3}$$

where

$$\Phi_0(u) = \frac{1}{2}\int (|\nabla u|^2 + (V(0) + 1)|u|^2)dx - \frac{\lambda}{2p}L(u) - \int F_2(u)dx \tag{2.4}$$

$$\Phi_\infty(u) = \frac{1}{2}\int (|\nabla u|^2 + (V_\infty + 1)|u|^2)dx - \frac{\lambda}{2p}L(u) - \int F_2(u)dx \tag{2.5}$$

and

$$\Psi(u) = \int F_1(u)dx. \tag{2.6}$$

The properties of F_1 and F_2, as demonstrated in [20] and [21], can be summarized as follows:

$$F_1, F_2 \in C^1(\mathbb{R}, \mathbb{R}). \tag{2.7}$$

For $\delta > 0$ small enough, F_1 is convex, even, $F_1(s) \geq 0$ for all $s \in \mathbb{R}$ and

$$F'_1(s)s \geq 0, \quad s \in \mathbb{R}. \tag{2.8}$$

For each fixed $q \in (2, 2^*)$, there is $C > 0$ such that

$$\left|F'_2(s)\right| \leq C|s|^{q-1}, \quad \forall s \in \mathbb{R}. \tag{2.9}$$

Utilizing the information provided earlier, it can be deduced that the functional Ψ possesses the properties of convexity and lower semicontinuity. Additionally, we can observe that the function Φ belongs to the class of C^1 functions.

As we've discussed earlier, solutions to equation (1.2) within a localized context can be addressed through conventional techniques. However, the situation undergoes a transformation when we expand our scope to encompass the entire space. Within this broader perspective, it becomes apparent that the functional Ψ lacks the characteristic of continuous differentiability (C^1). This particular case necessitates the application of a novel and separate critical point theorem. In the subsequent section, dedicated to the global case, it becomes essential to introduce definitions that were originally presented in the work referenced as [16].

Let J be a C^1 functional defined on Banach space X, we say that $\{u_n\}$ is a Palais-Smale sequence of J at c ($(PS)_c$ sequence, for short) if

$$J(u_n) \to c, \text{ and } \quad J'(u_n) \to 0, \quad \text{as } n \to +\infty \tag{2.10}$$

Let E be a Banach space, E' be the dual space of E and $\langle \cdot, \cdot \rangle$ be the duality paring between E' and E. Let $J : E \to \mathbb{R}$ be a functional of the form $J(u) = \Phi(u) + \Psi(u)$, where $\Phi \in C^1(E, \mathbb{R})$ and Ψ is convex and lower semicontinuous. Let us list some definitions:

1. The sub-differential $\partial J(u)$ of the functional J at a point $u \in H^1\left(\mathbb{R}^N\right)$ is the following set

$$\{w \in E' : \langle \Phi'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq \langle w, v - u \rangle, \forall v \in E\} \tag{2.11}$$

2. A critical point of J is a point $u \in E$ such that $J(u) < +\infty$ and $0 \in \partial J(u)$, i.e.,

$$\langle \Phi'(u), v - u \rangle + \Psi(v) - \Psi(u) \geq 0, \forall v \in E \tag{2.12}$$

3. A PS sequence at level d for J is a sequence $(u_n) \subset E$ such that $J(u_n) \to d$ and there is a numerical sequence $\tau_n \to 0^+$ with

$$\langle \Phi'(u_n), v - u_n \rangle + \Psi(v) - \Psi(u_n) \geq -\tau_n \|v - u_n\|, \quad \forall v \in E \tag{2.13}$$

4. The functional J satisfies the PS condition at level d ($(PS)_d$ condition, for short) if all PS sequences at level d has a convergent subsequence.

As [21] Lemma 2.2, J is of class C^1 in $H^1(\Omega)$ with Ω is a bounded domian. Hence we can construct the mountain pass structure and find the boundedness of the (PS) sequence without using the decomposition method in the local case, which is different from [12, 13, 20, 21].

In order to make the subsequent theorem proof involving the whole space situation clearer, we explain some necessary concepts here. Henceforward, for every $\omega \in D(J_0)$, the functional $J_0^1(w) : H_c^1\left(\mathbb{R}^3\right) \to \mathbb{R}$ given by

$$\langle J_0'(w), z \rangle = \langle \Phi_V'(w), z \rangle + \int F_1'(w) z, \quad \forall z \in H_c^1\left(\mathbb{R}^3\right)$$

and

$$\left\| J_0'(w) \right\| = \sup \left\{ \langle J_0'(w), z \rangle : z \in H_c^1\left(\mathbb{R}^3\right), \text{ and } \quad \|z\|_v \leq 1 \right\}.$$

If $\left\|J_0'(\omega)\right\|$ is finite, then $J_0'(w)$ can be extended to a bounded operator in $H^1(\mathbb{R}^3)$ and can be therefore be viewed as an element of $\left(H^1(\mathbb{R}^3)\right)'$.

If $\{u_n\} \subset D(J)\backslash\{0\}$ is a (PS) sequence for J_ε, then $J_\epsilon'(u_n)\,u_n = o_n(1)\,\|u_n\|_V$. If $\{u_n\}$ is bounded, we have

$$\begin{aligned} J_\epsilon(u_n) &= J_\epsilon(u_n) - \frac{1}{2}J_\epsilon'(u_n)\,u_n + o_n(1)\,\|u_n\|_V \\ &= \frac{1}{2}\int |u_n|^2 \;\mathrm{d}x + \frac{\lambda}{2}(1-\frac{1}{p})L(u_n) + o_n(1)\,\|u_n\|_V\,, \quad \forall n \in \mathbb{N}. \end{aligned}$$

3. The local case

In this section, we provide the proof of the existence of l nontrivial critical points for $J_{\epsilon,R}$ to equation (1.2) on a local case, which constitutes the preliminary step necessary for our main result. This serves as the foundational work leading up to our primary outcome.

Fix $R_0 > 0$ such that $z_i \in B_{R_0}(0)$ for all $i \in \{1,\cdots,l\}$. So for all $R > R_0$ and $u \in H^1\,(B_R(0))$,

$$J_{\epsilon,R}(u) = \frac{1}{2}\int_{B_R(0)}(|\nabla u|^2 + (V(\epsilon x)+1)u^2)dx - \frac{\lambda}{2p}L_R(u) - \frac{1}{2}\int_{B_R(0)} u^2\log u^2 dx.$$

For any $u, v \in H^1(B_R(0))$, it is easy to verify that $J_{\epsilon,R} \in C^1\left(H^1(B_R(0)),\mathbb{R}\right)$ and

$$J_{\epsilon,R}'(u)v = \int_{B_R(0)} \nabla u\cdot\nabla v dx + V(\epsilon x)uv dx - \lambda\int_{B_R(0)}(I_\alpha * |u|^p)|u|^{p-1}v dx - \int_{B_R(0)} uv\log u^2 dx.$$

The local space $H^1(B_R(0))$ is endow with the norm

$$\|u\|_V = \left(\int_{B_R(0)}(|\nabla u|^2 + (V(\epsilon x)+1)u^2)dx\right)^{\frac{1}{2}}$$

which is also a norm in $H^1(\mathbb{R}^3)$.

According to the definition of V-norm and H^1-norm, we have the following inequality

$$C_1\|u\|_{H^1} \le \left(\int(|\nabla u|^2 + (V(\epsilon x)+1)u^2)dx - \lambda L(u)^{\frac{1}{2}}\right)^{\frac{1}{2}} \le \|u\|_V \le C_2\|u\|_{H^1}.$$

One can see that V-norm is equivalent to H^1-norm.

In the subsequent analysis, we denote $\Sigma_{\epsilon,R}$ as the Nehari manifold correspond to $J_{\epsilon,R}$, which can be defined as follows:

$$\begin{aligned} \Sigma_{\epsilon,R} &= \left\{u \in H^1(B)\backslash\left\{0\right\}, \quad J_{\epsilon,R}'(u)u = 0\right\} \\ &= \left\{u \in H^1(B)\backslash\{0\}, \quad J_{\epsilon,R}(u) = \frac{1}{2}\int_{B_R(0)} u^2 + \frac{\lambda}{2}(1-\frac{1}{p})L_R(u)\right\}. \end{aligned}$$

For all $\epsilon > 0$, $R > R_0$, $J_{\epsilon,R}$ has the Mountain Pass geometry.

Proof. (i) Recall that

$$J_{\epsilon,R}(u) = \frac{1}{2}\int_{B_R(0)} (|\nabla u|^2 + (V(\epsilon x) + 1)u^2)dx - \frac{\lambda}{2p}L_R(u) - \frac{1}{2}\int_{B_R(0)} u^2 \log u^2 dx. \tag{3.1}$$

Following by the Hardy-Littlewood-Sobolev inequality and Sobolev imbedding, we obtain

$$L_B(u) \le \iint \frac{|u(x)|^p|u(y)|^p}{|x-y|^{N-\alpha}}dxdy \le \left(\int |u|^{\frac{2Np}{N+\alpha}}dx\right)^{\frac{N+\alpha}{N}} \le C\|u\|_V^{2p}, \tag{3.2}$$

where $\frac{N+\alpha}{N} < p < \frac{N+\alpha}{N-2}$. And for $q > 2$ small and $u > 0$, we have

$$\int u^2 \log u^2 dx \le C_q \int |u|^q \le \|u\|_V^q. \tag{3.3}$$

Hence, by (3.1),(3.2) and (3.3), it follows that

$$J_{\epsilon,R}(u) \ge \frac{1}{2}\|u\|_V^2 - \lambda C_1\|u\|_V^{2p} - C_2\|u\|_V^q > C > 0,$$

for a constant $C > 0$, and $\|u\|_V > 0$ small enough.
(ii) Fix $u \in D(J)\backslash\{0\}$ with $\operatorname{supp} u \subset B_R(0)$, and for $s > 0$, $\lambda > 0$, we have

$$\begin{aligned}
J_{\epsilon,R}(su) =& \frac{1}{2}\int_{B_R(0)} (s^2|\nabla u|^2 + s^2(V(\varepsilon x) + 1)u^2)dx - \frac{\lambda}{2p}s^{2p}L_R(u) - \frac{1}{2}s^2\log s^2 \int_{B_R(0)} u^2 dx \\
&- \frac{1}{2}s^2 \int_{B_R(0)} u^2 \log u^2 dx \\
\le& s^2\left(\frac{1}{2}\int_{B_R(0)} (|\nabla u|^2 + (V(\varepsilon x) + 1)u^2)dx - \log s \int_{B_R(0)} u^2 dx - \frac{1}{2}\int_{B_R(0)} u^2 \log u^2 dx\right).
\end{aligned}$$

Because of the boundness of $J_{\epsilon,R}$, there exist three bounded terms in the right side of the above inequality, except for the third term. Therefore, we obtain that $J_{\epsilon,R}(u) \to -\infty$ as $s \to +\infty$. So there exists $s_0 > 0$ independent of $\epsilon > 0$ small enough and $R > R_0$ such that $J_{\epsilon,R}(s_0 u) < 0$.

All (PS) sequence of $J_{\epsilon,R}$ are bounded in $H^1(B_R(0))$.

Proof. Let $\{u_n\} \subset H^1(B_R(0))$ be a $(PS)_d$ sequaence. Then,

$$\begin{aligned}
|u_n|^2_{L^2(B_R(0))} + \lambda(1 - \frac{1}{p})L_R(u_n) &\le 2J_{\epsilon,R}(u_n) - J'_{\epsilon,R}(u_n)\,u_n \\
&= 2d + o_n(1) + o_n(1)\,\|u_n\|_V \\
&\le C + o_n(1)\,\|u_n\|_V\,.
\end{aligned} \tag{3.4}$$

for some $C > 0$. And we ultilize the following logarithmic Sobolev inequality [11],

$$\int u^2 \log u^2 \le \frac{a^2}{\pi}\|\nabla u\|^2_{L^2(\mathbb{R}^N)} + \left(\log \|u\|^2_{L^2(\mathbb{R}^N)} - N(1 + \log a)\right)\|u\|^2_{L^2(\mathbb{R}^N)} \tag{3.5}$$

for all $a > 0$. By taking $\frac{a^2}{\pi} = \frac{1}{2}$, $\xi \in (0,1)$ and combining (3.4) and(3.5) we get

$$\int_{B_R(0)} u_n^2 \log u_n^2 \le \frac{1}{4}\|\nabla u_n\|_2^2 + C\left(1 + \|u_n\|_V\right)^{1+\xi}. \tag{3.6}$$

Above all, for some $\xi \in (0,1)$,

$$\begin{aligned} d + o_n(1) = J_{\epsilon,R}(u_n) &= \frac{1}{2}\int_{B_R(0)} |\nabla u_n|^2 + \frac{1}{2}\int_{B_R(0)} \left(V(\epsilon x) + 1\right) u_n^2 - \frac{\lambda}{2p} L_R(u_n) \\ &\quad - \frac{1}{2}\int_{B_R(0)} u_n^2 \log u_n^2 \\ &\ge C\|u_n\|_V^2 - \left(1 + \|u_n\|_V\right)^{1+\xi} - \frac{\lambda}{2p} L_R(u_n). \end{aligned}$$

By (3.4) we have $\frac{\lambda}{2p} L_R(u_n) \le \frac{\lambda}{2}(1 - \frac{1}{p}) L_R(u_n) \le C + o_n(1)\|u_n\|_V$, $\alpha \in (\frac{N}{2}, N)$; $p \in (2, \frac{N+\alpha}{N-2})$ therefore it implies that

$$C\|u_n\|_V^2 \le C\left(1 + \|u_n\|_V\right)^{1+\xi} + C + o_n(1)\|u_n\|_V,$$

which means $\|u_n\|_V \le C$, i.e. (u_n) is bounded in $H^1(B_R(0))$.

Fix $u_0 \ne 0$, $u_0 \in H^1(B_R(0))$ and $\int u_0^2 \log u_0^2 dx > -\infty$. According to

$$c_{\epsilon,R} = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} J_{\epsilon,R}(\gamma(t)) \le \sup_{t>0} J_{\epsilon,R}(tu_0) = D_0.$$

where the definition of the path set γ is given in the lemma 3 and D_0 is a uniform constant. Hence we obtain $\{u_n\}$ is also bounded in $H^1(\mathbb{R}^3)$.

Now, for a fixed $u \in D(J_0)\backslash\{0\}$, and $t > 0$. Define the function

$$t \to \phi(t) := J_\epsilon(tu).$$

Via computation, we have

$$\phi'(t) = t\left(\int (|\nabla u|^2 + V(\epsilon x)u^2)dx - \lambda t^{2p-2} L(u) - 2\log t \int u^2 dx - \int u^2 \log u^2 dx\right).$$

Setting $f(t) = \lambda a t^{2p-1} + 2b \log t$, for $a, b > 0$ and $p > 1$. In the following, we prove that there exists an unique critical point $\tilde{t}$, with $\tilde{t} > 0$, at which the function ϕ attains its maximum positive value.
1°. According to Mountain Pass Geometry, there exists $\tilde{t} > 0$ such that $f(\tilde{t}) = 0$, i.e. $\phi'(\tilde{t}) = 0$.
2°. Since $f'(t) = (2p-1)\lambda a t^{2p-2} + \frac{2b}{t} > 0$, we know that the function f is a monotonically increasing function, and furthermore, this means that ϕ reaches a positive maximum at the unique critical point $\tilde{t}$. Hence, for any $u \in D(J_\epsilon)\backslash\{0\}$, the intersection of every path $\{tu; t > 0\}$ forms a set

$$\Sigma_\epsilon = \left\{u \in D(J_\epsilon)\backslash\{0\}; J_\epsilon(u) = \frac{1}{2}\int u^2 dx + \frac{\lambda}{2}(1 - \frac{1}{p})L(u)\right\}$$

exactly at the unique point $\tilde{t}u$. Moreover, $\widetilde{t} = 1$ if and only if

$$u \in \Sigma_\epsilon \ (\tilde{t} = 1 \iff \phi'(\tilde{t}) = J_\epsilon'(\tilde{t}u)u = J_\epsilon'(u)u = 0).$$

Based on the energy levels shown above, the following results are obtained. For $\epsilon \geq 0$,

$$c_\epsilon = \inf_{u \in \Sigma_\epsilon} J_\epsilon(u). \tag{3.7}$$

Proof. Let

$$\Gamma := \{\gamma \in C([0,1], H^1(\mathbb{R}^3)) : \gamma(0) = 0, J(\gamma(1)) < 0\}$$

we can define the mountain pass energy level

$$c := \inf_{\eta \in \Gamma} \sup_{t \in [0,1]} J(\eta(t)).$$

Let $u \in \Sigma_\epsilon$, we consider $J_\epsilon(t_0 u) < 0$ for some $t_0 > 0$. Then for the continuous path $\gamma_\epsilon(t) = t \cdot t_0 u$, we have

$$\inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} J_\epsilon(\gamma_\epsilon(t)) = c_\epsilon \leq \max_{t \in [0,1]} J_\epsilon(\gamma_\epsilon(t)) \leq \max_{t \geqslant 0} J_\epsilon(tu) = J_\epsilon(u).$$

Hence

$$c_\epsilon \leq \inf_{u \in \Sigma_\epsilon} J_\epsilon(u). \tag{3.8}$$

On the other hand, we will prove that $c_\epsilon \geq \inf_{u \in \Sigma_\epsilon} J_\epsilon(u)$. Take a (PS) sequence $\{u_n\} \subset H^1(\mathbb{R}^3)$ for J_ϵ. By Lemma 3, (u_n) is bounded in $H^1(\mathbb{R}^3)$. We claim $\|u_n\|_2 \nrightarrow 0$. By contradiction, if $\|u_n\|_2 \to 0$, using interpolation, $\|u_n\|_q \to 0$, for any $q \in [2, 2^*)$. Because $\left|F_2'(s)\right| \leq C|s|^{q-1}$, then

$$\int F_2'(u_n) u_n \to 0,$$

and using Hardy-Littlewood-Sobolev inequality again, we obtain $L(u_n) \to 0$. Recall that

$$\begin{aligned}
\|u_n\|_V^2 + \int F_1'(u_n) u_n dx &= J_\epsilon'(u_n) u_n + \lambda L(u_n) + \int u_n^2 dx + \int F_2'(u_n) u_n dx \\
&= o_n(1) \|u_n\|_V + \lambda L(u_n) + \int u_n^2 dx + \int F_2'(u_n) u_n dx \\
&= o_n(1),
\end{aligned} \tag{3.9}$$

from where it follows that $\|u_n\|_V \to 0$ and $\int F_1'(u_n) u_n \to 0$.

Since F_1 is convex, even and $F_1(t) \geq F_1(0) = 0$, for all $t \in \mathbb{R}$, we derive that $0 \leq F_1(t) \leq F_1'(t)t$ for all $t \in \mathbb{R}$. Hence $F_1(u_n) \to 0$ in $L^1(\mathbb{R}^3)$. Then $J_\epsilon(u_n) \to J_\epsilon(0) = 0$, which contradicts to $c_\epsilon > 0$. Our claim is proved. Hence, there are constants b_1 and b_2 such that

$$0 < b_1 \leq \|u_n\|_2 \leq b_2. \tag{3.10}$$

Next, let $t_n \in (0,1)$, $t_n u_n \in \Sigma_\epsilon$, and recalling that

$$\begin{aligned}
J_\epsilon(t_n u_n) &= \frac{1}{2} \int |t_n u_n|^2 dx + \frac{\lambda}{2}(1 - \frac{1}{p}) L(t_n u_n) \\
&= \frac{1}{2} t_n^2 \int |\nabla u_n|^2 dx + (V(\epsilon x) + 1) u_n^2 dx - \frac{\lambda}{2p} t_n^{2p} L(u_n) - \frac{1}{2} t_n^2 \log t_n^2 \int u_n^2 dx \\
&- \frac{1}{2} t_n^2 \int u_n^2 \log u_n^2 dx.
\end{aligned} \tag{3.11}$$

and

$$J_\epsilon'(u_n)u_n = \int (|\nabla u_n|^2 + V(\epsilon x)u_n^2)dx - \lambda L(u_n) - \int u_n^2 \log u_n^2 dx.$$

Then we get

$$\lambda\left(t_n^{2p-2} - 1\right) L(u_n) + \log t_n^2 \int u_n^2 dx = J_\epsilon'(u_n)u_n = o_n(1)\|u_n\|_V.$$

According to (3.10) and $L(u) \geq 0$, this equation implies $t_n \to 1$. In addition, by (3.11) and Remark 2 we have

$$\begin{aligned}
\inf_{u\in\Sigma_\epsilon} J_\epsilon(u) \leq J_\epsilon(t_n u_n) &= \frac{t_n^2}{2}\int u_n^2 dx + \frac{\lambda}{2}(1-\frac{1}{p})t_n^{2p}L(u_n) \\
&\leq t_n^2\left(\frac{1}{2}\int u_n^2 dx + \frac{\lambda}{2}(1-\frac{1}{p})L(u_n)\right) \\
&= t_n^2\left(J_\epsilon(u_n) + o_n(1)\|u_n\|_V\right).
\end{aligned}$$

Therefore, taking the limit we get

$$\inf_{u\in\Sigma_\epsilon} J_\epsilon(u) \leq c_\epsilon.$$

The functional $J_{\epsilon,R}$ satisfies the (PS) condition.

Proof. Take a (PS) sequence $\{u_n\} \subset H^1(B_R(0))$, it means that

$$J_{\epsilon,R}(u_n) \to d,$$

$$J_{\epsilon,R}'(u_n)u_n = o_n(1)\|u_n\|_V.$$

By Lemma 3, we know there exists $\{u_n\} \subset H^1(B_R(0))$, and a subsequence of u_n, which still denoted by itself such that $\|u_n\|_V$, i.e.

$$u_n \rightharpoonup u \ \text{ in } \ H^1(B_R(0)),$$

$$u_n \to u \ \text{ in } \ L^q(B_R(0)), \forall q \in [1, 2^*),$$

$$u_n \to u \ \text{ a.e. in } \ B_R(0).$$

From [13], we set $f(t) = t\log t^2$, $F(t) = \int_0^t f(s)ds = \frac{1}{2}\left(t^2\log t^2 - t^2\right)$ for all $t \in \mathbb{R}$ and for $p \in (2, 2^*)$, there is $C > 0$ such that

$$|f(t)| \leq C\left(1 + |t|^{p-1}\right), \ \forall t \in \mathbb{R}$$

and

$$|F(t)| \leq C(1 + |t|^p), \ \forall t \in \mathbb{R}.$$

In addition, by definition of the norm in $H^1(B_R(0))$, we get

$$\|u_n - u\|_V^2 = \int |\nabla(u_n - u)|^2 dx + (V(\epsilon x) + 1)|u_n - u|^2 dx,$$

$$J_{\epsilon,R}'(u_n)(u_n - u) = \int \nabla u_n \nabla(u_n - u)dx + V(\epsilon x)u_n(u_n - u)dx - \lambda\int\left(I_\alpha * |u_n|^2\right)|u_n - u|u_n dx$$

$$
\begin{aligned}
&-\int (u_n - u)\, u_n \log u_n^2 dx \\
=&\int |\nabla (u_n - u)|^2\, dx + V(\epsilon x)\, |u_n - u|^2\, dx - \lambda \int \left(I_\alpha * |u_n - u|^2\right) |u_n - u|^2\, dx \\
&-\int f(u_n)\, |u_n - u|\, dx = o_n(1).
\end{aligned}
$$

Hence, it is easy to see that

$$
\begin{aligned}
\int |\nabla (u_n - u)|^2\, dx + V(\epsilon x)\, |u_n - u|^2\, dx =&\lambda \int \left(I_\alpha * |u_n - u|^2\right) |u_n - u|^2\, dx \\
&+ \int f(u_n)\, (u_n - u)\, dx + o_n(1) \\
=&o_n(1).
\end{aligned}
$$

It implies that

$$\|u_n - u\|_V \to 0,$$

which means the sequence $\{u_n\}$ satisfies (PS) condition.

In fact, Theorem 3 concerns the existence of multiple solutions for equation (1.2) on a ball, which is crucial for the study of the existence of multiple solutions on the entire space as we desire. In order to prove this crucial result, we first present several lemmas. Next, we use the tricks in [13], by constructing l small balls and finding the center of mass, it plays a key role in the proof of the following theorem.

Fix $\rho_0 > 0$ so that it satisfies $\overline{B_{\rho_0}(z_i)} \cap \overline{B_{\rho_0}\left(z_j\right)} = \phi$ for $i \neq j, i, j \in \{1, \cdots, l\}$ and $\bigcup_{i=1}^{l} B_{\rho_0}(z_i) \subset B_{R_0}(0)$. Denote $K_{\frac{\rho_0}{2}} = \bigcup_{i=1}^{l} \overline{B_{\frac{\rho_0}{2}}(z_i)}$, and define the functional $Q_\varepsilon : H^1\left(\mathbb{R}^3\right) \backslash \{0\} \to \mathbb{R}^3$ by

$$Q_\varepsilon(u) = \frac{\int \chi(\varepsilon x) g(\varepsilon x) |u|^2 dx}{\int g(\varepsilon x) |u|^2 dx}$$

where $\chi : \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\chi(x) = \begin{cases} x, & |x| \leq R_0. \\ R_0 \frac{x}{|x|}, & |x| > R_0. \end{cases}$ and $g : \mathbb{R}^3 \to \mathbb{R}^3$ is a radial positive continuous function with

$$g(z_i) = 1, \;\; i \in \{1, \cdots, l\} \;\text{ and }\; g(x) \to 0, \;\text{as}\; |x| \to +\infty.$$

The next lemma provides a useful way to generate $(PS)_c$ sequence associated with J_ϵ. There exist $\alpha_0 > 0$, $\epsilon_0 > 0$, and $R_0 > 0$ such that $\varepsilon_1 \in (0, \varepsilon_0)$ small enough and $R_1 > R_0$ large enough, if $u \in \Sigma_{\varepsilon,R}$ and $J_{\varepsilon,R}(u) \leq c_0 + \alpha_0$, then $Q_\varepsilon(u) \in K_{\frac{\rho_0}{2}}$ for any $\varepsilon \in (0, \varepsilon_1)$ and $R \geq R_1$.

Proof. We prove this lemma by contradiction. If there is $\alpha_n \to 0$, $\varepsilon_n \to 0$ and $R_n \to \infty$, $u_n \in \Sigma_{\varepsilon_n,R_n}$ satisfies

$$J_{\varepsilon_n,R_n}(u) \leq c_0 + \alpha_n,$$

but

$$Q_\varepsilon(u_n) \notin K_{\frac{\rho_0}{2}}.$$

By definition of c_0 and Lemma 3, $c_0 \leq c_{\varepsilon_n,R_n}$, it is easy to see that

$$c_0 \leq c_{\varepsilon_n,R_n} \leq J_{\varepsilon_n,R_n}(u_n) \leq c_0 + \alpha_n$$

which means $J_{\varepsilon_n R_n}(u_n) = c_{\varepsilon_n,R_n} + o_n(1)$. Denote the functional $\Psi_{\varepsilon_n,R_n} : H^1(B_{R_n}(0)) \to \mathbb{R}$ by

$$\Psi_{\varepsilon_n,R_n}(u) = J_{\varepsilon_n,R_n}(u) - \frac{1}{2}\int_{B_{R_n}(0)} |u|^2 - \frac{\lambda}{2}(1-\frac{1}{p})L_R(u).$$

It implies that

$$\Sigma_{\varepsilon_n,R_n} = \left\{u \in H^1(B_R(0)) \backslash \{0\} : \Psi_{\varepsilon_n,R_n}(u) = 0\right\}.$$

Via computation, we obtain

$$\Psi'_{\varepsilon_n,R_n}(u)u = -\int |u|^2 - \lambda(p-1)L(u) \leq -\beta, \quad \forall n \in \mathbb{N},$$

where $\beta > 0$ to guarantee $c_{\varepsilon_n,R_n} > 0$. Without loss of generality, we have the above conditions. We can then proceed to apply the Ekeland Variational Principle from Theorem 8.5 in [22], assuming that

$$\left\|J'_{\varepsilon_n,R_n}(u_n)\right\| \to \infty, \quad \text{as} \quad n \to \infty.$$

Now, from $J_{\varepsilon_n,R_n}(u_n) = \frac{1}{2}\int_{B_{R_n}(0)} |u_n|^2\,dx + \frac{\lambda}{2}(1-\frac{1}{p})L_{R_n}(u_n) \geq c_0 > 0$, we have $\liminf_{n\to\infty} R_n > 0$. And according to Section 6 in [12] , there are two cases:

1. $u_n \to u \neq 0$ in $L^2\left(\mathbb{R}^N\right)$, and $u \in H^1\left(\mathbb{R}^N\right)$.
2. There exists $(y_n) \subset \mathbb{R}^N$ such that $v_n = u_n(\cdot + y_n) \longrightarrow v \neq 0$ in $L^2\left(\mathbb{R}^N\right)$, and $v \in H^1\left(\mathbb{R}^N\right)$.

For case (1), recall that our assumption $\varepsilon \to 0$, $\chi(0) = 0$ and $g(0) = 1$

$$Q_{\varepsilon_n}(u_n) = \frac{\int \chi(\varepsilon_n x)\, g(\varepsilon_n x)\,|u_n|^2\,dx}{\int g(\varepsilon_n x)\,|u_n|^2\,dx} \to \frac{\int \chi(0)g(0)\,|u_n|^2\,dx}{\int g(0)\,|u_n|^2\,dx} = 0 \in K_{\frac{\rho_0}{2}}.$$

This contradicts to $Q_{\varepsilon_n} \notin K_{\frac{\rho_0}{2}}$.

For case (2), there are two different situations. If $|\varepsilon_n y_n| \to +\infty$, then $J'_\infty(v)v \leq 0$. Thus, for $s \in (0,1]$ such that $sv \in \Sigma_\infty$,

$$\begin{aligned}
2c_\infty \leq 2J_\infty(sv) &= 2J_\infty(sv) - J'_\infty(sv)sv \\
&= \int |sv|^2 + \lambda(1-\frac{1}{p})\iint s^{2p}\frac{|v|^p(x)|v|^p(y)}{|x-y|^{N-\alpha}}dxdy \\
&\leq \int |v|^2 + \lambda(1-\frac{1}{p})\iint \frac{|v|^p(x)|v|^p(y)}{|x-y|^{N-\alpha}}dxdy \\
&\leq \liminf_{n\to+\infty}\int |v_n|^2 + \lambda(1-\frac{1}{p})\iint \frac{|v_n|^p(x)|v_n|^p(y)}{|x-y|^{N-\alpha}}dxdy \\
&= \liminf_{n\to+\infty}\int |u_n|^2 + \lambda(1-\frac{1}{p})\iint \frac{|u_n|^p(x)|u_n|^p(y)}{|x-y|^{N-\alpha}}dxdy \\
&= \lim_{n\to\infty} 2J_{\varepsilon_n,R_n}(u_n) = 2c_0,
\end{aligned}$$

which contradicts $c_0 < c_\infty$. If $\varepsilon_n y_n \to y$ for some $y \in \mathbb{R}^N$, and some subsequence. In this case, the functional $J_V : H^1(\mathbb{R}^N\Big) \to \mathbb{R}$ is given by

$$J_V(u) = \frac{1}{2}\int (|\nabla u|^2 + (V(y)+1)u^2)dx - \frac{\lambda}{2p}\iint \frac{|u_n|^p(x)|u_n|^p(y)}{|x-y|^{N-\alpha}}dxdy - \frac{1}{2}\int u^2 \log u^2 dx,$$

and c_V is the moutain pass level of J_V. Similar as before,

$$c_V = \inf_{u \in \Sigma_V} J_V(u),$$

where

$$\Sigma_V = \left\{ u \in D(J_V)\setminus\{0\} : J_V(u) = \frac{1}{2}\int u^2 + \frac{\lambda}{2}(1-\frac{1}{p})\iint \frac{|u|^p(x)|u|^p(y)}{|x-y|^{N-\alpha}}dxdy \right\}.$$

If $V(y) > 1 = \min_i V(x_i)$, $i \in \{1, \cdots, l\}$, then

$$c_V > c_0,$$

but according to the previous arguments

$$c_V \leq c_0,$$

which is a contradiction. So $V(y) = 1$ and $y = z_i$ for $i \in \{1, \cdots, l\}$.

$$\begin{aligned} Q_{\varepsilon_n}(u_n) = \frac{\int \chi(\varepsilon_n x)\, g(\varepsilon_n x)\, |u_n|^2\, dx}{\int g(\varepsilon_n x)\, |u_n|^2\, dx} &= \frac{\int \chi(\varepsilon_n (x+y_n))\, g(\varepsilon_n (x+y_n))\, |v_n|^2\, dx}{\int g(\varepsilon_n (x+y_n))\, |v_n|^2\, dx} \\ \to \frac{\int \chi(zi) g(zi)|v|^2 dx}{\int g(zi)|v|^2 dx} &= z_i \in K_{\frac{\rho_0}{2}}. \end{aligned}$$

This is contrary to our initial hypothesis, and the proof is done.

In the following, for simplicity, we indicate the following notations.

$$\begin{aligned} \Omega^i_{\varepsilon,R} &\triangleq \{u \in \Sigma_{\varepsilon,R} : |Q_\varepsilon(u) - z_i| < \rho_0\}, \\ \partial\Omega^i_{\varepsilon,R} &\triangleq \{u \in \Sigma_{\varepsilon,R} : |Q_\varepsilon(u) - z_i| = \rho_0\}, \\ \alpha^i_{\varepsilon,R} &\triangleq \inf_{u \in \Omega^i_{\varepsilon,R}} J_{\varepsilon,R}(u), \\ \tilde{\alpha}^i_{\varepsilon,R} &\triangleq \inf_{u \in \partial\Omega^i_{\varepsilon,R}} J_{\varepsilon,R}(u). \end{aligned}$$

For $\gamma \in \left(\frac{c_\infty - c_0}{8}, \frac{c_\infty - c_0}{2}\right)$, there exists $\varepsilon_2 \in (0, \varepsilon_1)$ small enough such that

$$\alpha^i_{\varepsilon,R} < c_0 + \gamma \quad \text{and} \quad \alpha^i_{\varepsilon,R} < \tilde{\alpha}^i_{\varepsilon,R}$$

for all $\varepsilon \in (0, \varepsilon_2)$, and $R \geq R_1(\varepsilon) > R_0$.

Proof. Let $u \in H^1\left(\mathbb{R}^3\right)$ be a ground state solution of J_0, that is for $u \in \Sigma_0$,

$$J_0(u) = \inf_{u\in\Sigma_0} J_0(u) = c_0, \quad \text{and} \quad J_0'(u) = 0.$$

For any $i \in \{1, \cdots, l\}$, there exists $\varepsilon_1 > 0$ such that

$$\left|Q_\varepsilon\left(u\left(\cdot - \frac{z_i}{\varepsilon}\right)\right) - z_i\right| < \rho, \quad \forall\varepsilon \in (0, \varepsilon_1).$$

Fix $R > R_1 = R_1(\varepsilon)$ and $t_{\varepsilon,R} > 0$ such that $u^i_{\varepsilon,R}(x) = t_{\varepsilon,R}\varphi_R(x)u\left(x - \frac{z_i}{\varepsilon}\right) \in \Sigma_{\varepsilon,R}$,

$$\left|Q_\varepsilon\left(u^i_{\varepsilon,R}\right) - z_i\right| < \rho, \quad \forall\varepsilon \in (0, \varepsilon_1) \ \text{ and } \ R > R_1,$$

and

$$J_{\varepsilon,R}\left(u^i_{\varepsilon,R}\right) \le c_0 + \frac{\alpha_0}{8}, \quad \forall\varepsilon \in (0, \varepsilon_1), \quad R > R_1, \tag{3.12}$$

where $\varphi_R(x) = \varphi\left(\frac{x}{R}\right)$ with $\varphi \in C_0^\infty\left(\mathbb{R}^3\right)$, $0 \le \varphi(x) \le 1$ for all $x \in \mathbb{R}^3$, $\varphi(x) = 1$ for $x \in B_{\frac{1}{2}}(0)$ and $\varphi(x) = 0$ for $x \in B_1^c(0)$. So

$$u^i_{\varepsilon,R} \in \Omega^i_{\varepsilon,R} \quad \forall\varepsilon \in (0, \varepsilon_2) \quad \text{and} \quad R > R_1.$$

Take the infimum for (3.12), thanks to $\alpha_0 < \frac{c_\infty - c_0}{2}$, $J_{\varepsilon,R} \le c_0 + \alpha_0 < \frac{c_\infty + c_0}{2}$, we get

$$\alpha^i_{\varepsilon,R} < c_0 + \frac{\alpha_0}{4} < c_0 + \gamma. \tag{3.13}$$

Now let $\frac{c_\infty - c_0}{8} < \gamma < \frac{c_\infty - c_0}{2}$, then the first inequality is done. Next, if $u \in \partial\Omega^i_{\varepsilon,R}$, then there is

$$u \in \Sigma_{\varepsilon,R} \quad \text{and} \quad |Q_\varepsilon(u) - z_i| = \rho_0 > \frac{\rho_0}{2},$$

hence $Q_\varepsilon(u) \notin K_{\frac{\rho_0}{2}}$. By Lemma 3, we have

$$J_{\varepsilon,R}(u) > c_0 + \alpha_0 \tag{3.14}$$

for $u \in \partial\Omega^i_{\varepsilon,R}$ and $\varepsilon \in (0, \varepsilon_2)$, $R \ge R_1$. Take the infimum for (3.14) we obtain

$$\tilde{\alpha}_{\varepsilon 1 R} = \inf_{\partial\Omega_{\varepsilon,R}} J_{\varepsilon,R}(u) \ge c_0 + \alpha_0, \quad \forall\varepsilon \in (0, \varepsilon_2), \ \ R \ge R_1. \tag{3.15}$$

Above all, from (3.13) and (3.15)

$$\alpha^i_{\varepsilon,R} < \tilde{\alpha}^i_{\varepsilon,R} \quad \text{for} \quad \varepsilon \in (0, \varepsilon_2), \quad \text{and} \quad R \ge R_1,$$

where $\varepsilon_2 \in (0, \varepsilon_1)$.

For $\varepsilon_* \in (0, \varepsilon_2)$ small enough and $R_1 = R_1(\varepsilon) > R_0$ large enough, there exist at least l nontrival critical points of $J_{\varepsilon,R}$ for $\varepsilon \in (0, \varepsilon_0)$ and $R \ge R_1$. Moreover, all of the solutions are positive.

Proof. From Lemma 3, for $\varepsilon_* \in (0, \varepsilon_2)$ small enough and $R_1 > R_0$ large enough, there is

$$\alpha^i_{\varepsilon,R} < \tilde{\alpha}^i_{\varepsilon,R} \quad \text{for} \ \ \varepsilon \in (0, \varepsilon^*) \quad \text{for} \ \ R \geq R_1.$$

As stated Theorem 2.1 in [23], the inequalities mentioned above enable us to employ Ekeland's variational principle to establish the $(PS)_{\alpha^i_{\varepsilon,R}}$ sequence $\left(u^i_n\right) \subset \Omega^i_{\varepsilon,R}$ for $J_{\varepsilon,R}$. Following by Lemma 3, since $\alpha^i_{\varepsilon,R} < c_0 + \gamma$, there is u^i such that $u^i_n \to u^i$ in $H^1\left(B_R(0)\right)$. Then

$$u^i \in \Omega^i_{\varepsilon,R}, \quad J_{\varepsilon,R}\left(u^i\right) = \alpha^i_{\varepsilon,R}, \quad J'_{\varepsilon,R}\left(u^i\right) = 0.$$

Recall that

$$\overline{B_{\rho_0}(z_i)} \cap \overline{B_{\rho_0}(z_j)} \neq \phi, \quad i \neq j,$$

and

$$Q_\varepsilon\left(u^i\right) \in \overline{B_{\rho_0}(z_i)} \quad \left(Q_\varepsilon \in K_{\frac{\rho}{2}} = \bigcup_{i=1}^{l} \overline{B_{\frac{\rho}{2}}(z_i)}\right).$$

We have $u^i \neq u^j, i \neq j, i, j \in \{1, \cdots, l\}$. If we decrease γ and increase R_1 when necessary, we can assume that

$$2c_{\varepsilon,R} < c_0 + \gamma.$$

for $\varepsilon \in (0, \varepsilon^*), R \geq R_1$. So all of the solutions do not charge sign, and because the function $f(u) = u \log u^2$ is odd, we make them nonnegative. The maximum principle implies that any solution to a given equation or system of equations within the open ball $B_R(0)$ will necessarily be positive throughout the entire ball, provided that it is positive on the boundary.

4. Existence of solution for the original equation

In this section, we prove the existence of solution for the original equation (1.2).
For $v \in H^1\left(B_{R_n}(0)\right)$, $u^i_n = u^i_{\varepsilon,R_n}$ be a solution obtained in Theorem 3.

$$\int_{B_{R_n}} \nabla u^i_n \nabla v + V(\varepsilon x) u^i_n v = \lambda \int_{B_{R_n}} (I_\alpha * \left|u^i_n\right|^p) |u^i_n|^{p-1} v dx + \int_{B_{R_n}} u^i_n \log \left|u^i_n\right|^2 v dx,$$

$$J_{\varepsilon,R_n}\left(u^i_n\right) = \alpha^i_{\varepsilon,R_n}, \quad \forall n \in \mathbb{N}.$$

There exists $u^i \in H^1\left(\mathbb{R}^3\right)$ satisfies $u^i_n \rightharpoonup u^i$ in $H^1(\mathbb{R}^3)$ and $u^i \neq 0$, $i \in \{1, \cdots, l\}$.

Proof. From Lemma 3, we know that $\{\alpha^i_{\varepsilon,R_n}\}$ is a bounded sequence,

$$J_{\varepsilon,R_n}\left(u^i_n\right) = \alpha^i_{\varepsilon,R_n} < c_0 + \gamma$$

which implies that $\{u^i_n\}$ is a bounded sequence. So we can assume that $u^i_n \rightharpoonup u^i$ for some $u^i \in H^1\left(\mathbb{R}^3\right)$. Next, we prove $u^i \neq 0$. In the following, we use $\{u_n\}$ and $\{\alpha_n\}$ to denote $\{u^i_n\}$ and $\{\alpha^i_{\varepsilon,R_n}\}$ for convenience.

To continue, let us utilize the Concentration Compactness Principle, originally introduced by Lions [13], applied to the following sequence.

$$\rho_n(x) := \frac{|u_n(x)|^2}{\|u_n\|_2^2}, \quad \forall x \in \mathbb{R}^3.$$

This principle guarantees that one and only one of the following statements is true for a subsequence for $\{\rho_n\}$, which we will still refer to as $\{\rho_n\}$:
($Vanishing$) For all $K > 0$, one has:

$$\lim_{n\to+\infty} \sup_{y\in\mathbb{R}^N} \int_{B_K(y)} \rho_n dx = 0; \tag{4.1}$$

($Compactness$) There exists a sequence $\{y_n\}$ in $\mathbb{R}^3$ with the property that for all $\varepsilon > 0$, there exists $K > 0$ such that for all $n \in \mathbb{N}$, one has:

$$\int_{B_K(y_n)} \rho_n dx \geq 1 - \eta; \tag{4.2}$$

($Dichotomy$) There exists $\{y_n\} \subset \mathbb{R}^N$, $\alpha \in (0, 1)$, $K_1 > 0$, $K_n \to +\infty$ such that the functions $\rho_{1,n}(x) = \chi_{B_{K_1}(y_n)}(x)\rho_n(x)$ and $\rho_{2,n}(x) := \chi_{B^c_{K_n}(y_n)}(x)\rho_n(x)$ satisfy:

$$\int \rho_{1,n} dx \to \alpha \quad \text{and} \quad \int \rho_{2,n} dx \to 1 - \alpha. \tag{4.3}$$

Our goal is to demonstrate that the sequence $\{\rho_n\}$ satisfies the Compactness condition, and to achieve this, we will exclude the other two possibilities. By doing so, we will arrive at a contradiction, thus proving the proposition.

The vanishing case (4.1) can not occur, otherwise we deduce that $\|u_n\|_p \to 0$, and consequently $\int F_2'(u_n)\, u_n < \infty$. By employing the same reasoning as in the previous section, it can be proven that $u_n \to 0$ in $H^1(\mathbb{R}^3)$. However, this contradicts the fact that $\alpha_n \geq c_1$ for all $n \in \mathbb{N}$, as stated in Lemma 3.

The Dichotomy case (4.3) can not occur. Let us assume that the dichotomy case holds, under this assumption, we claim that the sequence $\{y_n\}$ is unbounded. If this were not the case and $\{y_n\}$ were bounded, then in that situation,utilizing the fact that $\|u_n\|_{L^2(\mathbb{R}^3)} \nrightarrow 0$, the first convergence in (4.3) would lead to

$$\int_{B_{K_1}(y_n)} |u_n|^2\, dx = |u_n|_2^2 \int_{\mathbb{R}^3} \rho_{1,n} dx \geqslant \delta,$$

for some $\delta > 0$ and n large enough. Therefore, taking $R' > 0$ such that $B_{K_1}(y_n) \subset B_{R'}(0)$ for all $n \in \mathbb{N}$, it follows that $\int_{B_{R'}(0)} |u_n|^2 dx \geq \delta, \quad$ for all n sufficiently large.$Because$u$_n \to 0$ in $L^2(B_{R'}(0))$, the inequality above is impossible. As a result, $\{y_n\}$ is an unbounded sequence. In the following, denote:

$$v_n(x) := u_n(x + y_n), \quad x \in \mathbb{R}^3.$$

Since the boundness of the sequence $(v_n) \subset H^1(\mathbb{R}^3\big)$ and up to subsequence, we may assume that $v_n \rightharpoonup v$. By the first part of (4.3), $v \not\equiv 0$ holds.
Claim4.1. $F_1'(v)v \in L^1\left(\mathbb{R}^3\right)$ and $J_\infty'(v)v \leq 0$. For $\eta \in C_0^\infty(\mathbb{R}^3)$, $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B_1(0)$ and $\eta \equiv 0$ in $B_2(0)^c$, we define $\eta_R := \eta\left(\frac{\cdot}{R}\right)$ and $v = \eta_R(\cdot - y_n)\, u_n$, we get

$$\int \nabla v_n \nabla(\eta_R v_n)\, dx + (V(\varepsilon(x + y_n)) + 1)\, v_n^2 \eta_R dx + \int F_1'(v_n)\, v_n \eta_R dx$$

$$= \int F_2'(v_n)\, v_n \eta_R dx + \lambda \int (I_\alpha * |v_n|^p)\, |v_n|^p \eta_R dx + o_n(1).$$

If we fix R and go to the limit in the above equation when $n \to \infty$, we get

$$\int |\nabla v|^2 \eta_R dx + v \nabla \eta_R \cdot \nabla v dx + (V_\infty + 1)\, v^2 \eta_R dx + \int F_1'(v) v \eta_R dx$$
$$\leq \int F_2'(v) v \eta_R dx + \lambda \int (I_\alpha * |v|^p)\, |v|^p \eta_R dx$$

where $|\nabla \eta_R| \leq \frac{2}{R}$, using that $F_1'(t)t \geq 0$ for all $t \in \mathbb{R}$, and Fatou's lemma as $R \to +\infty$, we obtain

$$\int |\nabla v|^2 dx + (V_\infty + 1)\, v^2 dx - \lambda \int (I_\alpha * |v|^p)\, |v|^p dx + \int F_1'(v) v dx$$
$$- \int F_2'(v) v dx \leq 0,$$

that is $J_\infty'(v)v \leq 0$.

On this account, there exists $t_\infty \in (0, 1]$ such that $t_\infty v \in \Sigma_\infty$, then

$$\begin{aligned}
c_\infty \leq J_\infty(t_\infty v) &= \frac{t_\infty^2}{2} \int |v|^2 dx + \frac{\lambda}{2}(1 - \frac{1}{p}) t_\infty^{2p} L(v) \\
&\leq \liminf_{n \to +\infty} \left[\frac{1}{2} \int |v_n|^2\, dx + \frac{\lambda}{2}(1 - \frac{1}{p}) L(v_n) \right] \\
&\leq \limsup_{n \to +\infty} \left[\frac{1}{2} \int |u_n|^2\, dx + \frac{\lambda}{2}(1 - \frac{1}{p}) L(u_n) \right] \\
&= \limsup_{n \to +\infty} J_{\varepsilon_n, R_n}(u_n) \\
&= \limsup_{n \to \infty} \alpha_n \leq c_0 + \gamma.
\end{aligned}$$

But we have $\gamma < c_\infty - c_0$, it is absurd. Hence, there is no dichotomy, and in fact compactness must hold. We make the last requirement to achieve our aim.

Claim4.2. The sequence of points $\{y_n\} \subset \mathbb{R}^3$ in (4.2) is bounded.

To establish this claim, we employ a proof by contradiction by assuming that the sequence of $\{y_n\}$ is bounded. However, by considering a subsequence, we observe that $|y_n| \to +\infty$. Following a similar approach as in the case of the Dichotomy, where $\{y_n\}$ was unbounded, we eventually arrive at the inequality $c_0 + \gamma \geq c_\infty$.

For a given $\eta > 0$, there is $R > 0$ such that

$$\int_{B_R^c(0)} \rho_n dx < \eta, \quad \forall n \in \mathbb{N},$$

that is

$$\int_{B_R^c(0)} |u_n|^2\, dx \leq \eta\, |u_n|_2^2 \leq \eta \sup_{n \in \mathbb{N}} |u_n|_2^2 = b\eta.$$

Therefore, for $R_1 \geq \max\{R, R'\}$, since $u_n \to 0$ in $L^2(B_{R_1}(0))$, there is $n_0 \in \mathbb{N}$ large enough such that

$$\int_{B_{R_1}(0)} |u_n|^2\, dx \leq \eta, \quad \forall n \geq n_0.$$

Thereby, we conslude

$$\int |u_n|^2\, dx \leq \eta + \int_{B^c_{R_1}(0)} |u_n|^2\, dx \leq \eta + b\eta \leq C\eta,$$

where $C \nsim \eta$. Due to the arbitrary nature of η, we can deduce that $u_n \to 0$ in $L^2\left(\mathbb{R}^3\right)$. By interpolation on the Lebesgue spaces and $\{u_n\}$ is bounded in $H^1(\mathbb{R}^3)$, it follows that

$$u_n \to 0 \quad \text{in} \quad L^p\left(\mathbb{R}^3\right), \quad 2 \leq p < 2^*.$$

Using the trick that for some $p > 1$ small, $t \log t \leq Ct^p$, it implies that

$$\int u_n^2 \log u_n^2 \to 0.$$

For $p \in (\frac{3+\alpha}{3}, 3+\alpha)$, the sequence $\{\|u_n\|_p\}_{n\in\mathbb{N}}$ converges to $\|u\|_p$ in the sense of measures, $\{u_n\}_{n\in N}$ converges to u almost everywhere, the sequence $\{I_\alpha * |u_n|_p\}_{n\in\mathbb{N}}$is bounded in $L^2(\mathbb{R}^3)$ and $u \neq 0$.
From Proposition 4.8 in [24], since $u_n \in D(J)\backslash\{0\}$ then we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} I_\alpha * |u_n|^p)|u_n|^p - (I_\alpha * |u_n - u|^p)\,|u_n - u|^p = \int (I_\alpha * |u|^p)|u|^p. \tag{4.4}$$

Above all, $J_{\varepsilon,R_n}(u_n) = \alpha_n \to 0$, which contradicts $\alpha_n \geq c_\varepsilon > 0$, for all $n \in \mathbb{N}$.

Proposition 4 yields a direct corollary as follows. For $\varepsilon \in (0, \varepsilon^*)$ small, considering each sequence $\{u_n^i\} \subset H^1\left(\mathbb{R}^3\right)$ as stated in Proposition 4, we have $u^i \neq 0$ and $J'_\varepsilon\left(u^i\right)v = 0$ for all $v \in C_0^\infty\left(\mathbb{R}^3\right)$, i.e. J_ε has a nontrival weak solution u^i. Moreover, for $i \in \{1, \cdots, l\}$,

$$Q_\varepsilon\left(u_n^i\right) \longrightarrow Q_\varepsilon\left(u^i\right). \tag{4.5}$$

And since

$$Q_\varepsilon\left(u_n^i\right) \in \overline{B_{\rho_0}(z_i)}, \quad \forall n \in \mathbb{N},$$

we have

$$Q_\varepsilon\left(u^i\right) \in \overline{B_{\rho_0}(z_i)}. \tag{4.6}$$

Proof. By Proposition 4, $u^i \neq 0$, $i \in \{1, \cdots, l\}$ and $u_n^i \to u^i$ in $L^p_{1oc}\left(\mathbb{R}^3\right)$ for $p \in [2, 2^*)$, we obtain that

$$\int u_n^i \log\left|u_n^i\right|^2 v dx \to \int u^i \log\left|u^i\right|^2 v dx, \quad \forall v \in C_0^\infty\left(\mathbb{R}^3\right)$$

Besides, as in Proposition 4 and (4.4), we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} (I_\alpha * |u_n^i|^p)|u_n^i|^{p-1}v - (I_\alpha * |u_n^i - u^i|^p)|u_n^i - u^i|^{p-1}v = \int_{\mathbb{R}^3} (I_\alpha * |u^i|^p)|u^i|^{p-1}v,$$

for all $v \in C_0^\infty(\mathbb{R}^3)$. And since

$$\int \left(\nabla u_n^i \cdot \nabla v + (V(\varepsilon x) + 1)u_n^i v\right) dx \to \int (\nabla u^i \cdot \nabla v + V(\varepsilon x) + 1)u^i v)dx,$$

for all $v \in C_0^\infty\left(\mathbb{R}^3\right)$. We conclude that $J'_\varepsilon\left(u^i\right)v = 0$ for all $v \in C_0^\infty\left(\mathbb{R}^3\right)$. By definition of g we have $g(x) \to 0$ as $|x| \to +\infty$, it is clear that

$$\int \chi(\varepsilon x)g(\varepsilon x)\left|u_n^i\right|^2 dx \longrightarrow \int \chi(\varepsilon x)g(\varepsilon x)\left|u^i\right|^2 dx$$

and

$$\int g(\varepsilon x)\left|u_n^i\right|^2 dx \to \int g(\varepsilon x)\left|u^i\right|^2 dx.$$

Under the condition that these two limits hold, (4.5) and (4.6) are guaranteed.

Next, we give a proof of Theorem 1, that is, there exist l solutions $u^i \in H^1\left(\mathbb{R}^3\right)\backslash\{0\}$.

Proof of Theorem 1.

According to Corollary 4, for $i \in \{1, \cdots, l\}$ and $\varepsilon \in (0, \varepsilon_*)$, there exists a solution $u^i \in H^1\left(\mathbb{R}^3\right)\backslash\{0\}$ for problem (1.2) such that

$$Q_\varepsilon\left(u^i\right) \in \overline{B_{\rho_0}(z_i)}.$$

Because we have

$$\overline{B_{\rho_0}(z_i)} \cap \overline{B_{\rho_0}(z_j)} = \phi, \quad i \neq j.$$

Then it implies that $u^i \neq u^j$ for $i \neq j$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflict of interest.

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