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Research article

Normalized solutions to nonautonomous Kirchhoff equation

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Abstract: In this paper, we studied the existence of normalized solutions to the following Kirchhoff equation with a perturbation:

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2\,dx\right)\Delta u+\lambda u=|u|^{p-2}u+h(x)\,|u|^{q-2}\,u, & \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N}|u|^2\,dx=c, \quad u\in H^1(\mathbb{R}^N), \end{cases}$$

where $1\le N\le 3, a,b,c>0, 1\le q<2, \lambda\in\mathbb{R}$. We treated three cases:
(i)When $2<p<2+\frac{4}{N}, h(x)\ge 0$, we obtained the existence of a global constraint minimizer.
(ii)When $2+\frac{8}{N}<p<2^*, h(x)\ge 0$, we proved the existence of a mountain pass solution.
(iii)When $2+\frac{8}{N}<p<2^*, h(x)\le 0$, we established the existence of a bound state solution.

Keywords: nonautonomous Kirchhoff equations; normalized solutions; bound state solution; L^2-critical exponent
Mathematics Subject Classification: 35A15, 35J60, 35J20

1. Introduction

In this paper, we consider the existence of solutions with prescribed L^2-norm to the following Kirchhoff problem with a perturbation

$$\begin{cases} -\left(a+b\int_{\mathbb{R}^N}|\nabla u|^2dx\right)\Delta u+\lambda u=|u|^{p-2}u+h(x)|u|^{q-2}u, & \text{in } \mathbb{R}^N,\\ \int_{\mathbb{R}^N}|u|^2dx=c, \quad u\in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $1\le N\le 3, a,b,c>0, p\in(2,2^*), q\in[1,2), h(x):\mathbb{R}^N\to\mathbb{R}$ is a potential, $2^*=6$ if $N=3$, and $2^*=+\infty$ if $N=1,2$. Based on these observations, we establish the existence of normalized solutions under different assumptions on $h(x)$.

The energy functional of Eq.(1.1) is defined by

$$I(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} h(x)|u|^q dx \tag{1.2}$$

constrained on the L^2-spheres in $H^1(\mathbb{R}^N)$:

$$S_c = \{u \in H^1(\mathbb{R}^N) : \|u\|_2^2 = c > 0\}.$$

In 1883, Kirchhoff [1] first proposed the following nonlinear wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0,$$

which extends the original wave equation by describing the transversal oscillations of a stretched string and, particularly, by considering the subsequent change in string length caused by oscillations. Thereafter, there was a boom in the study of the Kirchhoff-type equation. We can refer to [2–4] for the physical background about Kirchhoff problem.

Mathematically, Eq.(1.1) is not a pointwise identity as a result of the emergence of the term $(b\int_{\mathbb{R}^N} |\nabla u|^2 dx)\Delta u$. This causes some mathematical difficulties. In the renowned paper [5], J.L. Lions raised an abstract framework that has received much attention. There are two ways to study the Kirchhoff-type equation. The first approach is to consider fixing the parameter $\lambda \in \mathbb{R}$. In this case, there are a lot of results, which have been widely studied by using variational methods. We can refer to [6–9] and the references therein. Another way is to fix the L^2-norm. In this case, the desired solutions have a priori prescribed L^2-norm, which are usually referred to as normalized solutions in the literature; that is, for any fixed $c > 0$, we take $(u_c, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ as a normalized solution with $\|u_c\|_2^2 = c$, λ_c is a Lagrange multiplier. From a physical perspective, the L^2-prescribed norm represents the number of particles of each component in Bose-Einstein condensates or the power supply in a nonlinear optics framework. In addition, the L^2-prescribed norm can provide a better insight on the dynamical properties, like orbital stability or instability, and can describe attractive Bose-Einstein condensates.

For the local case, i.e., $b = 0$, Eq.(1.1) reduces to the general Schrödinger type:

$$\begin{cases} -\Delta u + \lambda u = f(x,u), & \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c, & u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.3}$$

which dates back to the groundbreaking work by Stuart. In [10], Stuart tackled the problem (1.3) for $f(x,u) = |u|^{p-2}u$ and $p \in (2, 2+\frac{4}{N})$ (L^2-subcritical case); here, $2+\frac{4}{N}$ is called the L^2-critical exponent. For L^2-subcritical case, the minimization method is the conventional method to find normalized solutions. When f is L^2-supercritical growth, a groundbreaking work in the L^2-supercritical case was accomplished by Jeanjean [11]. Jeanjean developed a novel argument related to the mountain pass geometry by the stretched functional. Bartsch and Soave [12, 13] also proposed a new approach by using a minimax principle based on the homotopy stable family to prove the existence of normalized solutions for the problem (1.3). Moreover, Soave in [14] studied the combined nonlinearity case $f(x,u) = |u|^{p-2}u + \mu|u|^{q-2}u$, $2 < q \le 2 + \frac{4}{N} \le p < 2^*$ and $q < p$, where $2^* = \infty$ if $N \le 2$ and $2^* = \frac{2N}{N-2}$ if $N \ge 3$. Soave showed that nonlinear terms with different power strongly affects the geometry of the functional and the existence and properties of ground states.

When $f(x,u) = a(x)f(u)$, the solutions to the nonautonomous problem were first studied by Chen and Tang [15]. Compared with the autonomous problems, the main challenge of the problem is constructing a (PS) sequence with an additional property to recover the compactness. Very recently, Chen and Zou [16] studied the following problem with a perturbation

$$\begin{cases} -\Delta u + \lambda u = |u|^{p-2}u + h(x), \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c, \quad u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.4}$$

where $h(x) \geq 0$. For $p \in (2, 2+\frac{4}{N})$ and an arbitrarily positive perturbation, Chen and Zou proved that there exists a global minimizer with negative energy. The existence of a mountain pass solution with positive energy for $p \in (2+\frac{4}{N}, 2^*)$ was studied. We can refer to [17–19] for more details.

For the nonlocal case, i.e., $b > 0$, the more general form of Eq.(1.1) is the following equation

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + \lambda u = f(x,u), \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c, \quad u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.5}$$

which has attracted considerable attention. When $f(x,u) = |u|^{p-2}u$ (i.e., the limited problem of Eq.(1.1)), the problem (1.5) turns to

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + \lambda u = |u|^{p-2}u, \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N} |u|^2 dx = c, \quad u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.6}$$

where $a, b, c > 0$ are constants, $1 \leq N \leq 3$, and $p \in (2, 2^*)$. The energy functional of (1.6) is

$$I_\infty(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx. \tag{1.7}$$

By the Gagliardo-Nirenberg inequality [20] for any $p \in (2, 2^*)$

$$\|u\|_p \leq C_{N,p}\|\nabla u\|_2^{\gamma_p}\|u\|_2^{1-\gamma_p} \tag{1.8}$$

where $\gamma_p = \frac{N(p-2)}{2p}$, we can get L^2-critical exponent $\bar{p} = 2 + \frac{8}{N}$ of the Kirchhoff problem. It is well known that Ye [21] obtained the sharp existence of global constraint minimizers for Eq.(1.6) in the case of $p \in (2, \bar{p})$. When $p \in (2+\frac{4}{N}, \bar{p})$, Ye proved a local minimizer, which is a critical point of $I_\infty|_{S_c}$. By considering a global minimization problem

$$l_{\infty,c} := \inf_{S_c} I_\infty(u), \tag{1.9}$$

we have

$$\begin{cases} l_{\infty,c} \in (-\infty, 0], \quad if \quad p \in (2, \bar{p}), \\ l_{\infty,c} = -\infty, \quad if \quad p \in (\bar{p}, 2^*), \end{cases} \tag{1.10}$$

for any given $c > 0$. We can see that the minimization method is not feasible for $p \in (\bar{p}, 2^*)$. Then, Ye proved the existence of normalized solutions by taking advantage of the Pohozaev constraint method in the case of $p \in (\bar{p}, 2^*)$. For the L^2-critical case of $\bar{p} = 2 + \frac{8}{N}$, Ye [22] showed the existence and mass concentration of critical points. Using some simple energy estimates instead of the concentration-compactness principles introduced in [21], Zeng studied the existence and uniqueness of normalized solutions for $p \in (2, 2^*)$ in [23].

Additionally, Li, Luo, and Yang [24] proved the existence and asymptotic properties of solutions to the following equation with combined nonlinearity

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)\Delta u + \lambda u = |u|^{p-2}u + \mu|u|^{q-2}u, \quad \text{in } \mathbb{R}^3, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = c, \quad u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.11}$$

where $a, b, c, \mu > 0$, $2 < q < \frac{14}{3} < p \leq 6$ or $\frac{14}{3} < q < p \leq 6$. They showed a multiplicity result for the case of $2 < q < \frac{10}{3}$ and $\frac{14}{3} < p < 6$ and obtained the existence of ground state normalized solutions for $2 < q < \frac{10}{3} < p = 6$ or $\frac{14}{3} < q < p \leq 6$. They also showed some asymptotic results on the obtained solutions. For the case $\mu \leq 0$, in [25], Carri*ã*o, Miyagaki, and Vicente studied the ground states existence of Eq.(1.11) for $2 < q < 2^*, p = 2^*$ or $2 < q \leq \bar{p} < p < 2^*$. For the nonautonomous problem, when $f(x, u) = |u|^{p-2}u + V(x)|u|^{q-2}u$, $N = 3$, $p = \frac{14}{3}$, $q = 4$ and $V \in L^\infty_{loc}(\mathbb{R}^3)$, Ye [26] considered the existence of minimizers to the nonautonomous problem. Moreover, $V(x)$ satisfies

$$V(x) \geq 0, \quad \lim_{|x|\to\infty} V(x) = 0.$$

By the concentration compactness principle, if $b < b_0$, Ye showed that there exists $a_0, c_0 > 0$ such that the above problem has a minimizer for all $a < a_0$ and $c < c_0$. Additionally, when $f(x, u) = K(x)f(u)$, Chen and Tang [27] considered the existence of ground state solutions, where $K(x) \in C(\mathbb{R}^3, \mathbb{R}^+)$ and $f(u)$ is L^2-supercritical. When $2 + \frac{4}{N} < p < 2 + \frac{8}{N}$, the geometric structure of the energy functional is more complex, especially when $h(x) > 0$, and there are very few works studying this range with potential. Other results about normalized solutions of Kirchhoff equation in a more general form can be found in [28–31].

Motivated by the results above, when μ of Eq.(1.11) is replaced by a potential function $h(x)$ and $1 \leq q < 2$, there are no results in studying normalized solutions of such nonautonomous Kirchhoff equations with a small perturbation. In the present paper, we first obtain the normalized solution of this type of equation, which can be seen as an extension of some known results in the literature.

Let us now outline the main strategy to prove the three results of this paper under different assumptions on $h(x)$. First, we treat the mass-subcritical case $2 < p < 2 + \frac{4}{N}$: for any $c > 0$, we set

$$l_c := \inf_{S_c} I(u). \tag{1.12}$$

It is standard that the minimizers of l_c are critical points of $I|_{S_c}$. We introduce the following assumptions on $h(x)$.

($\mathbf{h_1}$) $h \in L^{\frac{2}{2-q}}(\mathbb{R}^N)$ and $h(x) > 0$ on a set with positive measure.

Now we state the main results of this paper:

Theorem 1.1. Suppose $1 \le N \le 3$, $2 < p < 2 + \frac{4}{N}$ and $h(x) \ge 0$ satisfies $(\mathbf{h_1})$. Then, for all $c > 0$, l_c has a minimizer, hence Eq.(1.1) has a normalized ground state solution.

Remark 1.1. Notice that the minimizer obtained in Theorem 1.1 is a global minimizer rather than a local minimizer. It is easy to find that the energy functional is coercive on S_c, which hints that each minimizing sequence $\{u_n\}$ is bounded on S_c. The main difficulty of proof is to show that the minimizing sequence $\{u_n\}$ converges strongly to $u \ne 0$ in $H^1(\mathbb{R}^N)$. The key step is to establish the inequality $l_{c_1+c_2} \le l_{c_1} + l_{\infty,c_2}$ for $c_1, c_2 > 0$ (see Lemma 2.2), which is crucial to recover the compactness.

Next, while addressing the L^2-supercritical case, the functional is unbounded from below on S_c, thus the minimizing approach on S_c is not valid anymore. Ye [21] proved that $l_{\infty,c} = -\infty$ for all $c > 0$ if $p \in \left(2 + \frac{8}{N}, 2^*\right)$, and proved the existence of one normalized solution by a suitable submanifold of S_c. In this paper, after the appearance of a very small perturbation term, we want to show that the energy functional I has a mountain pass geometry and show the existence of a mountain pass solution with positive energy level for $p \in \left(2 + \frac{8}{N}, 2^*\right)$. We require the perturbation $h(x)$ to have a higher regularity. We need to assume that:

$$(\mathbf{h_2})\ \ h \in L^{\frac{p}{p-q}}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N),\ \ \langle \nabla h, x\rangle \in L^{\frac{2}{2-q}}(\mathbb{R}^N) \text{ and } h(x) \ge 0.$$

We have the following result.

Theorem 1.2. Suppose $1 \le N \le 3$, $2 + \frac{8}{N} < p < 2^*$ and $h(x)$ satisfies $(\mathbf{h_2})$. Let $c > 0$ be fixed. Moreover,

$$\|h\|_{\frac{p}{p-q}} < \frac{aq(p\gamma_p - 2)}{2C_{N,p}^q \gamma_p (p-q)} \left(\frac{ap(2 - q\gamma_p)}{2\gamma_p (p-q) C_{N,p}^p} \right)^{\frac{2-q\gamma_p}{p\gamma_p - 2}} c^{-\frac{(1-\gamma_p)(p-q)}{p\gamma_p - 2}}, \tag{1.13}$$

$$\|\nabla h \cdot x\|_{\frac{2}{2-q}} < \frac{q(2p - Np + 2N)}{p-2} m_c c^{-\frac{q}{2}}. \tag{1.14}$$

Then, Eq.(1.1) has a mountain pass solution u at a positive energy level.

Remark 1.2. We are going to use the minimax characterization to find a critical point. Although the mountain pass geometry of the functional on S_c can be obtained easily, unfortunately the boundedness of the obtained (PS) sequence is not yet clear. In this paper, we adopt a similar idea to [11] and construct an auxiliary map $\tilde{I}(t, u) := I(t \star u)$, which on $\mathbb{R} \times S_c$ has the same type of geometric structure as I on S_c. Besides, the (PS) sequence of I satisfies the additional condition (see Lemma 3.5), which is the key ingredient to obtain the boundedness of the (PS) sequence.

Finally, we will discuss $h(x) \le 0$, and the problem becomes more delicate and difficult. Although the mountain pass structure by Jeanjean [11] is destroyed, Bartsch et al. [32] established a new variational principle exploiting the Pohozaev identity. For convenience, we define $\bar{h}(x) := -h(x) \ge 0$. Next, we state our basic assumptions on $\bar{h}(x)$.

$(\mathbf{h_3})$ $\bar{h}(x) \in L^{\frac{2}{2-q}}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$, $\langle \nabla \bar{h}(x), x\rangle \in L^{\frac{2}{2-q}}(\mathbb{R}^N)$ and $\bar{h}(x) \ge 0$. For some constants $\Upsilon > 0$, $\bar{h}(x)$ satisfies

$$\left| x \cdot \nabla \bar{h}(x) \right| \le \Upsilon \bar{h}(x).$$

Theorem 1.3. Assume $1 \leq N \leq 3$, $2 + \frac{8}{N} < p < 2^*$. If $(\mathbf{h_3})$ holds and $\bar{h}(x)$ satisfies

$$0 < \|\bar{h}\|_{\frac{2}{2-q}} < \min\left\{1, \frac{2p(1-\gamma_p)}{2(p-q)+(p-2)\Upsilon}\right\} \cdot \frac{qm_c}{c^{\frac{q}{2}}}. \tag{1.15}$$

Then Eq.(1.1) has a couple of solutions $(u,\lambda) \in H^1(\mathbb{R}^N) \times \mathbb{R}$ and $\lambda > 0$.

Remark 1.3. Indeed, when $h(x) \leq 0$, the problem is made more difficult by the simultaneous appearance of a negative potential and nonlocal term. We refer to Bartsch et al. [32] constructing a suitable linking geometry method to obtain the existence of bound state solutions with high Morse index. The crucial step is to estimate the minimax level $m_c < L_{h,c} < 2m_c$ (see Lemma 4.3 and Lemma 4.5) to recover the compactness.

Notations: We introduce some notations that will clarify what follows:

- $H^1(\mathbb{R}^N)$ is the usual Sobolev space with the norm $\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2\, dx\right)^{\frac{1}{2}}$.
- $L^p(\mathbb{R}^N)$ with $p \in [1,\infty)$ is the Lebesgue space with the norm $\|u\|_p = \left(\int_{\mathbb{R}^N} |u|^p\, dx\right)^{\frac{1}{p}}$.
- The arrows $' \rightharpoonup '$ and $' \to '$ denote the weak convergence and strong convergence, respectively.
- C, C_i denote positive constants, which may vary from line to line.
- $(t \star u)(x) := t^{\frac{N}{2}} u(tx)$ for $t \in \mathbb{R}^+$ and $u \in H^1(\mathbb{R}^N)$.

2. Proof of Theorem 1.1

In this section, for $2 < p < 2 + \frac{4}{N}$ and $h(x) \geq 0$ we prove Theorem 1.1. By the Gagliardo-Nirenberg inequality (1.8), the Hölder inequality, and the assumption $(\mathbf{h_1})$, we have

$$\begin{aligned} I(u) &= \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{p}\|u\|_p^p - \frac{1}{q}\int_{\mathbb{R}^N} h(x)|u|^q dx \\ &\geq \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{p}C_{N,p}^p \|\nabla u\|_2^{p\gamma_p}\|u\|_2^{p(1-\gamma_p)} - \frac{1}{q}\|h\|_{\frac{2}{2-q}}\|u\|_2^q, \end{aligned} \tag{2.1}$$

thus I is bounded from below on S_c since $0 < p\gamma_p < 2$.

For $1 \leq N \leq 3$ and $2 < p < 2 + \frac{4}{N}$, the existence and uniqueness of positive normalized solutions of the limited problem (1.6) have been studied in [21]. In order to find the minimizer of I on S_c, first we state some fundamental properties of $l_{\infty,c}$, which will be crucial to recover the compactness later on. The proof of the next lemma can be found in [28, Theorem 1.1 and Lemma 2.5].

Lemma 2.1. Suppose $1 \leq N \leq 3$ and $2 < p < 2 + \frac{4}{N}$. Then, for all $c > 0$, we have
(i) the strict sub-additivity for $l_{\infty,c}$, i.e.,

$$l_{\infty,c_1+c_2} < l_{\infty,c_1} + l_{\infty,c_2} \quad \text{for} \quad c_1, c_2 > 0;$$

(ii) the limited problem (1.6) has a couple of ground state solutions $(u_\infty, \lambda_c) \in H^1(\mathbb{R}^N) \times \mathbb{R}$, i.e.,

$$l_{\infty,c} = \inf_{S_c} I_\infty(u) = I_\infty(u_\infty) < 0.$$

Next, we introduce the inequality $l_{c_1+c_2} \le l_{c_1} + l_{\infty,c_2}$, which plays a crucial role in proving the convergence of the minimizing sequence.

Lemma 2.2. Suppose $2 < p < 2 + \frac{4}{N}$ and $h(x)$ satisfies ($\mathbf{h_1}$), then the following holds
(i)$-\infty < l_c < l_{\infty,c} < 0$ for $c > 0$;
(ii)$l_{c_1+c_2} \le l_{c_1} + l_{\infty,c_2}$ for $c_1, c_2 > 0$.

Proof. (i) It is obvious that $l_c > -\infty$ by (2.1). Moreover, by Lemma 2.1, we have

$$\begin{aligned} l_c &\le I(u_\infty) \\ &= \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u_\infty|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u_\infty|^2 dx\right)^2 - \frac{1}{p}\int_{\mathbb{R}^N} |u_\infty|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} h|u_\infty||^q dx \\ &< I_\infty(u_\infty) \\ &= l_{\infty,c} < 0, \end{aligned}$$

since $u_\infty > 0$ and $h(x)$ satisfies ($\mathbf{h_1}$).

(ii) For any $\varepsilon > 0$, $c = c_1 + c_2$, we can find $\varphi_\varepsilon, \psi_\varepsilon \in C_0^\infty(\mathbb{R}^N)$ such that

$$\begin{aligned} \varphi_\varepsilon \in S_{c_1}, \quad I(\varphi_\varepsilon) < l_{c_1} + \frac{\varepsilon}{2}, \\ \psi_\varepsilon \in S_{c_2}, \quad I_\infty(\psi_\varepsilon) < l_{\infty,c_2} + \frac{\varepsilon}{2}. \end{aligned}$$

Let $u_{\varepsilon,n}(x) := \varphi_\varepsilon(x) + \psi_\varepsilon(x - n\boldsymbol{e}_1)$, where $\boldsymbol{e}_1$ is the unit vector $(1, 0, \cdots)$ in $\mathbb{R}^N$. Since φ_ε and ψ_ε have compact support, we see that $u_{\varepsilon,n} \in S_c$ and

$$l_c \le I(u_{\varepsilon,n}) = I(\varphi_\varepsilon) + I(\psi_\varepsilon(x - n\boldsymbol{e}_1)),$$

for large n. Moreover, thanks to $h \in L^{\frac{2}{2-q}}(\mathbb{R}^N)$, we have that $\int_{\mathbb{R}^N} h(x)\psi_\varepsilon^q(x - n\boldsymbol{e}_1)dx \to 0$ as $n \to \infty$, hence $I(\psi_\varepsilon(\cdot - n\boldsymbol{e}_1)) \to I_\infty(\psi_\varepsilon)$ as $n \to \infty$. It follows that

$$\begin{aligned} l_c &\le \limsup_{n\to\infty} I(u_{\varepsilon,n}) \\ &= \limsup_{n\to\infty}(I(\varphi_\varepsilon) + I(\psi_\varepsilon(\cdot - n\boldsymbol{e}_1))) \\ &= I(\varphi_\varepsilon) + I_\infty(\psi_\varepsilon) \\ &< l_{c_1} + l_{\infty,c_2} + \varepsilon. \end{aligned}$$

Passing to the limit, thus $l_c \le l_{c_1} + l_{\infty,c_2}$ since $\varepsilon > 0$ is arbitrary. □

Let $\{u_n\} \subset S_c$ be a minimizing sequence for l_c. By (2.1), we know that $I(u)$ is coercive on S_c and deduce that $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. Thus, there exists a subsequence such that $u_n \rightharpoonup u_0$ and

$$I(u_0) \le \liminf_{n\to\infty} I(u_n) = l_c, \quad c_1 := \|u_0\|_2^2 \le \|u_n\|_2^2 = c.$$

We need to prove $I(u_0) = l_c$ and $\|u_0\|_2^2 = c$. Now we argue by contradiction to prove this.

Lemma 2.3. Suppose $2 < p < 2 + \frac{4}{N}$ and $h(x)$ satisfies $(\mathbf{h_1})$. Then, every minimizing sequence for l_c has a strong convergent subsequence in $L^2(\mathbb{R}^N)$.

Proof. We argue by contradiction and assume that $c_1 < c$. We divide the proof into four steps.

Step 1: There exists $\{y_n\} \subset \mathbb{R}^N$ and $\mu_0 \in H^1(\mathbb{R}^N)\backslash\{0\}$ such that

$$|y_n| \to \infty, \quad u_n(\cdot + y_n) \rightharpoonup \mu_0 \quad \text{ in } H^1(\mathbb{R}^N). \tag{2.2}$$

First, we show by contradiction that

$$\delta_0 := \liminf_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |u_n - u_0|^2 dx > 0, \tag{2.3}$$

where $B_1(y) = \{x \in \mathbb{R}^N : |x - y| \leq 1\}$. Suppose, on the contrary, that $\delta_0 = 0$. Then, $u_n \to u_0$ strongly in $L^p(\mathbb{R}^N)$. Since $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N), h \in L^{\frac{2}{2-q}}(\mathbb{R}^N)$, we see that $\int_{\mathbb{R}^N} h|u_n|^q dx \to \int_{\mathbb{R}^N} h|u_0|^q dx$. Combined with Lemma 2.1 (ii), for $c - c_1 > 0$, we have that

$$\begin{aligned} l_c &= I(u_n) + o(1) \\ &= I(u_0) + I(u_n - u_0) + o(1) \\ &= I(u_0) + \frac{a}{2}\int_{\mathbb{R}^N} |\nabla (u_n - u_0)|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla (u_n - u_0)|^2 dx\right)^2 + o(1) \\ &> l_{c_1} + l_{\infty, c-c_1}, \end{aligned}$$

which is a contradiction with Lemma 2.2 (ii). Therefore, (2.3) holds. From (2.3) and $u_n \to u_0$ in $L^2_{loc}(\mathbb{R}^N)$, we can find $\{y_n\} \subset \mathbb{R}^N$ such that $\int_{B_1(y_n)} |u_n - u_0|^2 dx \to c_0 > 0$ and $|y_n| \to \infty$. Let $u_n(\cdot + y_n) \rightharpoonup \mu_0$ weakly in $H^1(\mathbb{R}^N)$. Note that $\mu_0 \neq 0$ since $c_0 > 0$. Therefore, $\{y_n\}$ and μ_0 satisfy (2.2). Thus, the proof of Step 1 is complete.

Step 2: We show that $\{y_n\}$ and (u_0, μ_0) satisfy

$$\lim_{n\to\infty} \|u_n - u_0 - \mu_0(\cdot - y_n)\|_2^2 = 0. \tag{2.4}$$

Since $|y_n| \to \infty$, we have that

$$\begin{aligned} \|u_n - u_0 - \mu_0(\cdot - y_n)\|_2^2 =& \|u_n\|_2^2 + \|u_0\|_2^2 + \|\mu_0\|_2^2 \\ & - 2\langle u_n, u_0\rangle_{L^2} - 2\langle u_n(\cdot + y_n), \mu_0\rangle_{L^2} + o(1) \\ =& \|u_n\|_2^2 - \|u_0\|_2^2 - \|\mu_0\|_2^2 + o(1). \end{aligned} \tag{2.5}$$

According to (2.5), we could let $\delta_1 := \lim_{n\to\infty} \|u_n - u_0 - \mu_0(\cdot - y_n)\|_2^2$. Then, we have $\delta_1 = c - c_1 - c_2$, where $c_2 := \|\mu_0\|_2^2$. We want to show that $\delta_1 = 0$. Suppose on the contrary that $\delta_1 > 0$, by direct calculations we have

$$\begin{aligned} & \|\nabla u_n\|_2^2 - \|\nabla u_0\|_2^2 - \|\nabla \mu_0(\cdot - y_n)\|_2^2 - \|\nabla(u_n - u_0 - \mu_0(\cdot - y_n))\|_2^2 \\ &= -2\|\nabla u_0\|_2^2 - 2\|\nabla \mu_0\|_2^2 + 2\langle \nabla u_n, \nabla u_0\rangle_{L^2} + 2\langle \nabla u_n(\cdot + y_n), \nabla \mu_0\rangle_{L^2} \\ &= o(1). \end{aligned} \tag{2.6}$$

From the Brezis-Lieb Lemma, we have

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n|^p \, dx = \int_{\mathbb{R}^N} |u_0|^p \, dx + \int_{\mathbb{R}^N} |\mu_0 (\cdot - y_n)|^p \, dx \\ + \int_{\mathbb{R}^N} |u_n - u_0 - \mu_0 (\cdot - y_n)|^p \, dx + o(1).\end{aligned} \tag{2.7}$$

Similarly,

$$\begin{aligned}\int_{\mathbb{R}^N} h \, |u_n|^q \, \mathrm{d}x = \int_{\mathbb{R}^N} h \, |u_0|^q \, dx + \int_{\mathbb{R}^N} h \, |\mu_0 (\cdot - y_n)|^q \, dx \\ + \int_{\mathbb{R}^N} h \, |(u_n - u_0 - \mu_0 (\cdot - y_n))|^q \, dx + o(1).\end{aligned} \tag{2.8}$$

Combining (2.6)–(2.8), we have

$$I (u_n) - I (u_0) - I (\mu_0 (\cdot - y_n)) - I (u_n - u_0 - \mu_0 (\cdot - y_n)) = o(1). \tag{2.9}$$

Since $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$, $|y_n| \to \infty$ and $h \in L^{\frac{2}{2-q}} \left(\mathbb{R}^N\right)$, we have

$$\int_{\mathbb{R}^N} h \, |u_n - u_0 - \mu_0 (\cdot - y_n)|^q \, dx \to 0. \tag{2.10}$$

Recalling that $l_{\infty,c}$ is continuous with respect to $c > 0$ (see [33], Theorem 2.1), we have that

$$\begin{aligned}&\liminf_{n\to\infty} I (u_n - u_0 - \mu_0 (\cdot - y_n)) \\ &\quad = \liminf_{n\to\infty} I_\infty (u_n - u_0 - \mu_0 (\cdot - y_n)) \\ &\quad \geq l_{\infty,\delta_1},\end{aligned} \tag{2.11}$$

and

$$\liminf_{n\to\infty} I (\mu_0 (\cdot - y_n)) \geq l_{\infty,c_2}. \tag{2.12}$$

Hence by (2.9)–(2.12), we have

$$l_c \geq l_{c_1} + l_{\infty,c_2} + l_{\infty,\delta_1}. \tag{2.13}$$

However, using Lemma 2.1 (i), for any $c_2, \delta_1 > 0$, there exists $l_{\infty,c_2+\delta_1} < l_{\infty,c_2} + l_{\infty,\delta_1}$. Hence, we also have

$$\begin{aligned}l_c &\geq l_{c_1} + l_{\infty,c_2} + l_{\infty,\delta_1} \\ &> l_{c_1} + l_{\infty,c_2+\delta_1} \\ &\geq l_{c_1+c_2+\delta_1} \\ &= l_c.\end{aligned} \tag{2.14}$$

This gives a contradiction and thus we have that $\delta_1 = 0$.

Step 3: Moreover, the following holds

$$I (u_0) = l_{c_1}, \ \ I_\infty (\mu_0) = l_{\infty,c_2}, \tag{2.15}$$

and

$$l_c = l_{c_1} + l_{\infty,c_2}. \tag{2.16}$$

By (2.9)–(2.12) and $\delta_1 = 0$, we have that

$$\begin{aligned} l_c &= \lim_{n\to\infty} I(u_n) \\ &= \liminf_{n\to\infty} \left(I(u_0) + I(\mu_0(\cdot + y_n))\right) \\ &\geq I(u_0) + I_\infty(\mu_0) \\ &\geq l_{c_1} + l_{\infty,c_2}. \end{aligned} \tag{2.17}$$

Combined with Lemma 2.2 (ii), we see that $l_c = l_{c_1} + l_{\infty,c_2}$. $I(u_0) = l_{c_1}$ and $I_\infty(\mu_0) = l_{\infty,c_2}$. Thus, Step 3 is proved.

Step 4: Now, we prove the precompactness of minimizing sequence, i.e., $u_n \to u_0$ in $L^2(\mathbb{R}^N)$. We can suppose that $\{u_n\}$ are nonnegative. Using the strong maximum principle, we have $u_0, \mu_0 > 0$ and $h(x) > 0$ on a set with positive measure, we have that

$$\int_{\mathbb{R}^N} h\left|\sqrt{u_0^2 + \mu_0^2}\right|^q \mathrm{d}x > \int_{\mathbb{R}^N} h\,|u_0|^q \;\mathrm{d}x.$$

Combine with the two following inequalities:

$$\int_{\mathbb{R}^N} \left|\nabla\sqrt{u_0^2 + \mu_0^2}\right|^2 dx \leq \int_{\mathbb{R}^N} (|\nabla u_0|^2 + |\nabla \mu_0|^2) dx, \tag{2.18}$$

$$\int_{\mathbb{R}^N} \left|\sqrt{u_0^2 + \mu_0^2}\right|^p dx \geq \int_{\mathbb{R}^N} (|u_0|^p + |\mu_0|^p)\, dx. \tag{2.19}$$

So we have

$$\begin{aligned} l_c &\leq I\left(\sqrt{u_0^2 + \mu_0^2}\right) \\ &= \frac{a}{2}\int_{\mathbb{R}^N} \left|\nabla\sqrt{u_0^2 + \mu_0^2}\right|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} \left|\nabla\sqrt{u_0^2 + \mu_0^2}\right|^2 dx\right)^2 \\ &\quad - \frac{1}{p}\int_{\mathbb{R}^N} \left|\sqrt{u_0^2 + \mu_0^2}\right|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} h\left|\sqrt{u_0^2 + \mu_0^2}\right|^q dx \\ &< I(u_0) + I_\infty(\mu_0) \\ &= l_{c_1} + l_{\infty,c-c_1} \\ &= l_c, \end{aligned} \tag{2.20}$$

which is a contradiction. Thus the proof of Lemma 2.3 is completed. □

Proof of Theorem 1.1. From Lemma 2.3, the minimizing sequence $\{u_n\}$ satisfies $u_n \to u_0$ in $L^2(\mathbb{R}^N)$ and $l_c = I(u_0)$, $c = c_1$. Since $\{u_n\} \subset S_c$ is the minimizing sequence of l_c, we have $dI|_{S_c}(u_n) \to 0$ and there exists a sequence of real numbers $\{\lambda_n\}$ such that

$$I'(u_n)[\varphi] + \lambda_n \int_{\mathbb{R}^N} u_n \varphi dx \to 0, \quad as\;\; n \to \infty, \tag{2.21}$$

for every $\varphi \in H^1(\mathbb{R}^N)$. Hence, by (2.21), we have that

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^N} |\nabla u_0|^2 dx\right)\Delta u_0 + \bar{\lambda} u_0 = |u_0|^{p-2}u_0 + h(x)|u_0|^{q-2}u_0 \quad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u_0|^2 dx = c. \end{cases} \tag{2.22}$$

Notice that $h(x) \geq 0$, then by the maximum principle, $u_0 > 0$, and we finish the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

In this section, we study the mass-supercritical and Sobolev-subcritical case: $2 + \frac{8}{N} < p < 2^*$, $1 \leq N \leq 3$, and $h(x)$ satisfies the assumption ($\mathbf{h_2}$). First, we show that the energy functional I possesses a mountain pass geometry, which implies the existence of the (PS) sequence. Next, we prove that the limit of the sequence of the Lagrange multipliers related to the (PS) sequence is positive. Then, by applying the splitting lemma, we recover the compactness for this sequence, which yields the existence of solutions for Eq.(1.1).

In order to study the behavior of (PS) sequence, we introduce the splitting lemma, which plays a crucial role in overcoming the lack of compactness. For $\lambda > 0$, we set

$$I_\lambda(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^N} \lambda u^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} h|u|^q dx$$

and

$$I_{\infty,\lambda}(u) = \frac{a}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2 dx\right)^2 + \frac{1}{2}\int_{\mathbb{R}^N} \lambda u^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx.$$

Lemma 3.1. Let $\{u_n\} \subset H^1(\mathbb{R}^N)$ be a (PS) sequence for I_λ such that $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$ and $\lim_{n\to\infty} \|\nabla u_n\|_2^2 = A^2$. Then, there exists an integer $k \geq 0$, k nontrivial solutions $\omega^1, \cdots, \omega^k \in H^1(\mathbb{R}^N)$ to the following problem

$$-(a + bA^2)\Delta\omega + \lambda\omega = |\omega|^{p-2}\omega, \tag{3.1}$$

and k sequences $\{y_n^j\} \subset \mathbb{R}^N, 1 \leq j \leq k$, such that as $n \to \infty, |y_n^j| \to \infty, |y_n^{j_1} - y_n^{j_2}| \to \infty$ for each $1 \leq j_1, j_2 \leq k, j_1 \neq j_2$, and

$$\left\| u_n - u - \sum_{j=1}^{k} \omega^j(\cdot - y_n^j) \right\| \to 0, \tag{3.2}$$

$$A^2 = \|\nabla u\|_2^2 + \sum_{j=1}^{k} \|\nabla\omega^j\|_2^2, \tag{3.3}$$

$$\|u_n\|_2^2 = \|u\|_2^2 + \sum_{j=1}^{k} \|w^j\|_2^2 + o(1), \tag{3.4}$$

and

$$I_\lambda(u_n) \to J_{h,\lambda}(u) + \sum_{j=1}^{k} J_{\infty,\lambda}(\omega^j), \tag{3.5}$$

as $n \to \infty$ where

$$J_{h,\lambda}(u) := \left(\frac{a}{2} + \frac{bA^2}{4}\right)\int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda}{2}\int_{\mathbb{R}^N} u^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |u|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} h|u|^q dx$$

and

$$J_{\infty,\lambda}(u) := \left(\frac{a}{2} + \frac{bA^2}{4}\right) \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{\lambda}{2} \int_{\mathbb{R}^N} u^2 dx - \frac{1}{p} \int_{\mathbb{R}^N} |u|^p dx.$$

Proof. The proof is similar to [34, Proposition 2.1] and [28, Lemma 1.6]; therefore, we omit it. □

Lemma 3.2. Let X be a Hilbert manifold and let $F \in C^1(X, \mathbb{R})$ be a given functional. Let $K \subseteq X$ be compact and consider a subset.

$$\mathcal{E} \subset \{E \subset X : E \text{ is compact}, K \subset E\},$$

which is invariant with respect to deformations leaving K fixed. Assume that

$$\max_{u \in K} F(u) < c := \inf_{E \in \mathcal{E}} \max_{u \in E} F(u) \in \mathbb{R}.$$

Let $\sigma_n \in \mathbb{R}$ be such that $\sigma_n \to 0$ and $E_n \in \mathcal{E}$ be a sequence such that

$$c \leq \max_{u \in E_n} F(u) < c + \sigma_n.$$

Then, there exists a sequence $v_n \in X$ such that
1. $c \leq F(v_n) < c + \sigma_n$,
2. $\|\nabla_X F(v_n)\| < \tilde{c}\sqrt{\sigma_n}$,
3. $\operatorname{dist}(v_n, E_n) < \tilde{c}\sqrt{\sigma_n}$,

for some constant $\tilde{c} > 0$.

We shall prove that I on S_c possesses a kind of mountain pass geometrical structure. To this aim, we establish two preliminary lemmas.

Lemma 3.3. Assume that $h \in L^{\frac{p}{p-q}}(\mathbb{R}^N)$ and let $u \in S_c$ be arbitrary but fixed. Then, we have:
(i) $I(t \star u) \to 0$ as $t \to 0$;
(ii) $I(t \star u) \to -\infty$ as $t \to +\infty$.

Proof. (i) By the Gagliardo-Nirenberg inequality (1.8), the Hölder inequality, and the assumption $(\mathbf{h_2})$, then we have that

$$\begin{aligned}
|I(t \star u)| &\leq \frac{a}{2} \int_{\mathbb{R}^N} |\nabla (t \star u)|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla (t \star u)|^2 dx \right)^2 \\
&\quad + \frac{1}{p} \int_{\mathbb{R}^N} |t \star u|^p dx + \frac{1}{q} \int_{\mathbb{R}^N} h |t \star u|^q dx \\
&\leq \frac{at^2}{2} \|\nabla u\|_2^2 + \frac{bt^4}{4} \|\nabla u\|_2^4 + \frac{t^{p\gamma_p}}{p} C_{N,p}^p c^{\frac{p - p\gamma_p}{2}} \|\nabla u\|_2^{p\gamma_p} + \frac{1}{q} t^{q\gamma_p} C_{N,p}^q c^{\frac{q(1-\gamma_p)}{2}} \|h\|_{\frac{p}{p-q}} \|\nabla u\|_2^{q\gamma_p} \\
&\to 0
\end{aligned}$$

as $t \to 0^+$, since $p\gamma_p, q\gamma_p > 0$.

(ii) Similarly, we have that

$$\begin{aligned}
I(t \star u) &\leq \frac{at^2}{2} \|\nabla u\|_2^2 + \frac{bt^4}{4} \|\nabla u\|_2^4 - \frac{1}{p} \int_{\mathbb{R}^N} |t \star u|^p dx + \frac{1}{q} \int_{\mathbb{R}^N} h |t \star u|^q dx \\
&\leq \frac{at^2}{2} \|\nabla u\|_2^2 + \frac{bt^4}{4} \|\nabla u\|_2^4 - \frac{t^{p\gamma_p}}{p} \int_{\mathbb{R}^N} |u|^p dx + \frac{1}{q} t^{q\gamma_p} C_{N,p}^q c^{\frac{q(1-\gamma_p)}{2}} \|h\|_{\frac{p}{p-q}} \|\nabla u\|_2^{q\gamma_p} \\
&\to -\infty
\end{aligned}$$

as $t \to +\infty$, since $p\gamma_p > 4$. $\square$

Again, using the Gagliardo-Nirenberg inequality and the Hölder inequality,

$$\begin{aligned} I(u) &\geq \frac{a}{2}\|\nabla u\|_2^2 + \frac{b}{4}\|\nabla u\|_2^4 - \frac{1}{p}C_{N,p}^p c^{\frac{p-p\gamma_p}{2}}\|\nabla u\|_2^{p\gamma_p} - \frac{1}{q}C_{N,p}^q c^{\frac{q(1-\gamma_p)}{2}}\|h\|_{\frac{p}{p-q}}\|\nabla u\|_2^{q\gamma_p} \\ &\geq \frac{a}{2}\|\nabla u\|_2^2 - \frac{1}{p}C_{N,p}^p c^{\frac{p-p\gamma_p}{2}}\|\nabla u\|_2^{p\gamma_p} - \frac{1}{q}C_{N,p}^q c^{\frac{q(1-\gamma_p)}{2}}\|h\|_{\frac{p}{p-q}}\|\nabla u\|_2^{q\gamma_p}. \end{aligned} \tag{3.6}$$

To understand the geometry of the functional I on S_c, it is useful to consider the function $\varphi : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$\varphi(t) := \frac{a}{2}t^2 - \frac{1}{p}C_{N,p}^p c^{\frac{p-p\gamma_p}{2}} t^{p\gamma_p} - \frac{1}{q}C_{N,p}^q c^{\frac{q(1-\gamma_p)}{2}}\|h\|_{\frac{p}{p-q}} t^{q\gamma_p}. \tag{3.7}$$

Since $0 < q\gamma_p < 2 < p\gamma_p$, we have that $\varphi(0^+) = 0^-$ and $\varphi(+\infty) = -\infty$. The role of assumption (1.13) is clarified by the following lemma.

Lemma 3.4. Under the assumption ($\mathbf{h_2}$), if (1.13) holds, then the function φ has a local strict minimum at negative level and a global strict maximum at positive level. Moreover, there exists $0 < R_1 < R_2$, both depending on c, such that $\varphi(R_1) = 0 = \varphi(R_2)$ and $\varphi(t) > 0$ if and only if $t \in (R_1, R_2)$.

Proof. For $t > 0$, we see that $\varphi(t) > 0$ if and only if

$$\psi(t) > \frac{1}{q}C_{N,p}^q c^{\frac{q(1-\gamma_p)}{2}}\|h\|_{\frac{p}{p-q}},$$

where

$$\psi(t) := \frac{a}{2}t^{2-q\gamma_p} - \frac{1}{p}C_{N,p}^p c^{\frac{p-p\gamma_p}{2}} t^{p\gamma_p - q\gamma_p}.$$

Observe that $p\gamma_p - q\gamma_p > 2 - q\gamma_p > 0$, then ψ has a unique critical point $\bar{t}$ on $(0, +\infty)$, which is a global maximum point at positive level. In fact, the expression of $\bar{t}$ is

$$\bar{t} = \left(\frac{ap(2-q\gamma_p)}{2\gamma_p(p-q)C_{N,p}^p c^{\frac{p-p\gamma_p}{2}}}\right)^{\frac{1}{p\gamma_p-2}},$$

and the maximum value of ψ is

$$\psi(\bar{t}) = \frac{a(p\gamma_p-2)}{2\gamma_p(p-q)}\left(\frac{ap(2-q\gamma_p)}{2\gamma_p(p-q)C_{N,p}^p}\right)^{\frac{2-q\gamma_p}{p\gamma_p-2}} c^{-\frac{p(1-\gamma_p)(2-q\gamma_p)}{2(p\gamma_p-2)}}. \tag{3.8}$$

Therefore, if (1.13) holds, then $\psi(\bar{t}) > \frac{1}{q}C_{N,p}^q c^{\frac{q(1-\gamma_p)}{2}}\|h\|_{\frac{p}{p-q}}$, thus the equation $\varphi = 0$ has two roots R_1, R_2 and φ is positive on (R_1, R_2). Moreover, φ has a global maximum point t_2 at positive level. According to the expression of φ, we can deduce that φ also has a local minimum point t_1 at negative level in $(0, R_1)$. $\square$

Set

$$\begin{aligned} A_\iota &:= \{u \in S_c : \|\nabla u\|_2 < \iota\}, \\ I^k &:= \{u \in S_c : I(u) < k\}. \end{aligned}$$

By Lemmas 3.3 and 3.4, there exists a $\iota_1 > 0$ small enough, such that

$$I(u) < \frac{1}{2}\varphi(t_2), \text{ for any } u \in A_{\iota_1}.$$

Moreover, $I^{\varphi(t_1)} \subset \{\|\nabla u\|_2 > R_2\}$ since $I(u) \geq \varphi(\|\nabla u\|_2)$. Now we can get a mountain pass structure of I on manifold S_c.

$$\Gamma := \{\gamma \in C([0,1], S_c) : \gamma(0) \in A_\iota, \gamma(1) \in I^{\varphi(t_1)}\}, \tag{3.9}$$

and the mountain pass value is

$$m_{h,c} := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I(\gamma(t)). \tag{3.10}$$

Remark 3.1.

$$I_\infty(v_c) = m_c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} I_\infty(\gamma(t))$$

where v_c satisfies

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^N} |\nabla v_c|^2 dx\right)\Delta v_c + \lambda v_c = |v_c|^{p-2}v_c \quad \text{ in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |v_c|^2 dx = c, \quad u \in H^1(\mathbb{R}^N), \end{cases}$$

i.e., the solution v_c of the problem (1.6) is a mountain pass critical point of I_∞ constrained on S_c. (see [35]). It is immediately seen that

$$m_{h,c} < m_c. \tag{3.11}$$

Lemma 3.5. Under the assumption ($\mathbf{h_2}$), suppose that h satisfies (1.14), then there exists a (PS) sequence $\{u_n\}$ of $I|_{S_c}$, which satisfies

$$I(u_n) \to m_{h,c}, \tag{3.12}$$

$$I' |_{S_c} (u_n) \to 0, \tag{3.13}$$

$$P(u_n) \to 0, \tag{3.14}$$

as $n \to \infty$, where

$$P(u) = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \gamma_p \int_{\mathbb{R}^N} |u|^p dx - \gamma_q \int_{\mathbb{R}^N} h|u|^q dx + \frac{1}{q}\int_{\mathbb{R}^N} \langle \nabla h, x\rangle |u|^q dx,$$

and

$$\lim_{n\to\infty} \|(u_n)^-\| = 0. \tag{3.15}$$

We remark that (3.13) means that there exists $\{\lambda_n\}_{n\geq 1}$, such that for any $\varphi \in C_0^\infty(\mathbb{R}^N)$, there holds

$$I'(u_n)[\varphi] + \lambda_n \int_{\mathbb{R}^N} u_n \varphi dx \to 0, \quad as \;\; n \to \infty. \tag{3.16}$$

Moreover, $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and the related Lagrange multipliers $\{\lambda_n\}$ in (3.16) are also bounded, up to a subsequence, $\lambda_n \to \bar{\lambda}$, with $\bar{\lambda} > 0$.

Proof. We divide the proof into three steps.

Step 1: Existence of the Palais-Smale sequence. The existence of the (PS) sequence that verifies (3.14) and (3.15) closely follows the arguments in [32], where the authors adapt some ideas from [11]. We recall the main strategy, referring to [32] for the details. A key tool is to set

$$\tilde{I}(t,u) := I(t \star u) \quad \text{for all } (t,u) \in \mathbb{R} \times H^1(\mathbb{R}^N).$$

The corresponding minimax structure of $\tilde{I}$ on $\mathbb{R} \times S_c$, as follows

$$\tilde{\Gamma} := \{\gamma = (\gamma_1, \gamma_2) \in C([0,1], \mathbb{R} \times S_c) : \gamma(0) \in (0, A_{\iota_1}), \gamma(1) \in (0, I^{\varphi(t_1)})\}, \tag{3.17}$$

and its minimax value is

$$\tilde{m}_{h,c} := \inf_{\gamma \in \tilde{\Gamma}} \max_{t \in [0,1]} \tilde{I}(\gamma(t)). \tag{3.18}$$

It turns out that $\tilde{m}_{h,c} = m_{h,c}$ and that, if $(t_n, v_n)_n$ is a $(PS)_c$ sequence for $\widetilde{I}$ with $t_n \to 0$, then $u_n = t_n \star v_n$ is a $(PS)_c$ sequence for I. Now, let us consider a sequence $\xi_n \in \Gamma$ such that

$$m_{h,c} \leq \max_{t \in [0,1]} I(\xi_n(t)) < m_{h,c} + \frac{1}{n}.$$

We observe that, since $I(u) = I(|u|)$ for every $u \in H^1(\mathbb{R}^N)$, we can take $\xi_n(t) \geq 0$ in $\mathbb{R}^N$, for every $t \in [0,1]$ and $n \in \mathbb{N}$. We are in a position to apply Lemma 3.2 to $\tilde{I}$ with

$$X := \mathbb{R} \times S_c, \quad K := \{(0, A_{\iota_1}), (0, I^{\varphi(t_1)})\}, \quad \mathcal{E} = \widetilde{\Gamma}, \quad E_n := \{(0, \xi_n(t)) : t \in [0,1]\}.$$

As a consequence, there exists a sequence $(t_n, v_n) \in \mathbb{R} \times S_c$ and $\tilde{c} > 0$ such that

$$\begin{aligned} m_{h,c} - \frac{1}{n} < \tilde{I}(t_n, v_n) &< m_{h,c} + \frac{1}{n}, \\ \min_{t \in [0,1]} \|(t_n, v_n) - (0, \xi_n(t))\|_{\mathbb{R} \times H^1(\mathbb{R}^N)} &< \frac{\tilde{c}}{\sqrt{n}}, \\ \|\nabla_{\mathbb{R} \times S_c} \widetilde{I}(t_n, v_n)\| &< \frac{\tilde{c}}{\sqrt{n}}. \end{aligned} \tag{3.19}$$

Now, we can define

$$u_n = t_n \star v_n.$$

We observe that, by differentiating $\tilde{I}$ with respect to t, we get the "almost" Pohozaev identity (3.14), differentiating with respect to the second variable on the tangent space to S_c, and by (3.19) and $\xi_n(t) \geq 0$ we get (3.15).

Step 2: Boundedness of the (PS) sequence.

By (3.12), for the (PS) sequence $\{u_n\} \subset S_c$, there holds

$$\begin{aligned} m_{h,c} &= I(u_n) + o(1) \\ &= \frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \frac{1}{p}\int_{\mathbb{R}^N} |u_n|^p dx - \frac{1}{q}\int_{\mathbb{R}^N} h|u_n|^q dx + o(1). \end{aligned} \tag{3.20}$$

Combining with (3.14),

$$\begin{aligned} m_{h,c} =&\frac{a(N(p-2)-4)}{2N(p-2)}\|\nabla u_n\|_2^2 + \frac{b(N(p-2)-8)}{4N(p-2)}\|\nabla u_n\|_2^4 - \frac{p-q}{q(p-2)}\int_{\mathbb{R}^N} h|u_n|^q dx \\ &- \frac{2}{qN(p-2)}\int_{\mathbb{R}^N}\langle \nabla h, x\rangle |u_n|^q dx + o(1) \\ \geq&\frac{a(N(p-2)-4)}{2N(p-2)}\|\nabla u_n\|_2^2 - \frac{p-q}{q(p-2)}\int_{\mathbb{R}^N} h|u_n|^q dx \\ &- \frac{2}{qN(p-2)}\int_{\mathbb{R}^N}\langle \nabla h, x\rangle |u_n|^q dx + o(1) \\ \geq&\frac{a(N(p-2)-4)}{2N(p-2)}\|\nabla u_n\|_2^2 - \frac{p-q}{q(p-2)}C_{N,p}^q c^{\frac{q(1-\gamma_p)}{2}}\|h\|_{\frac{p}{p-q}}\|\nabla u_n\|_2^{q\gamma_p} \\ &- \frac{2}{qN(p-2)}\|\nabla h\cdot x\|_{\frac{2}{2-q}} c^{\frac{q}{2}} + o(1). \end{aligned} \tag{3.21}$$

Thus $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ since $h \in L^{\frac{p}{p-q}}(\mathbb{R}^N)$ and $\|\nabla h \cdot x\|_{\frac{2}{2-q}} < \infty$.

Step3: Positivity of the Lagrange multiplier.

By taking u_n as a test function for (3.16), we obtain that

$$o(1)\|u_n\|_{H^1} = a\|\nabla u_n\|_2^2 + b\|\nabla u_n\|_2^4 - \|u_n\|_p^p - \int_{\mathbb{R}^N} h|u_n|^q + \lambda_n c.$$

So

$$|\lambda_n| = \frac{1}{c}\left| o(1)\|u_n\|_{H^1} - a\|\nabla u_n\|_2^2 - b\|\nabla u_n\|_2^4 + \|u_n\|_p^p + \int_{\mathbb{R}^N} h|u_n|^q \right| < +\infty.$$

Thus the Lagrange multipliers $\{\lambda_n\}$ are also bounded. Next, we show that $\{\lambda_n\}$ has a positive lower bound. In fact, according to (3.14) and (3.16),

$$\begin{aligned} \lambda_n c &= \lambda_n \int_{\mathbb{R}^N} |u_n|^2 dx \\ &= -a\|\nabla u_n\|_2^2 - b\|\nabla u_n\|_2^4 + \|u_n\|_p^p + \int_{\mathbb{R}^N} h|u_n|^q dx + o(1) \\ &= (1-\gamma_p)\|u_n\|_p^p + (1-\gamma_q)\int_{\mathbb{R}^N} h|u_n|^q dx + \frac{1}{q}\int_{\mathbb{R}^N}\langle \nabla h, x\rangle |u_n|^q dx + o(1). \end{aligned} \tag{3.22}$$

We also have that

$$\begin{aligned} m_{h,c} &= \frac{a}{2}\|\nabla u_n\|_2^2 + \frac{b}{4}\|\nabla u_n\|_2^4 - \frac{1}{p}\|u_n\|_p^p - \frac{1}{q}\int_{\mathbb{R}^N} h|u_n|^q dx + o(1) \\ &= -\frac{b}{4}\|\nabla u_n\|_2^4 + \frac{N(p-2)-4}{4p}\|u_n\|_p^p \\ &\quad + \frac{N(q-2)-4}{4q}\int_{\mathbb{R}^N} h|u_n|^q dx - \frac{1}{2q}\int_{\mathbb{R}^N}\langle \nabla h, x\rangle |u_n|^q dx + o(1). \end{aligned} \tag{3.23}$$

Then, combined with the assumption (1.14), we have that

$$\begin{aligned}
&\lambda_n c + o(1)\\
=&\frac{4p(1-\gamma_p)}{N(p-2)-4}m_{h,c} + \frac{bp(1-\gamma_p)}{N(p-2)-4}\|\nabla u_n\|_2^4 + \frac{2p-4}{q(N(p-2)-4)}\int_{\mathbb{R}^N}\langle\nabla h, x\rangle|u_n|^q dx\\
&+\left(\frac{2q-N(q-2)}{2q} + \frac{(2p-N(p-2))(4-N(q-2))}{2q(N(p-2)-4)}\right)\int_{\mathbb{R}^N} h|u_n|^q dx + o(1)\\
\geq&\frac{4p(1-\gamma_p)}{N(p-2)-4}m_{h,c} - \frac{2p-4}{q(N(p-2)-4)}\|\nabla h\cdot x\|_{\frac{2}{2-q}} c^{\frac{q}{2}} + o(1)
\end{aligned} \tag{3.24}$$

since

$$\|\nabla h\cdot x\|_{\frac{2}{2-q}} < \frac{q(2p-Np+2N)}{p-2}m_c c^{-\frac{q}{2}}.$$

Now we prove the convergence of the (PS) sequence $\{u_n\}$ and hence we complete the proof of Theorem 1.2.

Proof of Theorem 1.2. Next, we prove the existence of solutions of (1.1) with a positive energy level when $2+\frac{8}{N} < p < 2^*$. We consider the bounded (PS) sequence $\{u_n\}$ given by Lemma 3.5. Then, there exists $u \in H^1(\mathbb{R}^N)$ such that $u_n \rightharpoonup u$ due to the boundedness of $\{u_n\}$. We claim that $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$.

For any $\psi \in H^1(\mathbb{R}^N)$, $\{u_n\}$ satisfies

$$\begin{aligned}
a\int_{\mathbb{R}^N}\nabla u_n\nabla\psi dx + b\left(\int_{\mathbb{R}^N}|\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N}\nabla u_n\nabla\psi dx&\\
-\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\psi dx - \int_{\mathbb{R}^N}h(x)|u_n|^{q-2}u_n\psi dx&\\
= -\lambda_n\int_{\mathbb{R}^N}u_n\psi dx + o(1)\|\psi\|.&
\end{aligned}$$

Using the boundedness of $\{\lambda_n\}$ again, we obtain that

$$\begin{aligned}
a\int_{\mathbb{R}^N}\nabla u_n\nabla\psi dx + b\left(\int_{\mathbb{R}^N}|\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N}\nabla u_n\nabla\psi dx&\\
-\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\psi dx - \int_{\mathbb{R}^N}h(x)|u_n|^{q-2}u_n\psi dx&\\
= -\bar\lambda\int_{\mathbb{R}^N}u_n\psi dx + (\bar\lambda-\lambda_n)\int_{\mathbb{R}^N}u_n\psi dx + o(1)\|\psi\|.&
\end{aligned}$$

And hence

$$\begin{aligned}
&a\int_{\mathbb{R}^N}\nabla u_n\nabla\psi dx + b\left(\int_{\mathbb{R}^N}|\nabla u_n|^2 dx\right)\int_{\mathbb{R}^N}\nabla u_n\nabla\psi dx\\
&-\int_{\mathbb{R}^N}|u_n|^{p-2}u_n\psi dx - \int_{\mathbb{R}^N}h(x)|u_n|^{q-2}u_n\psi dx\\
&= -\bar\lambda\int_{\mathbb{R}^N}u_n\psi dx,
\end{aligned}$$

which implies that $\{u_n\}$ is a (PS) sequence for I_λ at level $m_{h,c} + \frac{\lambda}{2}c$, so that we can apply the Splitting Lemma 3.1, getting

$$u_n = u + \Sigma_{j=1}^{k} \omega^j(\cdot - y_n^j) + o(1).$$

Assume by contradiction that $k \geq 1$, or, equivalently, that $\|u\|_2^2 < c$. In addition, if $0 < \alpha < \beta$, then $m_\alpha > m_\beta$ and $J_{\infty,0}(\omega^j) \geq m_{\alpha_j}$ (see [28]). Therefore,

$$m_{h,c} + \frac{\lambda}{2}c = J_{h,0}(u) + \frac{\lambda}{2}\beta + \Sigma_{j=1}^{k} J_{\infty,0}(\omega^j) + \frac{\lambda}{2}\Sigma_{j=1}^{k}\alpha_j, \tag{3.25}$$

where $\beta := \|u\|_2^2$, $\alpha_j := \|\omega^j\|_2^2$. By (3.4), we have

$$c = \beta + \Sigma_{j=1}^{k}\alpha_j.$$

Thus, combined with (3.25), we obtain

$$m_{h,c} = J_{h,0}(u) + \Sigma_{j=1}^{k} J_{\infty,0}(\omega^j). \tag{3.26}$$

Since $J_{h,0}(u), J_{\infty,0}(\omega^j) \geq m_c$, we have $m_{h,c} \geq m_c$, which is a contradiction of (3.11). Thus $k = 0$. That is $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$ and u is a solution of Eq.(1.1). □

4. Proof of Theorem 1.3

In this section, we assume that $2 + \frac{8}{N} < p < 2^*$, $1 \leq N \leq 3$, $\bar{h}(x) = -h(x) \geq 0$, and $\bar{h}(x) \not\equiv 0$. By using a min-max argument, we can find the existence of normalized solutions of Eq.(1.1). First, we show that the energy functional corresponding to Eq.(1.1) has a linking geometry. For $s \in \mathbb{R}$ and $u \in H^1(\mathbb{R}^N)$, we introduce the scaling

$$s \star u(x) := e^{\frac{N}{2}s} u(e^s x),$$

which preserves the L^2-norm: $\|s * u\|_2 = \|u\|_2$ for all $s \in \mathbb{R}$. For $\mathbb{R} > 0$ and $s_1 < 0 < s_2$, which will be determined later, we set

$$Q := B_R \times [s_1, s_2] \subset \mathbb{R}^N \times \mathbb{R}$$

where $B_R = \{x \in \mathbb{R}^N : |x| \leq R\}$ is the closed ball of radius R around 0 in $\mathbb{R}^N$. For $c > 0$, define

$$\Gamma_c := \{\gamma : Q \to S_c \mid \gamma \in C(\mathbb{R}^N), \gamma(y, s) = s \star v_c(\cdot - y) \text{ for all } (y, s) \in \partial Q\},$$

where v_c satisfies

$$\begin{cases} -\left(a + b\displaystyle\int_{\mathbb{R}^N} |\nabla v_c|^2 dx\right)\Delta v_c + \lambda v_c = |v_c|^{p-2} v_c \quad \text{in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |v_c|^2 dx = c, \quad u \in H^1(\mathbb{R}^N). \end{cases}$$

We define

$$L_{h,c} := \inf_{\gamma \in \Gamma_c} \max_{(y,s)\in Q} I(\gamma(y, s)).$$

To prove that the energy functional I has a linking geometry, it is necessary to find the suitable $R > 0$, $s_1 < 0 < s_2$ such that

$$\sup_{\gamma\in\Gamma_c} \max_{(y,s)\in\partial Q} I(\gamma(y, s)) < L_{h,c}$$

at least for some suitable choice of Q. Now, we recall the notion of barycenter of a function $u \in H^1(\mathbb{R}^N)\backslash\{0\}$, which has been introduced in [36] and in [37]. Setting

$$\nu(u)(x) = \frac{1}{|B_1(0)|}\int_{B_1(x)} |u(y)|dy,$$

we observe that $\nu(u)$ is bounded and continuous, so the function

$$\hat{u}(x) = \left[\nu(u)(x) - \frac{1}{2}\max \nu(u)\right]^+$$

is well defined, continuous, and has compact support. Therefore, we can define $\beta : H^1(\mathbb{R}^N)\backslash\{0\} \to \mathbb{R}^N$ as

$$\beta(u) = \frac{1}{||\hat{u}||_1}\int_{\mathbb{R}^N} \hat{u}(x)xdx.$$

The map β is well defined, because $\hat{u}$ has compact support, and it is not difficult to verify that it enjoys the following properties:
(i) β is continuous in $H^1(\mathbb{R}^N)\backslash\{0\}$;
(ii) if u is a radial function, then $\beta(u) = 0$;
(iii) $\beta(tu) = \beta(u)$ for all $t \neq 0$ and for all $u \in H^1(\mathbb{R}^N)\backslash\{0\}$;
(iv) setting $u_z(x) = u(x - z)$ for $z \in \mathbb{R}^N$ and $u \in H^1(\mathbb{R}^N)\backslash\{0\}$ there holds $\beta(u_z) = \beta(u) + z$.
Now, we define

$$\begin{aligned}
\mathcal{D} &:= \{D \subset S_c : D \text{ is compact, connected, } s_1 \star v_c, s_2 \star v_c \in D\},\\
\mathcal{D}_0 &:= \{D \in \mathcal{D} : \beta(u) = 0 \text{ for all } u \in D\},\\
\mathcal{D}_r &:= \mathcal{D} \cap H^1_{\text{rad}}(\mathbb{R}^N),
\end{aligned}$$

and

$$\begin{aligned}
w_c^r &:= \inf_{D\in\mathcal{D}_r}\max_{u\in D} I_\infty(u),\\
w_c^0 &:= \inf_{D\in\mathcal{D}_0}\max_{u\in D} I_\infty(u),\\
w_c &:= \inf_{D\in\mathcal{D}}\max_{u\in D} I_\infty(u).
\end{aligned}$$

It has been proved in [28] that

$$m_c = \inf_{\sigma\in\Sigma_c}\max_{t\in[0,1]} I_\infty(\sigma(t))$$

where

$$\Sigma_c = t\{\sigma \in C([0,1], S_c) : \sigma(0) = s_1 \star v_c, \sigma(1) = s_2 \star v_c\}.$$

Lemma 4.1. $w_c^r = w_c^0 = w_c = m_c$.

Proof. Clearly $\mathcal{D}_r \subset \mathcal{D}_0 \subset \mathcal{D}$, so that $w_c^r \geq w_c^0 \geq w_c$. It remains to prove that $w_c \geq m_c$ and $m_c \geq w_c^r$.

Arguing by contradiction, we assume that $m_c > w_c$. Then, $\max_{u\in D} I_\infty(u) < m_c$ for some $D \in \mathcal{D}$, hence $\sup_{u\in U_\delta(D)} I_\infty(u) < m_c$ for some $\delta > 0$, here $U_\delta(D)$ is the δ-neighborhood of D. Observe that $U_\delta(D)$ is open and connected, so it is path-connected. Therefore, there exists a path $\sigma \in \Sigma_c$ such that $\max_{t\in[0,1]} I_\infty(\sigma(t)) < m_c$, a contradiction.

The inequality $m_c \geq w_c^r$ follows from the fact that the set $D := \{s \star v_c : s \in [s_1, s_2]\} \in \mathcal{D}_r$ satisfies

$$\max_{u \in D} I_\infty(u) = \max_{s \in [s_1, s_2]} I_\infty(s \star v_c) = m_c. \quad \square$$

Lemma 4.2. $L_c := \inf_{D \in \mathcal{D}_0} \max_{u \in D} I(u) > m_c$.
Proof. Using $\bar{h}(x) \geq 0$ and Lemma 4.1, we have

$$\max_{u \in D} I(u) \geq \max_{u \in D} I_\infty(u) \geq w_c^0 = m_c, \quad \text{for all } D \in \mathcal{D}_0. \tag{4.1}$$

Now, we argue by contradiction and assume that there exists a sequence $D_n \in \mathcal{D}_0$ such that

$$\max_{u \in D_n} I(u) \to m_c.$$

In view of (4.1), we also have

$$\max_{u \in D_n} I_\infty(u) \to m_c.$$

Adapting an argument from [11, Lemma 2.4], we consider the functional

$$\tilde{I}_\infty : H^1(\mathbb{R}^N) \times \mathbb{R} \to \mathbb{R}, \quad \tilde{I}_\infty(u, s) := I_\infty(s \star u)$$

constrained to $M := S_c \times \mathbb{R}$. We apply Lemma 3.2 with

$$K := \{(s_1 \star v_c, 0), (s_2 \star v_c, 0)\}$$

and

$$C := \{C \subset M : C \text{compact, connected, } K \subset C\}.$$

Observe that

$$\tilde{w}_c := \inf_{C \in C} \max_{(u,s) \in C} \tilde{I}_\infty(u, s) = w_c = m_c$$

because $\mathcal{D} \times \{0\} \subset C$, hence $w_c \geq \tilde{w}_c$, and for any $C \in C$ we have $D := \{s \star u : (u, s) \in C\} \in \mathcal{D}$ and

$$\max_{(u,s) \in C} \tilde{I}_\infty(u, s) = \max_{(u,s) \in C} I_\infty(s \star u) = \max_{v \in D} I_\infty(v),$$

hence $w_c \leq \tilde{w}_c$. Hence, Lemma 3.2 yields a sequence $(u_n, s_n) \in S_c \times \mathbb{R}$ such that
(1) $|\tilde{I}_\infty(u_n, s_n) - m_c| \to 0$ as $n \to \infty$;
(2) $\|\nabla_{S_c \times \mathbb{R}} \tilde{I}_\infty(u_n, s_n)\| \to 0$ as $n \to \infty$;
(3) $\operatorname{dist}((u_n, s_n), D_n \times \{0\}) \to 0$ as $n \to \infty$.
Then $v_n := s_n \star u_n \in S_c$ is a (PS) sequence for I_∞ on S_c at m_c, and there exists Lagrange multipliers $\lambda_n \in \mathbb{R}$ such that

$$\begin{aligned}
&I_\infty(v_n) \to m_c, \\
&a\|\nabla v_n\|_2^2 + b\|\nabla v_n\|_2^4 - \frac{N(p-2)}{2p}\|v_n\|_p^p \to 0, \\
&\|I'_\infty(v_n) + \lambda_n G'(v_n)\|_{(H^1(\mathbb{R}^N))^*} \to 0, \quad \text{where } G(u) = \frac{1}{2}\int_{\mathbb{R}^N} u^2 dx,
\end{aligned}$$

as $n \to \infty$. So, combining those properties, we can infer that

$$\frac{N(p-2)-4}{2N(p-2)}a\|\nabla v_n\|_2^2 + \frac{N(p-2)-8}{4N(p-2)}b\|\nabla v_n\|_2^4 \to m_c > 0, \text{ as } n \to \infty,$$

and

$$\begin{aligned}-\lambda_n c &= a\|\nabla v_n\|_2^2 + b\|\nabla v_n\|_2^4 - \|v_n\|_p^p \\ &= \frac{N(p-2)-2p}{2p}\|v_n\|_p^p = \frac{N(p-2)-2p}{N(p-2)}(a\|\nabla v_n\|_2^2 + b\|\nabla v_n\|_2^4).\end{aligned}$$

Therefore, $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and $\{\lambda_n\}$ is bounded in $\mathbb{R}$. We may assume that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^N)$, $\|\nabla v_n\|_2^2 \to A^2$, and $\lambda_n \to \lambda > 0$. In fact, $\{v_n\}$ is a (PS) sequence for $I_{\infty,\lambda}$ at level $m_c + \frac{\lambda}{2}c$. As a consequence of Lemma 3.1, v_n can be rewritten as

$$v_n = v + \sum_{j=1}^{k} w^j(\cdot - y_n^j) + o(1)$$

in $H^1(\mathbb{R}^N)$, where $k \geq 0$ and $w^j \neq 0$, v are solutions to

$$-(a + bA^2)\Delta w + \lambda w = |w|^{p-2}w$$

and $|y_n^j| \to \infty$. Moreover, we get

$$c = \|v\|_2^2 + \sum_{j=1}^{k} \|w^j\|_2^2 + o(1), \tag{4.2}$$

$$A^2 = \|\nabla v\|_2^2 + \sum_{j=1}^{k} \|\nabla w^j\|_2^2, \tag{4.3}$$

$$I_{\infty,\lambda}(v_n) \to J_{\infty,\lambda}(v) + \sum_{j=1}^{k} J_{\infty,\lambda}(w^j),$$

and hence,

$$m_c + \frac{\lambda}{2}c = J_{\infty,0}(v) + \frac{\lambda}{2}\|v\|_2^2 + \sum_{j=1}^{k} J_{\infty,0}(w^j) + \frac{\lambda}{2}\sum_{j=1}^{k}\|w^j\|_2^2 + o(1).$$

By (4.2), we have

$$m_c = J_{\infty,0}(v) + \sum_{j=1}^{k} J_{\infty,0}(w^j) + o(1).$$

If $v \neq 0$ and $k \geq 1$, we get $A^2 > \|\nabla v\|_2^2$ from (4.3), we have

$$\begin{aligned}J_{\infty,0}(v) &= \left(\frac{a}{2} + \frac{bA^2}{4}\right)\int_{\mathbb{R}^N} |\nabla v|^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |v|^p dx. \\ &> \frac{a}{2}\int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla v|^2 dx\right)^2 dx - \frac{1}{p}\int_{\mathbb{R}^N} |v|^p dx \\ &= I_\infty(v) \\ &\geq m_{\|v\|_2^2} \geq m_c.\end{aligned}$$

Similarly, we have $J_{\infty,0}(w^j) \geq m_c$. Thus,

$$m_c + o(1) \geq (k+1)m_c + o(1),$$

we get a contradiction. Therefore, $k = 1$ and $v = 0$, or $k = 0$ and $v \neq 0$. If $k = 1$ and $v = 0$, then $v_n(\cdot + y_n^1) + o(1) = w^1$. On the other hand, due to point (3) that $\mathrm{dist}((u_n, s_n), D_n \times \{0\}) \to 0$, we obtain

$$\beta(w^1) = \beta(v_n(\cdot + y_n^1)) + o(1) = y_n^1 + o(1),$$

which contradicts the fact that β is continuous and $|y_n^1| \to \infty$.

If $k = 0$ and $v \neq 0$, then $v_n \to v$ in $H^1(\mathbb{R}^N)$. Using again point (3), we also have $\beta(v) = 0$. Hence, by the uniqueness, $v_n \to \pm v_c$ in $H^1(\mathbb{R}^N)$. This implies

$$I(v_n) = I_\infty(v_n) + \frac{1}{q}\int_{\mathbb{R}^N} \bar{h}(x)|v_n|^q dx \to m_c + \frac{1}{q}\int_{\mathbb{R}^N} \bar{h}(x)|v_c|^q dx > m_c,$$

which is a contradiction. □

Lemma 4.3. For any $c > 0$, then $L_{h,c} \geq L_c$ holds.
Proof. Similar to [32, Proposition 3.5], so we omit it. □

Lemma 4.4. For any $c > 0$ and for any $\varepsilon > 0$, there exists $\bar{R} > 0$ and $\bar{s}_1 < 0 < \bar{s}_2$ such that for $Q = B_R \times [s_1, s_2]$ with $R \geq \bar{R}$, $s_1 \leq \bar{s}_1$, $s_2 \geq \bar{s}_2$ the following holds:

$$\max_{(y,s)\in\partial Q} I(s \star v_c(\cdot - y)) < m_c + \varepsilon.$$

Proof. We have

$$I(s \star v_c(\cdot - y)) = I_\infty(s \star v_c) + \frac{e^{\frac{qsN}{2}}}{q}\int_{\mathbb{R}^N} \bar{h}(x) v_c(e^s(x-y))^q dx$$

and

$$\begin{aligned} I_\infty(s \star v_c) &= \frac{e^{2s}}{2}\int_{\mathbb{R}^N} |\nabla v_c|^2 dx + \frac{e^{4s}}{4}\left(\int_{\mathbb{R}^N} |\nabla v_c|^2 dx\right)^2 - \frac{e^{\frac{N}{2}(p-2)s}}{p}\int_{\mathbb{R}^N} |v_c|^p dx \\ &= \begin{cases} O(-e^{\frac{N}{2}(p-2)s}) \to -\infty & \text{as } s \to \infty, \\ O(e^{2s}) \to 0 & \text{as } s \to -\infty. \end{cases} \end{aligned}$$

Moreover, there holds

$$\begin{aligned} \frac{e^{\frac{qsN}{2}}}{q}\int_{\mathbb{R}^N} \bar{h}(x) v_c^q(e^s(x-y))dx &\leq \frac{e^{\frac{qsN}{2}}}{q}\left(\int_{\mathbb{R}^N} \bar{h}^{\frac{2}{2-q}} dx\right)^{\frac{2-q}{2}}\left(\int_{\mathbb{R}^N} v_c^2(e^s(x-y))dx\right)^{\frac{q}{2}} \\ &= \frac{1}{q}\|\bar{h}\|_{\frac{2}{2-q}} c^{\frac{q}{2}} \end{aligned}$$

because $\bar{h}(x)$ satisfies (1.15), thus for all $s \in \mathbb{R}$, we have

$$\frac{e^{\frac{qsN}{2}}}{q}\int_{\mathbb{R}^N} \bar{h}(x) v_c^q(e^s(x-y))dx < m_c.$$

As a consequence, we deduce

$$\max_{y\in B_R, s\in\{s_1,s_2\}} I(s\star v_c(\cdot-y)) < m_c + +o(1)$$

provided $s_1<0$ is small enough and $s_2>0$ is large enough. Moreover, for $|y|=R$ large enough and $s\in[s_1,s_2]$, we choose $\alpha\in(0,1)$ such that $\alpha(1+e^{-s_1})<1$, so that we have

$$\begin{aligned}
&\frac{e^{\frac{qsN}{2}}}{q}\int_{\mathbb{R}^N}\bar h(x)v_c^q(e^s(x-y))dx\\
&\le \frac{e^{\frac{qsN}{2}}}{q}\int_{|x|>\alpha R}\bar h(x)v_c^q(e^s(x-y))dx+\frac{e^{\frac{qsN}{2}}}{q}\int_{|x-y|>\alpha Re^{-s}}\bar h(x)v_c^q(e^s(x-y))dx.
\end{aligned}$$

The first integral is bounded by

$$\begin{aligned}
\frac{e^{\frac{qsN}{2}}}{q}\int_{|x|>\alpha R}\bar h(x)v_c^q(e^s(x-y))dx &\le \frac{e^{\frac{qsN}{2}}}{q}\left(\int_{|x|>\alpha R}\bar h^{\frac{2}{2-q}}dx\right)^{\frac{2-q}{2}}\left(\int_{|x|>\alpha R}v_c^2(e^s(x-y))dx\right)^{\frac{q}{2}}\\
&\le \frac{1}{q}\left(\int_{|x|>\alpha R}\bar h^{\frac{2}{2-q}}dx\right)^{\frac{2-q}{2}}\left(\int_{\mathbb{R}^N}v_c^2dx\right)^{\frac{q}{2}}\to 0
\end{aligned}$$

as $R\to\infty$ and

$$\begin{aligned}
\frac{e^{\frac{qsN}{2}}}{q}\int_{|x-y|>\alpha Re^{-s}}\bar h(x)v_c^q(e^s(x-y))dx &\le \frac{1}{q}\left(\int_{|x-y|>\alpha Re^{-s}}\bar h^{\frac{2}{2-q}}dx\right)^{\frac{2-q}{2}}\left(\int_{|\xi|>\alpha R}v_c^2(\xi)d\xi\right)^{\frac{q}{2}}\\
&\le \frac{1}{q}\left(\int_{\mathbb{R}^N}\bar h^{\frac{2}{2-q}}dx\right)^{\frac{2-q}{2}}\left(\int_{|\xi|>\alpha R}v_c^2dx\right)^{\frac{q}{2}}\to 0
\end{aligned}$$

as $R\to\infty$, which concludes the proof. □

By Lemma 4.3 and 4.4, we may choose $R>0$ and $s_1<0<s_2$ such that

$$\max_{(y,s)\in\partial Q} I(s\star v_c(\cdot-y)) < L_{h,c}.$$

Therefore, I has a linking geometry and there exists a (PS) sequence at the level $L_{h,c}$. In order to estimate $L_{h,c}$, we have the following Lemma.

Lemma 4.5. If $|s_1|, s_2$ are large enough, then

$$L_{h,c} < 2m_c.$$

Proof. This follows from

$$\begin{aligned}
L_{h,c} &\le \max_{(y,s)\in Q}\left\{I_\infty(s\star v_c(\cdot-y))+\frac{1}{q}\int_{\mathbb{R}^N}\bar h(x)(s\star v_c)^q(x-y)dx\right\}\\
&\le m_c+\frac{1}{q}|\bar h|_{\frac{2}{2-q}}c^{\frac{q}{2}}\\
&< 2m_c
\end{aligned}$$

provided $|s_1|, s_2$ are large enough. □

By the Lemma 4.3 and Lemma 4.5, we can get

$$m_c < L_{h,c} < 2m_c.$$

Next, we construct a bounded (PS) sequence of I at $L_{h,c}$ by adopting the approach from [11] and Lemma 3.2. We define a auxiliary C^1 functional

$$\tilde{I}(u,s) := I(s \star u) \text{ for all } (u,s) \in H^1(\mathbb{R}^N) \times \mathbb{R},$$

$$\tilde{\Gamma}_c := \{\tilde{\gamma} : Q \to S_c \mid \tilde{\gamma} \in C(\mathbb{R}^N), \tilde{\gamma}(y,s) = s \star v_c(\cdot - y) \text{ for all } (y,s) \in \partial Q\},$$

and

$$\tilde{L}_{h,c} := \inf_{\tilde{\gamma} \in \tilde{\Gamma}_c} \max_{(y,s)\in Q} \tilde{I}(\tilde{\gamma}(y,s)).$$

Lemma 4.6. (1) $\tilde{L}_{h,c} = L_{h,c}$.
(2) If (u_n, s_n) is a (PS) sequence for $\tilde{I}$ at level $\tilde{L}_{h,c}$ and $s_n \to 0$, then $(s_n \star u_n)_n$ is a (PS) sequence for I at level $L_{h,c}$.
Proof. The proof is similar to that of [11] and is omitted. □

Lemma 4.7. Let $\tilde{g}_n \in \tilde{\Gamma}_c$ be a sequence such that

$$\max_{(y,s)\in Q} \tilde{I}(\tilde{g}_n(y,s)) \leq L_{h,c} + \frac{1}{n}.$$

Then, there exists a sequence $(u_n, s_n) \in S_c \times \mathbb{R}$ and $\tilde{c} > 0$ such that

$$L_{h,c} - \frac{1}{n} \leq \tilde{I}(u_n, s_n) \leq L_{h,c} + \frac{1}{n}$$

$$\min_{(y,s)\in Q} \|(u_n, s_n) - \tilde{g}_n(y,s)\|_{H^1(\mathbb{R}^N)\times\mathbb{R}} \leq \frac{\tilde{c}}{\sqrt{n}}$$

$$\|\nabla_{S_c\times\mathbb{R}} \tilde{I}(u_n, s_n)\| \leq \frac{\tilde{c}}{\sqrt{n}}.$$

The last inequality means:

$$\left| D\tilde{I}(u_n, s_n)[(z,s)] \right| \leq \frac{\tilde{c}}{\sqrt{n}} (\|z\|_{H^1(\mathbb{R}^N)} + |s|)$$

for all

$$(z,s) \in \left\{ (z,s) \in H^1(\mathbb{R}^N) \times \mathbb{R} : \int_{\mathbb{R}^N} z u_n dx = 0 \right\}$$

Proof. Apply Lemma 3.2 to $\tilde{I}$ with

$$X := S_c \times \mathbb{R}, \quad K := \{(s \star v_c(\cdot - y), 0) : (y,s) \in \partial Q\}, \quad \mathcal{E} = \widetilde{\Gamma}_c, \quad E_n := \{\tilde{g}_n(y,s) : (y,s) \in Q\}.$$

Lemma 4.8. Under the assumption ($\mathbf{h_3}$), then there exists a bounded (PS) sequence $\{v_n\}$ of $I|_{S_c}$, which satisfies

$$I(v_n) \to L_{h,c}. \tag{4.4}$$

$$I' |_{S_c} (v_n) \to 0, \tag{4.5}$$

$$P(v_n) \to 0, \tag{4.6}$$

as $n \to \infty$, where

$$P(u) = a\|\nabla u\|_2^2 + b\|\nabla u\|_2^4 - \gamma_p \int_{\mathbb{R}^N} |u|^p dx + \gamma_q \int_{\mathbb{R}^N} \bar{h}|u|^q dx - \frac{1}{q}\int_{\mathbb{R}^N} \langle \nabla \bar{h}, x\rangle |u|^q dx,$$

$$\lim_{n\to\infty} \|(v_n)^-\| = 0. \tag{4.7}$$

Moreover, the sequence of Lagrange multipliers satisfies, up to subsequence $\lambda_n \to \lambda > 0$.

Proof. First, the existence of the (PS) sequence that verifies (4.6) and (4.7) closely follows the arguments in Lemma 3.5. The proof is omitted.

Next, we prove $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$. By (4.4), for the (PS) sequence $\{v_n\} \subset S_c$, there holds

$$\begin{aligned} L_{h,c} &= I(v_n) + o(1) \\ &= \frac{a}{2}\|\nabla v_n\|_2^2 + \frac{b}{4}\|\nabla v_n\|_2^4 - \frac{1}{p}\int_{\mathbb{R}^N} |v_n|^p dx + \frac{1}{q}\int_{\mathbb{R}^N} \bar{h}|v_n|^q dx + o(1). \end{aligned} \tag{4.8}$$

Combining with (4.6),

$$\begin{aligned} L_{h,c} =& \frac{a(N(p-2)-4)}{2N(p-2)}\|\nabla v_n\|_2^2 + \frac{b(N(p-2)-8)}{4N(p-2)}\|\nabla v_n\|_2^4 + \frac{p-q}{q(p-2)}\int_{\mathbb{R}^N} \bar{h}|v_n|^q dx \\ &+ \frac{2}{qN(p-2)}\int_{\mathbb{R}^N} \langle \nabla \bar{h}, x\rangle |v_n|^q dx + o(1) \\ \geq& \frac{a(N(p-2)-4)}{2N(p-2)}\|\nabla v_n\|_2^2 + \frac{2}{qN(p-2)}\int_{\mathbb{R}^N} \langle \nabla \bar{h}, x\rangle |v_n|^q dx + o(1) \\ \geq& \frac{a(N(p-2)-4)}{2N(p-2)}\|\nabla v_n\|_2^2 - \frac{2}{qN(p-2)}\|\nabla \bar{h}\cdot x\|_{\frac{2}{2-q}} c^{\frac{q}{2}} + o(1). \end{aligned} \tag{4.9}$$

Thus $\{v_n\}$ is bounded in $H^1(\mathbb{R}^N)$, since $\|\nabla \bar{h}\cdot x\|_{\frac{2}{2-q}} < \infty$.

Then, we prove the positivity of the Lagrange multiplier in the same way as lemma 3.5. By (4.5), we obtain that

$$|\lambda_n| = \frac{1}{c}\left| o(1)\|v_n\|_{H^1} - a\|\nabla v_n\|_2^2 - b\|\nabla v_n\|_2^4 + \|v_n\|_p^p - \int_{\mathbb{R}^N} \bar{h}|v_n|^q \right| < +\infty.$$

Thus, the Lagrange multipliers $\{\lambda_n\}$ are also bounded. In fact, according to (4.5) and (4.6), we have that

$$\begin{aligned}
&\lambda_n c + o(1)\\
&= \frac{4p(1-\gamma_p)}{N(p-2)-4}L_{h,c} + \frac{bp(1-\gamma_p)}{N(p-2)-4}\|\nabla v_n\|_2^4 - \frac{4(p-q)}{q(N(p-2)-4)}\int_{\mathbb{R}^N}\bar{h}v_n^q dx\\
&- \frac{2p-4}{q(N(p-2)-4)}\int_{\mathbb{R}^N}\langle\nabla\bar{h}, x\rangle v_n^q dx + o(1)\\
&\geq \frac{4p(1-\gamma_p)}{N(p-2)-4}m_c - \frac{4(p-q)}{q(N(p-2)-4)}\|\bar{h}\|_{\frac{2}{2-q}}c^{\frac{q}{2}} - \frac{2p-4}{q(N(p-2)-4)}\Upsilon\|\bar{h}\|_{\frac{2}{2-q}}c^{\frac{q}{2}}\\
&= \frac{2}{N(p-2)-4}\left(2p(1-\gamma_p)m_c - \frac{2(p-q)}{q}\|\bar{h}\|_{\frac{2}{2-q}}c^{\frac{q}{2}} - \frac{p-2}{q}\Upsilon\|\bar{h}\|_{\frac{2}{2-q}}c^{\frac{q}{2}}\right)
\end{aligned} \tag{4.10}$$

thus $\lambda > 0$ provided

$$\frac{2(p-q)}{q}\|\bar{h}\|_{\frac{2}{2-q}}c^{\frac{q}{2}} + \frac{p-2}{q}\Upsilon\|\bar{h}\|_{\frac{2}{2-q}}c^{\frac{q}{2}} < 2p(1-\gamma_p)m_c.$$

So

$$\|\bar{h}\|_{\frac{2}{2-q}} < \frac{2p(1-\gamma_p)}{2(p-q)+(p-2)\Upsilon}\cdot\frac{qm_c}{c^{\frac{q}{2}}},$$

which is given in (1.15). □

Proof of Theorem 1.3. Since $\{v_n\}$ is bounded, after passing to a subsequence it converges weakly in $H^1(\mathbb{R}^N)$ to $v \in H^1(\mathbb{R}^N)$. By (4.7) and weak convergence, v is a nonnegative weak solution of

$$-(a+bA^2)\triangle v + \lambda v + \bar{h}(x)|v|^{q-2}v = |v|^{p-2}v \tag{4.11}$$

such that $\beta := \|v\|_2^2 \leq c$, where $A^2 := \lim_{n\to\infty}\|\nabla v_n\|_2^2$. We note that $\{v_n\}$ is a bounded (PS) sequence of I_λ at level $L_{h,c} + \frac{\lambda}{2}c$, therefore, by Lemma 3.1, there exists an integer $k \geq 0$, k non-trivial solutions $w^1, w^2, \ldots, w^k$ to the equation

$$-(a+bA^2)\triangle w + \lambda w = |w|^{p-2}w$$

and k sequences $\{y_n^j\} \in H^1(\mathbb{R}^N)$, $1 \leq j \leq k$, such that $|y_n^j| \to \infty$ as $n \to \infty$.

Moreover, we have

$$\begin{aligned}
&v_n - \sum_{j=1}^k w^j(\cdot - y_n^j) \to v \ \text{ in } \ H^1(\mathbb{R}^N),\\
&\|v_n\|_2^2 \to \|v\|_2^2 + \sum_{j=1}^k \|w^j\|_2^2, \quad A^2 = \|\nabla v\|_2^2 + \sum_{j=1}^k \|\nabla w^j\|_2^2,
\end{aligned} \tag{4.12}$$

and

$$I_\lambda(v_n) \to J_{h,\lambda}(v) + \sum_{j=1}^k J_{\infty,\lambda}(w^j) \tag{4.13}$$

as $n \to \infty$. It remains to show $k = 0$, so that $v_n \to v$ strongly in $H^1(\mathbb{R}^N)$ and we are done. Thus, by contradiction, we can assume that $k \geq 1$, or equivalently $\beta < c$.

First, we exclude the case $v = 0$. In fact, if $v = 0$ and $k = 1$, we have $w^1 > 0$ and $\|w^1\|_2^2 = c$ and $\|\nabla w^1\|_2^2 = A^2$ so that (4.13) would give $L_{h,c} = m_c$, which is not possible due to Lemma 5.3. On the other hand, if $k \geq 2$, we get $J_{\infty,0}(w^j) \geq m_{\alpha_j} \left(\alpha_j := \|w^j\|_2^2\right)$ and $m_{\alpha_j} > m_c$, thus $L_{h,c} > 2m_c$, which contradicts with Lemma 4.5.

Therefore, from now on, we will assume $v \neq 0$ and $k \geq 1$. From (4.13) and $I(v_n) \to L_{h,c}$, we deduce

$$L_{h,c} + \frac{\lambda}{2}c = J_{h,0}(v) + \frac{\lambda}{2}\beta + \sum_{j=1}^{k} J_{\infty,0}(w^j) + \sum_{j=1}^{k} \frac{\lambda}{2}\alpha_j.$$

Using (4.12), we have

$$L_{h,c} = J_{h,0}(v) + \sum_{j=1}^{k} J_{\infty,0}(w^j).$$

Then, from $A^2 > \|\nabla v\|_2^2$ and $\bar{h}(x) \geq 0$, we have

$$\begin{aligned} J_{h,0}(v) &= \left(\frac{a}{2} + \frac{bA^2}{4}\right) \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{1}{q} \int_{\mathbb{R}^N} \bar{h}|v|^q dx - \frac{1}{p} \int_{\mathbb{R}^N} |v|^p dx. \\ &\geq \frac{a}{2} \int_{\mathbb{R}^N} |\nabla v|^2 dx + \frac{b}{4} \left(\int_{\mathbb{R}^N} |\nabla v|^2 dx\right)^2 dx - \frac{1}{p} \int_{\mathbb{R}^N} |v|^p dx \\ &= I_\infty(v) \\ &\geq m_{\|v\|_2^2} \geq m_c. \end{aligned}$$

Similarly, we have $J_{\infty,0}(w^j) \geq m_c$. Thus,

$$m_c + o(1) \geq (k+1)m_c + o(1),$$

we get a contradiction. Thus $k = 0$ and $\{v_n\}$ converges strongly to v in $H^1(\mathbb{R}^N)$. □

Author contributions

Xin Qiu: Writing-original draft, Writing-review & editing; Zeng Qi Ou: Supervision, Formal Analysis; Ying Lv: Writing-review & editing, Methodology, Supervision.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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