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Research article

No-go theorems for r-matrices in symplectic geometry

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Abstract: If a triangular Lie algebra acts on a smooth manifold, it induces a Poisson bracket on it. In case this Poisson structure is actually symplectic, we show that this already implies the existence of a flat connection on any vector bundle over the manifold the Lie algebra acts on, in particular the tangent bundle. This implies, among other things, that $\mathbb{C}P^n$ and higher genus Pretzel surfaces cannot carry symplectic structures that are induced by triangular Lie algebras.

Keywords: symplectic geometry; Lie algebras; Yang-Baxter equation
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1. Introduction

Formal deformation quantization was introduced in [1] in order to give a consistent definition of the term *quantization*. A formal deformation quantization is an *associative* formal deformation à la Gerstenhaber [2] of the commutative algebra of smooth functions on a Poisson manifold (M, π), such that the first order of the commutator coincides with the Poisson bracket. The deformed product is usually referred to as a *star product*. The question of existence of star products for generic Poisson manifolds has been answered positively by Kontsevich in [3], but nevertheless there is a lack of concrete formulas for star products, since Kontsevich's approach is based on globalizing already complicated local formulas in a non-trivial way, see [4]. Another option is to quantize via so-called *Drinfel'd twists* which are certain elements in $(\mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}))[[\hbar]]$, where $\mathcal{U}(\mathfrak{g})$ denotes the universal enveloping algebra of the (finite dimensional) Lie algebra $\mathfrak{g}$ which acts on M and induces a star product via this action, see the textbook [5] and references therein for more details. These twists can be of the same complexity as a star product, but can also in some sense be more accessible, and there are recursive formulas known, see [6]. Moreover, a Drinfel'd twist does not only deform the commutative product, but also the symmetries into a so called *quantum group*, see [5], and thus from many angles it is nice to have a star product which is induced by a twist. There is one downside to this line of action: not every Poisson manifold can be quantized in this way, see [7, 8] for certain no-go theorems. The main problem using this approach, is that the given Poisson structure has to be induced by an r-matrix, the semi-classical limit of the Drinfel'd twist, which is a certain element in $\Lambda^2\mathfrak{g}$, via the action of a Lie algebra. If a Poisson structure is induced by an r-matrix, we can always find a Drinfel'd twist with the pre-described r-matrix as a semi-classical limit, see [6, 9], and this twist then induces a quantization of the Poisson structure.

We are using quantization mainly as a motivation, and we will only concentrate on the semi-classical version: Poisson structures induced by r-matrices. We refer the reader to Section 2 for details. Note that this situation is also desirable if one is not interested in quantization, since a lot of computations in symplectic geometry can be performed directly on the Lie algebra level which is finite dimensional. Moreover, r-matrices on their own gained popularity in the field of integrable systems, as they are special cases of infinitesimal counterparts of Poisson-Lie groups, see [5]. Additionally, r-matrices themselves can be seen as left (or right) invariant Poisson structures on Lie groups, and as such they possess a symplectic foliation. It turns out that the symplectic leaf through the unit of the Lie group is a subgroup with an invariant symplectic structure, a so-called symplectic Lie group. They have been studied extensively (see [10] for an introduction to symplectic Lie groups) and even have been classified up to dimension 4 and partly up to dimension 6 (at least on the Lie algebra level, [11–13]), but unfortunately are rather rare. So, already from this point of view, the situation where the Poisson bracket is induced by an r-matrix seems to be rare as well. Nevertheless, we focus on exactly this situation.

This note is devoted to find obstructions to the situation that a given Poisson bracket is induced by an r-matrix extending the ones found in [7] and [8]. We will focus on the case where the Poisson structure is actually symplectic, where we can show that the tangent bundle of the symplectic manifold has to carry a flat connection as a special case of an equivariant vector bundle (Theorem 3.6). We exclude some well known examples of symplectic manifolds with our obstruction.

2. Preliminaries

This section is meant to fix the notation for the rest of this note and only describes rather standard techniques and is spelled out in detail, for example, in [6].

For a Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$, we extend the Lie bracket $[\cdot,\cdot]\colon \Lambda^\bullet\mathfrak{g}\times\Lambda^\bullet\mathfrak{g}\to\Lambda^{\bullet+\bullet-1}\mathfrak{g}$ by

$$[X_1\wedge\cdots\wedge X_k, Y_1\wedge\cdots\wedge Y_\ell] := \sum_{i,j}(-1)^{i+j}[X_i,Y_j]\wedge X_1\wedge\cdots\overset{i}{\wedge}\cdots\wedge X_k\wedge Y_1\wedge\cdots\overset{j}{\wedge}\cdots\wedge Y_\ell,$$

where $\overset{i}{\wedge}$ and $\overset{j}{\wedge}$ denotes the omission of X_i and Y_j, respectively. Note that $(\Lambda^\bullet\mathfrak{g}, [\cdot,\cdot], \wedge)$ is a Gerstenhaber algebra as usual.

Definition 2.1. *Let $(\mathfrak{g}, [\cdot,\cdot])$ be a Lie algebra. An element $r\in\Lambda^2\mathfrak{g}$ is called r-matrix if*

$$[r,r] = 0.$$

In this case, the triple $(\mathfrak{g}, [\cdot,\cdot], r)$ is called triangular Lie algebra.

Remark 2.2. *An r-matrix r on a Lie algebra $\mathfrak{g}$ always induces a Lie bialgebra structure $\delta_r\colon \mathfrak{g}\to\Lambda^2\mathfrak{g}$. In fact, r satisfies the classical Yang-Baxter equation (CYBE). We will not use the bialgebra structure in this short note. For more details on this, we refer the reader to [14] and [5].*

Note that the condition $[r, r] = 0$ implies that

$$[r, \cdot]\colon \Lambda^\bullet \mathfrak{g} \to \Lambda^{\bullet+1}\mathfrak{g}$$

induces a differential, i.e. $[r, \cdot]^2 = 0$. If we use r to induce the contraction $r^\sharp\colon \mathfrak{g}^* \to \mathfrak{g}$ and extend it to $r^\sharp\colon \Lambda^\bullet \mathfrak{g}^* \to \Lambda^\bullet \mathfrak{g}$, it is a chain map

$$r^\sharp\colon (\Lambda^\bullet \mathfrak{g}^*, \delta_{\mathrm{CE}}) \to (\Lambda^\bullet \mathfrak{g}, [r, \cdot]),$$

where δ_{CE} is the Chevalley-Eilenberg differential with trivial representation. Let us denote by $\mathfrak{h}_r := \operatorname{im}(r^\sharp) \subseteq \mathfrak{g}$. It can be shown that this is a Lie subalgebra, see e.g. [6, 7]. If $\mathfrak{h}_r = \mathfrak{g}$, we call r non-degenerate. Note that $r \in \Lambda^2\mathfrak{h}_r \subseteq \Lambda^2\mathfrak{g}$ and $(\mathfrak{h}_r, [\cdot, \cdot], r)$ is a triangular Lie algebra with r being non-degenerate, see [15, Prop. 3.2-3.3]. For a non-degenerate r-matrix r on $\mathfrak{g}$, there is always a unique element $\omega \in \Lambda^2\mathfrak{g}^*$ with $r^\sharp(\omega) = r$ which has the following properties:

1. The contraction map $\omega^\flat\colon \mathfrak{g} \to \mathfrak{g}^*$ is an isomorphism, and $r^\sharp = (\omega^\flat)^{-1}$.
2. $\delta_{\mathrm{CE}}\omega = 0$.

A Lie algebras together with a 2-cocycle ω fulfilling condition 1 as above is called *symplectic Lie algebra*. Symplectic Lie algebras have been extensively studied and even have been completely classified up to dimension 4 and partly up to dimension 6, see [12, 13]. Moreover, it is clear that a symplectic Lie algebra is always triangular.

In this note, we are not particularly interested in triangular Lie algebras, but in their infinitesimal action on a smooth manifold, i.e. Lie algebra maps

$$\triangleright\colon \mathfrak{g} \to \Gamma^\infty(TM).$$

For an element $\xi \in \mathfrak{g}$ we call $\xi \triangleright \in \Gamma^\infty(TM)$ the fundamental vector field of ξ. We can also extend this action to a map $\triangleright\colon \Lambda^\bullet \mathfrak{g} \to \Gamma^\infty(\Lambda^\bullet TM)$ of Gerstenhaber algebras. An immediate consequence is the following corollary which already appeared in [14].

Corollary 2.3. *Let* $(\mathfrak{g}, [\cdot, \cdot], r)$ *be a triangular Lie algebra and let* $\triangleright\colon \mathfrak{g} \to \Gamma^\infty(TM)$ *be a Lie algebra action on a smooth manifold. Then*

$$\pi_r = r\triangleright \in \Gamma^\infty(\Lambda^2 TM)$$

is a Poisson structure. Moreover, π_r *is said to be induced by the r-matrix r. If we restrict the action to* $(\mathfrak{h}_r, [\cdot, \cdot], r)$ *the same Poisson structure is induced.*

This corollary shows in particular that if a Poisson structure is induced by an r-matrix, it is canonically induced by a symplectic Lie algebra. One of the first examples of this situation is of course if $\mathfrak{g} = \mathrm{Lie}(G)$ for some Lie group G. In this case, the action is just left translating and hence π_r is a left invariant Poisson structure on G and if r is non-degenerate this Poisson structure is actually symplectic and G is a symplectic Lie group, see [10]. Nevertheless, it should be mentioned that the induced Poisson structure is not invariant with respect to the action of $\mathfrak{g}$ in general, but one can induce a Lie bialgebra structure on $\mathfrak{g}$ as already mentioned in Remark 2.2 and this Lie bialgebra acts on the Poisson manifold in the sense of Drinfel'd (see [16]) and Lu and Weinstein (see [17]).

Remark 2.4. *Note that, even though a Poisson structure induced by an r-matrix is just built out of a bilinear combination of fundamental vector fields, the Lie algebra action does not have to restrict to the symplectic leaves of the Poisson structure. Recall that the symplectic leaves of a Poisson manifold (M, π) are the integral submanifolds of the (possibly singular) involutive distribution spanned by Hamiltonian vector fields $\{f, \cdot\} \in \Gamma^\infty(TM)$ for $f \in C^\infty(M)$. As an example, we consider the abelian Lie algebra $\mathbb{R}^2$ with the standard basis $\{e_1, e_2\}$ and the r-matrix $r = e_1 \wedge e_2$. We consider the action $\triangleright\colon \mathbb{R}^2 \to \Gamma^\infty(T\mathbb{R}^2)$ which is defined by*

$$e_1 \triangleright = \frac{\partial}{\partial x} \quad \text{and} \quad e_2 \triangleright = y\frac{\partial}{\partial y}.$$

The induced Poisson structure is given by $\pi_r = y\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$, which has the symplectic leaves: $\mathbb{R}^2 \setminus \{y = 0\}$, and every point on the x-axis is zero dimensional leaf. Obviously, $e_1 \triangleright$ is not tangential to $\{0\}$ for example.

3. Symplectic structures induced by r-matrices

We have seen in Section 2 that, if a triangular Lie algebra acts on a manifold, the r-matrix always induces a Poisson structure. Throughout this section, we will assume that this Poisson structure is actually symplectic and derive some topological obstruction to this situation.

In the following, we fix a finite-dimensional real triangular Lie algebra $(\mathfrak{g}, r)$ with non-degenerate $r \in \Lambda^2\mathfrak{g}$ with inverse denoted by $\Omega \in \Lambda^2\mathfrak{g}^*$, a basis $\{e_i\}_{i\in I}$ with dual $\{e^i\}_{i\in I}$, and an action $\triangleright\colon \mathfrak{g} \to \Gamma^\infty(TM)$ on a manifold, such that $\pi_r := r\triangleright \in \Gamma^\infty(\Lambda^2 TM)$ is actually symplectic with inverse $\omega \in \Gamma^\infty(\Lambda^2 T^*M)$. With the help of the non-degeneracies, we can define

$$\triangleright^*\colon \mathfrak{g}^* \ni \alpha \mapsto \omega^\flat(r^\sharp(\alpha)\triangleright) \in \Gamma^\infty(T^*M).$$

Remark 3.1. *As already discussed in Section 2 the assumption that r is non-degenerate is not a strong assumption, since we may always pass to a smaller Lie subalgebra in which the r-matrix is non-degenerate.*

Example 3.2.

- *Let $T^2 = S^1 \times S^1$ with the symplectic structure $\omega = \mathrm{d}x \wedge \mathrm{d}y$. In this case, we can induce this symplectic structure with the abelian Lie algebra $\mathfrak{g} = \mathbb{R}^2$ with the Lie algebra action $e_1 \mapsto \frac{\partial}{\partial x}$ and $e_2 \mapsto \frac{\partial}{\partial y}$ and the r-matrix $r = e_1 \wedge e_2$. Similarly, one can induce the symplectic structure on $(T^2)^n$ by the action of $\mathbb{R}^{2n}$. Note that $((T^2)^n, \omega)$ is a symplectic Lie group.*
- *We consider $\mathbb{R}^2 \setminus \{0\}$ together with $\omega = \mathrm{d}x \wedge \mathrm{d}y$. This symplectic structure can be induced by the two-dimensional non-abelian Lie algebra which is generated by E, F fulfilling $[E, F] = F$. The map defined by $E \mapsto -x\frac{\partial}{\partial x}$ and $F \mapsto \frac{1}{x}\frac{\partial}{\partial y}$ is a Lie algebra map, and the r-matrix $r = -E \wedge F$ induces the symplectic structure ω.*

Lemma 3.3. *The extension $\triangleright^*\colon \Lambda^\bullet\mathfrak{g}^* \to \Gamma^\infty(\Lambda^\bullet T^*M)$ is a chain map with respect to the Chevalley-Eilenberg differential $\delta_{CE}\colon \Lambda^\bullet\mathfrak{g}^* \to \Lambda^{\bullet+1}\mathfrak{g}^*$ and the usual de Rham differential $\mathrm{d}\colon \Gamma^\infty(\Lambda^\bullet T^*M) \to \Gamma^\infty(\Lambda^{\bullet+1}T^*M)$. Moreover, for a basis $\{e_i\}_{i\in I}$ of $\mathfrak{g}$ with dual $\{e^i\}_{i\in I} \subseteq \mathfrak{g}^*$, we get for the induced vector fields $X_i = e_i\triangleright$ and the corresponding 1-forms $\Theta^i = e^i\triangleright^*$ the following statements:*

1. $\mathrm{d}\Theta^i = -\frac{1}{2}C^i_{kl}\Theta^k \wedge \Theta^l$ *for* $[e_i, e_j] = C^k_{ij}e_k$.

2. *Every vector field* $X \in \Gamma^\infty(TM)$ *can be written as* $X = \Theta^i(X)X_i$ *and every* 1*-form* $\alpha = \alpha(X_i)\Theta^i$.
3. $\omega = \frac{1}{2}\Omega_{ij}\Theta^i \wedge \Theta^j$, *where* $\Omega = \frac{1}{2}\Omega_{ij}e^i \wedge e^j$.

Proof. Note that $\triangleright^*$ is a concatenation of three chain maps, namely

- $r^\sharp \colon (\Lambda^\bullet\mathfrak{g}^*, \delta_{\mathrm{CE}}) \to (\Lambda^\bullet\mathfrak{g}, [r, \cdot])$,
- $\triangleright \colon (\Lambda^\bullet\mathfrak{g}, [r, \cdot]) \to (\Gamma^\infty(\Lambda^\bullet TM), [\pi_r, \cdot])$ and
- $\omega^\flat \colon (\Gamma^\infty(\Lambda^\bullet TM), [\pi_r, \cdot]) \to (\Gamma^\infty(\Lambda^\bullet T^*M), \mathrm{d})$.

Thus, the map $\triangleright^*$ is a chain map as well. Using this, we get the statements:

1. We have that $\delta_{\mathrm{CE}}e^i = -\frac{1}{2}C^i_{kl}e^k \wedge e^l$ and then we use the chain map $\triangleright^*$.
2. For a 1-form $\alpha \in \Gamma^\infty(T^*M)$, using $\pi_r = \frac{1}{2}r^{ij}X_i \wedge X_j$ where r^{ij} are the coefficients of r with respect to the basis $\{e_i\}_{i\in I}$, we have

$$\begin{aligned}\pi_r^\sharp(\alpha) &= \alpha(X_i)r^{ij}X_j \\ &= \alpha(X_i)\pi_r^\sharp(\Theta^i) \\ &= \pi_r^\sharp(\alpha(X_i)\Theta^i),\end{aligned}$$

where we use that $\Theta^i = \omega^\flat(r^{ij}X_j)$. Since $\pi_r^\sharp$ is non-degenerate, we can conclude $\alpha = \alpha(X_i)\Theta^i$. For a vector field $X \in \Gamma^\infty(TM)$, we have for all $\beta \in \Gamma^\infty(T^*M)$

$$\beta(X) = (\beta(X_i)\Theta^i)(X) = \beta(\Theta^i(X)X_i)$$

and thus $X = \Theta^i(X)X_i$.

3. This follows directly from the previous parts.

Remark 3.4. *The preceding lemma shows that the action of the Lie algebra has to be* infinitesimally transitive*, i.e. on each tangent space the fundamental vector fields are generators since every tangent vector at a point* p *can be written as a sum of fundamental vector fields evaluated at* p *(statement 2 from Lemma 3.3). Note that this does not imply that they are a basis on every tangent space, since the action might fail to be infinitesimally free. In fact, Lemma 3.3 shows that the tangent bundle can be seen as a Lie subalgebroid of the action Lie algebroid* $\mathfrak{g} \times M \to M$ *via the map*

$$T_pM \ni X_p \mapsto \Theta^i_p(X_p)e_i \in \mathfrak{g}.$$

Moreover, $\mathfrak{g} \times M$ *is a* symplectic Lie algebroid *in the sense of [18], and* TM *becomes a symplectic Lie subalgebroid.*

Let us now assume that we additionally have a vector bundle $E \to M$ together with a Lie algebra map $\triangleright_{\mathrm{E}} \colon \mathfrak{g} \to \Gamma^\infty(DE)$, such that $\sigma(\xi\triangleright_{\mathrm{E}}) = \xi\triangleright$ for all $\xi \in \mathfrak{g}$. Here, we denote by $DE \to M$ the Atiyah algebroid of E, whose smooth sections are first-order differential operators from E to E with scalar symbol, and $\sigma \colon DE \to TM$ its anchor, i.e. the symbol map. We define

$$\nabla_X s := \Theta^i(X)e_i \triangleright_{\mathrm{E}} s$$

for $X \in \Gamma^\infty(TM)$ and $s \in \Gamma^\infty(E)$.

Lemma 3.5. *The map $\nabla\colon \Gamma^\infty(TM) \to \Gamma^\infty(DE)$ defines a flat connection.*

Proof. We first have to check that ∇ is in fact a connection. It is clearly function linear in the vector field part. Let $X \in \Gamma^\infty(TM)$, $s \in \Gamma^\infty(E)$, and $f \in C^\infty(M)$, then

$$\begin{aligned}\nabla_X fs &= \Theta^i(X)e_i \triangleright_{\mathrm{E}} fs\\ &= \Theta^i(X)(e_i \triangleright f)s + f\Theta^i(X)e_i \triangleright_{\mathrm{E}} s\\ &= X(f)s + f\nabla_X s,\end{aligned}$$

where we used $X = \Theta^i(X)X_i$ from Lemma 3.3. To see that this connection is flat, we choose $X, Y \in \Gamma^\infty(TM)$:

$$\begin{aligned}\nabla_X\nabla_Y s - \nabla_Y\nabla_X s &= \Theta^i(X)e_i \triangleright_{\mathrm{E}} (\Theta^j(Y)e_j \triangleright_{\mathrm{E}} s) - \Theta^i(Y)e_i \triangleright_{\mathrm{E}} (\Theta^j(X)e_j \triangleright_{\mathrm{E}} s)\\ &= X(\Theta^j(Y))e_j \triangleright_{\mathrm{E}} s + \Theta^i(X)\Theta^j(Y)e_i \triangleright_{\mathrm{E}} e_j \triangleright_{\mathrm{E}} s\\ &\quad - Y(\Theta^j(X))e_j \triangleright_{\mathrm{E}} s - \Theta^i(Y)\Theta^j(X)e_i \triangleright_{\mathrm{E}} e_j \triangleright_{\mathrm{E}} s\\ &= \mathrm{d}\Theta^j(X,Y)e_j \triangleright_{\mathrm{E}} s + \Theta^j([X,Y])e_j \triangleright_{\mathrm{E}} s + \Theta^i(X)\Theta^j(Y)[e_i, e_j] \triangleright_{\mathrm{E}} s\\ &= \mathrm{d}\Theta^j(X,Y)e_j \triangleright_{\mathrm{E}} s + \Theta^j([X,Y])e_j \triangleright_{\mathrm{E}} s + C^k_{ij}\Theta^i(X)\Theta^j(Y)e_k \triangleright_{\mathrm{E}} s\\ &= \Theta^j([X,Y])e_j \triangleright_{\mathrm{E}} s = \nabla_{[X,Y]}s,\end{aligned}$$

where the last step makes use of Lemma 3.3 statement 1.

The rest of this note exploits this simple observation in order to obtain examples of symplectic structures which cannot be induced by triangular Lie algebras. Let us start with the tangent bundle of our given symplectic manifold (M,ω): the map

$$e_j \triangleright_{\mathrm{TM}}\colon \xi \mapsto [\xi\triangleright, -] \in \Gamma^\infty(DTM),$$

fulfills all the requirements, and thus we conclude the following theorem.

Theorem 3.6. *Let $(\mathfrak{g}, [\cdot,\cdot], r)$ be a triangular Lie algebra with non-degenerate r, which induces a symplectic structure on a manifold M. Then there exists a (not necessarily symmetric) flat connection on the tangent bundle of M.*

Remark 3.7. *We want to stress that the implication of Theorem 3.6 is completely independent of the symplectic structure and the triangular Lie algebra, which allows us to find counter examples very easily without even referring to a specific Lie algebra action or a symplectic structure. For example, this excludes every manifold with non-vanishing Pontryagin class.*

Remark 3.8. *Finding obstructions for a Poisson structure being induced by an r-matrix seems to be much harder. In view of Remark 2.4, we cannot simply apply our obstruction leaf-wise to obtain similar obstructions on each leaf. Nevertheless, finding those to a bigger class of Poisson structures, such as symplectic fiber bundles and $\mathfrak{b}$-symplectic structures, is part of future work.*

Theorem 3.6 implies in particular the main counter examples in [7] and [8], but we are using much less advanced techniques.

Corollary 3.9. *Let $(\mathfrak{g}, [\cdot,\cdot], r)$ be a triangular Lie algebra, which induces a symplectic structure on a compact connected 2-dimensional manifold M. Then M is the torus.*

Proof. It is well known, see e.g. [19], that the only compact connected surface which admits a flat connection on the tangent bundle is the torus. By Example 3.2, we know that we can find a symplectic structure on T^2, which is induced by a triangular Lie algebra.

Since all complex projective spaces are simply connected and do not admit flat connections on their tangent bundles, we immediately can conclude the following corollary.

Corollary 3.10. *The complex projective space $\mathbb{C}P^n$ cannot possess any symplectic structure that is induced by a triangular Lie algebra for $n \geq 1$.*

Note that in [8] the authors only showed that the Fubini-Study symplectic structure on $\mathbb{C}P^n$ cannot be induced by an r-matrix defined on $\mathfrak{gl}_{n+1}(\mathbb{C})$.

Using the the same line of thought, one can produce many counter examples: (complex analytic) K3-surfaces are compact simply connected complex manifolds of complex dimension 2 with a trivial canonical line bundle. It is well known that K3-surfaces are Kähler manifolds and thus symplectic. Note, moreover, that their Pontryagin classes do not vanish, and thus we can conclude the following corollary.

Corollary 3.11. *A K3 surface (as a smooth manifold) cannot possess any symplectic structure that is induced by a triangular Lie algebra.*

As a last (class of) counter example(s), we show that we can produce non-compact ones. Let us assume that we have a manifold M with non-trivial first Pontryagin class. It follows that the first Pontryagin class of T^*M is also non-vanishing and thus we obtain the following corollary

Corollary 3.12. *For a manifold M with non-trivial first Pontryagin class, the canonical symplectic structure $\omega_{\mathrm{can}} \in \Omega^2(T^*M)$ cannot be induced by a triangular Lie algebra.*

As a last part of this section we want to consider an even more special case. Note that, in general, the connection on the tangent bundle constructed above can not be the Levi-Civita connection of a metric, since

$$\mathrm{Tor}^{\nabla}(X, Y) = \mathrm{d}\Theta^i(X, Y)X_i,$$

which generally does not vanish. If we assume that $\mathfrak{g}$ is abelian, we immediately see that

$$\mathrm{Tor}^{\nabla}(X, Y) = 0,$$

using 3.3 and the vanishing of the structure constants of the Lie algebra. Moreover, we have

$$\nabla_X X_k = \Theta^i(X)[X_i, X_k] = 0$$

for all $X \in \Gamma^\infty(TM)$, and hence the X_i's are parallel and thus $\nabla\pi_r = 0$, which implies $\nabla\omega = 0$.

Lemma 3.13. *Let (M, ω) be a connected symplectic manifold and assume that ω is induced by a abelian triangular Lie algebra. Then M admits a flat Kähler structure with symplectic form ω.*

Proof. Let X_i be the fundamental vector fields of a basis $\{e_i\}_{i \in I}$ of an abelian triangular Lie algebra of $(\mathfrak{g}, [\cdot, \cdot], r)$. Since the action is infinitesimally transitive (Lemma 3.3), we can choose $\{X_{i_\ell}\}_{1 \leq \ell \leq \dim(M)}$ such that they form a basis of T_pM at some $p \in M$. Using the connectedness of M and the flatness of the X_i's, we conclude that the choice $\{X_{i_\ell}\}_{1 \leq \ell \leq \dim(M)}$ are a basis at every point. We define a metric $g \in \Gamma^\infty(\mathrm{S}^2 T^* M)$ by declaring them orthonormal, and thus ∇ is the Levi-Civita connection of g. This makes (M, ω, g) a flat Kähler manifold.

Remark 3.14. *Note that all the symplectic manifolds we discussed in this short note cannot possess star products which are induced by Drinfel'd twists, simply because we already excluded the first order term, see [6, 7] for the exact statements.*

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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