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Research article

An example in Hamiltonian dynamics

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Abstract: We present an example of a three-degrees-of-freedom polynomial Hamilton function with a critical point characterized by indefinite quadratic part with a Morse index 2. This function generates a Hamiltonian system wherein all eigenvalues equal ±i, but it lacks small-amplitude periodic solutions with a period $\approx 2\pi$.

Keywords: Hamiltonian system; periodic solutions; Lyapunov theorem
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1. Introduction

In this paper, we study the Hamiltonian vector field (denoted also by X_H)

$$\dot{q}_j = \frac{\partial H}{\partial p_j}, \quad \dot{p}_j = -\frac{\partial H}{\partial q_j}, \tag{1.1}$$

$j = 1, \ldots, m$, generated by the following Hamilton function:

$$H = \frac{1}{2}\left(|z_1|^2 + |z_2|^2 - |z_3|^2\right) + \left(|z_2|^2 + |z_3|^2\right) \mathrm{Re}\,(z_2 z_3) + |z_1|^2 \,\mathrm{Re}\,(\bar{z}_1 (z_2 + \varepsilon \bar{z}_3)), \tag{1.2}$$

where $z_j = q_j + \mathrm{i} p_j$, $\mathrm{i} = \sqrt{-1}$, $m = 3$, and ε is a complex parameter.

Note that the linear part of system (1.1) at the origin $z = 0$ has eigenvalues ±i, each with multiplicity 3, but with trivial Jordan cells. Thus, all solutions of the corresponding linear system are 2π−periodic. Moreover, the quadratic part

$$F = \frac{1}{2}\left(|z_1|^2 + |z_2|^2 - |z_3|^2\right) \tag{1.3}$$

of H is indefinite, with the Morse index (the number of negative terms) equal 2.

Theorem 1. *System (1.1), with the Hamilton function (1.2), does not have periodic solutions near $z = 0$ of period $\approx 2\pi$ for ε in a neighborhood of the origin in $\mathbb{C} \simeq \mathbb{R}^2$ minus a finite collection of real analytic curves.*

Let us remind ourselves of the history of the problem of small-amplitude periodic solutions to autonomous differential systems. It has begun with Lyapunov theorems (see [1, 2]); we present the last of them.

Consider an autonomous differential system

$$\dot{x} = Ax + \dots, \ \ x \in (\mathbb{R}^n, 0), \tag{1.4}$$

with analytic right-hand sides such that the matrix A has pure imaginary eigenvalues

$$\pm \mathrm{i}\omega_1, \dots, \pm \mathrm{i}\omega_m, \ \ \omega_j > 0, \tag{1.5}$$

$m = \frac{n}{2}$, and one of the frequencies, say ω_1, is such that none of the other frequencies is is an integer multiple of it; thus

$$\omega_j/\omega_1 \notin \mathbb{Z}, \quad j \geq 2. \tag{1.6}$$

One can define formal obstructions to the existence of a 1–parameter family of periodic solutions of period $\approx 2\pi/\omega_1$. For this, one uses the so-called Poincaré–Dulac normal form. There exists a formal invariant surface $\mathcal{V}$ tangent to the invariant plane $\mathcal{E}$ for A with the eigenvalues $\pm\mathrm{i}\omega_1$ and one defines so-called Poincaré–Lyapunov focus quantities, which constitute obstructions to the existence of a formal first integral on $\mathcal{V}$.

Lyapunov proved that:

in this case, if all the focus quantities vanish, then there exists a family of periodic solutions $x = \phi(t; c)$, $c \in (\mathbb{R}_+, 0)$, *of period* $T(c) \to 2\pi/\omega_1$ *as* $c \to 0$, *depending analytically on c, and such that* $\phi(t; 0) \equiv 0$.

In the case $n = 2$, this result was independently proved by Poincaré [3] and is known as the Lyapunov–Poincaré theorem.

The Lyapunov theorem has attracted the attention of specialists in the Hamiltonian dynamics. Note that, in the Hamiltonian case, the above-mentioned obstructions are absent, because H restricted to the invariant plane $\mathcal{V}$ is a suitable first integral. Therefore, inequalities (1.6) are the only assumption of the Hamiltonian version of the Lyapunov theorem.

Assume that system (1.1) has equilibrium point $q = p = 0$ with the eigenvalues $\pm\mathrm{i}\omega_1, \dots, \pm\mathrm{i}\omega_m$, $\omega_j > 0$. Assuming $H(0) = 0$ the leading part of the Taylor expansion of the Hamilton function is

$$F = \sum \frac{1}{2} \epsilon_j \omega_j \left(q_j^2 + p_j^2\right), \tag{1.7}$$

where q_j, p_j are suitable canonical variables, i.e., with the Poisson brackets $\left\{p_i, q_j\right\} = \delta_{ij}$, and $\epsilon_j = \pm 1$ are well defined signs. *

*We have $\dot{f} = \{f, H\}$ in the case of a general Hamiltonian system with the symplectic structure defined by a Poisson bracket. In particular, for $z_j = q_j + \mathrm{i}p_j$ and $\bar{z}_j = q_j - \mathrm{i}p_j$, we have $\left\{z_j, z_k\right\} = \left\{\bar{z}_j, \bar{z}_k\right\} = 0$ and $\left\{z_j, \bar{z}_k\right\} = 2\mathrm{i}\delta_{jk}$. Thus $\dot{z}_j = \left\{z_j, \bar{z}_j\right\} \partial H/\partial \bar{z}_j = 2\mathrm{i} \cdot \partial H/\partial \bar{z}_j$. Also the resonant monomials $g = z^\alpha \bar{z}^\beta$ form the Birkhoff theorem satisfy $\{g, F\} = 0$.

D. Schmidt [4] studied 1–parameter families of periodic solutions for such systems with two degrees of freedom in the cases of resonant frequencies ω_1 and ω_2. His analysis was mainly focused on definite Hamiltonians, i.e., when $\epsilon_1 = \epsilon_2$, but he also considered the situation near the Lagrangian libration point in the restricted three-body problem, where the Hamilton function is indefinite. He did not refer to the Lyapunov theorem.

A. Weinstein [5] has applied the Lusternik–Schnirelmann category to prove that:

if the quadratic part F of H is positive definite, i.e., all $\epsilon_j = 1$ in Eq. (1.7), then the vector field X_H has at least $m = \frac{n}{2}$ 1–parameter families of periodic solutions.

In [5], it was stated that each hypersurface $\{H = c\}$, $c > 0$, contains at least m periodic trajectories; but, in the analytic case, these trajectories form families parametrized by c.

The positive definiteness of the F condition is important, because we have the following example from the book [6, Example 9.2] by J. Mawhin and J. Willem.

Example 1. *Let*

$$H = \frac{1}{2}\left(|z_1|^2 - |z_2|^2\right) + |z|^2 \operatorname{Re}(z_1 z_2). \tag{1.8}$$

It generates the system

$$\dot{z}_1 = \mathrm{i}z_1 (1 + 2\operatorname{Re}(z_1 z_2)) + \mathrm{i}\,|z|^2\,\bar{z}_2, \quad \dot{z}_2 = -\mathrm{i}z_2 (1 - 2\operatorname{Re}(z_1 z_2)) + \mathrm{i}\,|z|^2\,\bar{z}_1.$$

One finds

$$\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Im}(z_1 z_2) = 2\,[\operatorname{Re}(z_1 z_2)]^2 + |z|^4,$$

which excludes the existence of nontrivial periodic solutions.

Next, J. Moser has somehow specified Weinstein's result, but his statements were not precise and his own example [7, Example 2] was confusing; see also my discussion of Moser's approach in Remark 1 in the next section.

Next, this subject was brought up by specialists in nonlinear functional analysis. In particular, A. Szulkin [8] considered the case when one of the frequencies, say ω_1, of multiplicity k, is such that condition (1.6) holds for all $\omega_j \neq \omega_1$ and

$$\sum_{\omega_j=\omega_1} \epsilon_j \neq 0, \tag{1.9}$$

i.e., the Morse index $m^-(F_1)$ (= the number of minuses) of the quadratic form $F_1 = F$ restricted to the invariant subspace $\mathcal{E}_1$ associated with the eigenvalues $\pm\mathrm{i}\omega_1$ differs from $m^+(F_1) = m^-(-F_1)$. He claimed that:

then there exists a sequence $\{\gamma_n(t)\}$ of non-constant periodic solutions to the system $\dot{x} = X_H(x)$ tending to $\gamma(t) \equiv 0$ of periods tending to $2\pi/\omega_1$.†

†His (not very long) proof is quite technical, i.e., with many homological groups in an infinite dimensional context.

There is also another paper [9] by E. N. Dancer and S. Rybicki, where the Szulkin's statement is confirmed and generalized. The corresponding 'bifurcations' are investigated in the context of $\mathbb{S}^1$– invariant Hamiltonian systems. There some $\mathbb{S}^1$–equivariant indices and degrees are defined and used.

In the case of the quadratic part of the Hamiltonian (1.2), we have $m^-(F) = 2$ and $m^+(F) = 4$; hence, our Theorem 1 contrasts with the Szulkin's claim.

Szulkin's statement was used by other specialists in this field: E. Pérez-Chavela, A. Gołębiewska, S. Rybicki, D. Strzelecki, and A. Ureña, see [10, 11] for example. Fortunately, those results are correct, because the corresponding Hamiltonians restricted to the center manifold are positive-definite.

Finally, it worth mentioning the paper of S. van Straten [12], where the number of 1–parameter families is calculated in the cases of Hamiltonians of the form

$$H = \frac{1}{2}|z|^2 + H_{2d}(z, \bar{z}),$$

where H_{2d} is a generic homogeneous polynomial of degree $2d$ in the Birkhoff normal form.

In the next section, we recall some tools developed in [2], e.g., we replace our dynamical problem with a suitable algebraic problem, and in the third section, we complete the proof of Theorem 1.

2. Birkhoff normal form, twisted Poincaré map and symplectic reduction

The aim of this section is to reformulate the problem of small-amplitude periodic solutions to some algebraic problem.

Proposition 1. *The small-amplitude periodic solutions of system (1.1) of period $\approx 2\pi$ with H given in Eq. (1.2) correspond to small solutions of the following system:*

$$\begin{aligned} r^2 \mathrm{Im}(u + \varepsilon\bar{v}) &= 0 \\ u\mathrm{Re}(2uv - 3r(u + \varepsilon\bar{v})) + \left(|u|^2 + |v|^2\right)\bar{v} + r^3 &= 0 \\ v\mathrm{Re}(2uv + 3r(u + \varepsilon\bar{v})) + \left(|u|^2 + |v|^2\right)\bar{u} + \varepsilon r^3 &= 0 \end{aligned} \tag{2.1}$$

for

$$r > 0$$

and complex u and v.

Before proving this statement, we present some tools introduced in [2].

2.1. Birkhoff normal form

Recall that the Hamiltonian system generated by a Hamilton function with the quadratic part (1.7) takes the form

$$\dot{z}_j = \lambda_j z_j + \ldots = \mathrm{i}\epsilon_j\omega_j z_j + \ldots, \ \dot{v}_j = -\lambda_j v_j + \ldots = -\mathrm{i}\epsilon_j\omega_j v_j + \ldots,$$

where $z_j = q_j + \mathrm{i}p_j$ and $v_j = q_j - \mathrm{i}p_j$; in the real domain, we have $v_j = \bar{z}_j$. We assume that the right-hand sides are analytic.

G. Birkhoff [13] proved that there exists a formal symplectic change $(z, v) \longmapsto (Z, V)$ which reduces the Hamilton function to the following **Birkhoff normal form**:

$$F(Z, V) + \sum a_{k;l} Z^k V^l,$$

where the summation runs over the pairs $(k; l) = (k_1, \ldots, k_m; l_1, \ldots, l_m) \in \mathbb{Z}^m_{\geq 0} \times \mathbb{Z}^m_{\geq 0}$ such that the **resonant relations**

$$(k - l, \lambda) = \sum \left(k_j - l_j\right) \lambda_j = 0$$

hold, and $|k| + |l| \geq 3$.

Note that the Hamiltonian (1.1) is in the Birkhoff normal form.

From the dynamical point of view, the property of H being in the Birkhoff normal form means that the Hamiltonian flow $\left\{g^t_{X_H}\right\}$ commutes with the Hamiltonian flow $\left\{g^s_{X_F}\right\}$ generated by the quadratic part F of H; thus, $\{H, F\} = 0$. The second flow is the following:

$$z_j \longmapsto \mathrm{e}^{\mathrm{i}\epsilon_j \omega_j s} z_j, \; j = 1, \ldots, m.$$

2.2. Return system

Assume firstly that

$$\omega_1 = \omega_2 = \ldots = \omega_m = \omega, \tag{2.2}$$

i.e., all solutions to the corresponding linear system defined by F are $2\pi/\omega$−periodic, and

$$\epsilon_1 = 1.$$

Moreover, assume that the Birkhoff normal form is analytic (or that the analytic Hamilton function is in the Birkhoff normal form). We look for periodic solutions close to periodic solutions of the linear approximation of period $\approx 2\pi/\omega$.

Such a periodic solution, of period $T = 2\pi/\eta$ with $\eta \approx \omega$, is of the form $z_j(t) \approx c_j \mathrm{e}^{\mathrm{i}\epsilon_j \eta t}$, where c_j are small constants, not all of which equal zero. Assume that $c_1 \neq 0$. The angle $\theta = \arg z_1$ is of the form $\theta(t) \approx \theta_0 + \epsilon_1 \eta t + \theta_1(t)$ (where $\theta_1(t)$ is small, $2\pi/\eta$−periodic) and varies along the whole $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$. Therefore, the phase curves are graphs of functions of θ :

$$|z_1| = r = r(\theta), \; z_j = z_j(\theta), \; (j > 1),$$

which are 2π−periodic.

We define the **Poincaré return map**

$$P : S \longmapsto S,$$

where $S = (\mathbb{R}_+, 0) \times \left(\mathbb{C}^{m-1}, 0\right)$ is the Poincaré section. This amounts to putting $z_1 = r\mathrm{e}^{\mathrm{i}\theta}$, writing down equations for $\mathrm{d}r/\mathrm{d}\theta = \dot{r}/\dot{\theta}$ and for $\mathrm{d}z_j/\mathrm{d}\theta$, $j > 1$, (elimination of time), and evaluating the solution after the new time 2π of a corresponding initial value problem. Since $\dot{\theta} \approx \epsilon_1 \omega \neq 0$, the return time to the section S is $\approx 2\pi/\omega$, and the map P is well defined.

The fixed points of this map correspond to periodic orbits of the Hamiltonian system of period $\approx 2\pi/\omega$. Other periodic orbits of P correspond to periodic orbits of X_H of period being approximately a multiple of $2\pi/\omega$.

Moreover, due to the analyticity of the right-hand sides of the differential equations for r and z_j's, the equilibrium points of the return system are of two types:

equal $(r, z) = (0, 0)$, i.e., there are no nontrivial periodic solutions; or

a real analytic subvariety of $\left(\mathbb{R}_+ \times \mathbb{C}^{m-1}, (0,0)\right)$ of positive dimension, usually a 1–dimensional curve.

Now we slightly change our point of view. We apply the following change:

$$z_1 = re^{i\theta}, \ z_j = w_j e^{i\epsilon_j\theta}, \ \ (j > 1), \tag{2.3}$$

where $r \geq 0$ and $w_j \in (\mathbb{C}, 0)$. This is the same as the action of the flow $\left\{g^s_{X_F}\right\}$ generated by the quadratic part F of H (see the previous section). In fact, the variables r and w_j are invariant for the flow $\left\{g^s_{X_F}\right\}$.

The property of commuting of the two Hamiltonian flows (as H is in the Birkhoff normal form) implies that corresponding differential equations, after elimination of the time, take the form

$$\frac{\mathrm{d}r}{\mathrm{d}\theta} = \Pi(r, w, \bar{w}), \ \frac{\mathrm{d}w_j}{\mathrm{d}\theta} = \Lambda_j(r, w, \bar{w}), \tag{2.4}$$

i.e., the right-hand sides do not depend on θ. We call system (2.4) the **return system**. We define the **twisted Poincaré map** P^{tw} via solutions of Eqs. (2.4) after the new time 2π.

Of course, the fixed points of the twisted Poincaré map are the fixed points of the twisted Poincaré map. Among them are the equilibrium points of the return system, i.e., defined by the equations $\Pi = \Lambda_2 = \ldots = \Lambda_m = 0$.

Assume now that the hyperplane $\{z_1 = 0\}$ is invariant; this means that the right-hand side of the equation for $\dot{z}_1$ lies in the ideal generated by z_1 and $\bar{z}_1$. For the return system, it means that $\dot{r}$ is divided by r and $\dot{\theta} = \omega + \ldots$; thus, $\Pi(r, w) = r\widetilde{\Pi}(r, w)$, with analytic $\widetilde{\Pi}$. (The opposite case is more complicated.)

We claim that:

Under the invariance of the hyperplane $\{z_1 = 0\}$ assumption, the fixed points, being the equilibrium points of the return system, are the only fixed points of the twisted Poincaré map in a neighborhood of the origin.

Indeed, other fixed points of P^{tw} would correspond to (nontrivial) closed phase curves of the return vector field; moreover, of period $2\pi/k$ for an integer k. In fact, such periodic curves should lie in analytic families, $\{\gamma_c\}_{c\in(\mathbb{R}_+,0)}$ (by the analyticity of the right-hand sides). The period of γ_c is calculated as follows:

$$\mathcal{T}(c) = \int_{\gamma_c} \mathrm{d}\theta = \int_{\gamma_c} \frac{\mathrm{d}r}{\Pi(r, w)}.$$

But $\Pi(r, w) = r\widetilde{\Pi}(r, w)$ and is of high order, ≥ 2. If the variables $r, \ w_j, \ \bar{w}_j$ at γ_c are of given orders of c, e.g., $r \sim c^{\alpha} \to 0$, then $\mathcal{T}(c) \sim c^{-\beta} \to \infty$ as $c \to 0$.

By the way, such solutions would not approximate the corresponding solutions of the linear system (like in the Weinstein theorem).

Recall that the solutions to the system $\Pi = \Lambda_2 = \ldots = \Lambda_m = 0$ are of two types:

(i) equal $(r, z) = (0, 0)$; or

(ii) a real analytic variety of positive dimension, usually a 1–dimensional curve.

Example 2. *(Example 1 revisited). The Hamiltonian (1.8) is in Birkhoff normal form. The change (2.3) means*

$$z_1 = re^{i\theta},\ z_2 = we^{-i\theta}.$$

We have the following return system:

$$\frac{dr}{d\theta} = \frac{-r\left(r^2+|w|^2\right)\mathrm{Im}w}{r+\left(3r^2+|w|^2\right)\mathrm{Re}w},\quad \frac{dw}{d\theta} = i\frac{\left(5r^2+|w|^2\right)w\mathrm{Re}w + r^2\left(r^2+|w|^2\right)}{r+\left(3r^2+|w|^2\right)\mathrm{Re}w}.$$

Its equilibrium points are defined by: either $r = 0$, and then $w = 0$; or $\mathrm{Im}w = 0$*, i.e.,* $\mathrm{Re}w = w$*, and hence $\left(5r^2+|w|^2\right)|w|^2 + r^2\left(r^2+|w|^2\right) = 0$, and then again $r = w = 0$. So, there are no nontrivial equilibrium points.*

But the differential system from Example 1 does not have invariant planes; so, one needs an additional argument provided in Example 1.

2.3. *Proof of Proposition 1*

Recall that the Hamiltonian (1.1) is in Birkhoff normal form. It generates the system

$$\begin{aligned} \dot z_1 &= i\left\{z_1\left(1+2\mathrm{Re}\bar z_1\left(z_2+\varepsilon\bar z_3\right)\right) + |z_1|^2\left(z_2+\varepsilon\bar z_3\right)\right\}, \\ \dot z_2 &= i\left\{z_2\left(1+2\mathrm{Re}\left(z_2z_3\right)\right) + \left(|z_2|^2+|z_3|^2\right)\bar z_3 + |z_1|^2 z_1\right\}, \\ \dot z_3 &= i\left\{z_3\left(-1+2\mathrm{Re}\left(z_2z_3\right)\right) + \left(|z_2|^2+|z_3|^2\right)\bar z_2 + \varepsilon|z_1|^2\bar z_1\right\}. \end{aligned} \tag{2.5}$$

One can see that neither of the coordinate 4–spaces $\{z_2 = 0\}$, $\{z_3 = 0\}$ nor of the coordinate planes $\left\{z_j = z_k = 0\right\}$ is invariant, but the subspace $\{z_1 = 0\}$ is invariant.

However, system (2.5) restricted to the subspace $\{z_1 = 0\}$, i.e., the last two equations, is the same as the system from Example 1, which is without periodic solutions.

Therefore, we can introduce the variables $r \geq 0$, θ, u, v (analogues of the variables (2.3)) via the formulas

$$z_1 = re^{i\theta},\ z_2 = ue^{i\theta},\ z_3 = ve^{-i\theta},$$

or $r = |z_1|$, $\theta = \arg z_1$, $u = \bar z_1 z_2/|z_1|$, $v = z_1z_3/|z_1|$. In fact, we have $r > 0$.

We get

$$\dot\theta = 1 + 3r\mathrm{Re}\left(u+\varepsilon\bar v\right) \tag{2.6}$$

and

$$\begin{aligned} \dot r &= -r^2\mathrm{Im}\left(u+\varepsilon\bar v\right), \\ \dot u &= i\left\{u\mathrm{Re}\left(2uv-3r\left(u+\varepsilon\bar v\right)\right) + \left(|u|^2+|v|^2\right)\bar v + r^3\right\}, \\ \dot v &= i\left\{v\mathrm{Re}\left(2uv+3r\left(u+\varepsilon\bar v\right)\right) + \left(|u|^2+|v|^2\right)\bar u + \varepsilon r^3\right\}. \end{aligned} \tag{2.7}$$

By the arguments given in Section 2.2 the small-amplitude periodic solutions to our Hamiltonian system of period $\approx 2\pi$are in one-to-one correspondence with the equilibrium points of the above system.

But the right-hand sides of system (2.7) are the left-hand sides of Eqs. (2.1).

Note also that the period of a periodic solution $z_1(t) \approx r_0 e^{it}$, $z_2(t) \approx u_0 e^{it}$, $z_2(t) \approx v_0 e^{-it}$, corresponding to an eventual equilibrium point (r_0, u_0, v_0), equals

$$T = \int_0^{2\pi} \frac{d\theta}{1 + 3r(\theta)\,\mathrm{Re}\,(u(\theta) + \varepsilon\bar{v}(\theta))} = \frac{2\pi}{1 + 3r_0 \mathrm{Re}\,(u_0 + \varepsilon\bar{v}_0)},$$

where $r(\theta) \equiv r_0$, $u(\theta) \equiv u_0$ and $v(\theta) \equiv v_0$ are solutions to the corresponding return system with the initial condition $r(0) = r_0, u(0) = u_0, v(0) = v_0$.

Finally, since the right-hand sides of Eqs. (2.7) are homogeneous, eventual equilibrium points are not isolated. They should form 1–dimensional straight semi-lines, corresponding to 1–parameter families of periodic solutions to the Hamiltonian system. □

2.4. *Symplectic reduction*

We complete this section with a short discussion of the additional elements of the novel approach to the problem started in [2], which are potentially interesting to the reader.

In [2, Proposition 6], it was proved that a Hamiltonian in the Birkhoff normal form, under assumptions (2.2), is invariant with respect to the following action of circle $\mathbb{S}^1$:

$$z = (z_1, \ldots, z_m) \longmapsto \sigma^{\phi}(z) = \left(e^{i\epsilon_1\phi} z_1, \ldots, e^{i\epsilon_m\phi} z_m\right), \quad 0 \le \phi \le 2\pi. \tag{2.8}$$

Action (2.8) is symplectic, it is a periodic phase flow generated by the Hamilton function $F(z, \bar{z})$, i.e., the homogeneous quadratic part of H.

We deal with the classical phenomenon called **symplectic reduction** (see [14]). The function F, called **momentum mapping**, is the first integral for the Hamiltonian vector field X_H system. So, we take the invariant manifolds

$$M_f = \{F(z, \bar{z}) = f\}, \tag{2.9}$$

and their quotients

$$N_f = M_f / \mathbb{S}^1 \tag{2.10}$$

of dimension $2m - 2$. The latter varieties are smooth and equipped with a natural symplectic structure and support vector fields Y_f obtained from the Hamiltonian vector field X_H. Each vector field Y_f is Hamiltonian with the Hamilton function $\pi_* H = \pi_* F + \pi_* G$, where $\pi : M_f \longmapsto N_f$ is the projection and

$$G = H - F \tag{2.11}$$

contain higher order terms.

The variables $(r, w) = \left(r^{(1)}, w^{(1)}\right)$ from Eqs. (2.3) form a local chart in the quotient variety N_f. Other local charts, $\left(r^{(l)}, w^{(l)}\right)$, $l > 1$, are defined via the formulas

$$z_l = r^{(l)} e^{i\theta}, \; z_j = w_j^{(l)} e^{i\epsilon_j\theta/\epsilon_l}, \; (j \ne l),$$

where $r^{(l)} \ge 0$ and $w_j^{(l)} \in (\mathbb{C}, 0)$. With each such chart, we associate a corresponding return system, like system (2.4), whose equilibrium locus either reduces to $\left(r^{(l)}, w^{(l)}\right) = (0, 0)$ or is a real analytic set of the positive dimension (corresponding to some families of periodic solutions with period $\approx 2\pi/\omega$).

In [2, Proposition 8], it was proved that the periodic orbits of X_H in M_f of period $\approx 2\pi/\omega$ correspond to the critical points of the function $\pi_* G$ on N_f.

In the case of a definite (say positive definite) momentum map F, the quotient varieties N_f, $f > 0$, are complex projective spaces $\simeq \mathbb{P}^{m-1}$. Here, the number of critical points of $\pi_* G$ on N_f is estimated from below using the Schnirelmann–Lusternik category; this is the estimate from the Weinstein theorem. [‡] Moreover, we also have the Poincaré–Hopf formula at our disposal.

But in the case of indefinite Hamiltonians, one cannot use the above topological tools. Here, the varieties N_f are non-compact, and it is easy to find vector fields without singular points on them.

2.5. Generalizations

Following [5, Proof of Theorem 2.1], we consider the following **equivalence relation on the set** $\{\omega_1, \ldots, \omega_m\}$ of frequencies:

$$\omega_i \sim \omega_j \text{ iff } \omega_i/\omega_j \in \mathbb{Q}.$$

Assume firstly that there is only one equivalent class and H is in analytic Birkhoff normal form.

So, we can write

$$\omega_j = p_j \omega_0, \quad p_j \in \mathbb{N}, \quad \gcd(p_1, \ldots, p_m) = 1. \tag{2.12}$$

In [2, Proposition 6], it was proved that in this case the Birkhoff normal form is invariant with respect to the following action of the circle $\mathbb{S}^1$ (generalization of action (2.8)):

$$z = (z_1, \ldots, z_m) \longmapsto \sigma^{\phi}(z) = \left(\mathrm{e}^{\mathrm{i}\epsilon_1 p_1 \phi} z_1, \ldots, \mathrm{e}^{\mathrm{i}\epsilon_m p_m \phi} z_m\right), \quad 0 \le \phi \le 2\pi. \tag{2.13}$$

This action is also symplectic and its periodic phase flow is generated by the Hamilton function $F(z, \bar{z})$, the homogeneous quadratic part of H.

Again, we deal with the **symplectic reduction**. The function F, called **momentum mapping**, is the first integral for the vector field X_H. So, we take the invariant manifolds

$$M_f = \{F(z, \bar{z}) = f\},$$

and their quotients

$$N_f = M_f/\mathbb{S}^1$$

of dimension $2m - 2$. The latter varieties are equipped with a natural symplectic structure and support vector fields Y_f obtained from the Hamiltonian vector field X_H. Each vector field Y_f is Hamiltonian with the Hamilton function $\pi_* H = \pi_* F + \pi_* G$, where $\pi : M_f \longmapsto N_f$ is the projection and

$$G = H - F$$

contain higher order terms.

But now the quotient varieties N_f may be singular, but with with normal singularities (quotients of $\left(\mathbb{C}^k, 0\right)$ by an action of a finite group). For example, in the cases of positive definite F, the sets M_f, $f >$

[‡]Weinstein skillfully constructed a function on the level hypersurface $L_h = \{H(z, \bar{z}) = h\}$, $h > 0$, which has critical locus at the set of periodic phase curves of X_H in L_h. His construction is not direct and involves many technical details. The approach from [2] is more direct.

0, are diffeomorphic with $\mathbb{S}^{2m-1}$ and their quotients are the weighted projective spaces. Nevertheless, due to normality, the corresponding functions π_*F, π_*G, and the vector field Y_f are well defined.

We have the local charts defined by

$$z_l = r^{(l)}\mathrm{e}^{\mathrm{i}\theta}, \ z_j = w_j^{(l)}\mathrm{e}^{\mathrm{i}\epsilon_j p_j\theta/\epsilon_l p_l}, \ (j \neq l),$$

and corresponding return systems whose equilibrium points correspond to periodic orbits of X_H with period $\approx 2\pi/\omega_k$.

Those equilibrium points correspond to the critical points of the function π_*G on N_f. Again, in the case of compact N_f, the number of critical points of π_*G on N_f is estimated from below using the Schnirelmann–Lusternik category. In the non-compact case we do not have such tools.

Consider now the case of several equivalent classes for the collection of frequencies.

For each **equivalence class** C_ν we have a linear subspace $\mathcal{E}_\nu$ invariant for the linear part of the system, but we can say more. In [2, Proposition 3], it was proved that for each such class, there exists a formal invariant submanifold $\mathcal{V}_\nu$ tangent to $\mathcal{E}_\nu$ at the origin.

Remark 1. *Here I would like to comment on Moser's statement mentioned in the introduction. Let us recall it (compare [7, Theorem 4]):*

'Assume that $\mathbb{R}^{2m} = \mathcal{E} \oplus \mathcal{F}$, where $\mathcal{E}$ and $\mathcal{F}$ are invariant subspaces of the matrix A defining the linear part of X_H, such that all solutions in $\mathcal{E}$ of the linear system have the same period $T > 0$, while no nontrivial solution in $\mathcal{F}$ has this period. Assume also that the quadratic part F of H restricted to $\mathcal{E}$ is positive definite. Then, on each energy surface $\{H = c\}$, $c > 0$ and small, the number of periodic orbits of X_H is at least $\frac{1}{2}\dim\mathcal{E}$.'

In [2], I have expressed the opinion that this statement must be wrong. My argument relied upon an analysis of Moser's example ([7, Example 2]) with the Hamiltonian $H = \frac{1}{2}\left(|z_1|^2 - j\,|z_2|^2\right) + \mathrm{Re}(z_1 z_2^j)$, which is not in the Birkhoff normal form and leads to wrong statements. Recently I realized that the latter example was given with a mistake, and the correct Hamiltonian is

$$H = \frac{1}{2}\left\{j\,|z_1|^2 - |z_2|^2 + z_1 z_2^j + \bar{z}_1\bar{z}_2^j\right\},$$

where $j \geq 2$ is an integer; note that it is in the Birkhoff normal form. We have

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Im}(z_1 z_2^j) = |z_2|^{2j} + j^2\,|z_1|^2\,|z_2|^{2j-2}.$$

The quadratic part is positive definite on the plane $\{z_2 = 0\}$ with periodic solutions for the linear system of period $2\pi/j$; other periodic solutions in the plane $\{z_1 = 0\}$ have period 2π. But the corresponding nonlinear system has only $2\pi/j$–periodic solutions in $\{z_2 = 0\}$.

Next, I have assumed that the period T in Moser's theorem is the minimal period, but the reviewer of this work has pointed out that I could be wrong. Plausibly, Moser had in mind an invariant subspace $\mathcal{E}$ corresponding to one of Weinstein's equivalence classes of frequencies. Otherwise, he could not claim the consequence of Weinstein's theorem from his statement.

Indeed, consider the case with three frequencies: $\omega_1 = 2$, $\omega_2 = 3$, and $\omega_3 = 6$. Then we have two invariant linear subspaces: $\mathcal{E}_1$ associated with ω_1 and ω_3 and $\mathcal{E}_2$ associated with ω_2 and ω_3; all *solutions of the linear system in $\mathcal{E}_1$ have period $T_1 = \pi$, and all solutions in $\mathcal{E}_2$ have period $T_2 = 2\pi/3$. One can show that there exist corresponding formal invariant subspaces $\mathcal{V}_1$ and $\mathcal{V}_2$ for X_H; moreover, we have the invariant subspaces $\mathcal{E}_3 = \mathcal{E}_1 \cap \mathcal{E}_2$ and $\mathcal{V}_3 = \mathcal{V}_1 \cap \mathcal{V}_2$. From the original statement of Moser's theorem it follows that there exist many periodic orbits in $\mathcal{V}_1$ and in $\mathcal{V}_2$, but all of them could lie in $\mathcal{V}_3$ (provided $\dim \mathcal{E}_3 > 2$).*

Finally, I want to note that in [2, Theorem 5], I have specified the Weinstein theorem. It is associated with the ordering of the different frequencies in one of Weinsten's equivalence classes.

Remark 2. *(a correction) I would like to use this opportunity to make a correction to my previous paper [2]. Namely, Propositions 3 and 4 in Section 6, about the analytic property of the invariant submanifolds, are not true, at least without an additional assumption. That assumption is the center condition, i.e., that there is a family of periodic solutions at the formal level. Of course, everything is OK when the Poincaré–Dulac–Birkhoff normal forms are analytic.*

In the general case, in the proofs of Theorem 4 (in Section 8.2), Theorem 5 (in Section 8.3), and Theorem 7 (in Section 8.1), one should first use the approximation argument. It relies on the fact that the general (topological) properties of an analytic curve, defined by the fixed-point equation for the twisted Poincaré map, are determined by the polynomial approximation of this map. Therefore, we can approximate the corresponding system by a truncated Poincaré–Dulac or Birkhoff normal form. Then the corresponding invariant manifolds become analytic.

Moreover, in [2, Eq. (6.2)], the action of $\mathbb{S}^1$ (in the case of one equivalence class) was defined incorrectly; the correct formula is Eq. (2.13)

3. Proof of Theorem 1

We shall show that system (2.1) does not have nontrivial solutions.

Since Eqs. (2.1) are homogeneous and we assume $r \neq 0$, we put

$$r = 1$$

(dehomogenization); thus, we replace u with ru and v with rv. We get the algebraic system

$$\begin{aligned} f &:= \operatorname{Im}(u + \varepsilon\bar{v}) = 0, \\ g &:= u\{2\operatorname{Re} uv - 3u - 3\varepsilon\bar{v}\} + \left(|u|^2 + |v|^2\right)\bar{v} + 1 = 0, \\ h &:= v\{2\operatorname{Re} uv + 3u + 3\varepsilon\bar{v}\} + \left(|u|^2 + |v|^2\right)\bar{u} + \varepsilon = 0. \end{aligned} \tag{3.1}$$

We treat Eqs. (3.1) as a system of five real algebraic equations on six real variables $\operatorname{Re}\varepsilon$, $\operatorname{Im}\varepsilon$, $\operatorname{Re}u$, $\operatorname{Im}u$, $\operatorname{Re}v$, $\operatorname{Im}v$, i.e., in $\mathbb{R}^6$. It defines an algebraic variety $C \subset \mathbb{R}^6$. The projection of the variety C to the $\varepsilon-$plane is an semi-algebraic variety $D \subset \mathbb{R}^2$ consisting of those parameters for which there exists a 1−parameter family of periodic solutions to the perturbed Hamiltonian system near $z(t) \approx \left(re^{it}, rue^{it}, rve^{-it}\right)$. For ε's outside D, there are no such periodic solutions. Our goal is to prove that the intersection of D with a neighborhood of $\varepsilon = 0$ is 1–dimensional, a union of germs of irreducible curves (its components).

For this, it is enough to show that the part of the variety C above a neighborhood of $\varepsilon = 0$ is a 1−dimensional algebraic curve, a union of irreducible local curves (components).

Assume firstly that

$$\varepsilon = 0. \tag{3.2}$$

Lemma 1. *Under assumption (3.2), system (3.1) has three solutions:*
$\tilde{u}_0 = 0, \tilde{v}_0 = -1$*;*
$\tilde{u}_{1,2} = \pm\sqrt{\frac{3}{8}\left(4\sqrt{6}-9\right)} \approx \pm 0.54702,\ \tilde{v}_{1,2} = \frac{1}{4}\left(2-\sqrt{6}\right) \approx -0.11237.$

Proof. Indeed, by the first of Eqs. (3.1) we can assume that u is real. Then the third of these equations factorizes,

$$u\left\{2v\mathrm{Re}v + 3v + u^2 + |v|^2\right\} = 0.$$

If $u = 0$ then the second of Eqs. (3.1) gives the solution $(\tilde{u}_0, \tilde{v}_0)$.

Otherwise, v is also real and we have

$$u^2 = -3v(v+1).$$

Then the second equation gives $3u^2(v-1)+v^3+1 = 0$, i.e., $-8v^3+9v+1 = -(v+1)\left(-8v+8v^2-1\right) = 0$, with the additional values $v = \frac{1}{4}\left(2 \pm \sqrt{6}\right)$. But only for $v = \tilde{v}_1 = \tilde{v}_2 = \frac{1}{4}\left(2-\sqrt{6}\right)$ the quantity $-3v(v+1)$ is positive and gives $u = \tilde{u}_{1,2} = \pm\sqrt{-3\tilde{v}_1(\tilde{v}_1+1)}$ form the thesis of the lemma. □

Let now

$$\varepsilon = \varepsilon_1 + \mathrm{i}\varepsilon_2 \neq 0. \tag{3.3}$$

We claim that no solution to system (3.1) bifurcates as ε approaches zero along a generic ray.

Lemma 2. *No solution bifurcates from infinity. Namely, there exist* $\varepsilon_0 > 0$ *and* $R > 0$ *such that system (3.1), with* $|\varepsilon| < \varepsilon_0$*, does not have solutions in* $\{R < |u| + |v| < \infty\}$.

Proof. Let us sum up the left-hand side of the second of Eqs. (3.1) multiplied by v and the third multiplied by u. We get

$$4uv\mathrm{Re}(uv) + \left(|u|^2 + |v|^2\right)^2 + \ldots,$$

where the dots mean lower-degree terms. This expression is separated from 0 for large $|u| + |v|$ and small $|\varepsilon|$. □

Therefore, any eventual solution could bifurcate only from one of the points $\left(\tilde{u}_j, \tilde{v}_j\right)$.

Lemma 3. *No solution bifurcates from* $\left(\tilde{u}_j, \tilde{v}_j\right)$*,* $j = 0, 1, 2$*, as* ε *varies in a neighborhood of* $\varepsilon = 0$ *outside a finite number of real analytic curves. Namely, for any* $\delta > 0$*, there exists* $\varepsilon_0 = \varepsilon_0(\delta) > 0$ *and a finite collection of germs* $(D_k, 0) \subset (\mathbb{C}, 0)$ *of real analytic curves such that, if* $0 < |\varepsilon| < \varepsilon_0$ *and* $\varepsilon \notin \bigcup D_k$*, then system (3.1) does not have solutions in* $\bigcup\limits_j \left\{\left|u - \tilde{u}_j\right| + \left|v - \tilde{v}_j\right| < \delta\right\}$.

Proof. Putting

$$u = \tilde{u}_j + u_1 + \mathrm{i}u_2, v = \tilde{v}_j + v_1 + \mathrm{i}v_2, \tag{3.4}$$

with $\varepsilon = \varepsilon_1 + \mathrm{i}\varepsilon_2$, we can treat Eqs. (3.1), near $\left(\tilde{u}_j, \tilde{v}_j, 0\right)$ as a system of five real analytic equations in six real variables $u_{1,2}$, $v_{1,2}$ and $\varepsilon_{1,2}$, i.e., in $\left(\mathbb{R}^6, 0\right)$.

One solves system (3.1) by successive approximations; first, considering the linear approximation of system (3.1), we find initial (linear) terms of the Puiseux type expansions of a corresponding component C_k. Next, one finds several further terms using higher-order terms in system (3.1) and the linear approximation of C_k, etc. In the expansion of the curves C_1 and C_2, the corresponding coefficients will not be exact, only approximate, like the values of $\tilde{u}_{1,2}$ and $\tilde{v}_{1,2}$ in Lemma 1.

In **Case** $(\tilde{u}_0, \tilde{v}_0)$ we arrive at the following system:

$$f = u_2 - \varepsilon_2 + v_1\varepsilon_2 - v_2\varepsilon_1 = 0,$$

$$\begin{gathered} g = 3v_1 - 5u_1^2 + 2u_2^2 + 3\varepsilon_1 u_1 - 3\varepsilon_2 u_2 \\ +\mathrm{i}\left(-v_2 - 7u_1u_2 + 3\varepsilon_2 u_1 + 3\varepsilon_1 u_2\right) + \ldots = 0, \\ h = 4\varepsilon_1 - u_1 + u_1u_2^2 + u_1^3 + \mathrm{i}\left(4\varepsilon_2 - 4u_2 - u_2^3 - u_1^2 u_2\right) + \ldots = 0, \end{gathered}$$

where the dots in the second equation mean quadratic and cubic terms containing v_j and the dots in the third equation mean cubic terms with v_j. The linear parts of these equations define the plane

$$\varepsilon_1 = \frac{1}{4}u_1, \ \varepsilon_2 = u_2, \ v_1 = v_2 = 0;$$

note that f and $\mathrm{Im}h$ have proportional linear parts. This plane is parametrized by u_1 and u_2; we can assume that ε_i and v_j are functions of $u_{1,2}$.

Taking into account nonlinear terms in the second and third equations, with $\varepsilon_1 \approx \frac{1}{4}u_1$ and $\varepsilon_2 \approx u_2$, we get

$$\begin{array}{ll} v_1 = \frac{17}{12}u_1^2 - \frac{2}{3}u_2^2 + \ldots, & v_2 = -\frac{13}{4}u_1u_2 + \ldots, \\ \varepsilon_1 = \frac{1}{4}u_1\left(1 - |u|^2\right) + \ldots, & \varepsilon_2 = u_2\left(1 + \frac{1}{4}|u|^2\right) + \ldots, \end{array} \tag{3.5}$$

where now the dots mean higher-order terms in u. Then the first equation implies

$$f = \frac{1}{48}u_2\left(95u_1^2 - 44u_2^2\right) + \ldots = 0. \tag{3.6}$$

Eqs (3.5)–(3.6) define the curve C_0. It has three components: one is defined by $u_2 = O\left(u_1^2\right)$ (and Eqs. (3.5)), one by $u_2 = \sqrt{44/95}u_1 + O\left(u_1^2\right)$ and one by $u_2 = -\sqrt{44/95}u_1 + O(u_1^2)$. The projection D_0 of C_0 has at most three components, defined by:

$$\varepsilon_2 = O\left(\varepsilon_1^2\right), \ \varepsilon_2 = 4\sqrt{\frac{95}{44}}\varepsilon_1 + O\left(\varepsilon_1^2\right), \ \varepsilon_2 = -4\sqrt{\frac{95}{44}}\varepsilon_1 + O\left(\varepsilon_1^2\right). \tag{3.7}$$

Consider **Case** $(\tilde{u}_1, \tilde{v}_1) \approx (0.54702, -0.11237)$. By abuse of notation, we put

$$\varepsilon = \varepsilon_1 + \mathrm{i}\varepsilon_2, \ \epsilon = \bar{\varepsilon} \tag{3.8}$$

and replace u, $\bar{u}$, v, and $\bar{v}$, by

$$\begin{aligned} u &\to 0.54702 + u, \quad & \bar{u} &\to 0.54702 + w, \\ v &\to -0.11237 + v, \quad & \bar{v} &\to -0.11237 + t, \end{aligned} \tag{3.9}$$

respectively.

The notation $\approx$ above and below will mean that the coefficients are not exact, only approximate. Dealing with the exact values given in Lemma 1 would lead only to much more complicated expressions without affecting the conclusions.

We have $f = f_1 + f_2$, $g = g_1 + g_2 + \dots$, $h = h_1 + h_2 + \dots$, where f_j, g_j, h_j are homogeneous polynomials of degree j. Namely,

$$\begin{aligned} 2f_1 &\approx u - w + 0.11237\,\epsilon - 0.11237\,\varepsilon, \\ 2f_2 &= \varepsilon t - \epsilon v, \\ g_1 &\approx 0.62372\,t - 3.5280\,u + 0.31186\,v - 0.12294\,w + 0.18441\,\varepsilon, \\ g_2 &\approx -1.6411\,t\varepsilon + 0.33711\,u\varepsilon + 1.094\,tu - 0.22474\,tv + 1.094\,tw + 1.094\,uv \\ &\quad -0.22474\,uw - 0.11237\,t^2 - 3.1124\,u^2, \\ h_1 &\approx 1.3952\,v - 2.2 \times 10^{-2}\,u - 0.12294\,t + 0.62372\,w + 1.0379\,\varepsilon, \\ h_2 &\approx -0.33711\,t\varepsilon - 0.33711\,v\varepsilon - 0.22474\,tw + 2.7753\,uv + 1.094\,uw \\ &\quad -0.22474\,vw + 0.54702\,v^2 + 0.54702\,w^2. \end{aligned}$$

First, we solve the corresponding linear equations. From $2f_1 = 0$, we express u as linear function of $w, v, t, \varepsilon, \epsilon$, and substitute it to $g_1 = 0$; then we get w as a linear function of $v, t, \varepsilon, \epsilon$, and also u becomes expressed via $v, t, \varepsilon, \epsilon$. Finally, we substitute these u and w to $h_1 = 0$; we get a linear complex equation for v, $t = \bar{v}$; by comparing the real and imaginary parts, we get

$$v \approx -0.74923\,\varepsilon_1 - 0.63564\,\mathrm{i}\varepsilon_2 + O\left(|\varepsilon|^2\right).$$

Then we find

$$u \approx -0.14148\,\varepsilon_1 + 0.11237\,\mathrm{i}\varepsilon_2 + O\left(|\varepsilon|^2\right),\ w \approx -0.14148\,\varepsilon_1 - 0.11237\,\mathrm{i}\varepsilon_2 + O\left(|\varepsilon|^2\right).$$

We see that u and v become functions of ε_1 and ε_2.

Moreover, this suggests that $w = \bar{u}$ in a linear approximation. If w were not equal $\bar{u}$ for all ε_j then the condition $w = \bar{u}$ would imply a linear restriction for ε_j's.

Next, we expand the solutions v, u and w in powers of ε_j modulo $O\left(|\varepsilon|^3\right)$. For this, we firstly express f_2, g_2, and h_2 via ε_j, and then we repeat calculations for v, u, and w. We get

$$\begin{aligned} 2f_2 &\approx -954.11\,\mathrm{i}\varepsilon_1\varepsilon_2 + O\left(|\varepsilon|^3\right), \\ g_2 &\approx 65688.\,\varepsilon_1^2 + 751.70\,\mathrm{i}\varepsilon_1\,\varepsilon_2 + 1.0745\,\varepsilon_2^2 + O\left(|\varepsilon|^3\right), \\ h_2 &\approx 2.8566 \times 10^5\,\varepsilon_1^2 + 712.06\,\mathrm{i}\varepsilon_1\varepsilon_2 - 1.5879 \times 10^{-2}\,\varepsilon_2^2 + O\left(|\varepsilon|^3\right). \end{aligned}$$

Next, we find

$$v \approx -0.74923\,\varepsilon_1 - 0.63564\,\mathrm{i}\varepsilon_2 - 2.0793 \times 10^5\,\varepsilon_1^2 - 0.11242\,\varepsilon_2^2 - 176.82\,\mathrm{i}\varepsilon_1\varepsilon_2$$

$$
\begin{aligned}
& +O\left(|\varepsilon|^3\right), \\
u &\approx -0.14148\,\varepsilon_1 + 0.11237\,\mathrm{i}\varepsilon_2 - 35292.\,\varepsilon_1^2 + 0.2655\,\varepsilon_2^2 + 253.11\,\mathrm{i}\varepsilon_1\varepsilon_2 \\
& +O\left(|\varepsilon|^3\right), \\
w &\approx -0.14148\,\varepsilon_1 - 0.11237\,\mathrm{i}\varepsilon_2 - 35292.\,\varepsilon_1^2 + 0.2655\,\varepsilon_2^2 - 701.00\,\mathrm{i}\varepsilon_1\varepsilon_2 \\
& +O\left(|\varepsilon|^3\right).
\end{aligned}
$$

We see that the real parts of $\bar{u}$ and w agree, but the imaginary parts disagree:

$$
\bar{u} - w \approx 447.89\,\mathrm{i}\varepsilon_1\varepsilon_2 + O\left(|\varepsilon|^3\right). \tag{3.10}
$$

This suggests that the projected curve D_1 has two components, one defined by $\{\varepsilon_1 \approx 0\}$ and, the other defined by $\{\varepsilon_2 \approx 0\}$; also, the curve C_1 has two components. (If also the real parts of u and w disagreed, then we would have another restriction on ε_j's.)

Consider now **Case** $(\tilde{u}_2, \tilde{v}_2) \approx (-0.54702, -0.11237)$. We follow the method from the previous case. Using notations analogous to (3.8)–(3.9), we get

$$
\begin{aligned}
2f_1 &\approx u - w - 0.11237\,\epsilon + 0.11237\,\varepsilon,\ 2f_2 = \varepsilon t - \epsilon v, \\
g_1 &\approx 0.62372\,t + 3.5280\,u + 0.31186\,v + 0.12294\,w - 0.18441\,\varepsilon, \\
g_2 &\approx 1.6411\,t\varepsilon + 0.33711\,u\varepsilon - 1.094\,tu - 0.22474\,tv - 1.094\,tw - 1.094\,uv \\
& \quad -0.22474\,uw - 0.11237\,t^2 - 3.1124\,u^2, \\
h_1 &\approx 0.12294\,t - 0.62371\,u - 1.3952\,v + 2.5254\times 10^{-2}\,w + 1.0379\,\varepsilon, \\
h_2 &\approx -0.33711\,t\varepsilon - 0.33711\,v\varepsilon - 1.094\,tv - 0.22474\,tw + 2.7753\,uv \\
& \quad -0.22474\,vw - 0.54702\,v^2 + 0.54702\,w^2.
\end{aligned}
$$

The solution of the linear equations gives

$$
v \approx 0.90062\,\varepsilon_1 + 0.63563\,\mathrm{i}\varepsilon_2 + O\left(|\varepsilon|^2\right)
$$

and

$$
u \approx -0.18028\,\varepsilon_1 + 0.11237\,\mathrm{i}\varepsilon_2 + O\left(|\varepsilon|^2\right),\ w \approx -0.18028\,\varepsilon_1 - 0.11237\,\mathrm{i}\varepsilon_2 + O\left(|\varepsilon|^2\right).
$$

Again, we find that u and v become functions of ε_1 and ε_2 and that $w \approx \bar{u}$ in linear approximation.

Substituting the above values into quadratic parts of our equations, we find

$$
\begin{aligned}
2f_2 &\approx 0.52998\,\mathrm{i}\varepsilon_1\varepsilon_2 + O\left(|\varepsilon|^3\right), \\
g_2 &\approx 1.5682\,\varepsilon_1^2 + 0.43066\,\mathrm{i}\varepsilon_1\varepsilon_2 + 1.0745\,\varepsilon_2^2 + O\left(|\varepsilon|^3\right), \\
h_2 &\approx -2.2981\,\varepsilon_1^2 - 1.203\,\mathrm{i}\varepsilon_1\varepsilon_2 - 0.42613\,\varepsilon_2^2 + O\left(|\varepsilon|^3\right).
\end{aligned}
$$

Then the solutions up to $O\left(|\varepsilon|^3\right)$ are the following:

$$
v \approx 0.90062\,\varepsilon_1 + 0.63563\,\mathrm{i}\varepsilon_2 - 1.8241\,\varepsilon_1^2 - 0.22343\,\varepsilon_2^2
$$

$$
\begin{aligned}
& -0.70624\ \mathrm{i}\varepsilon_1\varepsilon_2 + O\left(|\varepsilon|^3\right), \\
u &\approx -0.18028\ \varepsilon_1 + 0.11237\ \mathrm{i}\varepsilon_2 + 3.7904\times 10^{-2}\ \varepsilon_1^2 - 0.23705\ \varepsilon_2^2 \\
& -0.19613\ \mathrm{i}\varepsilon_1\varepsilon_2 + O\left(|\varepsilon|^3\right), \\
w &\approx -0.18028\ \varepsilon_1 - 0.11237\ \mathrm{i}\varepsilon_2 + 3.7904\times 10^{-2}\ \varepsilon_1^2 - 0.23705\ \varepsilon_2^2 \\
& +0.33385\ \mathrm{i}\varepsilon_1\varepsilon_2 + O\left(|\varepsilon|^3\right).
\end{aligned}
$$

We see that the real parts of w and $\bar{u}$ agree, but

$$\bar{u} - w \approx -0.13772\ \mathrm{i}\varepsilon_1\varepsilon_2 + O\left(|\varepsilon|^3\right). \tag{3.11}$$

Like in the previous case, the curve C_2 and its projection D_2 have two components. □

Use of AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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