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Research article

Spatial twisted central configuration for Newtonian $(2N+1)$-body problem

Liang Ding[1,2,*], **Jinrong Wang**[1] **and Jinlong Wei**[3]

[1] Department of Mathematics, Guizhou University, Guiyang, 550025, P.R. China
[2] School of Data Science and Information Engineering, Guizhou Minzu University, Guiyang, 550025, P.R. China
[3] School of Statistics and Mathematics, Zhongnan University of Economics and Law, Wuhan, 430073, P.R. China

* **Correspondence:** E-mail: ding2016liang@gzmu.edu.cn, lovemathlovemath@126.com.

Abstract: For a spatial twisted central configuration of the Newtonian $(2N+1)$-body problem where $2N$ masses are at the vertices of two paralleled regular N-polygons with distance $h > 0$, and the twist angle between the two regular N-polygons is $0 \le \theta < 2\pi$, we study the sufficient and necessary conditions for the existence of the spatial twisted central configuration. Additionally, we obtain the uniqueness of the spatial twisted central configuration.

Keywords: Newtonian $(2N+1)$-body problem; spatial central configuration; regular N-polygons; twist angle
Mathematics Subject Classification: 70F10, 70F15

1. Introduction

For the spatial Newtonian n-body problem, the equations of motion for the n masses $m_k > 0$ and positions $x_k \in \mathbb{R}^3$ with $k \in \{1, \ldots, n\}$ can be described by Newton's second law and Newton's universal gravitation law:

$$m_k \ddot{x}_k = \frac{\partial\Big(\sum\limits_{1 \le s < j \le n} \dfrac{m_j m_s}{|x_j - x_s|}\Big)}{\partial x_k}.$$

Define

$$\Omega = \{x : \; x = (x_1, x_2, \ldots, x_n) \in (\mathbb{R}^3)^n\},$$

and let

$$\triangle = \bigcup_{1 \le j \ne s \le n} \{x = (x_1, x_2, \ldots, x_n) \,|\, x_j = x_s, \; 1 \le j \ne s \le n\}$$

be the collision set. As usual, the set $\Omega \setminus \triangle$ is called the **configuration space**. First, we introduce the definition of central configuration for the Newtonian n-body problem (see [1]).

Definition 1.1. Given n masses $m_k > 0$ with positions $q_k \in \mathbb{R}^3$, $k = 1, \ldots, n$, we say a configuration $q \in \Omega \setminus \triangle$ is a **central configuration** at some moment if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \sum\limits_{\substack{j \neq k \\ 1 \leq j \leq n}} \frac{m_j m_k (q_j - q_k)}{|q_j - q_k|^3} = -\lambda m_k (q_k - x_0), \qquad k = 1, 2, \ldots, n, \\ x_0 = \dfrac{\sum\limits_{1 \leq k \leq n} m_k q_k}{\sum\limits_{1 \leq k \leq n} m_k}. \end{cases} \tag{1.1}$$

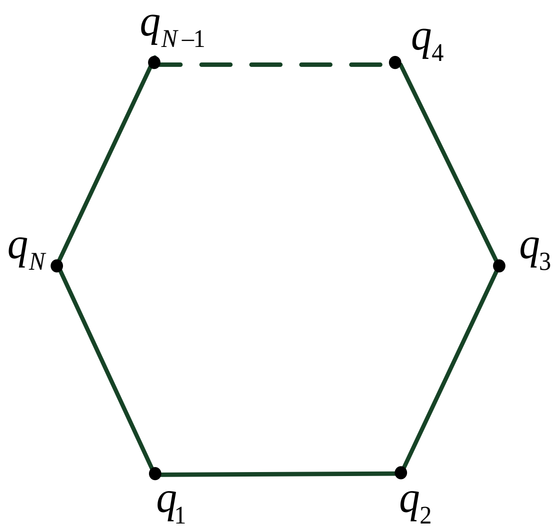

Figure 1. Planar N-body problem.

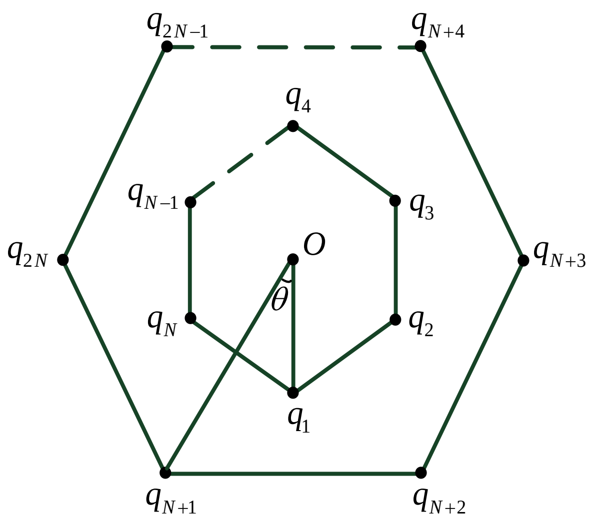

Figure 2. Planar $2N$-body problem.

Central configurations play a very important role in the study of the Newtonian n-body problem, and especially, central configurations can lead to rigid-motion solutions and homothetically collapsing solutions [1]. Central configurations of the Newtonian three-body ($n = 3$) problem with any given three masses have long been known, and there are always exactly two kinds of central configurations: Euler collinear central configuration and Lagrange equilateral-triangle central configuration [2, 3]. For a planar Newtonian N-body problem with $n = N \geq 4$, Perko and Walter [4] proved that if N masses are located at the vertices of a regular N-polygon (see Figure 1), then they can form a regular polygonal central configuration if and only if all the values of N masses are equal to each other. For more results of planar central configuration with one regular N-polygon, one can refer to [5–8].

For a planar central configuration with $n = 2N$ and $N \geq 2$ such that two regular N-polygons are concentric and that $2N$ equal masses are placed at the vertices of the two regular N-polygons (see Figure 2), Zhang and Zhou [9] proved that the values of masses in each separate regular N-polygon were equal. We say that p regular N-polygons with $p \geq 2$ are nested if they are coplanar and have the same number of vertices N and the same center, and the positions of the vertices of the innermost regular N-polygon $\mathbf{R}_j^{(1)}$ and those of the remaining p-1 regular N-polygons $\mathbf{R}_j^{(k)}$ with any

$k \in \{2, \ldots, p\}$ satisfy the relation that $\mathbf{R}_j^{(p)} = s_1\mathbf{R}_j^{(p-1)} = s_2\mathbf{R}_j^{(p-2)} = \ldots = s_{p-1}\mathbf{R}_j^{(1)}$ for some scale factors $s_{p-1} > s_{p-2} > \ldots > s_1 > 1$ and for all $j = 1, 2, \ldots, N$. For the central configuration such that two regular N-polygons are nested, masses on different regular N-polygons may be different, and Moeckel and Simó [10] proved that for every mass ratio b between the two masses, there were exactly two planar central configurations. Also, for the case of $n = 2N$ such that N equal masses are placed at the vertices of one regular N-polygon and the remaining N equal masses are placed at the vertices of the other regular N-polygon, which is rotated exactly at an angle $\theta = \pi/N$ with respect to the former regular N-polygon, Barrabés and Cors [11] proved the existence of the planar central configuration with any value of the mass ratio. For the case of $n = pN$ and $p \geq 2$, Corbera, Delgado, and Llibre [12] proved the existence of the nested central configuration such that pN masses were at the vertices of the p nested regular N-polygon with a common center. Moreover, all the masses on the same regular N-polygon were equal, but masses on a different regular N-polygon could be different. For the case of $n = pN + gN$ with $p \geq 1$ and $g \geq 1$, Zhao and Chen [13] proved the existence of planar central configurations such that p regular N-polygons were nested, and g regular N-polygons were rotated exactly at an angle π/N with respect to the other ones. For more details in this direction, we refer to [14–21] and the references therein.

Note that for a planar central configuration with $n = N + 1$, Chen and Luo [22] proved that if N masses are located at the vertices of one regular N-polygon and the position of the (N+1)-th mass is on the plane containing the regular N-polygon (see Figure 3), then all the values of N masses located at the vertices of the regular N-polygon are equal to each other. For a spatial central configuration with $n = N + 1$ and the (N+1)-th mass off the plane containing the regular N-polygon (see Figure 4), Ouyang, Xie, and Zhang [23] showed that the distance between the (N+1)-th mass and the regular N-polygon was unique.

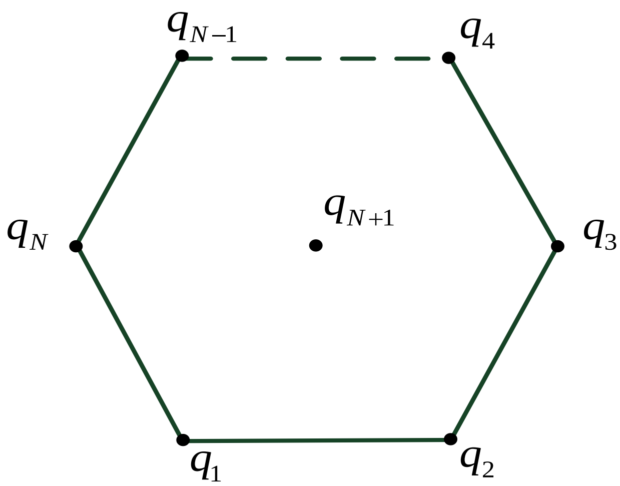

Figure 3. Planar (N+1)-body problem.

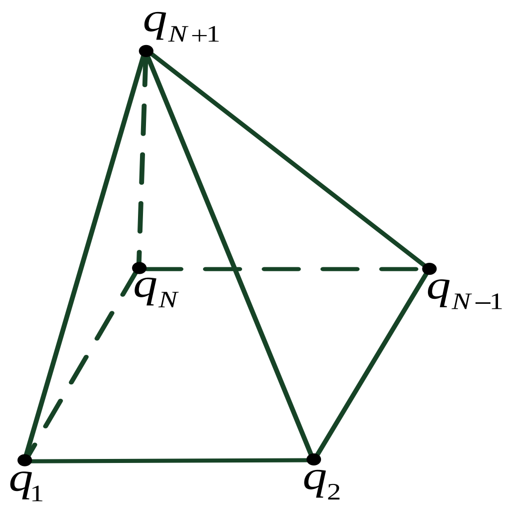

Figure 4. Spatial (N+1)-body problem.

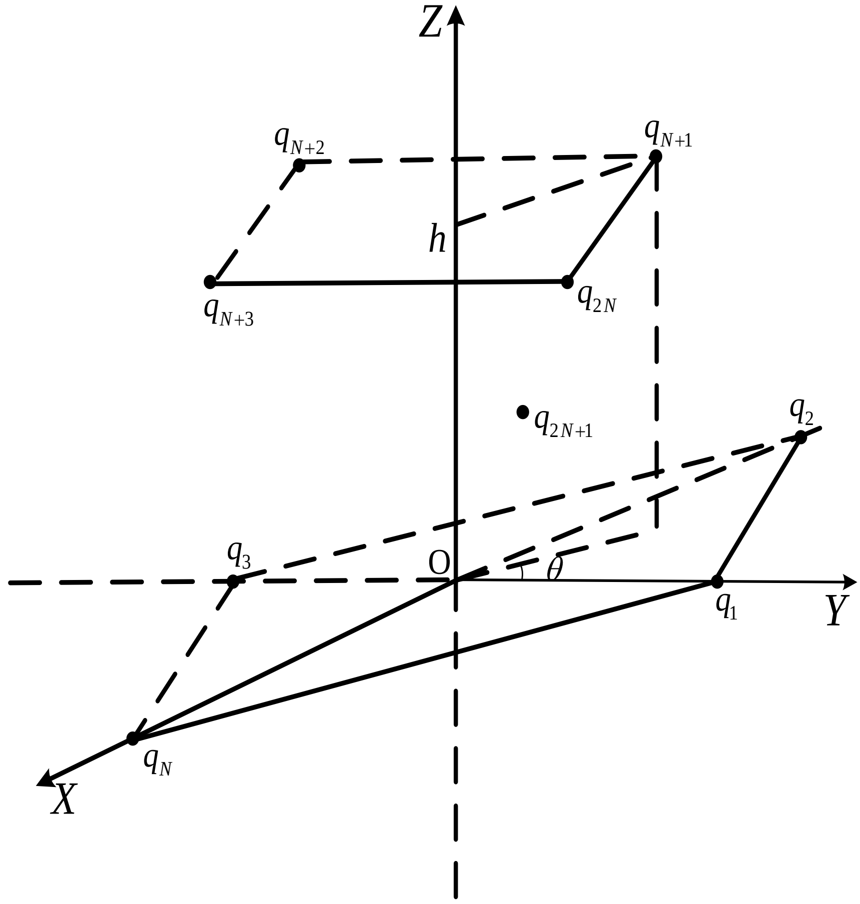

Figure 5. Spatial $(2N+1)$-body problem.

In this paper, we consider the spatial central configuration of a Newtonian $(2N+1)$-body problem in $\mathbb{R}^3$ formed by $2N$ masses placed at the vertices of two paralleled regular N-polygons with distance $h > 0$. It is assumed that the lower layer regular N-polygon lies in a horizontal plane, and the upper regular N-polygon parallels the lower one in $\mathbb{R}^3$ with distance h, and the z-axis passes through both centers of the two regular N-polygons (see Figure 5). For convenience, when choosing the coordinates, we treat $\mathbb{R}^3$ as the direct product of the complex plane and real axis. For the positions of the $2N+1$ masses $q = (q_1, q_2, \ldots, q_{2N}, q_{2N+1}) \in \Omega\backslash\triangle$, we have

$$\begin{cases} q_k = (\rho_k,\ 0), \quad k = 1, \ldots, N, \\ q_l = (a\rho_l e^{i\theta},\ h), \quad 0 \le \theta < 2\pi, \quad l = N+1, \ldots, 2N, \quad a > 0, \quad h > 0, \end{cases} \tag{1.2}$$

where a is the ratio of the sizes of the two regular N-polygons, ρ_d is the $d(\text{mod } N)$-th complex root of unity, i.e., $\rho_k = e^{i\theta_k}$ with $k = 1, 2, \ldots, N$, and $\rho_l = e^{i\theta_l}$ with $l = N+1, N+2, \ldots, 2N$ and $\theta_d = 2d\pi/N$ with $d \in \mathbb{Z}$. Here, we define θ as the twist angle between the two paralleled regular N-polygons with distance $h > 0$. Moreover, for the position of the $(2N+1)$-th mass and the barycenter $x_0 = (c_0,\ h_0)$, we define

$$
\begin{cases}
q_{2N+1} = (a_1 e^{i\alpha},\ h_{2N+1}),\ \ a_1 \geq 0,\ \ 0 \leq \alpha < 2\pi,\ \ -\infty < h_{2N+1} < +\infty, \\
c_0 = \dfrac{\sum\limits_{1\leq k\leq N} m_k\rho_k + \sum\limits_{N+1\leq l\leq 2N} a m_l \rho_l e^{i\theta} + a_1 m_{2N+1} e^{i\alpha}}{m_{2N+1} + \sum\limits_{1\leq k\leq N} m_k + \sum\limits_{N+1\leq l\leq 2N} m_l}, \\
h_0 = \dfrac{\sum\limits_{N+1\leq l\leq 2N} m_l h + m_{2N+1} h_{2N+1}}{m_{2N+1} + \sum\limits_{1\leq k\leq N} m_k + \sum\limits_{N+1\leq l\leq 2N} m_l}.
\end{cases}
\tag{1.3}
$$

Then, for the spatial twisted configuration with $n = 2N + 1$ and the notations (1.2)–(1.3), we have the following results.

For the existence, we have the following:

Theorem 1.1. *Suppose the values of N masses with $N \geq 2$ located at the vertices of one regular N-polygon are equal to each other, and all of the sides in the two regular N-polygons have the same size. Define the position of q_{2N+1} as (1.3). Then, the $2N+1$ masses form a central configuration if and only if all the values of the $2N$ masses located at the vertices of the two regular N-polygons are equal to each other, $a_1 = 0$ and $h_{2N+1} = h/2$, and the twist angle is $\theta = s\pi/N$ with $s \in \{0, 1, \ldots, 2N - 1\}$.*

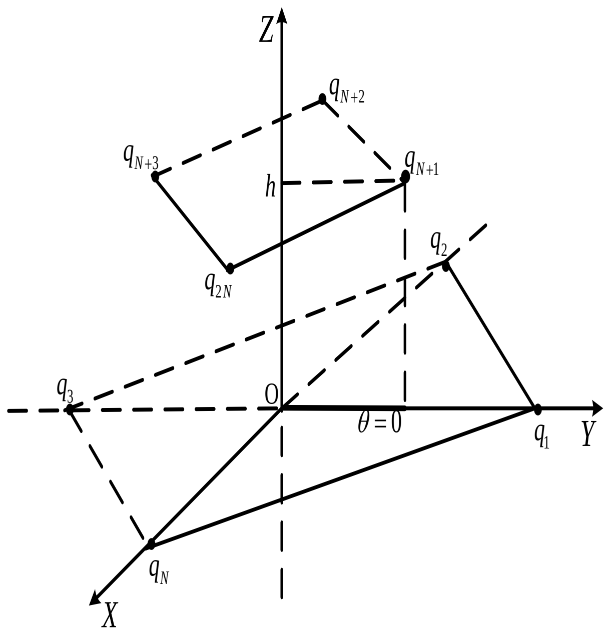

Figure 6. Spatial $2N$-body problem with $\theta = 0$.

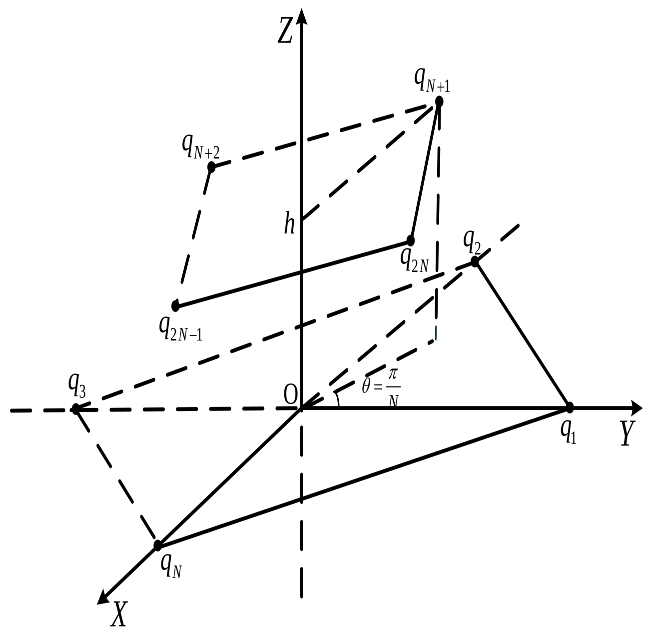

Figure 7. Spatial $2N$-body problem with $\theta = \pi/N$.

Remark 1.1. *For the spatial twisted central configuration of the Newtonian $2N$-body problem, under the assumption that the values of masses in each separate regular N-polygon were equal, Yu and Zhang [24] proved that the twist angle must be $\theta = 0$ or $\theta = \pi/N$ (see Figures 6 and 7). Meanwhile,* *in Theorem 1.1, we consider the spatial twisted central configuration of the Newtonian (2N+1)-body problem. Under the assumptions that the values of the N masses located at the vertices of one regular N-polygon are equal and that all of the sides in the two regular N-polygons have the same size, we not only obtain the values of the twist angle; but also prove that all* $2N$ *masses must be equal. Moreover, we know the position of the (2N+1)-th mass is* $(0, 0, h/2)$.

For the uniqueness, we have the following:

Theorem 1.2. *For the spatial twisted central configuration, if the values of the N masses located at the vertices of one regular N-polygon are equal to each other and all of the sides in the two regular N-polygons have the same size, then for any* $N \geq 2$*, both the distance between the two regular N-polygons and the position of the (2N+1)-th mass are unique.*

2. Preliminaries

Lemma 2.1. *[24, Lemma 2.9] For any* $a > 0$*, any* $\gamma \in (-\infty, +\infty)$*, any* $h > 0$*, and any* $N \geq 2$*, let*

$$f(\gamma) = \sum_{1 \leq j \leq N} \frac{a \sin(\theta_j + \gamma)}{[1 + a^2 - 2a\cos(\theta_j + \gamma) + h^2]^{\frac{3}{2}}}. \tag{2.1}$$

Then,

$$f(\frac{\pi}{N}) = 0, \quad f(-\gamma) = -f(\gamma) \text{ and } f(\gamma + \frac{2\pi}{N}) = f(\gamma).$$

Remark 2.1. *In Lemma 2.1, if we choose* $a = 1$ *and* $\gamma = \pi/N$ *where* $N \geq 2$*, then*

$$\sum_{1 \leq j \leq N} \frac{\sin(\theta_j - \frac{\pi}{N})}{[2 - 2\cos(\theta_j - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} = \sum_{1 \leq j \leq N} \frac{\sin(\theta_j + \frac{\pi}{N})}{[2 - 2\cos(\theta_j + \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} = 0, \quad \text{where } h > 0.$$

Lemma 2.2. *[24, Lemma 2.10] If* $\gamma \in (0, \pi/N)$ *with* $N \geq 2$*, then for any* $a > 0$ *and any* $h > 0$*, we have* $f(\gamma) > 0$.

Lemma 2.3. *If* $\theta = s\pi/N$ *with* $s \in \{0, 1, \ldots, 2N - 1\}$ *and* $N \geq 2$*, then for any* $k' \in \{1, 2, \ldots, N\}$*, any* $l' \in \{N + 1, N + 2, \ldots, 2N\}$*, and any* $h > 0$*, we have*

$$\begin{cases} \sum\limits_{1 \leq k \leq N} \frac{e^{i(\theta_{k-l'} - \theta)} - 1}{[|e^{i(\theta_{k-l'} - \theta)} - 1|^2 + h^2]^{\frac{3}{2}}} = \sum\limits_{N+1 \leq l \leq 2N} \frac{e^{i(\theta_{l-k'} + \theta)} - 1}{[|e^{i(\theta_{l-k'} + \theta)} - 1|^2 + h^2]^{\frac{3}{2}}} = \sum\limits_{1 \leq j \leq N} \frac{e^{i(\theta_j + \theta)} - 1}{[|e^{i(\theta_j + \theta)} - 1|^2 + h^2]^{\frac{3}{2}}} \in \mathbb{R}, \\ \sum\limits_{1 \leq k \leq N} \frac{h}{[|e^{i(\theta_{k-l'} - \theta)} - 1|^2 + h^2]^{\frac{3}{2}}} = \sum\limits_{N+1 \leq l \leq 2N} \frac{h}{[|e^{i(\theta_{l-k'} + \theta)} - 1|^2 + h^2]^{\frac{3}{2}}} = \sum\limits_{1 \leq j \leq N} \frac{h}{[|e^{i(\theta_j + \theta)} - 1|^2 + h^2]^{\frac{3}{2}}}. \end{cases} \tag{2.2}$$

Proof. Let $\hat{\mu} \in \{1, 2, \ldots, N\}$, $\tilde{\mu} \in \{0, 1, \ldots, N - 1\}$, $\tilde{\alpha} \in (-2\pi, 2\pi)$, and $\kappa \in \{0, 1\}$. We define a mapping σ by

$$\begin{cases} \{\kappa N + 1, \kappa N + 2, \ldots, \kappa N + N\} \xrightarrow{\sigma} \{\tilde{\alpha}, \frac{2\pi}{N} + \tilde{\alpha}, ..., \frac{2(N-1)\pi}{N} + \tilde{\alpha}\}, \\ \sigma(\mu) = \Big[(\mu - \hat{\mu} + \tilde{\mu})(\text{mod}\, N)\Big]\frac{2\pi}{N} + \tilde{\alpha}, \quad \forall\, \mu \in \{\kappa N + 1, \kappa N + 2, \ldots, \kappa N + N\}. \end{cases} \tag{2.3}$$

Notice that both $\{\kappa N+1, \kappa N+2, \ldots, \kappa N+N\}$ and $\{\tilde{\alpha}, (2\pi)/N+\tilde{\alpha}, \ldots, 2(N-1)\pi/N+\tilde{\alpha}\}$ are finite sets; the mapping σ is a surjection. Let us show σ is an injective mapping. The proof for $\kappa = 0$ is similar to $\kappa = 1$; we only check for $\kappa = 1$. Let $\mu_1 \neq \mu_2$ and $\mu_1, \mu_2 \in \{N+1, N+2, \ldots, 2N\}$. If $\sigma(\mu_1) = \sigma(\mu_2)$, then there exist $s_1,\ s_2 \in \mathbb{Z}$ such that

$$(\mu_1 - \hat{\mu} + \tilde{\mu}) + s_1 N = (\mu_2 - \hat{\mu} + \tilde{\mu}) + s_2 N. \tag{2.4}$$

Hence, $\mu_1 - \mu_2 = (s_2 - s_1)N$. By the facts that $-N < \mu_1 - \mu_2 < N$ and $s_2 - s_1 \in \mathbb{Z}$, $s_2 = s_1$, and thus $\mu_1 = \mu_2$, which is a contradiction. Therefore, σ is injective, which implies that σ is a bijection.

Similarly, for $l \in \{N+1, N+2, \ldots, 2N\}$, $\tilde{s} \in \{0, 1, \ldots, N-1\}$, and $\theta \in [0,\ 2\pi)$, if we define another mapping σ_1 by

$$\begin{cases} \{1, 2, \ldots, N\} \stackrel{\sigma_1}{\longrightarrow} \{\theta, \frac{2\pi}{N} + \theta, \ldots, \frac{2(N-1)\pi}{N} + \theta\}, \\ \sigma_1(k') = \Big[(l - k' + \tilde{s})(\operatorname{mod} N)\Big]\frac{2\pi}{N} + \theta, \quad \forall\, k' \in \{1, 2, \ldots, N\}, \end{cases} \tag{2.5}$$

then σ_1 is a bijection as well. Together with the fact that $\theta = s\pi/N$ with $s \in \{0, 1, \ldots, 2N-1\}$ is equivalent to $\theta = 2\tilde{s}\pi/N$ or $\theta = 2\tilde{s}\pi/N + \pi/N$ with $\tilde{s} \in \{0, 1, \ldots, N-1\}$, let us show (2.2) holds by considering the following two cases.

Case 1. $\theta = 2\tilde{s}\pi/N$ with $\tilde{s} \in \{0, 1, \ldots, N-1\}$.

Observing that $\theta_j = 2j\pi/N$ with $j \in \{1, 2, \ldots, N\}$,

$$\sum_{1 \le j \le N} \frac{\sin\theta_j}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}} = \sum_{1 \le j \le N} \frac{\sin 2\pi\frac{N-j}{N}}{[2 - 2\cos 2\pi\frac{N-j}{N} + h^2]^{\frac{3}{2}}} = -\sum_{1 \le j \le N} \frac{\sin\theta_j}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}},$$

which implies that

$$\sum_{1 \le j \le N} \frac{\sin\theta_j}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}} = 0. \tag{2.6}$$

Let $k' \in \{1, 2, \ldots, N\}$ and $\tilde{s} \in \{0, 1, \ldots, N-1\}$. In (2.3), if we choose $\mu = l$, $\hat{\mu} = k'$, $\tilde{\mu} = \tilde{s}$, $\kappa = 1$, and $\tilde{\alpha} = 0$, then the mapping

$$\begin{cases} \{N+1, N+2, \ldots, 2N\} \stackrel{\sigma}{\longrightarrow} \{0, \frac{2\pi}{N}, \ldots, \frac{2(N-1)\pi}{N}\}, \\ \sigma(l) = \Big[(l - k' + \tilde{s})(\operatorname{mod} N)\Big]\frac{2\pi}{N}, \quad \forall\, l \in \{N+1, N+2, \ldots, 2N\} \end{cases}$$

is a bijection. Combining (2.6), and $\theta_d = 2\pi d/N$ with $d \in \mathbb{Z}$ and

$$\begin{cases} \cos(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s}) = \cos\Big(\frac{2\pi}{N}[(l - k' + \tilde{s})(\operatorname{mod} N)]\Big), \\ \sin(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s}) = \sin\Big(\frac{2\pi}{N}[(l - k' + \tilde{s})(\operatorname{mod} N)]\Big) \end{cases}$$

for any $k' \in \{1, 2, \ldots, N\}$ and any $\tilde{s} \in \{0, 1, \ldots, N-1\}$, we have

$$\begin{cases} \sum\limits_{N+1 \le l \le 2N} \frac{\sin(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s})}{[2 - 2\cos(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s}) + h^2]^{\frac{3}{2}}} = \sum\limits_{N+1 \le l \le 2N} \frac{\sin\theta_{l-k'+\tilde{s}}}{[2 - 2\cos\theta_{l-k'+\tilde{s}} + h^2]^{\frac{3}{2}}} = \sum\limits_{1 \le j \le N} \frac{\sin\theta_j}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}} = 0, \\ \sum\limits_{N+1 \le l \le 2N} \frac{\cos(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s}) - 1}{[2 - 2\cos(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s}) + h^2]^{\frac{3}{2}}} = \sum\limits_{N+1 \le l \le 2N} \frac{\cos(\theta_{l-k'+\tilde{s}}) - 1}{[2 - 2\cos(\theta_{l-k'+\tilde{s}}) + h^2]^{\frac{3}{2}}} = \sum\limits_{1 \le j \le N} \frac{\cos\theta_j - 1}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}}, \\ \sum\limits_{N+1 \le l \le 2N} \frac{h}{[2 - 2\cos(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s}) + h^2]^{\frac{3}{2}}} = \sum\limits_{N+1 \le l \le 2N} \frac{h}{[2 - 2\cos(\theta_{l-k'+\tilde{s}}) + h^2]^{\frac{3}{2}}} = \sum\limits_{1 \le j \le N} \frac{h}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}}. \end{cases} \tag{2.7}$$

On the other hand, in (2.3), for any $l' \in \{N+1, N+2, \ldots, 2N\}$ and any $\tilde{s} \in \{0, 1, \ldots, N-1\}$, if we let $\mu = k$, $\hat{\mu} = l'$, $\tilde{\mu} = N - \tilde{s}$, $\kappa = 0$, and $\tilde{\alpha} = 0$, then we know that the mapping

$$\begin{cases} \{1, 2, \ldots, N\} \xrightarrow{\sigma} \{0, \frac{2\pi}{N}, \ldots, \frac{2(N-1)\pi}{N}\}, \\ \sigma(k) = \Big[(k - l' + (N - \tilde{s}))(\operatorname{mod} N)\Big]\frac{2\pi}{N} = \Big[(k - l' - \tilde{s})(\operatorname{mod} N)\Big]\frac{2\pi}{N}, \quad \forall\, k \in \{1, 2, \ldots, N\} \end{cases}$$

is a bijection as well. Thus, similar to the procedure of obtaining (2.7), for any $l' \in \{N+1, N+2, \ldots, 2N\}$, any $\tilde{s} \in \{0, 1, \ldots, N-1\}$, and any $h > 0$, we have

$$\begin{cases} \sum\limits_{1 \le k \le N} \frac{\sin(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s})}{[2 - 2\cos(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s}) + h^2]^{\frac{3}{2}}} = \sum\limits_{1 \le j \le N} \frac{\sin\theta_j}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}} = 0, \\ \sum\limits_{1 \le k \le N} \frac{\cos(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s}) - 1}{[2 - 2\cos(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s}) + h^2]^{\frac{3}{2}}} = \sum\limits_{1 \le j \le N} \frac{\cos\theta_j - 1}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}}, \\ \sum\limits_{1 \le k \le N} \frac{h}{[2 - 2\cos(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s}) + h^2]^{\frac{3}{2}}} = \sum\limits_{1 \le j \le N} \frac{h}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}}. \end{cases} \tag{2.8}$$

Employing (2.7), (2.8), and $\theta_d = 2\pi d/N$ with $d \in \mathbb{Z}$, we have

$$\begin{aligned} &\sum_{N+1 \le l \le 2N} \frac{e^{i(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s})} - 1}{[|e^{i(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s})} - 1|^2 + h^2]^{\frac{3}{2}}} = \sum_{N+1 \le l \le 2N} \frac{[\cos(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s}) - 1] + i\sin(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s})}{[2 - 2\cos(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s}) + h^2]^{\frac{3}{2}}} \\ = &\sum_{1 \le j \le N} \frac{(\cos\theta_j - 1) + i\sin\theta_j}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}} = \sum_{1 \le j \le N} \frac{\cos\theta_j - 1}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}} \\ = &\sum_{1 \le k \le N} \frac{[\cos(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s}) - 1] + i\sin(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s})}{[2 - 2\cos(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s}) + h^2]^{\frac{3}{2}}} \\ = &\sum_{1 \le k \le N} \frac{e^{i(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s})} - 1}{[|e^{i(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s})} - 1|^2 + h^2]^{\frac{3}{2}}} \in \mathbb{R} \end{aligned}$$

and

$$\begin{aligned} &\sum_{N+1 \le l \le 2N} \frac{h}{[|e^{i(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s})} - 1|^2 + h^2]^{\frac{3}{2}}} = \sum_{N+1 \le l \le 2N} \frac{h}{[2 - 2\cos(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s}) + h^2]^{\frac{3}{2}}} \\ = &\sum_{1 \le j \le N} \frac{h}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}} = \sum_{1 \le j \le N} \frac{h}{[2 - 2\cos\theta_j + h^2]^{\frac{3}{2}}} = \sum_{1 \le k \le N} \frac{h}{[2 - 2\cos(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s}) + h^2]^{\frac{3}{2}}} \\ = &\sum_{1 \le k \le N} \frac{h}{[|e^{i(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s})} - 1|^2 + h^2]^{\frac{3}{2}}}, \end{aligned}$$

where $\tilde{s} \in \{0, 1, \ldots, N-1\}$, $k' \in \{1, 2, \ldots, N\}$, and $l' \in \{N+1, N+2, \ldots, 2N\}$. Thus, (2.2) holds for the case of $\theta = 2\tilde{s}\pi/N$ with $\tilde{s} \in \{0, 1, 2, \ldots, N-1\}$.

Case 2. $\theta = 2\tilde{s}\pi/N + \pi/N$ with $\tilde{s} \in \{0, 1, 2, \ldots, N-1\}$.

For any $l' \in \{N+1, N+2, \ldots, 2N\}$ and any $\tilde{s} \in \{0, 1, \ldots, N-1\}$, in (2.3), if we choose $\mu = k$, $\hat{\mu} = l'$, $\tilde{\mu} = N - \tilde{s}$, $\kappa = 0$, and $\tilde{\alpha} = -\pi/N$, then the mapping

$$\begin{cases} \{1, 2, \ldots, N\} \xrightarrow{\sigma} \Big\{-\frac{\pi}{N}, \frac{2\pi}{N} - \frac{\pi}{N}, \ldots, \frac{2(N-1)\pi}{N} - \frac{\pi}{N}\Big\}, \\ \sigma(k) = \Big[(k - l' + (N - \tilde{s}))(\operatorname{mod} N)\Big]\frac{2\pi}{N} - \frac{\pi}{N} = \Big[(k - l' - \tilde{s})(\operatorname{mod} N)\Big]\frac{2\pi}{N} - \frac{\pi}{N}, \ \forall\, k \in \{1, 2, \ldots, N\} \end{cases}$$

is a bijection. Then, we have

$$\begin{cases} \sum\limits_{1\le k\le N} \frac{\sin(\theta_{k-l'}-\frac{2\pi}{N}\tilde{s}-\frac{\pi}{N})}{[2-2\cos(\theta_{k-l'}-\frac{2\pi}{N}\tilde{s}-\frac{\pi}{N})+h^2]^{\frac{3}{2}}} = \sum\limits_{1\le k\le N} \frac{\sin(\theta_{k-l'-\tilde{s}}-\frac{\pi}{N})}{[2-2\cos(\theta_{k-l'-\tilde{s}}-\frac{\pi}{N})+h^2]^{\frac{3}{2}}} = \sum\limits_{1\le j\le N} \frac{\sin(\theta_j-\frac{\pi}{N})}{[2-2\cos(\theta_j-\frac{\pi}{N})+h^2]^{\frac{3}{2}}}, \\ \sum\limits_{1\le k\le N} \frac{\cos(\theta_{k-l'}-\frac{2\pi}{N}\tilde{s}-\frac{\pi}{N})-1}{[2-2\cos(\theta_{k-l'}-\frac{2\pi}{N}\tilde{s}-\frac{\pi}{N})+h^2]^{\frac{3}{2}}} = \sum\limits_{1\le k\le N} \frac{\cos(\theta_{k-l'-\tilde{s}}-\frac{\pi}{N})-1}{[2-2\cos(\theta_{k-l'-\tilde{s}}-\frac{\pi}{N})+h^2]^{\frac{3}{2}}} = \sum\limits_{1\le j\le N} \frac{\cos(\theta_j-\frac{\pi}{N})-1}{[2-2\cos(\theta_j-\frac{\pi}{N})+h^2]^{\frac{3}{2}}}, \end{cases} \tag{2.9}$$

where $\tilde{s} \in \{0, 1, \ldots, N-1\}$, $l' \in \{N+1, N+2, \ldots, 2N\}$, and $h > 0$.

By the first equation of (2.9) and Remark 2.1, for any $\tilde{s} \in \{0, 1, \ldots, N-1\}$, any $l' \in \{N+1, N+2, \ldots, 2N\}$, and any $h > 0$, we see that

$$\begin{aligned} &\sum_{1\le k\le N} \frac{\sin(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s} - \frac{\pi}{N})}{[2 - 2\cos(\theta_{k-l'} + \frac{2\pi}{N}\tilde{s} - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} = \sum_{1\le j\le N} \frac{\sin(\theta_j - \frac{\pi}{N})}{[2 - 2\cos(\theta_j - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} \\ = &\sum_{1\le j\le N} \frac{\sin(\theta_j + \frac{\pi}{N})}{[2 - 2\cos(\theta_j + \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} = 0. \end{aligned} \tag{2.10}$$

Moreover, in (2.5), for any $l \in \{N+1, N+2, \ldots, 2N\}$ and any $\tilde{s} \in \{0, 1, \ldots, N-1\}$, if we choose $\theta = \pi/N$, then the mapping

$$\begin{cases} \{1, 2, \ldots, N\} \xrightarrow{\sigma_1} \left\{\frac{\pi}{N}, \frac{2\pi}{N} + \frac{\pi}{N}, \ldots, \frac{2(N-1)\pi}{N} + \frac{\pi}{N}\right\}, \\ \sigma_1(k') = \Big[(l - k' + \tilde{s})(\operatorname{mod} N)\Big]\frac{2\pi}{N} + \frac{\pi}{N}, \quad \forall\, k' \in \{1, 2, \ldots, N\} \end{cases}$$

is a bijection. Hence, combining (2.10), this leads to

$$\begin{cases} \sum\limits_{1\le k'\le N} \frac{\sin(\theta_{l-k'}+\frac{2\pi}{N}\tilde{s}+\frac{\pi}{N})}{[2-2\cos(\theta_{l-k'}+\frac{2\pi}{N}\tilde{s}+\frac{\pi}{N})+h^2]^{\frac{3}{2}}} = \sum\limits_{1\le k'\le N} \frac{\sin(\theta_{l-k'+\tilde{s}}+\frac{\pi}{N})}{[2-2\cos(\theta_{l-k'+\tilde{s}}+\frac{\pi}{N})+h^2]^{\frac{3}{2}}} = \sum\limits_{1\le j\le N} \frac{\sin(\theta_j+\frac{\pi}{N})}{[2-2\cos(\theta_j+\frac{\pi}{N})+h^2]^{\frac{3}{2}}} = 0, \\ \sum\limits_{1\le k'\le N} \frac{\cos(\theta_{l-k'}+\frac{2\pi}{N}\tilde{s}+\frac{\pi}{N})-1}{[2-2\cos(\theta_{l-k'}+\frac{2\pi}{N}\tilde{s}+\frac{\pi}{N})+h^2]^{\frac{3}{2}}} = \sum\limits_{1\le k'\le N} \frac{\cos(\theta_{l-k'+\tilde{s}}+\frac{\pi}{N})-1}{[2-2\cos(\theta_{l-k'+\tilde{s}}+\frac{\pi}{N})+h^2]^{\frac{3}{2}}} = \sum\limits_{1\le j\le N} \frac{\cos(\theta_j+\frac{\pi}{N})-1}{[2-2\cos(\theta_j+\frac{\pi}{N})+h^2]^{\frac{3}{2}}}. \end{cases} \tag{2.11}$$

Furthermore, one can verify that

$$\begin{aligned} &\sum_{1\le j\le N} \frac{\cos(\theta_j + \frac{\pi}{N}) - 1}{[2 - 2\cos(\theta_j + \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} - \sum_{1\le j\le N} \frac{\cos(\theta_j - \frac{\pi}{N}) - 1}{[2 - 2\cos(\theta_j - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} \\ = &\sum_{2\le j\le N+1} \frac{\cos(\theta_j - \frac{\pi}{N}) - 1}{[2 - 2\cos(\theta_j - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} - \sum_{1\le j\le N} \frac{\cos(\theta_j - \frac{\pi}{N}) - 1}{[2 - 2\cos(\theta_j - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} \\ = &\sum_{1\le j\le N} \frac{\cos(\theta_j - \frac{\pi}{N}) - 1}{[2 - 2\cos(\theta_j - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} - \sum_{1\le j\le N} \frac{\cos(\theta_j - \frac{\pi}{N}) - 1}{[2 - 2\cos(\theta_j - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} = 0, \end{aligned}$$

and this implies that

$$\sum_{1\le j\le N} \frac{\cos(\theta_j + \frac{\pi}{N}) - 1}{[2 - 2\cos(\theta_j + \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} = \sum_{1\le j\le N} \frac{\cos(\theta_j - \frac{\pi}{N}) - 1}{[2 - 2\cos(\theta_j - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}}. \tag{2.12}$$

Employing (2.9), (2.10), (2.11), and (2.12), for any $l, l' \in \{N+1, N+2, \dots, 2N\}$ and any $\tilde{s} \in \{0, 1, \dots, N-1\}$, we have

$$\begin{cases} \sum\limits_{1\le k\le N} \frac{\cos(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s} - \frac{\pi}{N}) - 1}{[2 - 2\cos\theta_{(k-l'} - \frac{2\pi}{N}\tilde{s} - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} = \sum\limits_{1\le k\le N} \frac{\cos(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s} + \frac{\pi}{N}) - 1}{[2 - 2\cos\theta_{(l-k'} + \frac{2\pi}{N}\tilde{s} + \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} = \sum\limits_{1\le j\le N} \frac{\cos(\theta_j - \frac{\pi}{N}) - 1}{[2 - 2\cos(\theta_j - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}}, \\ \sum\limits_{1\le k\le N} \frac{\sin(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s} - \frac{\pi}{N})}{[2 - 2\cos(\theta_{k-l'} - \frac{2\pi}{N}\tilde{s} - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} = \sum\limits_{1\le k\le N} \frac{\sin(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s} + \frac{\pi}{N})}{[2 - 2\cos(\theta_{l-k'} + \frac{2\pi}{N}\tilde{s} + \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} = \sum\limits_{1\le j\le N} \frac{\sin(\theta_j - \frac{\pi}{N})}{[2 - 2\cos(\theta_j - \frac{\pi}{N}) + h^2]^{\frac{3}{2}}} = 0. \end{cases}$$

Thus, (2.2) holds for the case of $\theta = 2\tilde{s}\pi/N + \pi/N$, where $\tilde{s} \in \{0, 1, 2, \dots, N-1\}$.

By **Cases 1–2**, we arrive at the conclusion that (2.2) holds for $\theta = s\pi/N$ with $s \in \{0, 1, \dots, 2N-1\}$. □

Remark 2.2. *Similar to dealing with the mappings σ and σ_1, for any $k' \in \{1, 2, \dots, N\}$ and any $l' \in \{N+1, N+2, \dots, 2N\}$ with $N \ge 2$, if one defines σ_2 and σ_3 by*

$$\begin{cases} \{1, 2, \dots, N\} \setminus \{k'\} \xrightarrow{\sigma_2} \{\frac{2\pi}{N}, \frac{4\pi}{N}, \dots, \frac{2(N-1)\pi}{N}\}, \\ \sigma_2(k) = \Big[(k-k')(\bmod N)\Big]\frac{2\pi}{N}, \quad \forall\, k \in \{1, 2, \dots, N\}\setminus\{k'\}, \end{cases}$$

and

$$\begin{cases} \{N+1, N+2, \dots, 2N\}\setminus\{l'\} \xrightarrow{\sigma_3} \{\frac{2\pi}{N}, \frac{4\pi}{N}, \dots, \frac{2(N-1)\pi}{N}\}, \\ \sigma_3(l) = \Big[(l-l')(\bmod N)\Big]\frac{2\pi}{N}, \quad \forall\, l \in \{N+1, N+2, \dots, 2N\}\setminus\{l'\}, \end{cases}$$

then σ_2 and σ_3 are bijections.

Lemma 2.4. *For any $\theta \in \mathbb{R}$, $N \ge 2$, $a > 0$, $h > 0$, and $m > 0$, we have*

$$\sum_{1\le k\le N} \frac{-hm}{[|e^{i\theta_k} - ae^{i(\theta_{l'}+\theta)}|^2 + h^2]^{\frac{3}{2}}} \equiv \text{constant}, \quad \forall\, l' \in \{N+1, N+2, \dots, 2N\}. \tag{2.13}$$

Proof. In fact, (2.13) is equivalent to

$$\sum_{1\le k\le N} \frac{-h}{[|e^{i(\theta_{k-l'}-\theta)} - a|^2 + h^2]^{\frac{3}{2}}} \equiv \text{constant}, \quad \forall\, l' \in \{N+1, N+2, \dots, 2N\}.$$

It is easy to see that $k - l' \in \mathbb{Z}$. Moreover, in (2.3), for any $l' \in \{N+1, N+2, \dots, 2N\}$ and any $\theta \in [0,\ 2\pi)$, if we let $\mu = k$, $\hat{\mu} = l' - N$, $\tilde{\mu} = 0$, $\kappa = 0$, and $\alpha = \theta$, then the mapping

$$\begin{cases} \{1, 2, \dots, N\} \xrightarrow{\sigma} \{\theta, \frac{2\pi}{N} + \theta, \dots, \frac{2(N-1)\pi}{N} + \theta\}, \\ \sigma(k) = \Big[(k - (l' - N))(\bmod N)\Big]\frac{2\pi}{N} + \theta = \Big[(k-l')(\bmod N)\Big]\frac{2\pi}{N} + \theta, \quad \forall\, k \in \{1, 2, \dots, N\} \end{cases}$$

is a bijection, which implies that

$$\sum_{1\le k\le N} \frac{-h}{[|e^{i(\theta_{k-l'}-\theta)} - a|^2 + h^2]^{\frac{3}{2}}} = \sum_{1\le j\le N} \frac{-h}{[|e^{i(\theta_j-\theta)} - a|^2 + h^2]^{\frac{3}{2}}}, \quad \forall\, l' \in \{N+1, N+2, \dots, 2N\}.$$

Thus, Lemma 2.4 is true. □

Lemma 2.5. *[4, Page 304] For any $N \geq 2$,* $\sum_{1\leq j\leq N-1} (1 - e^{i\theta_j})/|1 - e^{i\theta_j}|^3 = [\sum_{1\leq j\leq N-1} \csc(j\pi/N)]/4$.

Lemma 2.6. *[20, Pages 1431 and 1437] For any $N \geq 2$, the inequality*

$$\sum_{1\leq j\leq N} \frac{1 + \cos(\theta_j + \frac{\pi}{N})}{[2 - 2\cos(\theta_j + \frac{\pi}{N})]^{\frac{3}{2}}} - \sum_{1\leq j\leq N-1} \frac{1 - e^{i\theta_j}}{|1 - e^{i\theta_j}|^{\frac{3}{2}}} > 0$$

holds.

Now, we introduce the definition of circulant matrix and state its properties.

Definition 2.1. *[25, Pages 65–66] A matrix $\tilde{C} = (\tilde{c}_{kj})_{N\times N}$ is circulant if $\tilde{c}_{\hat{k},\hat{j}} = \tilde{c}_{\hat{k}-1,\hat{j}-1}$ where $1 \leq k, j, \hat{k}, \hat{j} \leq N$ and $N \geq 2$.*

In Definition 2.1, we take the circulant matrix $\tilde{C}$ as the following:

$$\tilde{C} =: C = (c_{k,j}), \quad \text{where} \quad c_{k,j} = \begin{cases} \frac{1-\rho_{k-j}}{|1-\rho_{k-j}|^3}, & k \neq j, \\ 0, & k = j. \end{cases} \tag{2.14}$$

We have some properties for the special circulant matrix C.

Lemma 2.7. *[4, Page 303] The circulant matrix C has the same forms of the eigenvalues $\lambda_j(C)$ and the corresponding eigenvectors ξ_j; more precisely,*

$$\lambda_j(C) = \sum_{1\leq k\leq N} c_{1,k}\rho_{j-1}^{k-1}, \quad \xi_j = (\rho_{j-1}, \rho_{j-1}^2, \ldots, \rho_{j-1}^N)^T, \quad j = 1, 2, \ldots, N,$$

where $N \geq 2$ and $\rho_{j-1} = e^{i\theta_{j-1}} = e^{2(j-1)\pi i/N}$.

Lemma 2.8. *[4, Corollary and Lemma 12] For the eigenvalues of C with $j \neq N$ and $N \geq 4$, $\lambda_j \neq 0$ except that $\lambda_{(N+1)/2} = 0$ for odd N.*

Lemma 2.9. *[22, Proposition 2.2] The eigenvectors ξ_j ($j = 1, 2, \ldots, N$ and $N \geq 3$) of circulant matrix C form a basis of $\mathbb{C}^N$.*

Lemma 2.10. *[25, Page 65] Denote the conjugate transpose of v_k by $(\bar{v}_k)^T$. Then,*

$$(\bar{\xi}_k)^T\xi_j = \begin{cases} N, & k = j, \\ 0, & k \neq j, \end{cases} \qquad (\rho_{-1}, \rho_{-2}, \ldots, \rho_{-N})(\bar{\xi}_N)^T = N.$$

3. Proof of Theorem 1.1

3.1. To prove the necessity

Let $k' \in \{1, 2, \ldots, N\}$ and $l' \in \{N + 1, N + 2, \ldots, 2N\}$. By **Definition 1.1**, it suffices to study the following system:

$$\begin{cases} \frac{(q_{2N+1}-q_{k'})m_{2N+1}m_{k'}}{|q_{2N+1}-q_{k'}|^3} + \sum\limits_{N+1\leq l\leq 2N} \frac{(q_l-q_{k'})m_l m_{k'}}{|q_l-q_{k'}|^3} + \sum\limits_{1\leq k\neq k'\leq N} \frac{(q_k-q_{k'})m_k m_{k'}}{|q_k-q_{k'}|^3} = -\lambda m_{k'}(q_{k'} - x_0), \\ \frac{(q_{2N+1}-q_{l'})m_{2N+1}m_{l'}}{|q_{2N+1}-q_{l'}|^3} + \sum\limits_{1\leq k\leq N} \frac{(q_k-q_{l'})m_k m_{l'}}{|q_k-q_{l'}|^3} + \sum\limits_{N+1\leq l\neq l'\leq 2N} \frac{(q_l-q_{l'})m_l m_{l'}}{|q_k-q_{l'}|^3} = -\lambda m_{l'}(q_{l'} - x_0), \\ \sum\limits_{1\leq k\leq N} \frac{(q_k-q_{2N+1})m_k m_{2N+1}}{|q_k-q_{2N+1}|^3} + \sum\limits_{N+1\leq l\leq 2N} \frac{(q_l-q_{2N+1})m_l m_{2N+1}}{|q_l-q_{2N+1}|^3} = -\lambda m_{2N+1}(q_{2N+1} - x_0). \end{cases} \tag{3.1}$$

Thanks to (1.2), (1.3), and (3.1), the $2N+1$ masses form a central configuration if and only if

$$\begin{cases} \frac{(a_1e^{i\alpha}-e^{i\theta_{k'}},\, h_{2N+1})m_{2N+1}m_{k'}}{[|a_1e^{i\alpha}-e^{i\theta_{k'}}|^2+h_{2N+1}^2]^{\frac{3}{2}}} + \sum\limits_{N+1\le l\le 2N} \frac{(ae^{i(\theta_l+\theta)}-e^{i\theta_{k'}},\, h)m_lm_{k'}}{[|ae^{i(\theta_l+\theta)}-e^{i\theta_{k'}}|^2+h^2]^{\frac{3}{2}}} + \sum\limits_{1\le k\ne k'\le N} \frac{(e^{i\theta_k}-e^{i\theta_{k'}},\, 0)m_km_{k'}}{|e^{i\theta_k}-e^{i\theta_{k'}}|^3} = -\lambda m_{k'}(e^{i\theta_{k'}}-c_0,\, -h_0), \\ \frac{(a_1e^{i\alpha}-ae^{i(\theta_{l'}+\theta)},\, h_{2N+1}-h)m_{2N+1}m_{l'}}{[|a_1e^{i\alpha}-ae^{i(\theta_{l'}+\theta)}|^2+(h_{2N+1}-h)^2]^{\frac{3}{2}}} + \sum\limits_{1\le k\le N} \frac{(e^{i\theta_k}-ae^{i(\theta_{l'}+\theta)},\, -h)m_km_{l'}}{[|e^{i\theta_k}-ae^{i(\theta_{l'}+\theta)}|^2+h^2]^{\frac{3}{2}}} + \sum\limits_{N+1\le l\ne l'\le 2N} \frac{(ae^{i(\theta_l+\theta)}-ae^{i(\theta_{l'}+\theta)},\, 0)m_lm_{l'}}{|ae^{i(\theta_l+\theta)}-ae^{i(\theta_{l'}+\theta)}|^3} \\ \qquad = -\lambda m_{l'}(ae^{i(\theta_{l'}+\theta)}-c_0,\, h-h_0), \\ \sum\limits_{1\le k\le N} \frac{(e^{i\theta_k}-a_1e^{i\alpha},\, -h_{2N+1})m_km_{2N+1}}{[|e^{i\theta_k}-a_1e^{i\alpha}|^2+h_{2N+1}^2]^{\frac{3}{2}}} + \sum\limits_{N+1\le l\le 2N} \frac{(ae^{i(\theta_l+\theta)}-a_1e^{i\alpha},\, h-h_{2N+1})m_lm_{2N+1}}{[|ae^{i(\theta_l+\theta)}-a_1e^{i\alpha}|^2+(h-h_{2N+1})^2]^{\frac{3}{2}}} = -\lambda m_{2N+1}(a_1e^{i\alpha}-c_0,\, h_{2N+1}-h_0). \end{cases} \tag{3.2}$$

By the assumption that the values of N masses located at the vertices of one regular N-polygon are equal to each other, without loss of generality, we suppose that $m_1 = m_2 = \ldots = m_N := m > 0$, and we divide the proof of the necessity into four steps.

Step 1. We prove that $a_1 = 0$.

Employing $m_1 = m_2 = \ldots = m_N = m > 0$ and the second equation of (3.2), we have

$$\frac{(h_{2N+1}-h)m_{2N+1}}{[|a_1e^{i\alpha}-ae^{i(\theta_{l'}+\theta)}|^2+(h_{2N+1}-h)^2]^{\frac{3}{2}}} + \sum_{1\le k\le N} \frac{-hm}{[|e^{i\theta_k}-ae^{i(\theta_{l'}+\theta)}|^2+h^2]^{\frac{3}{2}}} = -\lambda(h-h_0), \tag{3.3}$$

where $l' \in \{N+1, N+2, \ldots, 2N\}$. Combining Lemma 2.4, (3.3),

$$h_0 = \frac{\sum\limits_{N+1\le l\le 2N} m_lh + m_{2N+1}h_{2N+1}}{m_{2N+1} + \sum\limits_{1\le k\le N} m_k + \sum\limits_{N+1\le l\le 2N} m_l},$$

and that λ is independent of the choice of l', we deduce that

$$\frac{(h_{2N+1}-h)m_{2N+1}}{[|a_1e^{i\alpha}-ae^{i(\theta_{l'}+\theta)}|^2+(h_{2N+1}-h)^2]^{\frac{3}{2}}} \equiv \text{constant}, \quad \forall\, l' \in \{N+1, N+2, \ldots, 2N\}.$$

Thus, for any $l' \in \{N+1, N+2, \ldots, 2N\}$, we have $|a_1e^{i\alpha}-ae^{i(\theta_{l'}+\theta)}|^2 \equiv$ constant, i.e.,

$$|[a_1\cos\alpha - a\cos(\theta_{l'}+\theta)] + i[a_1\sin\alpha - a\sin(\theta_{l'}+\theta)]|^2 \equiv \text{constant}, \quad \forall\, l' \in \{N+1, N+2, \ldots, 2N\}.$$

Then, one computes that

$$a_1a\cos(\theta_{l'}+\theta-\alpha) \equiv \text{ constant}, \quad \forall\, l' \in \{N+1, N+2, \ldots, 2N\}.$$

Since a represents the ratio of the sizes of the two regular N-polygons, $a > 0$. Hence, if $a_1 \ne 0$, then

$$\cos(\theta_{l'}+\theta-\alpha) \equiv \text{ constant}, \quad \forall\, l' \in \{N+1, N+2, \ldots, 2N\}. \tag{3.4}$$

In what follows, we assume that $a_1 \ne 0$, and we divide the proof of impossibility of $a_1 \ne 0$ into two cases: $N = 2$ and $N \ge 3$.

(i) $N = 2$:

In this case, $l' \in \{3, 4\}$, and $a_1 \neq 0$. Then, by (3.4), we have $\cos(3\pi + \theta - \alpha) = \cos(4\pi + \theta - \alpha)$, which implies that $\cos(\theta - \alpha) = 0$.

Under the assumption that $m_1 = m_2 = m$, we convert (1.2) and (1.3) into

$$\begin{cases} q_1 = (-1,\ 0), \quad q_2 = (1,\ 0), \\ q_3 = (a\rho_3 e^{i\theta},\ h) = (-ae^{i\theta},\ h), \;\; q_4 = (a\rho_4 e^{i\theta},\ h) = (ae^{i\theta},\ h),\ 0 \le \theta < 2\pi,\ a > 0,\ h \ge 0, \\ q_5 = (a_1 e^{i\alpha},\ h_5),\;\; a_1 \ge 0,\;\; 0 \le \alpha \le 2\pi,\;\; -\infty < h_5 < +\infty, \\ c_0 = \frac{ae^{i\theta}(m_4 - m_3) + a_1 m_5 e^{i\alpha}}{m_1 + m_2 + m_3 + m_4 + m_5} = \frac{ae^{i\theta}(m_4 - m_3) + a_1 m_5 e^{i\alpha}}{2m + m_3 + m_4 + m_5}, \\ h_0 = \frac{(m_3 + m_4)h + m_5 h_5}{m_1 + m_2 + m_3 + m_4 + m_5} = \frac{(m_3 + m_4)h + m_5 h_5}{2m + m_3 + m_4 + m_5}. \end{cases} \tag{3.5}$$

First, in (2.3), for any $\theta \in [0, 2\pi)$ and any $l' \in \{N + 1, N + 2, \ldots, 2N\}$ with $N \ge 2$, if we let $\mu = k$, $\hat{\mu} = l' - N$, $\tilde{\mu} = 0$, $\kappa = 0$, and $\tilde{\alpha} = -\theta \in (-2\pi, 0]$, then the mapping

$$\begin{cases} \{1, 2, \ldots, N\} \stackrel{\sigma}{\longrightarrow} \{-\theta, \frac{2\pi}{N} - \theta, \ldots, \frac{2(N-1)\pi}{N} - \theta\}, \\ \sigma(k) = \Big[(k - (l' - N))(\mathrm{mod}\, N)\Big]\frac{2\pi}{N} - \theta = \Big[(k - l')(\mathrm{mod}\, N)\Big]\frac{2\pi}{N} - \theta, \quad \forall\, k \in \{1, 2, \ldots, N\} \end{cases}$$

is a bijection. Then, by the second equation of (3.2) and $m_1 = m_2 = \ldots = m_N = m > 0$, we have

$$\begin{aligned} & \frac{(1 - \frac{a_1}{a} e^{i(\alpha - \theta - \theta_{l'})}) m_{2N+1}}{[|a - a_1 e^{i(\alpha - \theta - \theta_{l'})}|^2 + (h_{2N+1} - h)^2]^{\frac{3}{2}}} + \sum_{1 \le k \le N} \frac{(1 - \frac{e^{i(\theta_{k-l'} - \theta)}}{a}) m}{[|a - e^{i(\theta_{k-l'} - \theta)}| + h^2]^{\frac{3}{2}}} \\ + & \sum_{N+1 \le l \ne l' \le 2N} \frac{(1 - e^{i\theta_{l-l'}}) m_l}{a^3 |1 - e^{i\theta_{l-l'}}|^3} \\ = & \frac{(1 - \frac{a_1}{a} e^{i(\alpha - \theta - \theta_{l'})}) m_{2N+1}}{[|a - a_1 e^{i(\alpha - \theta - \theta_{l'})}|^2 + (h_{2N+1} - h)^2]^{\frac{3}{2}}} + \sum_{1 \le k \le N} \frac{(1 - \frac{e^{i(\theta_k - \theta)}}{a}) m}{[|a - e^{i(\theta_k - \theta)}|^2 + h^2]^{\frac{3}{2}}} \\ + & \sum_{N+1 \le l \ne l' \le 2N} \frac{(1 - e^{i\theta_{l-l'}}) m_l}{a^3 |1 - e^{i\theta_{l-l'}}|^3} \\ = & \lambda - \frac{\lambda}{a} c_0 e^{-i(\theta_{l'} + \theta)}, \quad \text{where} \;\; l' \in \{N + 1, N + 2, \ldots, 2N\}. \end{aligned} \tag{3.6}$$

Note that all of the sides in the two regular N-polygons have the same size, and a represents the ratio of the sizes of the two regular N-polygons, so $a = 1$. Choosing $\theta_{l'} = 4\pi$ with $l' = 4$, and then employing (3.6) with $a = 1$ and $N = 2$, we have

$$\begin{aligned} & \frac{(1 - a_1 e^{i(\alpha - \theta)}) m_5}{[|1 - a_1 e^{i(\alpha - \theta)}|^2 + (h_5 - h)^2]^{\frac{3}{2}}} + \sum_{1 \le k \le 2} \frac{(1 - e^{i(\theta_k - \theta)}) m}{[|1 - e^{i(\theta_k - \theta)}| + h^2]^{\frac{3}{2}}} + \sum_{3 \le l \ne l' \le 4} \frac{(1 - e^{i\theta_{l-l'}}) m_l}{|1 - e^{i\theta_{l-l'}}|^3} \\ = & \lambda - \lambda c_0 e^{-i\theta}, \quad \text{where} \;\; l' \in \{3, 4\}. \end{aligned} \tag{3.7}$$

Combining $N = 2$, (3.7) and the definition of circulant matrix C in (2.14),

$$CM = \tilde{b}_1 \tilde{\xi}_1 - \tilde{b}_2 \tilde{\xi}_2,$$

where

$$\tilde{b}_1 = \lambda - \frac{(1 - a_1 e^{i(\alpha-\theta)})m_5}{[|1 - a_1 e^{i(\alpha-\theta)}|^2 + (h_5 - h)^2]^{\frac{3}{2}}} - \sum_{1\le k\le 2} \frac{(1 - e^{i(\theta_k-\theta)})m}{[|1 - e^{i(\theta_k-\theta)}| + h^2]^{\frac{3}{2}}},$$

$$\tilde{b}_2 = \lambda e^{-i\theta} c_0 = \lambda e^{-i\theta} \frac{a e^{i\theta}(m_4 - m_3) + a_1 m_{2N+1} e^{i\alpha}}{m_1 + m_2 + m_3 + m_4 + m_5},$$

$M = (m_3, m_4)^T$, $\tilde{\xi}_1 = (1, 1)^T$, and $\tilde{\xi}_2 = (\rho_1, \rho_1^2)^T = (-1, 1)^T$. Thus, we have

$$\begin{pmatrix} \tilde{b}_1 - \tilde{b}_2\rho_1 \\ \tilde{b}_1 - \tilde{b}_2\rho_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1-\rho_{-1}}{|1-\rho_{-1}|^3} \\ \frac{1-\rho_1}{|1-\rho_1|^3} & 0 \end{pmatrix} \begin{pmatrix} m_3 \\ m_4 \end{pmatrix}.$$

Then, by $\rho_{-1} = \rho_{-1+2} = \rho_1 = -1$, one computes that $\tilde{b}_2 = (m_4 - m_3)/8$.

On the other hand, by (3.5), we have

$$\frac{m_4 - m_3}{8} = \tilde{b}_2 = \lambda e^{-i\theta} c_0 = \lambda \frac{a(m_4 - m_3) + a_1 m_{2N+1} e^{i(\alpha-\theta)}}{2m + m_3 + m_4 + m_5} \in \mathbb{R}.$$

In addition, according to lines 1-8 of page 109 of [7], we have $\lambda > 0$ for **Definition 1.1**. Combining $a \in \mathbb{R}$ and $a_1 \neq 0$, one computes that $Im(e^{i(\alpha-\theta)}) = 0$, i.e., $\sin(\alpha - \theta) = 0$, which contradicts with $\cos(\theta - \alpha) = 0$. Thus, $\cos(\theta - \alpha) = 0$ is impossible, which implies that for the spatial twisted central configuration with $N = 2$, we deduce the conclusion that $a_1 = 0$.

(ii) $N \geq 3$:

For (3.4), if we let $\beta = \theta - \alpha$ and choose $l' = N + 1, N + 2$, and $N + 3$, then

$$\begin{cases} \cos(\frac{4\pi}{N} + \beta) - \cos(\frac{2\pi}{N} + \beta) = 0, \\ \cos(\frac{6\pi}{N} + \beta) - \cos(\frac{2\pi}{N} + \beta) = 0, \end{cases}$$

and this is equivalent to

$$\begin{cases} \sin\frac{\pi}{N} \sin(\beta + \frac{3\pi}{N}) = 0, \\ \sin\frac{2\pi}{N} \sin(\beta + \frac{4\pi}{N}) = 0. \end{cases} \tag{3.8}$$

Observing that $N \geq 3$, $\sin(\pi/N) \neq 0$, and $\sin(2\pi/N) \neq 0$. Combining with (3.8), we have

$$\begin{cases} \beta = k_1\pi - \frac{3\pi}{N}, & k_1 \in \mathbb{Z}, \\ \beta = k_2\pi - \frac{4\pi}{N}, & k_2 \in \mathbb{Z}, \end{cases}$$

which implies that $k_1\pi - 3\pi/N = k_2\pi - 4\pi/N$. So, $(k_2 - k_1)\pi = \pi/N$ where positive integer $N \geq 3$, and this is impossible. Therefore, (3.4) does not hold. Then, for the spatial twisted central configuration with $N \geq 3$, we arrive at the conclusion that $a_1 = 0$, too.

Step 2. We prove that $\theta = s\pi/N$ with $s \in \{0, 1, \ldots, 2N - 1\}$, and we divide the proof into two sub-steps.

Step 2.1. We show that $m_{N+1} = m_{N+2} = \ldots = m_{2N}$.

In fact, inserting $a = 1$ and $a_1 = 0$ into (3.6), we have

$$\frac{m_{2N+1}}{[1+(h_{2N+1}-h)^2]^{\frac{3}{2}}} + \sum_{1\le k\le N} \frac{(1-e^{i(\theta_k-\theta)})m}{[|1-e^{i(\theta_k-\theta)}|^2+h^2]^{\frac{3}{2}}} + \sum_{N+1\le l\ne l'\le 2N} \frac{(1-e^{i\theta_{l-l'}})m_l}{|1-e^{i\theta_{l-l'}}|^3}$$
$$= \lambda - \lambda c_0 e^{-i(\theta_{l'}+\theta)}, \text{ where } l' \in \{N+1, N+2, \ldots, 2N\}. \tag{3.9}$$

Combining $N \ge 2$, $m_1 = m_2 = \ldots = m_N = m$, (3.9), $\rho_d = e^{i\theta_d}$ with $\theta_d = 2d\pi/N$ and $d \in \mathbb{Z}$, along with the definition of circulant matrix C in (2.14),

$$CM = b_1\xi_1 - b_2\xi_N, \tag{3.10}$$

where

$$b_1 = \lambda - \frac{m_{2N+1}}{[1+(h_{2N+1}-h)^2]^{\frac{3}{2}}} - \sum_{1\le k\le N} \frac{(1-e^{i(\theta_k-\theta)})m}{[|1-e^{i(\theta_k-\theta)}|+h^2]^{\frac{3}{2}}},$$

$$\begin{aligned} b_2 &= \lambda e^{-i\theta}c_0 = \lambda e^{-i\theta}\frac{\sum\limits_{1\le k\le N} m_k\rho_k + \sum\limits_{N+1\le l\le 2N} m_l\rho_l e^{i\theta}}{m_{2N+1} + \sum\limits_{1\le k\le N} m_k + \sum\limits_{N+1\le l\le 2N} m_l} \\ &= \frac{\lambda \sum\limits_{N+1\le l\le 2N} m_l\rho_l}{m_{2N+1} + \sum\limits_{1\le k\le N} m_k + \sum\limits_{N+1\le l\le 2N} m_l}, \end{aligned} \tag{3.11}$$

$M = (m_{N+1}, m_{N+2}, \ldots, m_{2N})^T$, $\xi_1 = (1, 1, \ldots, 1)^T$ and $\xi_N = (\rho_{N-1}, \rho_{N-1}^2, \ldots, \rho_{N-1}^N)^T$.

In the following, we divide the proof of $m_{N+1} = m_{N+2} = \ldots = m_{2N}$ into three cases: $N = 2$, $N = 3$, and $N \ge 4$.

(i) $N = 2$:

By (3.10) with $N = 2$,

$$\begin{pmatrix} b_1 - b_2\rho_1 \\ b_1 - b_2\rho_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1-\rho_{-1}}{|1-\rho_{-1}|^3} \\ \frac{1-\rho_1}{|1-\rho_1|^3} & 0 \end{pmatrix} \begin{pmatrix} m_3 \\ m_4 \end{pmatrix}.$$

Moreover, when $N = 2$, it is easy to see that $\rho_1 + \rho_2 = 0$ and $\rho_{-1} = \rho_1 = -1$. Thus,

$$2b_2 = \frac{1-\rho_{-1}}{|1-\rho_{-1}|^3}m_4 - \frac{1-\rho_1}{|1-\rho_1|^3}m_3 = \frac{1-\rho_1}{|1-\rho_1|^3}(m_4 - m_3),$$

which implies that $b_2 = (m_4 - m_3)/8$. Thus, when $N = 2$, then inserting $\rho_3 = -1$, and $\rho_4 = 1$ into (3.11) and combining with $b_2 = (m_4 - m_3)/8$, we have

$$b_2 = \frac{\lambda(m_4 - m_3)}{\sum\limits_{1\le k\le 5} m_k} = \frac{(m_4 - m_3)}{8}. \tag{3.12}$$

In what follows, we prove that for the spatial twisted central configuration with $N = 2$, $m_3 = m_4$, and we prove it by contradiction. We assume $m_3 \neq m_4$.

In fact, on the one hand, by $m_3 \neq m_4$, $m_1 = m_2 = m$, and (3.12), we have

$$\lambda = \frac{(2m + m_3 + m_4 + m_5)}{8}. \tag{3.13}$$

Moreover, thanks to $N = 2$, $a = 1$, $a_1 = 0$, and the fourth equation of (3.5), one computes that

$$c_0 = \frac{ae^{i\theta}(m_4 - m_3) + a_1 m_5 e^{i\alpha}}{2m + m_3 + m_4 + m_5} = \frac{e^{i\theta}(m_4 - m_3)}{2m + m_3 + m_4 + m_5}. \tag{3.14}$$

Summing the equations of the first part of (3.2) over $k' = 1$ and $k' = 2$, by $N = 2$, $a_1 = 0$, $m_1 = m_2 = m$, (3.13), and (3.14), we have

$$\begin{aligned}
&\sum_{1\le k'\le 2} \frac{-e^{i\theta_{k'}} m_5}{[1 + h_5^2]^{\frac{3}{2}}} + \sum_{1\le k'\le 2}\sum_{3\le l\le 4} \frac{(ae^{i(\theta_l+\theta)} - e^{i\theta_{k'}})m_l}{[|ae^{i(\theta_l+\theta)} - e^{i\theta_{k'}}|^2 + h^2]^{\frac{3}{2}}} + \sum_{1\le k'\le 2}\sum_{1\le k\neq k'\le 2} \frac{(e^{i\theta_k} - e^{i\theta_{k'}})m_k}{|e^{i\theta_k} - e^{i\theta_{k'}}|^3} \\
= \; & -\lambda \sum_{1\le k'\le 2} e^{i\theta_{k'}} + 2\lambda c_0 = \frac{e^{i\theta}(m_4 - m_3)}{4}.
\end{aligned}$$

Then, combining $e^{i\theta_1} = e^{i\theta_{-1}} = -1$ ($N = 2$), we have

$$\begin{aligned}
&0 + \sum_{1\le k'\le 2} e^{i\theta_{k'}} \sum_{3\le l\le 4} \frac{(ae^{i(\theta_{l-k'}+\theta)} - 1)m_l}{[|ae^{i(\theta_{l-k'}+\theta)} - 1|^2 + h^2]^{\frac{3}{2}}} + \sum_{1\le k'\le 2}\sum_{1\le k\neq k'\le 2} \frac{(e^{i\theta_k} - e^{i\theta_{k'}})m}{|e^{i\theta_k} - e^{i\theta_{k'}}|^3} \\
= \; & \sum_{1\le k'\le 2} e^{i\theta_{k'}} \sum_{3\le l\le 4} \frac{(ae^{i(\theta_l+\theta)} - 1)m_l}{[|ae^{i(\theta_l+\theta)} - 1|^2 + h^2]^{\frac{3}{2}}} + \sum_{1\le k'\le 2} e^{i\theta_{k'}} \sum_{1\le k\neq k'\le 2} \frac{(e^{i\theta_{k-k'}} - 1)m}{|e^{i\theta_{k-k'}} - 1|^3} \\
= \; & \frac{(e^{i\theta_1} - 1)m}{|e^{i\theta_1} - 1|^3} \sum_{1\le k'\le 2} e^{i\theta_{k'}} = 0 = \frac{e^{i\theta}(m_4 - m_3)}{4},
\end{aligned}$$

i.e., $m_3 = m_4$, and this contradicts with the assumption that $m_3 \neq m_4$. Hence, for the spatial twisted central configuration with $N = 2$, we deduce that $m_3 = m_4$.

(ii) $N = 3$:

By (3.10),

$$\begin{pmatrix} b_1 - b_2\rho_2 \\ b_1 - b_2\rho_1 \\ b_1 - b_2\rho_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1-\rho_{-1}}{|1-\rho_{-1}|^3} & \frac{1-\rho_{-2}}{|1-\rho_{-2}|^3} \\ \frac{1-\rho_1}{|1-\rho_1|^3} & 0 & \frac{1-\rho_2}{|1-\rho_2|^3} \\ \frac{1-\rho_2}{|1-\rho_2|^3} & \frac{1-\rho_1}{|1-\rho_1|^3} & 0 \end{pmatrix} \begin{pmatrix} m_4 \\ m_5 \\ m_6 \end{pmatrix}. \tag{3.15}$$

From $N = 3$, we have $\rho_{-1} = \rho_2$ and $\rho_{-2} = \rho_1$; thus, $\rho_1 + \rho_2 + \rho_3 = 0$. Together with (3.15) and

$$\begin{cases} Re(\frac{1-\rho_1}{|1-\rho_1|^3}) = Re(\frac{1-\rho_2}{|1-\rho_2|^3}), \\ Im(\frac{1-\rho_1}{|1-\rho_1|^3}) = -Im(\frac{1-\rho_2}{|1-\rho_2|^3}), \end{cases}$$

there is

$$3b_1 = 3b_1 - b_2(\rho_1 + \rho_2 + \rho_3) \;\; = \;\; \frac{1 - \rho_{-1}}{|1 - \rho_{-1}|^3}m_5 + \frac{1 - \rho_{-2}}{|1 - \rho_{-2}|^3}m_6 + \frac{1 - \rho_1}{|1 - \rho_1|^3}m_4 + \frac{1 - \rho_2}{|1 - \rho_2|^3}m_6$$

$$
\begin{aligned}
&+ \quad \frac{1-\rho_2}{|1-\rho_2|^3}m_4 + \frac{1-\rho_1}{|1-\rho_1|^3}m_5 \\
&= \quad (\frac{1-\rho_1}{|1-\rho_1|^3} + \frac{1-\rho_2}{|1-\rho_2|^3})m_4 + (\frac{1-\rho_1}{|1-\rho_1|^3} + \frac{1-\rho_2}{|1-\rho_2|^3})m_5 \\
&+ \quad (\frac{1-\rho_1}{|1-\rho_1|^3} + \frac{1-\rho_2}{|1-\rho_2|^3})m_6 \in \mathbb{R},
\end{aligned}
$$

which implies that $b_1 \in \mathbb{R}$.

On the other hand, for $N = 3$, Lemma 2.9 gives us information that there exist constants c_1, c_2, and c_3 such that $M = c_1\xi_1 + c_2\xi_2 + c_3\xi_3$ where $M = (m_4, m_5, m_6)^T$. Thus, combining (3.10), we obtain

$$
c_1\lambda_1(C)\xi_1 + c_2\lambda_2(C)\xi_2 + c_3\lambda_3(C)\xi_3 = b_1\xi_1 - b_2\xi_3. \tag{3.16}
$$

Then, it follows from (3.16) and Lemma 2.9 that $c_1\lambda_1(C)\xi_1 = b_1\xi_1$ and $c_3\lambda_3(C)\xi_3 = -b_2\xi_3$.

Employing Lemma 2.5, Lemma 2.7, $\rho_3 = 1$, $\rho_4 = \rho_1$, $\rho_1 + \rho_2 = -1$, and $|1-\rho_1| = |1-\rho_2|$, we have

$$
\begin{cases}
\lambda_1(C) = \frac{1-\rho_1}{|1-\rho_1|^3} + \frac{1-\rho_2}{|1-\rho_2|^3} = \sum\limits_{1\le j\le 2} \frac{1-e^{i\theta_j}}{|1-e^{i\theta_j}|^3} \in \mathbb{R}, \\
\lambda_3(C) = \frac{(1-\rho_1)\rho_2}{|1-\rho_1|^3} + \frac{(1-\rho_2)\rho_1}{|1-\rho_2|^3} = \frac{\rho_2-\rho_3+\rho_1-\rho_3}{|1-\rho_1|^3} = \frac{-3}{|1-\rho_1|^3} \in \mathbb{R}.
\end{cases} \tag{3.17}
$$

Then, thanks to $b_1 \in \mathbb{R}$, $\xi_1 = (1, 1, \ldots, 1)^T$, and $c_1\lambda_1(C)\xi_1 = b_1\xi_1$, one computes that $c_1 \in \mathbb{R}$. In what follows, we will prove that $c_2 \in \mathbb{R}$ and $c_3 \in \mathbb{R}$.

By $\xi_1 = (\rho_0, \rho_0, \rho_0)$, $\xi_2 = (\rho_1, \rho_2, \rho_3)$, $\xi_3 = (\rho_2, \rho_1, \rho_3)$, and $M = c_1\xi_1 + c_2\xi_2 + c_3\xi_3$ with $N = 3$, we have

$$
\begin{cases}
Im(c_1\rho_0 + c_2\rho_1 + c_3\rho_2) = 0, \\
Im(c_1\rho_0 + c_2\rho_2 + c_3\rho_1) = 0, \\
Im(c_1\rho_0 + c_2\rho_3 + c_3\rho_3) = 0.
\end{cases} \tag{3.18}
$$

Based upon $\rho_0 = \rho_3 = 1$, $c_1 \in \mathbb{R}$, and the third equation of (3.18), we have $c_2 + c_3 \in \mathbb{R}$.

Employing $\rho_0 = 1$, $c_1 \in \mathbb{R}$, and the first equation of (3.18), we see that $c_2\rho_1 + c_3\rho_2 \in \mathbb{R}$. Note that

$$
\begin{aligned}
c_2\rho_1 + c_3\rho_2 &= (c_2\cos\frac{2\pi}{3} + ic_2\sin\frac{2\pi}{3}) + (c_3\cos\frac{4\pi}{3} + ic_3\sin\frac{4\pi}{3}) \\
&= (c_2 + c_3)\cos\frac{2\pi}{3} + (c_2 - c_3)i\sin\frac{2\pi}{3},
\end{aligned}
$$

and then $c_2 = c_3$. Therefore, with the help of $c_2 + c_3 \in \mathbb{R}$, we have $c_3 \in \mathbb{R}$.

In virtue of (3.17), $\lambda_3(C) \in \mathbb{R}$. Moreover, combining $c_3 \in \mathbb{R}$ and $c_3\lambda_3(C)\xi_3 = -b_2\xi_3$ where $\xi_3 = (\rho_2, \rho_4, \rho_6)^T = (\rho_2, \rho_1, \rho_3)^T$, one computes that $b_2 \in \mathbb{R}$.

By now, for (3.15), by the accumulated facts $b_1 \in \mathbb{R}$, $b_2 \in \mathbb{R}$, $|1-\rho_1| = |1-\rho_2|$, $\rho_{-1} = \rho_2$, $\rho_{-2} = \rho_1$, and $Im(\rho_1) = -Im(\rho_2)$, we have $m_4 = m_5 = m_6$.

(iii) $N \ge 4$:

Lemma 2.9 gives us information that there exist constants $\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_N$ such that $M = \tilde{c}_1\xi_1 + \tilde{c}_2\xi_2 + \ldots + \tilde{c}_N\xi_N$ where $M = (m_{N+1}, m_{N+2}, \ldots, m_{2N})^T$. We can regard (3.10) as (3.11) of [26]; moreover, we regard C, b_1, and b_2 of this paper as A_α, $\sum_{k=1}^N m_k$, and $\sum_{k=1}^N m_k q_k$ of [26], respectively. Then, combining

$N \geq 4$ and Lemmas 2.8–2.10, similar to the procedure of Case 2.1 on pages 6–7 of [26], we obtain $m_{N+1} = m_{N+2} = \ldots = m_{2N}$.

Step 2.2. Based on **Step 1** and **Step 2.1**, we prove that $\theta = s\pi/N$ with $s \in \{0, 1, \ldots, 2N-1\}$.

Inserting $m_1 = m_2 = \ldots = m_N$, $a_1 = 0$, and $m_{N+1} = m_{N+2} = \ldots = m_{2N}$ into the second equality of (1.3), we have $c_0 = 0$. Then, with the help of the first equation of (3.2), for any $k' \in \{1, 2, \ldots, N\}$, we obtain

$$\frac{-m_{2N+1}}{[1+h_{2N+1}^2]^{\frac{3}{2}}} + \sum_{N+1\leq l\leq 2N} \frac{(ae^{i(\theta_{l-k'}+\theta)}-1)m_l}{[|ae^{i(\theta_{l-k'}+\theta)}-1|^2+h^2]^{\frac{3}{2}}} + \sum_{1\leq k\neq k'\leq N} \frac{(e^{i\theta_{k-k'}}-1)m_k}{|e^{i\theta_{k-k'}}-1|^3} = -\lambda \in \mathbb{R}. \tag{3.19}$$

For any $k' \in \{1, 2, \ldots, N\}$, it follows from Remark 2.2 that the mapping

$$\{1, 2, \ldots, N\}\backslash\{k'\} \xrightarrow{\sigma_2} \Big\{\frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, \frac{2(N-1)\pi}{N}\Big\},$$

where

$$\sigma_2(k) = \Big[(k-k')(\mathrm{mod}\, N)\Big]\frac{2\pi}{N}, \qquad \forall\, k \in \{1, 2, \ldots, N\}\backslash\{k'\},$$

is a bijection. Thus, by the procedure of obtaining (2.6), and $\theta_d = 2\pi d/N$ with $d \in \mathbb{Z}$, we have

$$\begin{cases} \sum\limits_{\substack{k\neq k'\\ 1\leq k\leq N}} \frac{\sin\theta_{k-k'}}{|2-2\cos\theta_{k-k'}|^3} = \sum\limits_{1\leq j\leq N-1} \frac{\sin\theta_j}{|2-2\cos\theta_j|^3} = 0,\ \forall\, k' \in \{1, 2, \ldots, N\}, \\ \sum\limits_{\substack{k\neq k'\\ 1\leq k\leq N}} \frac{\cos\theta_{k-k'}-1}{|2-2\cos\theta_{k-k'}|^3} = \sum\limits_{1\leq j\leq N-1} \frac{\cos\theta_j-1}{|2-2\cos\theta_j|^3},\ \forall\, k' \in \{1, 2, \ldots, N\}, \\ \sum\limits_{\substack{k\neq k'\\ 1\leq k\leq N}} \frac{1}{|2-2\cos\theta_{k-k'}|^3} = \sum\limits_{1\leq j\leq N-1} \frac{1}{|2-2\cos\theta_j|^3},\ \forall\, k' \in \{1, 2, \ldots, N\}, \end{cases} \tag{3.20}$$

and then

$$\sum_{1\leq k\neq k'\leq N} \frac{e^{i\theta_{k-k'}}-1}{|e^{i\theta_{k-k'}}-1|^3} \in \mathbb{R}. \tag{3.21}$$

Combining (3.19) with (3.21), we have

$$Im\Big(\sum_{N+1\leq l\leq 2N} \frac{ae^{i(\theta_{l-k'}+\theta)}-1}{[|ae^{i(\theta_{l-k'}+\theta)}-1|^2+h^2]^{\frac{3}{2}}}\Big) = 0, \text{ where } h > 0 \text{ and } k' \in \{1, 2, \ldots, N\}. \tag{3.22}$$

For any $l \in \{N+1, N+2, \ldots, 2N\}$, in (2.5), if we let $\tilde{s} = 0$, then the mapping

$$\begin{cases} \{1, 2, \ldots, N\} \xrightarrow{\sigma_1} \{\theta, \frac{2\pi}{N}+\theta, \ldots, \frac{2(N-1)\pi}{N}+\theta\}, \\ \sigma_1(k') = \Big[(l-k')(\mathrm{mod}\, N)\Big]\frac{2\pi}{N}+\theta, \qquad \forall\, k' \in \{1, 2, \ldots, N\} \end{cases}$$

is a bijection, too. Hence, we have

$$\begin{cases} \sum\limits_{1\leq k'\leq N} \frac{a\cos(\theta_{l-k'}+\theta)-1}{[1+a^2-2a\cos\theta_{(l-k'}+\frac{\pi}{N})+h^2]^{\frac{3}{2}}} = \sum\limits_{1\leq j\leq N} \frac{a\cos(\theta_j+\theta)-1}{[1+a^2-2a\cos(\theta_j+\theta)+h^2]^{\frac{3}{2}}}, \\ \sum\limits_{1\leq k'\leq N} \frac{a\sin(\theta_{l-k'}+\theta)}{[1+a^2-2a\cos(\theta_{l-k'}+\theta)+h^2]^{\frac{3}{2}}} = \sum\limits_{1\leq j\leq N} \frac{a\sin(\theta_j+\theta)}{[1+a^2-2a\cos(\theta_j+\theta)+h^2]^{\frac{3}{2}}}. \end{cases} \tag{3.23}$$

Using the definition of $f(\theta)$ in (2.1), (3.22), and the second equation of (3.23), we can see that

$$f(\theta) = \sum_{1\le j\le N} \frac{a\sin(\theta_j+\theta)}{[|1+a^2-2a\cos(\theta_j+\theta)|^2+h^2]^{\frac{3}{2}}} = 0. \tag{3.24}$$

On the one hand, if $\theta \in (2\tilde{s}\pi/N,\ 2\tilde{s}\pi/N+\pi/N)$ where $\tilde{s} \in \{0, 1, \dots, N-1\}$, then by Lemmas 2.1-2.2, there is $f(\theta) > 0$, which contradicts (3.24).

On the other hand, if $\theta \in (2\tilde{s}\pi/N+\pi/N,\ 2\tilde{s}\pi/N+2\pi/N)$ where $\tilde{s} \in \{0, 1, \dots, N-1\}$, then by Lemma 2.1, we have $f(\theta) = -f(-\theta) = -f(-\theta+2\pi/N)$ and $-\theta+2\pi/N \in (-2\tilde{s}\pi/N,\ -2\tilde{s}\pi/N+\pi/N)$. Therefore, it follows from Lemma 2.2 that $f(\theta) < 0$, which also contradicts (3.24).

Thus, combining $\theta \in [0,\ 2\pi)$, we conclude that the twist angle must be $\theta = 2\tilde{s}\pi/N$ or $\theta = 2\tilde{s}\pi/N + \pi/N$ with $\tilde{s} \in \{0, 1, \dots, N-1\}$, and then $\theta = s\pi/N$ with $s \in \{0, 1, \dots, 2N-1\}$.

Step 3. We show that $m_1 = m_2 = \dots = m_N = m_{N+1} = m_{N+2} = \dots = m_{2N}$.

Based on the first part of **Step 2**, we can assume that $m_{N+1} = m_{N+2} = \dots = m_{2N} := bm$ where constant $b > 0$. By the assumption that $m_1 = m_2 = \dots = m_N := m$, it suffices to show that the value of b can only take $b = 1$, and we prove it by contradiction. We assume that $b \neq 1$.

Thanks to $a = 1$, $m_1 = m_2 = \dots = m_N := m$, $a_1 = 0$, and $m_{N+1} = m_{N+2} = \dots = m_{2N} := bm$, for (3.2) we have

$$\begin{cases} \frac{Nm_{2N+1}h_{2N+1}}{[1+h_{2N+1}^2]^{\frac{3}{2}}} + \sum\limits_{N+1\le l\le 2N} \frac{bmh}{[|e^{i(\theta_l-k'+\theta)}-1|^2+h^2]^{\frac{3}{2}}} = \lambda h_0, \\ \frac{Nm_{2N+1}(h_{2N+1}-h)}{[1+(h_{2N+1}-h)^2]^{\frac{3}{2}}} - \sum\limits_{1\le k\le N} \frac{mh}{[|e^{i(\theta_k-l'-\theta)}-1|^2+h^2]^{\frac{3}{2}}} = \lambda(h_0-h), \\ \frac{Nmh_{2N+1}}{[1+h_{2N+1}^2]^{\frac{3}{2}}} + \frac{Nbm(h_{2N+1}-h)}{[1+(h-h_{2N+1})^2]^{\frac{3}{2}}} = \lambda(h_{2N+1}-h_0), \end{cases} \tag{3.25}$$

where $k' \in \{1, 2, \dots, N\}$ and $l' \in \{N+1, N+2, \dots, 2N\}$. Combining the first and second equations of (3.25) with (2.7),

$$\begin{cases} \frac{Nm_{2N+1}h_{2N+1}}{[1+h_{2N+1}^2]^{\frac{3}{2}}} + \sum\limits_{1\le j\le N} \frac{bmh}{[|e^{i(\theta_j+\theta)}-1|^2+h^2]^{\frac{3}{2}}} = \lambda h_0, \\ \frac{Nm_{2N+1}(h_{2N+1}-h)}{[1+(h_{2N+1}-h)^2]^{\frac{3}{2}}} - \sum\limits_{1\le j\le N} \frac{mh}{[|e^{i(\theta_j+\theta)}-1|^2+h^2]^{\frac{3}{2}}} = \lambda(h_0-h), \\ \frac{Nmh_{2N+1}}{[1+h_{2N+1}^2]^{\frac{3}{2}}} + \frac{Nbm(h_{2N+1}-h)}{[1+(h_{2N+1}-h)^2]^{\frac{3}{2}}} = \lambda(h_{2N+1}-h_0). \end{cases} \tag{3.26}$$

Let

$$\begin{cases} \hat{x} = \frac{Nh_{2N+1}}{[1+h_{2N+1}^2]^{\frac{3}{2}}}, \\ y = \frac{N(h_{2N+1}-h)}{[1+(h_{2N+1}-h)^2]^{\frac{3}{2}}}, \\ z = \sum\limits_{1\le j\le N} \frac{mh}{[|e^{i(\theta_j+\theta)}-1|^2+h^2]^{\frac{3}{2}}}. \end{cases} \tag{3.27}$$

Thus, (3.26) can be simplified into

$$\begin{cases} m_{2N+1}\hat{x} + bz = \lambda h_0, \\ m_{2N+1}y - z = \lambda(h_0-h), \\ m\hat{x} + bmy = \lambda(h_{2N+1}-h_0). \end{cases} \tag{3.28}$$

On the one hand, by (3.28), we see that

$$\begin{cases} m_{2N+1}\hat{x} - m_{2N+1}y + (b+1)z = \lambda h, \\ m\hat{x} + bmy = \lambda(h_{2N+1} - h_0), \end{cases} \tag{3.29}$$

and

$$\begin{cases} m_{2N+1}\hat{x} + m_{2N+1}y + (b-1)z = 2\lambda h_0 - \lambda h, \\ m\hat{x} + bmy = \lambda(h_{2N+1} - h_0). \end{cases} \tag{3.30}$$

Then, it follows from (3.29), (3.30), and $b \neq 1$ that

$$\begin{aligned} & \frac{-\lambda h_0 m_{2N+1} + \lambda m_{2N+1}h_{2N+1} - \lambda mh + m(b+1)z}{mm_{2N+1}(b+1)} = y \\ = \quad & \frac{-\lambda h_0 m_{2N+1} - 2\lambda m h_0 + \lambda m_{2N+1}h_{2N+1} + \lambda mh + m(b-1)z}{mm_{2N+1}(b-1)}. \end{aligned}$$

Thus,

$$h_0 = \frac{bmh + m_{2N+1}h_{2N+1}}{m_{2N+1} + m + bm}.$$

Combining with

$$h_0 = \frac{\sum\limits_{N+1\le l\le 2N} m_l h + m_{2N+1}h_{2N+1}}{m_{2N+1} + \sum\limits_{1\le k\le N} m_k + \sum\limits_{N+1\le l\le 2N} m_l} = \frac{bNmh + m_{2N+1}h_{2N+1}}{m_{2N+1} + Nm + bNm}, \tag{3.31}$$

we have

$$\frac{bmh + m_{2N+1}h_{2N+1}}{m_{2N+1} + m + bm} = \frac{bNmh + m_{2N+1}h_{2N+1}}{m_{2N+1} + Nm + bNm},$$

which implies that $(1+b)h_{2N+1} = bh$. Moreover, with the help of (3.31), one computes that $h_0 = h_{2N+1}$. Then, it follows from the second equation of (3.29) that $x = -by$. Hence, with the aid of (3.27) and $(1+b)h_{2N+1} = bh$, we have

$$\frac{\frac{bh}{1+b}}{[1 + \frac{b^2h^2}{(1+b)^2}]^{\frac{3}{2}}} = \frac{\frac{bh}{1+b}}{[1 + \frac{h^2}{(1+b)^2}]^{\frac{3}{2}}},$$

that is, $b = \pm 1$, which contradicts $b > 0$ and the assumption $b \neq 1$. So, $b = 1$, and we arrive at the conclusion that $m_1 = m_2 = \ldots = m_N = m_{N+1} = m_{N+2} = \ldots = m_{2N}$.

Step 4. We prove that $h_{2N+1} = h/2$.

In virtue of (3.30), we have

$$m(2\lambda h_0 - \lambda h) = m_{2N+1}\lambda(h_{2N+1} - h_0).$$

Note that lines 1–8 of page 109 of [7] show us that in **Definition 1.1**, λ must be a positive number. Then, combining the last equality of (1.3) and $m_1 = m_2 = \ldots = m_N = m_{N+1} = m_{N+2} = \ldots = m_{2N} = m$,

$$m(\frac{2Nmh + 2h_{2N+1}m_{2N+1}}{2Nm + m_{2N+1}} - h) = m_{2N+1}(-\frac{Nmh + h_{2N+1}m_{2N+1}}{2Nm + m_{2N+1}} + h_{2N+1}),$$

which implies that $N = 1$ or $h = 2h_{2N+1}$. Combining with $N \geq 2$, for the spatial twisted central configuration, we have $h_{2N+1} = h/2$. □

Remark 3.1. *The proof of **Step 1** is independent of the condition that $a = 1$. That is, for the twisted central configuration of the (2N+1)-body problem with the assumption that $m_1 = m_2 = \ldots = m_N$, and without the assumption that $a = 1$, we have $a_1 = 0$. That is, the (2N+1)-th mass must be in the vertical line of the two paralleled planes containing the two regular N-polygons, respectively, and the vertical line segment passes through the geometric centers of the two regular N-polygons.*

3.2. To prove the sufficiency

We divide the proof into two steps.

Step 1. Based on the assumptions that $\theta = s\pi/N$ with $s \in \{0, 1, \ldots, 2N-1\}$, $a = 1$, $h_{2N+1} = h/2$, $a_1 = 0$, and $m_1 = m_2 = \ldots = m_N = m_{N+1} = m_{N+2} = \ldots = m_{2N} = m$, we show that if there exists a constant $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \frac{m_{2N+1}}{[1+\frac{h^2}{4}]^{\frac{3}{2}}} + \sum\limits_{1\leq j\leq N} \frac{(1-e^{i(\theta_j+\theta)})m}{[|e^{i(\theta_j+\theta)}-1|^2+h^2]^{\frac{3}{2}}} + \sum\limits_{1\leq j\leq N} \frac{(1-e^{i\theta_j})m}{|e^{i\theta_j}-1|^3} = \lambda, \\ \frac{m_{2N+1}}{[1+\frac{h^2}{4}]^{\frac{3}{2}}} + \sum\limits_{1\leq j\leq N} \frac{2m}{[|e^{i(\theta_j+\theta)}-1|^2+h^2]^{\frac{3}{2}}} = \lambda, \end{cases} \tag{3.32}$$

then the $2N{+}1$ masses can form a central configuration.

In fact, by (1.3), in this situation we get

$$\begin{cases} c_0 = \frac{\sum\limits_{1\leq k\leq N} m\rho_k + me^{i\theta} \sum\limits_{N+1\leq l\leq 2N} \rho_l}{2Nm+m_{2N+1}} = 0, \\ h_0 = \frac{\sum\limits_{N+1\leq l\leq 2N} mh + m_{2N+1}h_{2N+1}}{2Nm+m_{2N+1}} = \frac{Nmh+m_{2N+1}\frac{h}{2}}{2Nm+m_{2N+1}} = \frac{h}{2}. \end{cases} \tag{3.33}$$

Employing $a = 1$, $h_{2N+1} = h/2$, $a_1 = 0$, $m_1 = m_2 = \ldots = m_N = m_{N+1} = m_{N+2} = \ldots = m_{2N} = m$, (3.33) and

$$\sum_{1\leq k\leq N} e^{i\theta_k} = \sum_{N+1\leq l\leq 2N} e^{i(\theta_l+\theta)} = 0,$$

we see that (3.2) holds if and only if

$$\begin{cases} \frac{(-e^{i\theta_{k'}}, \frac{h}{2})mm_{2N+1}}{[1+\frac{h^2}{4}]^{\frac{3}{2}}} + \sum\limits_{N+1\leq l\leq 2N} \frac{(e^{i(\theta_l+\theta)}-e^{i\theta_{k'}}, h)m^2}{[|e^{i(\theta_l+\theta)}-e^{i\theta_{k'}}|^2+h^2]^{\frac{3}{2}}} + \sum\limits_{1\leq k\neq k'\leq N} \frac{(e^{i\theta_k}-e^{i\theta_{k'}}, 0)m^2}{|e^{i\theta_k}-e^{i\theta_{k'}}|^3} = -\lambda m(e^{i\theta_{k'}}, -\frac{h}{2}), \\ \frac{(-e^{i(\theta_{l'}+\theta)}, -\frac{h}{2})mm_{2N+1}}{[1+\frac{h^2}{4}]^{\frac{3}{2}}} + \sum\limits_{1\leq k\leq N} \frac{(e^{i\theta_k}-e^{i(\theta_{l'}+\theta)}, -h)m^2}{[|e^{i\theta_k}-e^{i(\theta_{l'}+\theta)}|^2+h^2]^{\frac{3}{2}}} + \sum\limits_{N+1\leq l\neq l'\leq 2N} \frac{(e^{i(\theta_l+\theta)}-e^{i(\theta_{l'}+\theta)}, 0)m^2}{|e^{i(\theta_l+\theta)}-e^{i(\theta_{l'}+\theta)}|^3} = -\lambda m(e^{i(\theta_{l'}+\theta)}, \frac{h}{2}). \end{cases} \tag{3.34}$$

Therefore, it suffices to verify (3.34) holds. Clearly, (3.34) is equivalent to

$$\begin{cases} \frac{(-1,\frac{h}{2})m_{2N+1}}{[1+\frac{h^2}{4}]^{\frac{3}{2}}} + \sum\limits_{N+1\le l\le 2N} \frac{(e^{i(\theta_{l-k'}+\theta)}-1,h)m}{[|e^{i(\theta_{l-k'}+\theta)}-1|^2+h^2]^{\frac{3}{2}}} + \sum\limits_{1\le k\ne k'\le N} \frac{(e^{i\theta_{k-k'}}-1,0)m}{|e^{i\theta_{k-k'}}-1|^3} = -\lambda(1,\,-\frac{h}{2}), \\ \frac{(-1,-\frac{h}{2})m_{2N+1}}{[1+\frac{h^2}{4}]^{\frac{3}{2}}} + \sum\limits_{1\le k\le N} \frac{(e^{i(\theta_{k-l'}-\theta)}-1,-h)m}{[|e^{i(\theta_{k-l'}-\theta)}-1|^2+h^2]^{\frac{3}{2}}} + \sum\limits_{N+1\le l\ne l'\le 2N} \frac{(e^{i\theta_{l-l'}}-1,0)m}{|e^{i\theta_{l-l'}}-1|^3} = -\lambda(1,\,\frac{h}{2}). \end{cases} \tag{3.35}$$

On the other hand, for any $l' \in \{N+1, N+2, \ldots, 2N\}$, it follows from Remark 2.2 that the mapping

$$\{N+1, N+2, \ldots, 2N\} \xrightarrow{\sigma_3} \Big\{\frac{2\pi}{N}, \frac{4\pi}{N}, \ldots, \frac{2(N-1)\pi}{N}\Big\},$$

where

$$\sigma_3(l) = \Big[(l-l')(\mathrm{mod}\, N)\Big]\frac{2\pi}{N}, \qquad \forall\, l \in \{N+1, N+2, \ldots, 2N\}\backslash\{l'\},$$

is a bijection. Thus, by $\theta_d = 2\pi d/N$ with $d \in \mathbb{Z}$, we have

$$\begin{cases} \sum\limits_{\substack{l\ne l' \\ N+1\le l\le 2N}} \frac{\sin\theta_{l-l'}}{|2-2\cos\theta_{l-l'}|^3} = \sum\limits_{1\le j\le N-1} \frac{\sin\theta_j}{|2-2\cos\theta_j|^3},\ \forall\, l' \in \{N+1, N+2, \ldots, 2N\}, \\ \sum\limits_{\substack{l\ne l' \\ N+1\le l\le 2N}} \frac{\cos\theta_{l-l'}-1}{[|2-2\cos\theta_{l-l'}|^3} = \sum\limits_{1\le j\le N-1} \frac{\cos\theta_j-1}{|2-2\cos\theta_j|^3},\ \forall\, l' \in \{N+1, N+2, \ldots, 2N\}, \\ \sum\limits_{\substack{l\ne l' \\ N+1\le l\le 2N}} \frac{1}{|2-2\cos\theta_{l-l'}|^3} = \sum\limits_{1\le j\le N-1} \frac{1}{|2-2\cos\theta_j|^3},\ \forall\, l' \in \{N+1, N+2, \ldots, 2N\}. \end{cases}$$

Together with (3.20), for any $k' \in \{1, 2, \ldots, N\}$ and any $l' \in \{N+1, N+2, \ldots, 2N\}$, it follows that

$$\sum_{\substack{k\ne k' \\ 1\le k\le N}} \frac{(e^{i\theta_{k-k'}}-1)m}{|e^{i\theta_{k-k'}}-1|^3} = -\sum_{1\le j\le N-1} \frac{(1-e^{i\theta_j})m}{|e^{i\theta_j}-1|^3} = \sum_{\substack{l\ne l' \\ N+1\le l\le 2N}} \frac{(e^{i\theta_{l-l'}}-1)m}{|e^{i\theta_{l-l'}}-1|^3}. \tag{3.36}$$

Employing Lemma 2.3, (3.35), and (3.36), we conclude that if there exists a constant $\lambda \in \mathbb{R}$ such that (3.32) holds, then by **Definition 1.1**, the $2N{+}1$ masses form a central configuration.

Step 2. We prove the existence of the spatial twisted central configuration, i.e., we prove the existence of λ of **Step 1.**

Define the function g as follows:

$$g(h) = -\frac{1}{2}\sum_{1\le j\le N} \frac{(1+e^{i(\theta_j+\theta)})m}{[|e^{i(\theta_j+\theta)}-1|^2+h^2]^{\frac{3}{2}}} + \frac{1}{2}\sum_{1\le j\le N-1} \frac{(1-e^{i\theta_j})m}{|e^{i\theta_j}-1|^3}, \tag{3.37}$$

where $h > 0$ and $\theta = s\pi/N$ with $s \in \{0, 1, \ldots, 2N-1\}$. Thanks to Lemmas 2.3, 2.5, and (3.37), we see that $g(h) \in \mathbb{R}$, which implies that

$$g(h) = -\frac{1}{2}\sum_{1\le j\le N} \frac{(1+\cos(\theta_j+\theta))m}{[2+h^2-2\cos(\theta_j+\theta)]^{\frac{3}{2}}} + \frac{1}{2}\sum_{1\le j\le N-1} \frac{(1-\cos\theta_j)m}{|2-2\cos\theta_j|^{\frac{3}{2}}}. \tag{3.38}$$

In what follows, we prove that there exists $h = \bar{h}(N)$ such that $g(\bar{h}(N)) = 0$. Note that $\theta = s\pi/N$ with $s \in \{0, 1, \ldots, 2N-1\}$ is equivalent to $\theta = 2\tilde{s}\pi/N$ or $\theta = 2\tilde{s}\pi/N + \pi/N$ with $\tilde{s} \in \{0, 1, \ldots, N-1\}$. We divide the following proof into two cases.

Case 1. $\theta = 2\tilde{s}\pi/N$ with $\tilde{s} \in \{0, 1, \ldots, N-1\}$.

On the one hand, since $m > 0$, $\theta = 2\tilde{s}\pi/N$, and $1 + \cos\theta_j \geq 0$ with $j \in \mathbb{Z}$, we have

$$\begin{aligned} & -\frac{1}{2}\sum_{1\leq j\leq N}\frac{(1+\cos(\theta_j+\theta))m}{[2+h^2-2\cos(\theta_j+\theta)]^{\frac{3}{2}}} = -\frac{1}{2}\sum_{1+\tilde{s}\leq j+\tilde{s}\leq N+s}\frac{(1+\cos\theta_{j+\tilde{s}})m}{[2+h^2-2\cos\theta_{j+\tilde{s}}]^{\frac{3}{2}}} \\ = & -\frac{1}{2}\sum_{1\leq j\leq N}\frac{(1+\cos\theta_j)m}{[2+h^2-2\cos\theta_j]^{\frac{3}{2}}} < 0. \end{aligned} \tag{3.39}$$

Moreover, if $h \to 0^+$, then

$$-\frac{1}{2}\sum_{1\leq j\leq N}\frac{(1+\cos\theta_j)m}{[2+h^2-2\cos\theta_j]^{\frac{3}{2}}} \to -\infty,$$

which implies that when the twist angle is $\theta = 2\tilde{s}\pi/N$ with $\tilde{s} \in \{0, 1, \ldots, N-1\}$, there exists $h = h_1(N)$ such that $g(h_1(N)) < 0$.

On the other hand, notice that if $h \to +\infty$, then

$$-\frac{1}{2}\sum_{1\leq j\leq N}\frac{(1+\cos\theta_j)m}{[2+h^2-2\cos\theta_j]^{\frac{3}{2}}} \to 0.$$

Then, by (3.38)–(3.39), we see that when the twist angle is $\theta = 2\tilde{s}\pi/N$ with $\tilde{s} \in \{0, 1, \ldots, N-1\}$, there exists $h = h_2(N)$ such that $g(h_2(N)) > 0$.

Hence for the case of $\theta = 2\tilde{s}\pi/N$ with $\tilde{s} \in \{0, 1, \ldots, N-1\}$, employing the fact that g is a continuous function, there exists $h = \bar{h}(N)$ such that $g(\bar{h}(N)) = 0$.

Case 2. $\theta = 2\tilde{s}\pi/N + \pi/N$ with $\tilde{s} \in \{0, 1, \ldots, N-1\}$.

In this case, by (3.38), $\tilde{s} \in \{0, 1, \ldots, N-1\}$, and $\theta_d = 2\pi d/N$ with $d \in \mathbb{Z}$, we have

$$\begin{aligned} g(h) &= \frac{m}{2}\Big[-\sum_{1\leq j+\tilde{s}\leq N+\tilde{s}}\frac{1+\cos(\theta_{j+\tilde{s}}+\frac{\pi}{N})}{[2+h^2-2\cos(\theta_{j+\tilde{s}}+\frac{\pi}{N})]^{\frac{3}{2}}} + \sum_{1\leq j\leq N-1}\frac{1-\cos\theta_j}{|2-2\cos\theta_j|^{\frac{3}{2}}}\Big] \\ &= \frac{m}{2}\Big[-\sum_{1\leq j\leq N}\frac{1+\cos(\theta_j+\frac{\pi}{N})}{[2+h^2-2\cos(\theta_j+\frac{\pi}{N})]^{\frac{3}{2}}} + \sum_{1\leq j\leq N-1}\frac{1-\cos\theta_j}{|2-2\cos\theta_j|^{\frac{3}{2}}}\Big]. \end{aligned} \tag{3.40}$$

Then, it follows from (3.40) and Lemma 2.5 that

$$g(h) = \frac{m}{2}\Big[-\sum_{1\leq j\leq N}\frac{1+\cos(\theta_j+\frac{\pi}{N})}{[2+h^2-2\cos(\theta_j+\frac{\pi}{N})]^{\frac{3}{2}}} + \sum_{1\leq j\leq N-1}\frac{1-e^{i\theta_j}}{|1-e^{i\theta_j}|^{\frac{3}{2}}}\Big]. \tag{3.41}$$

If $h \to 0$, then with the help of (3.41) and Lemma 2.6, $\lim_{h\to 0} g(h) < 0$, which implies that when the twist angle is $\theta = 2\tilde{s}\pi/N + \pi/N$ with $\tilde{s} \in \{0, 1, \ldots, N-1\}$, there exists $h = h_3(N)$ such that $g(h_3(N)) < 0$.

On the other hand, if $h \to +\infty$, then

$$-\sum_{1\leq j\leq N}\frac{1+\cos(\theta_j+\frac{\pi}{N})}{[2+h^2-2\cos(\theta_j+\frac{\pi}{N})]^{\frac{3}{2}}} \to 0,$$

which implies that when the twist angle is $\theta = 2\tilde{s}\pi/N + \pi/N$ with $\tilde{s} \in \{0, 1, \ldots, N-1\}$, there exists $h = h_4(N)$ such that $g(h_4(N)) > 0$.

Therefore, for the case of $\theta = 2\tilde{s}\pi/N + \pi/N$ with $\tilde{s} \in \{0, 1, \ldots, N-1\}$, combining with the continuity of function g, there exists $h = \bar{h}(N)$ such that $g(\bar{h}(N)) = 0$.

By now, **Cases 1–2** show us that when $\theta = s\pi/N$ with $s \in \{0, 1, \ldots, 2N-1\}$, there exists $h = \bar{h}(N)$ such that $g(\bar{h}(N)) = 0$, which implies that

$$\frac{1}{2}\sum_{1\le j\le N} \frac{(1-e^{i(\theta_j+\theta)})m}{[|e^{i(\theta_j+\theta)}-1|^2 + (\bar{h}(N))^2]^{\frac{3}{2}}} + \frac{1}{2}\sum_{1\le j\le N-1} \frac{(1-e^{i\theta_j})m}{|e^{i\theta_j}-1|^3} = \sum_{1\le j\le N} \frac{m}{[|e^{i(\theta_j+\theta)}-1|^2 + (\bar{h}(N))^2]^{\frac{3}{2}}}.$$

Moreover, note that

$$\sum_{1\le j\le N} \frac{m}{[|e^{i(\theta_j+\theta)}-1|^2 + (\bar{h}(N))^2]^{\frac{3}{2}}} \in \mathbb{R}.$$

Then, there is a constant $\lambda \in \mathbb{R}$ such that

$$\begin{aligned} &\frac{1}{2}\sum_{1\le j\le N} \frac{(1-e^{i(\theta_j+\theta)})m}{[|e^{i(\theta_j+\theta)}-1|^2 + (\bar{h}(N))^2]^{\frac{3}{2}}} + \frac{1}{2}\sum_{1\le j\le N} \frac{(1-e^{i\theta_j})m}{|e^{i\theta_j}-1|^3} = \frac{1}{2}\lambda - \frac{1}{2}\frac{m_{2N+1}}{[1+\frac{(\bar{h}(N))^2}{4}]^{\frac{3}{2}}} \\ =\ &\sum_{1\le j\le N} \frac{m}{[|e^{i(\theta_j+\theta)}-1|^2 + (\bar{h}(N))^2]^{\frac{3}{2}}} > 0, \end{aligned}$$

i.e.,

$$\begin{cases} \frac{1}{2}\frac{m_{2N+1}}{[1+\frac{(\bar{h}(N))^2}{4}]^{\frac{3}{2}}} + \frac{1}{2}\sum\limits_{1\le j\le N} \frac{(1-e^{i(\theta_j+\theta)})m}{[|e^{i(\theta_j+\theta)}-1|^2+(\bar{h}(N))^2]^{\frac{3}{2}}} + \frac{1}{2}\sum\limits_{1\le j\le N} \frac{(1-e^{i\theta_j})m}{|e^{i\theta_j}-1|^3} = \frac{1}{2}\lambda, \\ \frac{1}{2}\frac{m_{2N+1}}{[1+\frac{(\bar{h}(N))^2}{4}]^{\frac{3}{2}}} + \sum\limits_{1\le j\le N} \frac{m}{[|e^{i(\theta_j+\theta)}-1|^2+(\bar{h}(N))^2]^{\frac{3}{2}}} = \frac{1}{2}\lambda, \end{cases}$$

which implies that there exists a constant $\lambda \in \mathbb{R}$ such that (3.32) holds. Then, by **Step 1**, the $2N+1$ masses form a central configuration. □

4. Proof of Theorem 1.2

For the spatial twisted central configuration, (3.2) holds. Moreover, note that $m_1 = m_2 = \ldots = m_N = m$ and $a = 1$. Then, all the assumptions of Theorem 1.1 are satisfied, so we have $a_1 = 0$, $h_{2N+1} = h/2$, and $m_1 = m_2 = \ldots = m_N = m_{N+1} = \ldots = m_{2N} = m$. Thus, $c_0 = 0$, $h_0 = h/2$, and $q_{2N+1} = (0 + 0i, h/2)$. Thus, in the following, it suffices to prove the uniqueness of h.

In fact, by (3.36) and the first equation of (3.2), for any $k' \in \{1, 2, \ldots, N\}$ and any $l' \in \{N+1, N+2, \ldots, 2N\}$, one computes that

$$\begin{cases} \frac{m_{2N+1}}{[1+\frac{h^2}{4}]^{\frac{3}{2}}}\frac{h}{2} + \sum\limits_{N+1\le l\le 2N} \frac{[(1-e^{i(\theta_{l-k'}+\theta)})m]}{[|1-e^{i(\theta_{l-k'}+\theta)}|^2+h^2]^{\frac{3}{2}}}\frac{h}{2} + \sum\limits_{1\le j\le N-1} \frac{[(1-e^{i\theta_j})m]}{|1-e^{i\theta_j}|^3}\frac{h}{2} = \lambda\frac{h}{2}, \\ \frac{\frac{h}{2}m_{2N+1}}{[1+\frac{h^2}{4}]^{\frac{3}{2}}} + \sum\limits_{1\le j\le N} \frac{mh}{[|1-e^{i(\theta_j+\theta)}|^2+h^2]^{\frac{3}{2}}} = \lambda h_0 = \lambda\frac{h}{2}, \end{cases}$$

where $h > 0$ and $\theta \in [0, 2\pi)$. Hence, it follows from Lemma 2.3 that

$$\begin{cases} \frac{m_{2N+1}\frac{h}{2}}{[1+\frac{h^2}{4}]^{\frac{3}{2}}} + \sum\limits_{1\le j\le N} \frac{(1-e^{i(\theta_j+\theta)})m\frac{h}{2}}{[|1-e^{i(\theta_j+\theta)}|^2+h^2]^{\frac{3}{2}}} + \sum\limits_{1\le j\le N-1} \frac{(1-e^{i\theta_j})m\frac{h}{2}}{|1-e^{i\theta_j}|^3} = \lambda\frac{h}{2}, \\ \frac{\frac{h}{2}m_{2N+1}}{[1+\frac{h^2}{4}]^{\frac{3}{2}}} + \sum\limits_{1\le j\le N} \frac{mh}{[|1-e^{i(\theta_j+\theta)}|^2+h^2]^{\frac{3}{2}}} = \lambda\frac{h}{2}. \end{cases} \tag{4.1}$$

Let

$$\begin{cases} \bar{x} = \sum\limits_{1\le j\le N-1} \frac{1-e^{i\theta_j}}{|1-e^{i\theta_j}|^3}, \\ \bar{y}(h) = \sum\limits_{1\le j\le N} \frac{\cos(\theta_j+\theta)}{[2-2\cos(\theta_j+\theta)+h^2]^{\frac{3}{2}}}, \\ \bar{z}(h) = \sum\limits_{1\le j\le N} \frac{1}{[2-2\cos(\theta_j+\theta)+h^2]^{\frac{3}{2}}}. \end{cases} \tag{4.2}$$

Combining $\lambda \in \mathbb{R}$, $h \in \mathbb{R}$, (4.1), and (4.2), we have

$$\begin{cases} \frac{\frac{h}{2}m_{2N+1}}{[1+\frac{h^2}{4}]^{\frac{3}{2}}} + m\frac{h}{2}[\bar{z}(h) - \bar{y}(h)] + m\frac{h}{2}\bar{x} = \lambda\frac{h}{2}, \\ \frac{\frac{h}{2}m_{2N+1}}{[1+\frac{h^2}{4}]^{\frac{3}{2}}} + mh\bar{z}(h) = \lambda\frac{h}{2}, \end{cases}$$

which implies that

$$m\frac{h}{2}[\bar{z}(h) - \bar{y}(h)] - mh\bar{z}(h) + m\frac{h}{2}\bar{x} = -\lambda\frac{h}{2},$$

and then we obtain $\bar{x} = \bar{y}(h) + \bar{z}(h)$. Hence, if the $2N+1$ masses form a central configuration, then the distance h must satisfy that $\bar{x} = \bar{y}(h) + \bar{z}(h)$. Next, we prove the uniqueness of the distance h.

We take $G(h) = \bar{y}(h) + \bar{z}(h) - \bar{x}$ for $h > 0$, and by (4.2), it is easy to verify that $G'(h) < 0$. Moreover, note that $m_1 = m_2 = \ldots = m_N = m$ and $a = 1$. Then, employing Theorem 1.1, we obtain that the existence of the central configuration implies that $\theta = s\pi/N$ with $s \in \{0, 1, \ldots, 2N-1\}$ holds. Then, due to the rotational symmetry of the central configuration, in order to prove Theorem 1.2, it suffices to consider the following two cases: $\theta = 0$ and $\theta = \pi/N$.

Case 1. $\theta = 0$.

Lemma 2.5 shows that

$$\bar{x} = \sum_{1\le j\le N-1} \frac{1-e^{i\theta_j}}{|1-e^{i\theta_j}|^3} \in \mathbb{R},$$

and then by (4.2),

$$\begin{aligned} G(h) &= \bar{y}(h) + \bar{z}(h) - \bar{x} = \sum_{1\le j\le N} \frac{1+\cos\theta_j}{[2-2\cos\theta_j+h^2]^{\frac{3}{2}}} - \sum_{1\le j\le N-1} \frac{1-\cos\theta_j}{[2-2\cos\theta_j]^{\frac{3}{2}}} \\ &= \sum_{1\le j\le N-1} \frac{1+\cos\theta_j}{[2-2\cos\theta_j+h^2]^{\frac{3}{2}}} + \frac{2}{h^3} - \sum_{1\le j\le N-1} \frac{1-\cos\theta_j}{[2-2\cos\theta_j]^{\frac{3}{2}}}. \end{aligned} \tag{4.3}$$

Employing (4.3), one verifies that there exist a small enough constant $h = \bar{h} > 0$ and a big enough constant $h = \tilde{h} > 0$ such that $G(\bar{h}) > 0$ and $G(\tilde{h}) < 0$. Then, combining the continuity and monotonicity of function G, for the case of $\theta = 0$, there is a unique $h = \hat{h} > 0$, such that $G(\hat{h}) = 0$, i.e., $\bar{x} = \bar{y}(\hat{h}) + \bar{z}(\hat{h})$.

Case 2. $\theta = \pi/N$.
Employing (4.2), we have

$$\begin{aligned}\bar{y}(h) + \bar{z}(h) &= \sum_{1\le j\le N} \frac{1+\cos(\theta_j+\theta)}{[2-2\cos(\theta_j+\theta)+h^2]^{\frac{3}{2}}} = \sum_{1\le j\le N} \frac{1+\cos(\theta_j+\frac{\pi}{N})}{[2-2\cos(\theta_j+\frac{\pi}{N})+h^2]^{\frac{3}{2}}}\\ &= \sum_{1\le j\le N} \frac{1+\cos(\theta_j+\frac{\pi}{N})}{[2-2\cos(\theta_j+\frac{\pi}{N})+h^2]^{\frac{3}{2}}}.\end{aligned}$$

By Lemma 2.6, $\lim_{h\to 0} G(h) > 0$. Furthermore, by Lemma 2.5, we see that $\lim_{h\to +\infty} G(h) < 0$. Thus, for the case of $\theta = \pi/N$, due to the continuity and monotonicity of function G, there is a unique $h = \check{h} > 0$, such that $G(\check{h}) = 0$, i.e., $\bar{x} = \bar{y}(\check{h}) + \bar{z}(\check{h})$.

Based upon **Cases 1–2**, there exists only one $h > 0$ such that $\bar{x} = \bar{y}(h) + \bar{z}(h)$, i.e., there is only one $h > 0$ such that the $2N+1$ masses form a central configuration. Moreover, combining the other two conclusions that $a_1 = 0$ and $h_{2N+1} = h/2$, we obtain that $q_{2N+1} = (a_1 e^{i\alpha}, h_{2N+1})$ is unique, i.e., there are only one positive distance h between the two paralleled regular N-polygons and only one position q_{2N+1} of the $(2N+1)$-th mass such that the $2N+1$ masses form a central configuration. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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