Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 16(2): 334–344.
DOI: 10.3934/cam.2024016
Received: 06 December 2023
Revised: 24 January 2024
Accepted: 26 March 2024
Published: 06 May 2024

Research article

Multiple solutions for quasi-linear elliptic equations with Berestycki-Lions type nonlinearity

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Abstract: We studied the modified nonlinear Schrödinger equation

$$-\Delta u-\frac{1}{2}\Delta(u^2)u=g(u)+h(x),\quad u\in H^1(\mathbb{R}^N),\tag{0.1}$$

where $N\geq 3$, $g\in C(\mathbb{R},\mathbb{R})$ is a nonlinear function of Berestycki-Lions type, and $h\not\equiv 0$ is a nonnegative function. When $\|h\|_{L^2(\mathbb{R}^N)}$ is suitably small, we proved that (0.1) possesses at least two positive solutions by variational approach, one of which is a ground state while the other is of mountain pass type.

Keywords: nonhomogeneous quasi-linear elliptic equation; ground state solution; mountain pass type solution; variational methods
Mathematics Subject Classification: 35J20, 35J62

1. Introduction

The nonlinear scalar field equation

$$-\Delta u=g(u)\quad\text{in }\mathbb{R}^N\tag{1.1}$$

has been widely studied by many authors. In the celebrated papers [1,2], H. Berestycki and P.-L. Lions proved that (1.1) has a positive ground state solution, which is radially symmetric and decreasing with respect to $r=|x|$, and also has infinitely many (possibly sign-changing) solutions when $N\geq 3$ and g satisfies the almost optimal assumptions:

(g_1) $g\in C(\mathbb{R},\ \mathbb{R})$ and g is odd;
(g_2) $-\infty<\liminf_{t\to 0^+}g(t)/t\leq\limsup_{t\to 0^+}g(t)/t=-\kappa<0$;
(g_3) $-\infty\leq\limsup_{t\to+\infty}g(t)/t^{2^*-1}\leq 0$, where $2^*=2N/(N-2)$;

(g_4) there is a constant $\zeta > 0$ such that $G(\zeta) := \int_0^\zeta g(t)\,dt > 0$.

The above classical result has already been generalized in many directions. See, e.g., [3,4] for nonradial solutions of (1.1), [5–8] for nonautonomous semi-linear problems, [9–11] for quasi-linear problems, and [12, 13] for nonlocal problems. In particular, the nonhomogeneous semi-linear elliptic equation

$$-\Delta u = g(u) + h(x) \quad \text{in } \mathbb{R}^N, \tag{1.2}$$

which can be seen as a perturbation of (1.1), was investigated in [6]. Using Ekeland's variational principle and the mountain pass theorem, the authors proved that (1.2) has at least two nontrivial solutions when $\|h\|_{L^2(\mathbb{R}^N)}$ is suitably small. We also refer to [14, 15] for related results.

Motivated by [1, 2, 6, 9], we study the modified nonlinear Schrödinger equation

$$-\Delta u - \frac{1}{2}\Delta(u^2)u = g(u) + h(x), \quad u \in H^1(\mathbb{R}^N), \tag{1.3}$$

where, again, $N \geq 3$, g is a nonlinear function of Berestycki-Lions type, and $h \not\equiv 0$ is a nonnegative function. It is well known that (1.3) models the time evolution of the condensate wave function in super-fluid film. It also appears in the theory of Heisenberg ferromagnet and magnons, in dissipative quantum mechanics, and in condensed matter theory. See [16–18] for details on the background of (1.3). To state our main result, we make the following assumptions on g and h:

(g_1') $g \in C(\mathbb{R}, \mathbb{R})$;
(g_3') $\lim_{t\to+\infty} g(t)/t^{2\cdot 2^*-1} = 0$;
(h) $h \in C^1(\mathbb{R}^N, \mathbb{R}) \cap L^2(\mathbb{R}^N)$, $h(x) = h(|x|) \gneqq 0$, and $\nabla h(x) \cdot x \in L^{2N/(N+2)}(\mathbb{R}^N)$.

Theorem 1.1. *Assume (g_1'), (g_2), (g_3'), and (g_4) hold, then there exists a constant $\delta > 0$ such that, for any function h satisfying (h) and $\|h\|_{L^2(\mathbb{R}^N)} < \delta$, (1.3) has at least two positive solutions, one of which is a ground state while the other is of mountain pass type.*

Remark 1.2. The positive number δ in Theorem 1.1 will be given explicitly in the proof of Lemma 3.1. As mentioned in [19], the critical exponent for (1.3) is not 2^* but $2 \cdot 2^*$. This is why we assume different growth condition (g_3') instead of (g_3) in Theorem 1.1.

Remark 1.3. In the proof of Theorem 1.1, we borrow some ideas from [6]. However, due to the appearance of $\Delta(u^2)u$ and growth condition on g, there is no approximate function space in which the energy functional of (1.3) is both well defined and satisfies the compactness condition. To overcome this difficulty, we will make a change of variables to transform (1.3) into a new semi-linear problem, then we adopt similar ideas as in [6] to verify the geometrical structure and compactness property of the reduced functional. Nevertheless, the analysis is more delicate because the reduced functional involves the transform function.

2. Variational Framework

Since positive solutions are of particular interest in this paper, we always assume with no restriction that $g(t) = -\kappa t$ for $t \leq 0$ in the following arguments, where $\kappa > 0$ is given in (g_2). In form, (1.3) is the Euler equation of the energy functional

$$\mathcal{E}(u) = \frac{1}{2}\int_{\mathbb{R}^N}(1+u^2)|\nabla u|^2\,dx - \int_{\mathbb{R}^N} G(u)\,dx - \int_{\mathbb{R}^N} h(x)u\,dx,$$

where $G(u) = \int_0^u g(t)\,dt$. However, standard variational methods cannot be applied directly because one lacks an appropriate working space in which $\mathcal{E}$ is both well-defined and enjoys compactness properties. In order to surmount this obstacle, we shall adopt a change of unknown to transform (1.3) into a semi-linear problem. Let $u = f(v)$ be the inverse function of

$$v = \int_0^u \sqrt{1+t^2}\,dt = \frac{1}{2}u\sqrt{1+u^2} + \frac{1}{2}\ln\left(u + \sqrt{1+u^2}\right).$$

We recall some basic properties of f in the next lemma (see [20, 21]).

Lemma 2.1. *f is odd and has the following properties:*

$$f'(0) = \lim_{t\to 0} f(t)/t = 1, \quad \lim_{t\to+\infty} f(t)/\sqrt{t} = \sqrt{2},$$

and

$$0 < f'(t) \le 1, \ \ |f(t)| \le \min\{|t|, \sqrt{2|t|}\}, \ \ \frac{1}{2}f^2(t) \le f(t)f'(t)t \le f^2(t), \ \ \textit{for } t \in \mathbb{R}.$$

Setting $u = f(v)$, we change the functional $\mathcal{E}$ into

$$J(v) := \mathcal{E}(f(v)) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \int_{\mathbb{R}^N} G(f(v))\,dx - \int_{\mathbb{R}^N} h(x)f(v)\,dx.$$

By Lemma 2.1, one sees that J is well-defined in the Sobolev space $H_r^1(\mathbb{R}^N)$ and is of class C^1. Moreover, if $v \in H_r^1(\mathbb{R}^N)$ is a critical point of J, then $u = f(v)$ is a positive solution of (1.3). Indeed, since $J'(v) = 0$, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^N} |\nabla v^-|^2\,dx - \int_{\mathbb{R}^N} g(f(v))f'(v)v^-\,dx - \int_{\mathbb{R}^N} h(x)f'(v)v^-\,dx \\ &= \int_{\mathbb{R}^N} |\nabla v^-|^2\,dx + \kappa\int_{\mathbb{R}^N} f(v^-)f'(v^-)v^-\,dx - \int_{\mathbb{R}^N} h(x)f'(v)v^-\,dx, \end{aligned}$$

where $v^- = \min\{v, 0\}$. By Lemma 2.1 again and (h),

$$\int_{\mathbb{R}^N} |\nabla v^-|^2\,dx = \int_{\mathbb{R}^N} f^2(v^-)\,dx = 0.$$

Using Lemma 2.1 once more and the Sobolev inequality, we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} (v^-)^2\,dx &\le C_1\int_{\Omega_1} f^2(v^-)\,dx + \int_{\Omega_2} |v^-|^{2^*}\,dx \\ &\le C_1\int_{\mathbb{R}^N} f^2(v^-)\,dx + C_2\left(\int_{\mathbb{R}^N} |\nabla v^-|^2\,dx\right)^{\frac{N}{N-2}}, \end{aligned}$$

where $\Omega_1 = \{x \,|\, |v^-(x)| \le 1\}$ and $\Omega_2 = \{x \,|\, |v^-(x)| > 1\}$, then $v^- = 0$, so $v \ge 0$ in $\mathbb{R}^N$. By (g_1'), (g_2), and (g_3'), there exists a constant $K > 0$ such that $|g(f(t))f'(t)| \le K(|t| + |t|^{2^*-1})$ for $t \in \mathbb{R}$. Since $J'(v) = 0$, one has

$$-\Delta v + K(1 + v^{2^*-2})v = g(f(v))f'(v) + K(v + v^{2^*-1}) + h(x)f'(v) \ge 0 \quad \text{in } \mathbb{R}^N.$$

By the elliptic regularity theory, [1, Radial Lemma A.II], and the strong maximum principle, we can prove that v is positive in $\mathbb{R}^N$. Now, a standard argument shows that $u = f(v)$ is a positive solution of (1.3). Therefore, to prove Theorem 1.1, it suffices to find two critical points of J in $H_r^1(\mathbb{R}^N)$. We shall fulfill this task by using Ekeland's variational principle and the mountain pass theorem.

Lemma 2.2. ([22, Theorem 1.1]) *Assume that* (X, d) *is a complete metric space and that* $I : X \to \mathbb{R} \cup \{+\infty\}$ *is lower semicontinuous, bounded from below, and not identical to* $+\infty$. *Let* $\epsilon > 0$ *be arbitrary given. If* $u \in X$ *satisfies* $I(u) \le \inf_X I + \epsilon$, *then there exists* $v \in X$ *such that*

$$I(v) \le I(u), \quad d(u, v) \le 1, \quad I(w) > I(v) - \epsilon d(v, w) \text{ for any } w \in X \setminus \{v\}.$$

The next lemma is an expression of the mountain pass theorem without the Palais-Smale condition, which is essentially due to A. Ambrosetti and P. Rabinowitz (see [23, Theorem 2.1]).

Lemma 2.3. *Let* X *be a Banach space and* $I \in C^1(X, \mathbb{R})$. *Assume* $I(0) = 0$ *and*

- *there exist* $\rho > 0$ *and* $\alpha > 0$ *such that* $I(u) \ge \alpha$ *if* $\|u\| = \rho$,
- *there exists a function* $\omega \in X$ *such that* $\|\omega\| > \rho$ *and* $I(\omega) \le 0$.

Let $\Gamma = \{\gamma \in C([0, 1], X) \,|\, \gamma(0) = 0, \gamma(1) = \omega\}$ *and set*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)) \ge \alpha,$$

then there exists $\{u_n\} \subset X$ *satisfying* $I(u_n) \to c$ *and* $I'(u_n) \to 0$ *in* X^* *as* $n \to \infty$.

3. Proof of Theorem 1.1

By (g_1'), (g_2), (g_3'), and Lemma 2.1, there exist $a, b > 0$ such that

$$G(f(t)) \le -at^2 + b|t|^{2^*}, \quad \text{for } t \in \mathbb{R} \tag{3.1}$$

and for any $\epsilon > 0$, there is $C_\epsilon > 0$ such that

$$|G(f(t))| \le C_\epsilon t^2 + \epsilon |t|^{2^*}, \quad \text{for } t \in \mathbb{R}. \tag{3.2}$$

We will work in $H_r^1(\mathbb{R}^N)$, a subspace of $H^1(\mathbb{R}^N)$ consisting of radially symmetric functions. Denote $B_\rho = \{v \in H_r^1(\mathbb{R}^N) \,|\, \|v\| \le \rho\}$ and $\partial B_\rho = \{v \in H_r^1(\mathbb{R}^N) \,|\, \|v\| = \rho\}$. We first study the geometrical structure of J in the next lemmas.

Lemma 3.1. *There exist* $\delta > 0$, $\rho > 0$, *and* $\alpha > 0$ *such that if* $\|h\|_{L^2(\mathbb{R}^N)} < \delta$, *then* $\inf_{\partial B_\rho} J \ge \alpha$.

Proof. It follows from (3.1), the Hölder inequality, and Lemma 2.1 that

$$\begin{aligned} J(v) &\ge C_1 \|v\|^2 - b \int_{\mathbb{R}^N} |v|^{2^*} dx - \|h\|_{L^2(\mathbb{R}^N)} \|v\|_{L^2(\mathbb{R}^N)} \\ &\ge C_1 \|v\|^2 - C_2 \|v\|^{2^*} - \|h\|_{L^2(\mathbb{R}^N)} \|v\| \\ &= \|v\| \left(C_1 \|v\| - C_2 \|v\|^{2^*-1} - \|h\|_{L^2(\mathbb{R}^N)} \right), \end{aligned}$$

where $C_j > 0$ for $j = 1, 2$. We choose

$$\delta = 2 \left(\frac{N-2}{C_2} \right)^{\frac{N-2}{4}} \left(\frac{C_1}{N+2} \right)^{\frac{N+2}{4}} > 0, \quad \rho = \left(\frac{C_1(N-2)}{C_2(N+2)} \right)^{\frac{N-2}{4}} > 0, \quad \alpha = \delta\rho > 0,$$

then $\inf_{\partial B_\rho} J \ge \rho(2\delta - \|h\|_{L^2(\mathbb{R}^N)}) \ge \alpha$, provided that $\|h\|_{L^2(\mathbb{R}^N)} < \delta$. The proof is finished.

Lemma 3.2. *Set*

$$m_0 := \inf_{B_\rho} J,$$

where $\rho > 0$ is as in Lemma 3.1. We have $m_0 \in (-\infty, 0)$.

Proof. It is clear that $m_0 > -\infty$. Since $h \gneqq 0$ in $\mathbb{R}^N$, one can find a function $\varphi \in H_r^1(\mathbb{R}^N)$ such that

$$0 \le \varphi \le 1 \quad \text{and} \quad \int_{\mathbb{R}^N} h(x)\varphi\,dx > 0.$$

By (3.2) and Lemma 2.1, there exists $C > 0$ such that

$$\begin{aligned}
\limsup_{t\to 0^+} \frac{J(t\varphi)}{t} &= \limsup_{t\to 0^+} \left(\frac{t}{2} \int_{\mathbb{R}^N} |\nabla\varphi|^2\,dx - \frac{1}{t} \int_{\mathbb{R}^N} G(f(t\varphi))\,dx - \frac{1}{t} \int_{\mathbb{R}^N} h(x) f(t\varphi)\,dx \right) \\
&\le \limsup_{t\to 0^+} \left(C_\epsilon t \int_{\mathbb{R}^N} \varphi^2\,dx + \epsilon t^{2^*-1} \int_{\mathbb{R}^N} |\varphi|^{2^*}\,dx - C \int_{\mathbb{R}^N} h(x)\varphi\,dx \right) \\
&= -C \int_{\mathbb{R}^N} h(x)\varphi\,dx < 0.
\end{aligned}$$

Let $t > 0$ be sufficiently small such that $\|t\varphi\| < \rho$ and $J(t\varphi) < 0$, then we have $m_0 < 0$ as desired.

Lemma 3.3. *There exists $\omega \in H_r^1(\mathbb{R}^N) \setminus B_\rho$ such that $J(\omega) < 0$, where $\rho > 0$ is as in Lemma 3.1.*

Proof. Let $\tau > 0$ be such that $f(\tau) = \zeta$, where $\zeta > 0$ is given in (g_4). We define, as in [1],

$$\omega_R(x) = \begin{cases} \tau, & \text{if } |x| < R, \\ \tau(R + 1 - |x|), & \text{if } R \le |x| \le R + 1, \\ 0, & \text{if } |x| > R + 1, \end{cases}$$

where $R > 1$ will be determined later, then $\omega_R \in H_r^1(\mathbb{R}^N)$ and a simple calculation shows that

$$\int_{\mathbb{R}^N} |\nabla\omega_R|^2\,dx \le C_1 R^{N-1} \quad \text{and} \quad \int_{\mathbb{R}^N} G(f(\omega_R))\,dx \ge C_2 R^N - C_3 R^{N-1}, \tag{3.3}$$

where $C_1, C_2, C_3 > 0$ are independent of R. Set $\omega_{R,t} = \omega_R(\cdot/t)$ for $t > 0$. By (h) and (3.3), one has

$$J(\omega_{R,t}) \le C_1 R^{N-1} t^{N-2} - \left(C_2 R^N - C_3 R^{N-1} \right) t^N.$$

Choosing $R > 1$ and $t > 0$ sufficiently large, we have $\|\omega_{R,t}\| > \rho$ and $J(\omega_{R,t}) < 0$.

Next, we investigate the compactness property of the functional J.

Lemma 3.4. *Any bounded Palais-Smale sequence of J in $H_r^1(\mathbb{R}^N)$ has a convergent subsequence.*

Proof. Let $\{v_n\} \subset H_r^1(\mathbb{R}^N)$ be a sequence satisfying $\|v_n\| \le C$, $J(v_n) \le C$ and $J'(v_n) \to 0$ in $(H_r^1(\mathbb{R}^N))^*$ as $n \to \infty$. We assume by extracting a subsequence that $v_n \rightharpoonup v$ weakly in $H_r^1(\mathbb{R}^N)$, $v_n \to v$ strongly in $L^p(\mathbb{R}^N)$ for $2 < p < 2^*$, and $v_n \to v$ a.e. in $\mathbb{R}^N$, then $f'(v_n)(v_n - v) \rightharpoonup 0$ weakly in $L^2(\mathbb{R}^N)$ and, henceforth, by (h) and Lemma 2.1,

$$\int_{\mathbb{R}^N} h(x)(f'(v_n) - f'(v))(v_n - v)\,dx = o_n(1), \quad \text{as } n \to \infty. \tag{3.4}$$

Here and after, $o_n(1)$ stands for a generic infinitesimal as $n \to \infty$. Using (3.4) leads to

$$\begin{aligned} o_n(1) &= \langle J'(v_n) - J'(v), v_n - v\rangle \\ &= \int_{\mathbb{R}^N} |\nabla(v_n - v)|^2\, dx - \int_{\mathbb{R}^N} (g(f(v_n))f'(v_n) - g(f(v))f'(v))(v_n - v)\, dx + o_n(1) \\ &\geq \min\{1, \kappa\}\|v_n - v\|^2 - \int_{\mathbb{R}^N} (g(f(v_n))f'(v_n) + \kappa v_n - g(f(v))f'(v) - \kappa v)(v_n - v)\, dx + o_n(1). \end{aligned}$$

To conclude our proof, it suffices to show that

$$\int_{\mathbb{R}^N} (g(f(v_n))f'(v_n) + \kappa v_n - g(f(v))f'(v) - \kappa v)(v_n - v)\, dx \leq o_n(1), \quad \text{as } n \to \infty. \tag{3.5}$$

Let us define

$$g_1(t) = \begin{cases} (g(f(t))f'(t) + \kappa t)^+, & \text{if } t \geq 0, \\ (g(f(t))f'(t) + \kappa t)^-, & \text{if } t \leq 0, \end{cases}$$

and $g_2(t) = g(f(t))f'(t) + \kappa t - g_1(t)$ for $t \in \mathbb{R}$, then

$$\lim_{t\to 0} \frac{g_1(t)}{t} = \lim_{t\to\infty} \frac{g_1(t)}{|t|^{2^*-1}} = 0 \tag{3.6}$$

and

$$g_2(t)t \leq 0, \quad |g_2(t)| \leq C(|t| + |t|^{2^*-1}), \quad \text{for } t \in \mathbb{R}. \tag{3.7}$$

By (3.6), for any $\epsilon > 0$ and $p \in (2, 2^*)$, there is a constant $C_{\epsilon,p} > 0$ such that

$$|g_1(t)| \leq \epsilon(|t| + |t|^{2^*-1}) + C_{\epsilon,p}|t|^{p-1}, \quad \text{for } t \in \mathbb{R},$$

which, combined with $v_n \to v$ strongly in $L^p(\mathbb{R}^N)$, implies that

$$\int_{\mathbb{R}^N} (g_1(v_n) - g_1(v))(v_n - v)\, dx = o_n(1), \quad \text{as } n \to \infty. \tag{3.8}$$

Since $v_n \rightharpoonup v$ weakly in $H^1_r(\mathbb{R}^N)$, one has

$$\int_{\mathbb{R}^N} g_2(v)(v_n - v)\, dx = o_n(1), \quad \text{as } n \to \infty. \tag{3.9}$$

Clearly, (3.7) and Fatou's lemma imply that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} g_2(v_n)v_n\, dx \leq \int_{\mathbb{R}^N} g_2(v)v\, dx. \tag{3.10}$$

By (3.7) and the dominated convergence theorem, it is easy to verify that

$$\int_{\mathbb{R}^N} g_2(v_n)v\, dx = \int_{\mathbb{R}^N} g_2(v)v\, dx + o_n(1), \quad \text{as } n \to \infty. \tag{3.11}$$

Combining (3.9)–(3.11), we have

$$\int_{\mathbb{R}^N} (g_2(v_n) - g_2(v))(v_n - v)\, dx \leq o_n(1), \quad \text{as } n \to \infty, \tag{3.12}$$

then (3.5) is a direct consequence of (3.8) and (3.12). The proof is complete.

Lemma 3.5. *If $v \in H_r^1(\mathbb{R}^N)$ is a critical point of J, then $P(v) = 0$, where*

$$P(v) = \frac{N-2}{2}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - N\int_{\mathbb{R}^N} (G(f(v)) + h(x)f(v))\,dx - \int_{\mathbb{R}^N} (\nabla h(x)\cdot x)f(v)\,dx.$$

Proof. Since the argument is standard, we omit the details.

Lemma 3.6. *If $v \in H_r^1(\mathbb{R}^N)$ is a critical point of J, then*

$$J(v) \geq -\frac{1}{4NS}\|\nabla h(x)\cdot x\|^2_{L^{2N/(N+2)}(\mathbb{R}^N)},$$

where $S > 0$ is the best constant of the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$.

Proof. Let $v \in H_r^1(\mathbb{R}^N)$ be a critical point of J, then $P(v) = 0$ by Lemma 3.5, and

$$J(v) = J(v) - \frac{1}{N}P(v) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2\,dx + \frac{1}{N}\int_{\mathbb{R}^N} (\nabla h(x)\cdot x)f(v)\,dx.$$

By the Hölder inequality and Lemma 2.1, one has

$$\begin{aligned} J(v) &\geq \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \frac{1}{N}\|\nabla h(x)\cdot x\|_{L^{2N/(N+2)}(\mathbb{R}^N)}\|v\|_{L^{2^*}(\mathbb{R}^N)} \\ &\geq \frac{1}{N}\int_{\mathbb{R}^N} |\nabla v|^2\,dx - \frac{1}{NS^{1/2}}\|\nabla h(x)\cdot x\|_{L^{2N/(N+2)}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N} |\nabla v|^2\,dx\right)^{\frac{1}{2}} \\ &\geq -\frac{1}{4NS}\|\nabla h(x)\cdot x\|^2_{L^{2N/(N+2)}(\mathbb{R}^N)}. \end{aligned}$$

The proof is complete.

Proof of Theorem 1.1. Let $\delta > 0$ be as in Lemma 3.1 and assume that h satisfies (h) and $\|h\|_{L^2(\mathbb{R}^N)} < \delta$. **We first establish the existence of a positive ground state solution.** In view of Lemmas 2.2 and 3.2, there exists a sequence $\{v_n\} \subset B_\rho$ such that $m_0 \leq J(v_n) \leq m_0 + \frac{1}{n}$ and

$$J(w) \geq J(v_n) - \frac{1}{n}\|w - v_n\|, \quad \text{for any } w \in B_\rho. \tag{3.13}$$

By Lemmas 3.1 and 3.2, we may assume with no loss of generality that $\|v_n\| < \rho$ for all $n \in \mathbb{N}$. For any $\varphi \in H_r^1(\mathbb{R}^N)$ with $\|\varphi\| = 1$ and any small positive t, we see from (3.13) that $\frac{J(v_n+t\varphi)-J(v_n)}{t} \geq -\frac{1}{n}$. Letting $t \to 0$, we have $\langle J'(v_n), \varphi\rangle \geq -\frac{1}{n}$. Replacing φ by $-\varphi$, we also have $\langle J'(v_n), \varphi\rangle \leq \frac{1}{n}$ and, henceforth, $J'(v_n) \to 0$ in $(H_r^1(\mathbb{R}^N))^*$ as $n \to \infty$. Therefore, $\{v_n\}$ is a bounded Palais-Smale sequence of J in $H_r^1(\mathbb{R}^N)$ at the level m_0. It follows from Lemma 3.4 that there exists $v \in B_\rho$ such that $v_n \to v$ strongly in $H_r^1(\mathbb{R}^N)$ up to a subsequence, so v is a nontrivial critical point of J.

The above argument shows that $\mathcal{K} = \{v \in H_r^1(\mathbb{R}^N) \,|\, J'(v) = 0\} \neq \emptyset$. Now, we define

$$c_0 := \inf_{v\in\mathcal{K}} J(v).$$

Thus, $c_0 \in (-\infty, 0)$ by Lemmas 3.2 and 3.6. Let $\{\hat{v}_n\} \subset \mathcal{K}$ be a minimizing sequence for c_0, then

$$c_0 + o_n(1) = J(\hat{v}_n) - \frac{1}{N}P(\hat{v}_n) = \frac{1}{N}\int_{\mathbb{R}^N} |\nabla \hat{v}_n|^2\,dx + \frac{1}{N}\int_{\mathbb{R}^N} (\nabla h(x)\cdot x)f(\hat{v}_n)\,dx. \tag{3.14}$$

Using the Hölder inequality and (h), one has

$$\begin{aligned} c_0 + o_n(1) &\geq \frac{1}{N}\int_{\mathbb{R}^N} |\nabla \hat{v}_n|^2\,dx - \frac{1}{N}\|\nabla h(x)\cdot x\|_{L^{2N/(N+2)}(\mathbb{R}^N)}\|\hat{v}_n\|_{L^{2^*}(\mathbb{R}^N)} \\ &\geq \frac{1}{N}\int_{\mathbb{R}^N} |\nabla \hat{v}_n|^2\,dx - \frac{1}{NS^{1/2}}\|\nabla h(x)\cdot x\|_{L^{2N/(N+2)}(\mathbb{R}^N)}\left(\int_{\mathbb{R}^N}|\nabla \hat{v}_n|^2\,dx\right)^{\frac{1}{2}}. \end{aligned} \tag{3.15}$$

This implies $\{\|\nabla \hat{v}_n\|_{L^2(\mathbb{R}^N)}\}$ is bounded. In view of (3.1), we have

$$\begin{aligned} \frac{1}{2}\int_{\mathbb{R}^N}|\nabla \hat{v}_n|^2\,dx + a\int_{\mathbb{R}^N}\hat{v}_n^2\,dx &\leq J(\hat{v}_n) + b\int_{\mathbb{R}^N}|\hat{v}_n|^{2^*}\,dx + \|h\|_{L^2(\mathbb{R}^N)}\|\hat{v}_n\|_{L^2(\mathbb{R}^N)} \\ &\leq c_0 + o_n(1) + C\left(\int_{\mathbb{R}^N}|\nabla \hat{v}_n|^2\,dx\right)^{\frac{N}{N-2}} + \|h\|_{L^2(\mathbb{R}^N)}\|\hat{v}_n\|_{L^2(\mathbb{R}^N)}, \end{aligned} \tag{3.16}$$

then $\{\hat{v}_n\}$ is bounded in $H_r^1(\mathbb{R}^N)$. Using Lemma 3.4 again, we see that there exists $\hat{v} \in H_r^1(\mathbb{R}^N)$ such that $\hat{v}_n \to \hat{v}$ strongly in $H_r^1(\mathbb{R}^N)$ up to a subsequence, so $\hat{v}$ is a nontrivial critical point of J. The arguments in Section 2 indicate that $\hat{u} = f(\hat{v})$ is a positive solution of (1.3) and $\mathcal{E}(\hat{u}) = J(\hat{v}) = c_0 < 0$.

Next, we prove the existence of a mountain pass type solution. By Lemmas 3.1 and 3.3,

$$J(0) = 0, \quad \omega \in H_r^1(\mathbb{R}^N)\setminus B_\rho, \quad \inf_{\partial B_\rho} J \geq \alpha > 0 > J(\omega).$$

Let $\Gamma = \{\gamma \in C([0,1], H_r^1(\mathbb{R}^N))\,|\,\gamma(0) = 0, \gamma(1) = \omega\}$ and define the minimax value

$$c_1 := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} J(\gamma(t)) \geq \alpha > 0. \tag{3.17}$$

By Lemma 2.3, there exists a Palais-Smale sequence of J at the level c_1. However, it seems impossible to verify the boundedness of such a Palais-Smale sequence. To overcome this difficulty, we shall adopt an idea originated in [24]. Define a map $\Phi : \mathbb{R}\times H_r^1(\mathbb{R}^N) \to H_r^1(\mathbb{R}^N)$ by $\Phi(\theta, v)(x) = v(e^{-\theta}x)$. We introduce an auxiliary functional $J\circ\Phi : \mathbb{R}\times H_r^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$(J\circ\Phi)(\theta, v) = \frac{e^{(N-2)\theta}}{2}\int_{\mathbb{R}^N}|\nabla v|^2\,dx - e^{N\theta}\int_{\mathbb{R}^N} G(f(v))\,dx - e^{N\theta}\int_{\mathbb{R}^N} h(e^\theta x)f(v)\,dx.$$

Clearly, $J\circ\Phi \in C^1(\mathbb{R}\times H_r^1(\mathbb{R}^N), \mathbb{R})$ and $(J\circ\Phi)(0, v) = J(v)$ for $v \in H_r^1(\mathbb{R}^N)$. It is easy to verify that

$$c_1 = \inf_{\tilde{\gamma}\in\tilde{\Gamma}}\max_{t\in[0,1]}(J\circ\Phi)(\tilde{\gamma}(t)),$$

where $\tilde{\Gamma} = \{\tilde{\gamma} \in C([0,1], \mathbb{R}\times H_r^1(\mathbb{R}^N))\,|\,\tilde{\gamma}(0) = (0,0), \tilde{\gamma}(1) = (0,\omega)\}$. By (3.17), for each $n \in \mathbb{N}$, there is $\gamma_n \in \Gamma$ such that $\max_{t\in[0,1]} J(\gamma_n(t)) < c_1 + \frac{1}{n}$. Setting $\tilde{\gamma}_n = (0, \gamma_n)$, we have $\tilde{\gamma}_n \in \tilde{\Gamma}$ and $\max_{t\in[0,1]}(J\circ\Phi)(\tilde{\gamma}_n(t)) = \max_{t\in[0,1]} J(\gamma_n(t)) < c_1 + \frac{1}{n}$. Using similar arguments as in [25, Lemma 4.3] or by [26, Theorem 2.8], there exists a sequence $\{(\theta_n, v_n)\} \subset \mathbb{R}\times H_r^1(\mathbb{R}^N)$ such that

$$(J\circ\Phi)(\theta_n, v_n) \to c_1, \quad \operatorname{dist}((\theta_n, v_n), \tilde{\gamma}_n[0,1]) \to 0, \quad (J\circ\Phi)'(\theta_n, v_n) \to 0 \text{ in } (\mathbb{R}\times H_r^1(\mathbb{R}^N))^*$$

as $n \to \infty$, then it must be $\theta_n \to 0$ as $n \to \infty$. Setting $\tilde{v}_n = \Phi(\theta_n, v_n)$, we see that

$$J(\tilde{v}_n) = (J\circ\Phi)(\theta_n, v_n) \to c_1, \quad P(\tilde{v}_n) = \langle (J\circ\Phi)'(\theta_n, v_n), (1,0)\rangle \to 0, \quad \text{as } n \to \infty$$

and

$$\langle J'(\tilde{v}_n), \varphi \rangle = \langle (J \circ \Phi)'(\theta_n, v_n), (0, \Phi(-\theta_n, \varphi)) \rangle, \quad \text{for any } \varphi \in H_r^1(\mathbb{R}^N),$$

then, since $\theta_n \to 0$, we have $J'(\tilde{v}_n) \to 0$ in $(H_r^1(\mathbb{R}^N))^*$. Similar arguments as in (3.14)−(3.16) indicate that $\{\tilde{v}_n\} \subset H_r^1(\mathbb{R}^N)$ is a bounded Palais-Smale sequence of J at the level c_1. Using Lemma 3.4 once more, for some $\tilde{v} \in H_r^1(\mathbb{R}^N)$, we have $\tilde{v}_n \to \tilde{v}$ strongly in $H_r^1(\mathbb{R}^N)$ up to a subsequence, so $\tilde{v}$ is a nontrivial critical point of J. The arguments in Section 2 ensure that $\tilde{u} = f(\tilde{v})$ is a positive solution of (1.3) and $\mathcal{E}(\tilde{u}) = J(\tilde{v}) = c_1 > 0$. The proof is finished.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

H. Liu is supported by NSFC (Grant No. 12171204).

Conflict of interest

The authors declare there is no conflict of interest.

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