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Research article

Nonexistence of asymptotically free solutions for nonlinear Schrödinger system

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Abstract: In this paper, the Cauchy problem for the nonlinear Schrödinger system

$$\begin{cases} i\partial_t u_1(x,t) = \Delta u_1(x,t) - |u_1(x,t)|^{p-1}u_1(x,t) - |u_2(x,t)|^{p-1}u_1(x,t), \\ i\partial_t u_2(x,t) = \Delta u_2(x,t) - |u_2(x,t)|^{p-1}u_2(x,t) - |u_1(x,t)|^{p-1}u_2(x,t), \end{cases}$$

was investigated in d space dimensions. For $1 < p \leq 1 + 2/d$, the nonexistence of asymptotically free solutions for the nonlinear Schrödinger system was proved based on mathematical analysis and scattering theory methods. The novelty of this paper was to give the proof of pseudo-conformal identity on the nonlinear Schrödinger system. The present results improved and complemented these of Bisognin, Sepúlveda, and Vera(Appl. Numer. Math. **59**(9)(2009): 2285–2302), in which they only proved the nonexistence of asymptotically free solutions when $d = 1,\ p = 3$.

Keywords: nonlinear Schrödinger system; asymptotically free solutions; large time behavior; pseudo-conformal identity; scattering
Mathematics Subject Classification: 35Q55, 35P25, 35B40.

1. Introduction

In this paper, we consider the following nonlinear Schrödinger system

$$\begin{cases} i\partial_t u_1(x,t) = \Delta u_1(x,t) - |u_1(x,t)|^{p-1}u_1(x,t) - |u_2(x,t)|^{p-1}u_1(x,t), \\ i\partial_t u_2(x,t) = \Delta u_2(x,t) - |u_2(x,t)|^{p-1}u_2(x,t) - |u_1(x,t)|^{p-1}u_2(x,t), \end{cases} \tag{1.1}$$

and the corresponding free (linear) system

$$\begin{cases} i\partial_t v_1(x,t) = \Delta v_1(x,t), \\ i\partial_t v_2(x,t) = \Delta v_2(x,t). \end{cases} \tag{1.2}$$

Here, $x \in \mathbb{R}^d (d \geq 1),\ t \in \mathbb{R},\ 1 < p \leq 1 + 2/d$.

It is well known that there are a lot of results on solutions for the nonlinear Schrödinger equation. For instance, for nonlinear Schrödinger equation with harmonic potential

$$i\partial_t u(x,t) + \Delta u(x,t) - |x|^2 u(x,t) + |u(x,t)|^{p-1} u(x,t) = 0,$$

Shu and Zhang [1] derived a sharp criterion for blow-up and global existence of the solutions by constructing a cross-constrained variational problem and invariant manifolds of the evolution flow. Their results were improved by Xu and his co-authors [2, 3]. More precisely, Xu and Liu [2] pointed out the self-contradiction. Xu and Xu [3] derived different sharp criterion and different invariant manifolds that separate the global solutions and blow-up solutions by comparing the different cross-constrained problems. Moreover, they illustrated that some manifolds are empty and compared the three cross-constrained problems and the three depths of the potential wells. The potential well was also used to study the nonlinear Schrödinger equation with more general nonlinearities in [4], in which the global existence and nonexistence where at only the low initial energy level. It is certainly beyond the scope of the present paper to give a comprehensive review for the nonlinear Schrödinger equation. In this regard, we would like to give some references such as [5–12]. Barab [13] considered the perturbed (nonlinear) Schrödinger equation

$$i\partial_t u(x,t) = \Delta u(x,t) - g|u(x,t)|^{p-1} u(x,t)$$

and the corresponding free (linear) equation

$$i\partial_t v(x,t) = \Delta v(x,t).$$

He proved that for a nontrivial, smooth solution $u(x,t)$, if $d = 1$ and $2 < p \leq 3$, then there does not exist any finite energy free solution $v(x,t)$ such that $\|u(x,t) - v(x,t)\|_2 \to 0$ as $t \to +\infty$. This result is an extension to that one of Strauss [14] in which the same result was proven for $1 < p \leq 2$. Both Barab [13] and Strauss [14] applied the general idea that was originally used by Glassey [15] to prove the analogous result for the nonlinear Klein-Gordon equation to prove their theorems. Tsutsumi and Yajima [16] considered the nonlinear Schrödinger equation with power interactions

$$i\partial_t u(x,t) = -\frac{1}{2}\Delta u(x,t) + \lambda |u(x,t)|^{p-1} u(x,t)$$

in $\mathbb{R}^d,\ d \geq 2,\ \lambda > 0$. They proved that for any $u_0(x) \in \Sigma$ with $\Sigma = \{u \in L^2(\mathbb{R}^d); \|u\|_2 + \|\nabla u\|_2 + \|xu\|_2 < \infty\}$, there exists a unique $u_\pm \in L^2(\mathbb{R}^d)$ such that the solution $u(x,t)$ with $u(x,0) = u_0(x)$ has the free asymptote $u_\pm$ as $t \to \pm\infty$:

$$\lim_{t \to \pm\infty} \|u(x,t) - e^{\frac{1}{2} it\Delta} u_\pm\|_2 = 0$$

when $1 + 2/d < p < 1 + 4/(d-2)$. Guo and Tan [17] studied the asymptotic behavior of nonlinear Schrödinger equations with magnetic effect. By following the idea of [13, 16], they proved the nonexistence of the nontrivial free asymptotic solutions for $1 < p \leq 1 + 2/d$ and the existence of the nontrivial free asymptotic solutions for $1 + 2/d < p < 1 + 4/d,\ d = 2, 3$ under certain conditions, respectively.

For some systems of Schrödinger equations, some results were also obtained. Hayashi, Li, and Naumkina [18] considered the following system of nonlinear Schrödinger equations with quadratic nonlinearities in two space dimensions

$$\begin{cases} i\partial_t u_1(x,t) + \frac{1}{2m_1}\Delta u_1(x,t) = \gamma \overline{u_1}(x,t) u_2(x,t), \\ i\partial_t u_2(x,t) + \frac{1}{2m_2}\Delta u_2(x,t) = u_1^2(x,t), \end{cases}$$

where γ is a given complex number with $|\gamma| = 1$. They obtained time decay estimates of small solutions and nonexistence of the usual scattering states for a system. Moreover, they proved stability in time of small solutions in the neighborhood of solutions to a suitable approximate equation. More related results can be found in [19, 20]. Nakamura, Shimomura, and Tonegawa [21] investigated the Cauchy problem at infinite initial time of the following coupled system of the Schrödinger equation with cubic nonlinearities in one space dimension

$$\begin{cases} i\partial_t u_1(x,t) + \frac{1}{2m_1}\partial_x^2 u_1(x,t) = F_1(u_1(x,t), u_2(x,t)), \\ i\partial_t u_2(x,t) + \frac{1}{2m_2}\partial_x^2 u_2(x,t) = F_2(u_1(x,t), u_2(x,t)). \end{cases}$$

By constructing modified wave operators for small final data, they studied the global existence and the large time behavior. Cheng, Guo, et al. [22] addressed the global well-posedness and scattering of the two-dimensional cubic focusing nonlinear Schrödinger system. Hayashi, Li, and Naumkin [23] considered the nonlinear Schrödinger system

$$\begin{cases} -i\partial_t u_1(x,t) + \frac{1}{2}\Delta u_1(x,t) = F(u_1(x,t), u_2(x,t)), \\ -i\partial_t u_2(x,t) + \frac{1}{2}\Delta u_2(x,t) = F(u_1(x,t), u_2(x,t)), \end{cases}$$

in d space dimensions, where

$$F(u_1(x,t), u_2(x,t)) = -2^{-p} i\lambda |u_1(x,t) - u_2(x,t)|^{p-1} \left(u_1(x,t) - u_2(x,t)\right)$$

is a p-th order local or nonlocal nonlinearity smooth up to order p, with $1 < p \leq 1 + 2/d$ for $d \geq 2$ and $1 < p \leq 2$ for $d = 1$. They proved nonexistence of asymptotically free solutions in the critical and subcritical cases. In this paper, we will prove that there does not exist any finite energy asymptotically free solution of the system (1.1) for $d \geq 1,\ 1 < p \leq 1 + 2/d$. Pseudo-conformal identity on the nonlinear Schrödinger system for $d \geq 1,\ 1 < p \leq 1 + 2/d$ is proven first in this paper, and based on pseudo-conformal identity, we obtain decay estimates of perturbed solutions (see Lemma 2.4). Our results improve and complement that of Bisognin, Sepúlveda, and Vera [24], in which they only proved the nonexistence of asymptotically free solutions when $d = 1,\ p = 3$ for (1.1) by following an idea of Glassey [15].

The outline of the paper is as follows. In Section 2, we shall give some useful lemmas, which plays a pivotal role in proving the main results . In Section 3, we first give and prove nonexistence of asymptotically free solutions for (1.1) if $d \geq 2,\ 1 < p \leq 1+2/d$, and $d = 1,\ 1 < p \leq 2$(see Theorem 3.1). Second, we present the nonexistence of asymptotically free solutions for (1.1) if $d = 1,\ 2 < p \leq 3$(see Theorem 3.2).

2. Preliminaries

Throughout this paper, for each $q \in [1, \infty)$, we denote by the norm $\|u\|_q$ the usual spatial $L^q(\mathbb{R}^d)$-norm and the dual variable is denoted by q' so that $1/q + 1/q' = 1$. The alphabet c is a generic positive constant, which may be different in various positions. For convenience, denote

$$L^q := L^q(\mathbb{R}^d), \quad H^1 := H^1(\mathbb{R}^d),$$

$$\int \cdot dx := \int_{\mathbb{R}^d} \cdot dx, \quad u := u(t) = u(x,t), \quad \|u\|_\infty := \sup ess_{x\in\mathbb{R}^d} |u(x)|.$$

Definition 2.1. A solution $(u_1,\ u_2)$ to (1.1) is asymptotically free if there exist L^2-solutions $(v_{1\pm},\ v_{2\pm})$, decaying sufficiently rapidly, such that

$$\left\|u_1(t)-v_{1\pm}(t)\right\|_2+\left\|u_2(t)-v_{2\pm}(t)\right\|_2\to 0,\quad \text{as } t\to\pm\infty.$$

Remark 2.2. In this paper, we only focus on the solutions for $t>0$. The case $t<0$ can be handled similarly.

Before going further, let us give some preliminary lemmas, which are used to prove our main results. The first lemma is easily proved by following Lemma 2 in [13]. Here we omit the process.

Lemma 2.3. *If* $(v_1,\ v_2)$ *is a smooth solution to* (1.2) *with* $0\neq v_1(x,0)\in L^1\cap L^2,\ 0\neq v_2(x,0)\in L^1\cap L^2,\ 2\leq q\leq\infty$, *then*

(*i*) *there exists a constant* $c=c(\|v_1(0)\|_{q'},\|v_2(0)\|_{q'})$ *such that*

$$\|v_1(t)\|_q+\|v_2(t)\|_q\leq ct^{-d(q-2)/2q},\quad \forall t>0.$$

(*ii*) *there exist positive constants* $B=B(d,q,v_1(0),v_2(0))$ *and* $T_0=T_0(v_1(0),v_2(0))$ *such that*

$$\|v_1(t)\|_q+\|v_2(t)\|_q\geq Bt^{-d(q-2)/2q},\quad \forall t\geq T_0.$$

When $q=\infty$, *the power of t is* $-d/2$.

Lemma 2.4. *If* $(u_1,\ u_2)$ *is a smooth solution to* (1.1) *with* $1<p\leq 1+4/d$, $u_1(x,0),\ u_2(x,0)\in H^1\cap L^{p+1}$, *and* $\|xu_1(x,0)\|_2+\|xu_2(x,0)\|_2<\infty$, *then there exists* $c>0$ *depending on initial data such that*

$$\int(|u_1|^{p+1}+|u_2|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p-1}|u_1|^2)dx\leq ct^{-d(p-1)/2},\quad \forall t>0.$$

Proof. We borrow some ideas from Lemma 3 in [13] to prove this lemma.

First, let us prove the following pseudo-conformal identity

$$\begin{aligned}&\frac{d}{dt}\int\Big[|xu_1-2it\nabla u_1|^2+|xu_2-2it\nabla u_2|^2\\&\quad+\frac{8t^2}{p+1}\Big(|u_1|^{p+1}+|u_2|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p-1}|u_1|^2\Big)\Big]dx\\&=\frac{4t[4-d(p-1)]}{p+1}\int\Big(|u_1|^{p+1}+|u_2|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p-1}|u_1|^2\Big)dx.\end{aligned}\tag{2.1}$$

Let $r=|x|,\ u_i^k=\partial_k u_i=\frac{\partial u_i}{\partial x_k}$ with $i=1,2,\ k=1,2,\cdots,d$, multiply (1.1) by $2r\overline{\partial_r u_1},\ 2r\overline{\partial_r u_2}$ with $\partial_r u_i=\frac{\partial u_i}{\partial r},\ i=1,2$, integrate the real part over $\mathbb{R}^d$, and the use integration by parts, then

$$\begin{aligned}&2\mathrm{Re}i\int\sum_k x_k(\overline{u_1^k}\partial_t u_1+\overline{u_2^k}\partial_t u_2)dx\\&=2\mathrm{Re}\int r(\overline{\partial_r u_1}\Delta u_1+\overline{\partial_r u_2}\Delta u_2)dx\\&\quad-\frac{2}{p+1}\int r\partial_r(|u_1|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p+1}+|u_2|^{p-1}|u_1|^2)dx\\&=(d-2)\int(|\nabla u_1|^2+|\nabla u_2|^2)dx\\&\quad+\frac{2d}{p+1}\int(|u_1|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p+1}+|u_2|^{p-1}|u_1|^2)dx.\end{aligned}\tag{2.2}$$

Note that

$$
\begin{aligned}
&2\mathrm{Re}i\int\sum_k x_k(\overline{u_1^k}\partial_t u_1+\overline{u_2^k}\partial_t u_2)dx\\
&=\mathrm{Re}\Big[i\int\sum_k x_k(\overline{u_1^k}\partial_t u_1-u_1^k\overline{\partial_t u_1}+\overline{u_2^k}\partial_t u_2-u_2^k\overline{\partial_t u_2})dx\Big]\\
&=\frac{d}{dt}\mathrm{Re}\Big[i\int\sum_k x_k\Big(\partial_t(\overline{u_1^k}u_1)-\partial_k(u_1\overline{\partial_t u_1})+\partial_t(\overline{u_2^k}u_2)-\partial_k(u_2\overline{\partial_t u_2})\Big)dx\Big]\\
&=\frac{d}{dt}\mathrm{Re}\Big[i\int r(\overline{\partial_r u_1}u_1+\overline{\partial_r u_2}u_2)dx\Big]+\mathrm{Re}\Big[id\int(u_1\overline{\partial_t u_1}+u_2\overline{\partial_t u_2})dx\Big].
\end{aligned}
$$

Substitute $i\overline{\partial_t u_1},\ i\overline{\partial_t u_2}$ in (1.1), then the above identity can be transferred to

$$
\begin{aligned}
&2\mathrm{Re}i\int\sum_k x_k(\overline{u_1^k}\partial_t u_1+\overline{u_2^k}\partial_t u_2)dx\\
&=\frac{d}{dt}\mathrm{Im}\Big[\int r(\partial_r u_1\overline{u_1}+\partial_r u_2\overline{u_2})dx\Big]\\
&\quad+d\int\Big(|\nabla u_1|^2+|\nabla u_2|^2+|u_1|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p+1}+|u_2|^{p-1}|u_1|^2\Big)dx.
\end{aligned}
\tag{2.3}
$$

Combine (2.2) and (2.3), then

$$
\begin{aligned}
&\frac{d}{dt}\mathrm{Im}\Big[\int r(\partial_r u_1\overline{u_1}+\partial_r u_2\overline{u_2})dx\Big]\\
&=-2\int(|\nabla u_1|^2+|\nabla u_2|^2)dx\\
&\quad-\frac{d(p-1)}{p+1}\int(|u_1|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p+1}+|u_2|^{p-1}|u_1|^2)dx.
\end{aligned}
\tag{2.4}
$$

Multiply by $2\overline{u_1},\ 2\overline{u_2}$ in (1.1) and take the imaginary part to obtain

$$
\frac{d}{dt}(|u_1(t)|^2+|u_2(t)|^2)=\nabla\cdot\mathrm{Im}[2(\overline{u_1}\nabla u_1+\overline{u_2}\nabla u_2)].
\tag{2.5}
$$

Multiply (2.5) by $|x|^2$ and integrate over $\mathbb{R}^d$ to achieve

$$
\frac{d}{dt}(|xu_1(t)|^2+|xu_2(t)|^2)=-4\mathrm{Im}\Big[\int r(\partial_r u_1\overline{u_1}+\partial_r u_2\overline{u_2})dx\Big].
\tag{2.6}
$$

Let us multiply (2.4) by $4t$ to give

$$
\begin{aligned}
&\frac{d}{dt}\Big\{4t\mathrm{Im}\Big[\int r(\partial_r u_1\overline{u_1}+\partial_r u_2\overline{u_2})dx\Big]\Big\}-4\mathrm{Im}\Big[\int r(\partial_r u_1\overline{u_1}+\partial_r u_2\overline{u_2})dx\Big]\\
&=\frac{d}{dt}\Big[-4t^2\int(|\nabla u_1|^2+|\nabla u_2|^2)dx\Big]+4t^2\frac{d}{dt}\int(|\nabla u_1|^2+|\nabla u_2|^2)dx\\
&\quad-\frac{4d(p-1)}{p+1}t\int(|u_1|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p+1}+|u_2|^{p-1}|u_1|^2)dx.
\end{aligned}
$$

Further, make full use of (2.6) and the law of conservation of energy, then

$$
\begin{aligned}
&\frac{d}{dt}\int\Big[|xu_1(t)|^2+|xu_2(t)|^2+4t^2(|\nabla u_1|^2+|\nabla u_2|^2)-\mathrm{Re}4tir(\partial_r u_1\overline{u_1}+\partial_r u_2\overline{u_2})\Big]dx\\
&=\frac{d}{dt}\Big[-\frac{8t^2}{p+1}\int(|u_1|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p+1}+|u_2|^{p-1}|u_1|^2)dx\Big]\\
&\quad+\frac{16t}{p+1}\int(|u_1|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p+1}+|u_2|^{p-1}|u_1|^2)dx\\
&\quad-\frac{4d(p-1)}{p+1}t\int(|u_1|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p+1}+|u_2|^{p-1}|u_1|^2)dx,
\end{aligned}
$$

which yields (2.1) directly.

Second, we aim to give the Gronwall argument. Integrate for (2.1) over $[0,t]$, then

$$
\begin{aligned}
&\frac{8t^2}{p+1}\int\Big(|u_1|^{p+1}+|u_2|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p-1}|u_1|^2\Big)dx\\
&\le\|xu_1(0)\|_2^2+\|xu_2(0)\|_2^2+\frac{4[4-d(p-1)]}{p+1}\\
&\quad\times\int_0^t\tau\int\Big(|u_1|^{p+1}+|u_2|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p-1}|u_1|^2\Big)dxd\tau.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&t^2\int\Big(|u_1|^{p+1}+|u_2|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p-1}|u_1|^2\Big)dx\\
&\le\lambda'+\frac{4-d(p-1)}{2}\int_1^t\tau\int\Big(|u_1|^{p+1}+|u_2|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p-1}|u_1|^2\Big)dxd\tau,
\end{aligned}
\tag{2.7}
$$

where

$$
\begin{aligned}
\lambda'=&\frac{p+1}{8}(\|xu_1(0)\|_2^2+\|xu_2(0)\|_2^2)+\frac{4-d(p-1)}{2}\\
&\times\int_0^1\tau\int\Big(|u_1|^{p+1}+|u_2|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p-1}|u_1|^2\Big)dxd\tau.
\end{aligned}
$$

The law of conservation of energy implies

$$
\begin{aligned}
&\|\nabla u_1\|_2^2+\|\nabla u_2\|_2^2+\frac{2}{p+1}\int\Big(|u_1|^{p+1}+|u_2|^{p+1}+|u_1|^{p-1}|u_2|^2+|u_2|^{p-1}|u_1|^2\Big)dx\\
&=\|\nabla u_1(0)\|_2^2+\|\nabla u_2(0)\|_2^2+\frac{2}{p+1}\\
&\quad\times\int\Big(|u_1(0)|^{p+1}+|u_2(0)|^{p+1}+|u_1(0)|^{p-1}|u_2(0)|^2+|u_2(0)|^{p-1}|u_1(0)|^2\Big)dx.
\end{aligned}
$$

It is not difficult to get that

$$
\lambda'\le\lambda:=c(p)\Big(\|xu_1(0)\|_2^2+\|xu_2(0)\|_2^2+\|\nabla u_1(0)\|_2^2+\|\nabla u_2(0)\|_2^2+\eta\Big),
$$

where
$$\eta := \int \Big(|u_1(0)|^{p+1} + |u_2(0)|^{p+1} + |u_1(0)|^{p-1}|u_2(0)|^2 + |u_2(0)|^{p-1}|u_1(0)|^2\Big)dx,$$
and $c(p)$ is a positive constant depending on p.

By recalling the condition in Lemma 2.4, (2.7) can be rewritten as
$$F(t) \le \lambda + \int_1^t \beta(\tau)F(\tau)d\tau,$$
where for $t \ge 1$,
$$F(t) = t^2 \int \Big(|u_1|^{p+1} + |u_2|^{p+1} + |u_1|^{p-1}|u_2|^2 + |u_2|^{p-1}|u_1|^2\Big)dx,$$
$$\beta(t) = \frac{4 - d(p-1)}{2t}.$$
Since F and β are continuous on $[1, \infty)$, Gronwall's lemma indicates that
$$t^2 \int \Big(|u_1|^{p+1} + |u_2|^{p+1} + |u_1|^{p-1}|u_2|^2 + |u_2|^{p-1}|u_1|^2\Big)dx \le \lambda e^{\int_1^t \frac{4-d(p-1)}{2\tau}d\tau}, \quad \forall t > 1.$$
We can simplify it to get that
$$\int \Big(|u_1|^{p+1} + |u_2|^{p+1} + |u_1|^{p-1}|u_2|^2 + |u_2|^{p-1}|u_1|^2\Big)dx \le \lambda t^{-\frac{d(p-1)}{2}}, \quad \forall t > 1.$$
Further, for all t, $\int \Big(|u_1|^{p+1} + |u_2|^{p+1} + |u_1|^{p-1}|u_2|^2 + |u_2|^{p-1}|u_1|^2\Big)dx$ is bounded uniformly. Therefore, there exists a constant
$$c = c(p, \|xu_1(0)\|_2, \|xu_2(0)\|_2, \|\nabla u_1(0)\|_2, \|\nabla u_2(0)\|_2, \|u_1(0)\|_{p+1}, \|u_2(0)\|_{p+1})$$
such that
$$\Big[\int \Big(|u_1|^{p+1} + |u_2|^{p+1} + |u_1|^{p-1}|u_2|^2 + |u_2|^{p-1}|u_1|^2\Big)dx\Big]^{\frac{1}{p+1}} \le ct^{-d(p-1)/2(p+1)}, \quad \forall t > 0.$$

Lemma 2.5. *If* $1 \le p < \infty$, $(e^{-it\Delta}h_1,\ e^{-it\Delta}h_2)$ *is a nontrivial free solution, then*
$$t^{\frac{(p-1)d}{2}} \int e^{-it\Delta}\Big(|h_1|^{p+1} + |h_2|^{p+1} + |h_1|^{p-1}|h_2|^2 + |h_2|^{p-1}|h_1|^2\Big)dx$$
is bounded away from zero for large t.

Proof. Hölder's inequality yields
$$\begin{aligned}&\Big(\int_{|x|<kt} e^{-it\Delta}(|h_1|^2 + |h_2|^2)dx\Big)^{\frac{p+1}{2}}\\ &\le (kt)^{\frac{d(p-1)}{2}} \int e^{-it\Delta}\Big(|h_1|^{p+1} + |h_2|^{p+1} + |h_1|^{p-1}|h_2|^2 + |h_2|^{p-1}|h_1|^2\Big)dx.\end{aligned}$$

Obviously, it suffices to bound the left side of the above inequality away from zero. Set

$$e^{-it\Delta}h_1 + e^{-it\Delta}h_2 = (4\pi it)^{-\frac{d}{2}} \int e^{|x-y|^2/4it} e^{-it\Delta}[h_1(y,0) + h_2(y,0)]dy,$$

by a direct computation, then

$$\lim_{t\to\infty} \int_{|x|<kt} (|e^{-it\Delta}h_1|^2 + |e^{-it\Delta}h_2|^2)dx = \int_{|\xi|<k/2} \left[|\mathscr{F}(e^{-it\Delta}h_1(\xi,0))|^2 + |\mathscr{F}(e^{-it\Delta}h_2(\xi,0))|^2\right]d\xi,$$

where $\mathscr{F}(e^{-it\Delta}h_1(\xi,0))$, $\mathscr{F}(e^{-it\Delta}h_2(\xi,0))$ are the Fourier transform of the initial datum.

It is noted that $(e^{-it\Delta}h_1,\ e^{-it\Delta}h_2)$ is nontrivial and there is a k for which the limit does not vanish. This proof is completed.

3. Proof of the main results

In this section, we prove the main results by considering the following two cases.

Case 1: $d \geq 2,\ 1 < p \leq 1 + 2/d$, and $d = 1,\ 1 < p \leq 2$.

Case 2: $d = 1,\ 2 < p \leq 3$.

First, let us prove nonexistence of asymptotically free solutions for (1.1) under case 1. The following theorem is obtained.

Theorem 3.1. *For $d \geq 2,\ 1 < p \leq 1 + 2/d$ and $d = 1,\ 1 < p \leq 2$, if $(u_1,\ u_2)$ is a solution of* (1.1)*, then for all $(h_1,\ h_2) \in L^2 \times L^2$,*

$$\left\|u_1(t) - e^{-it\Delta}h_1\right\|_2 + \left\|u_2(t) - e^{-it\Delta}h_2\right\|_2$$

does not go to zero as $t \to +\infty$.

Proof. Suppose that there exists $(h_1,\ h_2) \in L^2 \times L^2$ such that

$$\left\|u_1(t) - e^{-it\Delta}h_1\right\|_2 + \left\|u_2(t) - e^{-it\Delta}h_2\right\|_2 \to 0, \quad \text{as } t \to +\infty. \tag{3.1}$$

Since the operator $e^{-it\Delta}$ is dense and unitary in L^2, we can assume $h_1,\ h_2 \in S$ (Schwartz space) to get

$$\begin{aligned}
&\frac{d}{dt}\int \left[(e^{it\Delta}u_1)\overline{h_1} + (e^{it\Delta}u_2)\overline{h_2}\right] dx \\
&= \int \left[e^{it\Delta}\overline{h_1}i(|u_1|^{p-1}u_1 + |u_2|^{p-1}u_1) + e^{it\Delta}\overline{h_2}i(|u_1|^{p-1}u_2 + |u_2|^{p-1}u_2)\right] dx \\
&= \int \left[\overline{e^{-it\Delta}h_1}i(|u_1|^{p-1}u_1 + |u_2|^{p-1}u_1) + \overline{e^{-it\Delta}h_2}i(|u_1|^{p-1}u_2 + |u_2|^{p-1}u_2)\right] dx.
\end{aligned}$$

Therefore,

$$\int_0^T \int \left[\overline{e^{-it\Delta}h_1}i(|u_1|^{p-1}u_1 + |u_2|^{p-1}u_1) + \overline{e^{-it\Delta}h_2}i(|u_1|^{p-1}u_2 + |u_2|^{p-1}u_2)\right] dxdt \tag{3.2}$$

has a limit as $T \to \infty$. In fact, as $T \to \infty$, we obtain

$$\begin{aligned}
&\int \left[\overline{h_1}e^{iT\Delta}u_1(T) + \overline{h_2}e^{iT\Delta}u_2(T)\right] dx \\
&= \int (h_1\overline{h_1} + h_2\overline{h_2})dx + \int \left[\overline{h_1}(e^{iT\Delta}u_1(T) - h_1) + \overline{h_2}(e^{iT\Delta}u_2(T) - h_2)\right] dx \to 0.
\end{aligned}$$

On the other hand, one has

$$\begin{aligned}
&\Big|\int\Big[i(|u_1|^{p-1}u_1+|u_2|^{p-1}u_1)\overline{e^{-it\Delta}h_1}+i(|u_1|^{p-1}u_2+|u_2|^{p-1}u_2)\overline{e^{-it\Delta}h_2}\\
&\quad -ie^{-it\Delta}(|h_1|^{p+1}+|h_2|^{p+1}+|h_1|^{p-1}|h_2|^2+|h_2|^{p-1}|h_1|^2)\Big]dx\Big|\\
&\le(\|u_1\|_2+\|u_2\|_2+\|h_1\|_2+\|h_2\|_2)^{p-1}(\|h_1\|_2+\|h_2\|_2)^{2-p}\\
&\quad\times\Big(\|e^{-it\Delta}h_1\|_\infty+\|e^{-it\Delta}h_2\|_\infty\Big)^{p-1}\Big(\Big\|u_1(t)-e^{-it\Delta}h_1\Big\|_2+\Big\|u_2(t)-e^{-it\Delta}h_2\Big\|_2\Big).
\end{aligned}$$

Since

$$\|u_1\|_2+\|u_2\|_2\to\|h_1\|_2+\|h_2\|_2,\quad \text{as } t\to\infty,$$

$$\Big(\|e^{-it\Delta}h_1\|_\infty+\|e^{-it\Delta}h_2\|_\infty\Big)^{p-1}\le|t|^{-\frac{(p-1)d}{2}}(\|h_1\|_1+\|h_2\|_1)^{p-1},$$

(3.1) combined with

$$\int e^{-it\Delta}\Big(|h_1|^{p+1}+|h_2|^{p+1}+|h_1|^{p-1}|h_2|^2+|h_2|^{p-1}|h_1|^2\Big)dx\overset{Lemma\ 2.5}{\ge}ct^{-\frac{(p-1)d}{2}},$$

obtains

$$\int\Big[i(|u_1|^{p-1}u_1+|u_2|^{p-1}u_1)\overline{e^{-it\Delta}h_1}+i(|u_1|^{p-1}u_2+|u_2|^{p-1}u_2)\overline{e^{-it\Delta}h_2}\Big]dx\ge\frac{c}{2}t^{-\frac{(p-1)d}{2}}.$$

The righthand side of the above inequality is not integrable, which is a contradiction since (3.2) has a limit as $T\to\infty$.

This completes the proof of this theorem.

In the light of Definition 2.1, we easily get the nonexistence of asymptotically free solutions for (1.1) under case 1. In what follows, let us illustrate the nonexistence of asymptotically free solutions for (1.1) under case 2.

Theorem 3.2. *If $d=1,\ 2<p\le3$, then the only smooth, asymptotically free solution to* (1.1) *is identically zero.*

Proof. Assume $(u_1,\ u_2)$ is a smooth, asymptotically free solution to (1.1), then there exists a smooth L^2-solution $(v_1,\ v_2)$ of (1.2) such that

$$\|u_1(t)-v_1(t)\|_2+\|u_2(t)-v_2(t)\|_2\to0,\quad \text{as } t\to+\infty, \tag{3.3}$$

$$\|v_1(t)\|_\infty+\|v_2(t)\|_\infty=O(t^{-1/2}),\quad \text{as } t\to+\infty, \tag{3.4}$$

Let us combine the conservation of the L^2-norm and (3.3) to show

$$\|u_1(t)\|_2+\|u_2(t)\|_2=\|v_1(t)\|_2+\|v_2(t)\|_2\equiv A,\quad \forall t. \tag{3.5}$$

We are now in a position to prove

$$v_1(0)=0,\ v_2(0)=0.$$

Let us prove by contradiction. Assume $v_1(0) \neq 0,\ v_2(0) \neq 0$. Using (3.4), then

$$\|v_1(t)\|_\infty + \|v_2(t)\|_\infty \leq ct^{-1/2}, \quad \forall t > T_1 \geq T_0, \tag{3.6}$$

where T_0 is as in Lemma 2.3. For $t > T_1$, define

$$H(t) = \int [u_1\overline{v_1} + u_2\overline{v_2}]dx.$$

Differentiate H with respect to t, substitute from (1.1) and (1.2) for $\partial_t u_1,\ \partial_t u_2,\ \partial_t v_1$ and $\partial_t v_2$, respectively, and integrate by parts to get

$$N(t) := \frac{dH(t)}{dt} = \int \left[i\overline{v_1}(|u_1|^{p-1}u_1 + |u_2|^{p-1}u_1) + i\overline{v_2}(|u_1|^{p-1}u_2 + |u_2|^{p-1}u_2) \right] dx.$$

Let us add and subtract

$$i\int \left(|v_1|^{p+1} + |v_1|^{p-1}|v_2|^2 + |v_2|^{p+1} + |v_2|^{p-1}|v_1|^2 \right) dx,$$

and then take the imaginary part to give

$$\begin{aligned} \mathrm{Im}N(t) = &\int \left(|v_1|^{p+1} + |v_1|^{p-1}|v_2|^2 + |v_2|^{p+1} + |v_2|^{p-1}|v_1|^2 \right) dx \\ &+ \mathrm{Re}\int \left[\overline{v_1}(|u_1|^{p-1}u_1 + |u_2|^{p-1}u_1) + \overline{v_2}(|u_1|^{p-1}u_2 + |u_2|^{p-1}u_2) \right] dx \\ &- \mathrm{Re}\int \left(|v_1|^{p+1} + |v_1|^{p-1}|v_2|^2 + |v_2|^{p+1} + |v_2|^{p-1}|v_1|^2 \right) dx. \end{aligned}$$

For $v_1,\ v_2$, apply Lemma 2.3(ii), then

$$\mathrm{Im}N(t) \geq Bt^{-(p-1)/2} - I, \quad \forall t > T_1. \tag{3.7}$$

Here,

$$\begin{aligned} I = \Big| &\mathrm{Re}\int \left[\overline{v_1}(|u_1|^{p-1}u_1 + |u_2|^{p-1}u_1) + \overline{v_2}(|u_1|^{p-1}u_2 + |u_2|^{p-1}u_2) \right] dx \\ &- \mathrm{Re}\int \left(|v_1|^{p+1} + |v_1|^{p-1}|v_2|^2 + |v_2|^{p+1} + |v_2|^{p-1}|v_1|^2 \right) dx \Big|. \end{aligned}$$

Next, we need to prove that

$$I = o(t^{-(p-1)/2}), \quad \text{as } t \to \infty,$$

so that

$$\mathrm{Im}N(t) \geq ct^{-(p-1)/2} > 0$$

for all large t.

We use the Minkowski inequality and the mean value theorem to give

$$\begin{aligned}
I \leq & \Big| \int [(|u_1|^{p-1} + |u_2|^{p-1}) - (|v_1|^{p-1} + |v_2|^{p-1})](u_1\overline{v_1} + u_2\overline{v_2})dx \Big| \\
& + \Big| \int (|v_1|^{p-1} + |v_2|^{p-1})[\overline{v_1}(u_1 - v_1) + \overline{v_2}(u_2 - v_2)]dx \Big| \\
\leq & c \int (|u_1|^{p-2} + |u_2|^{p-2})(|u_1 - v_1| + |u_2 - v_2|)(|u_1| + |u_2|)(|v_1| + |v_2|)dx \\
& + c \int (|v_1|^{p-2} + |v_2|^{p-2})(|u_1 - v_1| + |u_2 - v_2|)(|u_1| + |u_2|)(|v_1| + |v_2|)dx \\
& + \int (|v_1|^{p-1} + |v_2|^{p-1})(|u_1 - v_1| + |u_2 - v_2|)(|v_1| + |v_2|)dx \\
:= & J_1 + J_2 + J_3.
\end{aligned}$$

For J_3, using Hölder's inequality, (3.6), and (3.5), one obtains

$$\begin{aligned}
J_3 & \leq c(\|v_1(t)\|_\infty^{p-1} + \|v_2(t)\|_\infty^{p-1})(\|u_1(t) - v_1(t)\|_2 + \|u_2(t) - v_2(t)\|_2)(\|v_1(t)\|_2 + \|v_2(t)\|_2) \\
& \leq ct^{-\frac{p-1}{2}}(\|u_1(t) - v_1(t)\|_2 + \|u_2(t) - v_2(t)\|_2).
\end{aligned}$$

Recall (3.3), then

$$J_3 = o(t^{-\frac{p-1}{2}}). \tag{3.8}$$

For J_1, using Hölder's inequality, Lemma 2.3(i), and Lemma 2.4, we get

$$\begin{aligned}
J_1 & \leq c(\|u_1(t) - v_1(t)\|_2 + \|u_2(t) - v_2(t)\|_2) \\
& \quad \times \Big[\int \Big(|u_1|^{p+1} + |u_2|^{p+1} + |u_1|^{p-1}|u_2|^2 + |u_2|^{p-1}|u_1|^2\Big)dx \Big]^{\frac{p-1}{p+1}} \\
& \quad \times (\|v_1\|_{2(p+1)/(3-p)} + \|v_2\|_{2(p+1)/(3-p)}) \\
& \leq c(\|u_1(t) - v_1(t)\|_2 + \|u_2(t) - v_2(t)\|_2)t^{-(p-1)/2}.
\end{aligned}$$

Recall (3.3), then

$$J_1 = o(t^{-\frac{p-1}{2}}).$$

Similarly,

$$J_2 = o(t^{-\frac{p-1}{2}}).$$

Recall (3.8), then

$$I \leq J_1 + J_2 + J_3 = o(t^{-\frac{p-1}{2}}), \quad \text{as } t \to \infty.$$

This estimate together with (3.7) implies that there exist $T > \max\{1, T_1\}$ and a positive constant C such that

$$\mathrm{Im}N(t) \geq Ct^{-(p-1)/2}, \quad \forall t \geq T.$$

Let us fix C and T, let K be a positive integer, and integrate this inequality over $T \leq t \leq KT$ to deduce

$$\int_T^{KT} \frac{d}{dt}\mathrm{Im}H(t)dt \geq \int_T^{KT} Ct^{-(p-1)/2}dt \geq \int_T^{KT} Ct^{-1}dt.$$

Therefore,

$$\mathrm{Im}H(KT) - \mathrm{Im}H(T) \geq C \ln K,$$

It follows from the definition of $H(t)$ and Schwarz's inequality that

$$\begin{aligned}|\mathrm{Im}H(t)| \leq |H(t)| &= \left| \int [u_1(t)\overline{v_1}(t) + u_2(t)\overline{v_2}(t)]dx \right| \\ &\leq (\|u_1(t)\|_2 + \|u_2(t)\|_2)(\|v_1(t)\|_2 + \|v_2(t)\|_2) \overset{(3.5)}{=} A^2, \quad \forall t > T.\end{aligned}$$

Further,

$$C \ln K \leq |\mathrm{Im}H(KT)| + |\mathrm{Im}H(T)| \leq 2A^2.$$

Let us choose $K > e^{2A^2/C}$ to show contradiction. In conclusion, $v_1(0) = 0,\ v_2(0) = 0$. Hence, by (3.5), we get that $u_1(t) = 0,\ u_2(t) = 0$ in L^2 for all t. The smoothness of $u_1,\ u_2$ implies $u_1(x,t) = 0,\ u_2(x,t) = 0$.

The proof of this theorem is completed.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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