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Research article

Multiplicity of the large periodic solutions to a super-linear wave equation with general variable coefficient

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Abstract: In this paper, we were concerned with the multiplicity of the large periodic solutions to a super-linear wave equation with a general variable coefficient. In general, the variable coefficient $\rho(\cdot)$ needs to be satisfied ess inf $\eta_\rho(\cdot) > 0$ with $\eta_\rho(\cdot) = \frac{1}{2}\frac{\rho''}{\rho} - \frac{1}{4}(\frac{\rho'}{\rho})^2$. Especially, the case $\eta_\rho(\cdot) = 0$ is presented as an open problem in [Trans. Amer. Math. 349: 2015–2048, 1997]. Here, without any restrictions on $\eta_\rho(\cdot)$, we established the multiplicity of large periodic solutions for the Dirichlet-Neumann boundary condition and Dirichlet-Robin boundary condition when the period $T = 2\pi\frac{2a-1}{b}$ with $a, b \in \mathbb{N}^+$. The key ingredient of the proof is the combination of the variational method and an approximation argument. Since the sign of $\eta_\rho(\cdot)$ can change, our results can be applied to the classical wave equation.

Keywords: super-linear wave equation; large periodic solutions; existence; variational method
Mathematics Subject Classification: 35B10, 35L71

1. Introduction

We consider the nonlinear wave equation

$$\rho(x)u_{tt} - (\rho(x)u_x)_x = f(t, x, u), \quad (t, x) \in (0, T) \times (0, \pi), \tag{1.1}$$

together with time-periodic condition

$$u(0, x) = u(T, x), \quad u_t(0, x) = u_t(T, x), \tag{1.2}$$

where $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$ is T periodic with respect to t and the period T is determined by

$$T = \frac{2a-1}{b}2\pi, \quad \text{for } a, b \in \mathbb{N}^+. \tag{1.3}$$

In addition to (1.2), Equation (1.1) is subject to the boundary condition

$$\alpha_1 u(t,0) + \beta_1 u_x(t,0) = 0, \quad \alpha_2 u(t,\pi) + \beta_2 u_x(t,\pi) = 0, \tag{1.4}$$

where the coefficients α_i, β_i for $i = 1, 2$ satisfy

$$\alpha_i^2 + \beta_i^2 \neq 0, \ \ \beta_1\beta_2 = 0 \ \text{ and } \ \beta_1^2 + \beta_2^2 \neq 0, \tag{1.5}$$

which contains the Dirichlet-Neumann boundary condition (e.g., $\alpha_1 \neq 0, \beta_1 = 0, \alpha_2 = 0, \beta_2 \neq 0$) and the Dirichlet-Robin boundary condition (e.g., $\alpha_1 \neq 0, \beta_1 = 0, \alpha_2 \neq 0, \beta_2 \neq 0$).

Equation (1.1) originated from the following equation

$$\omega(z)u_{tt} - (\nu(z)u_z)_z = 0, \tag{1.6}$$

which is used to describe the forced vibrations of a nonhomogeneous string and the propagation of seismic waves in nonisotropic media (see [1–9]). Here, u represents the vertical displacement of the seismic wave, $\omega(\cdot)$ denotes the rock density, and $\nu(\cdot)$ is the elasticity coefficient. By means of transformation of variables $x = \int_0^z \left(\frac{\omega(s)}{\nu(s)}\right)^{1/2} \mathrm{d}s$, Equation (1.6) is simplified as

$$\rho(x)u_{tt} - (\rho(x)u_x)_x = 0,$$

where $\rho = (\omega\nu)^{1/2}$ denotes the impedance function.

Equation (1.1) degenerates to the classical wave equation when $\rho(\cdot) \equiv C$. Since the 1960s, much work has focused on periodic solutions of classical wave equations (see [10–16]). For recent results on Hamiltonian systems, see [17, 18], and on higher-dimensional problems, see [19–21]. For the Euler equation, see [22]. In addition, for stability results, see [23, 24], and for blow-up solutions, see [25, 26]. Many of the works are based on the spectrum made up of the eigenvalues $n^2 - m^2$ with $n \in \mathbb{N}, m \in \mathbb{Z}$ for the frequency $\omega \in \mathbb{Q}$, for example, [27–31]. This property ensures that the desired compact conditions hold. However, for the frequency $\omega \in \mathbb{R}\backslash\mathbb{Q}$, the "small divisor problem" raised naturally in realistic models, such as the wave equations and the beam equations. The tools to solve this problem are the Nash-Morse iteration and KAM (Kolmogorov-Arnold-Moser) theory (see [16, 32]).

In recent decades, the nonlinear wave equations with variable coefficient have attracted broad interests. For the nonlinearity satisfying Lipschitz continuity, Barbu and Pavel in [2] used the monotonicity method to establish a periodic solution under the assumption $\operatorname{ess\,inf} \eta_\rho(\cdot) > 0$ with $\eta_\rho(\cdot) = \frac{1}{2}\frac{\rho''}{\rho} - \frac{1}{4}(\frac{\rho'}{\rho})^2$. The appearance of the function $\eta_\rho(\cdot)$ is due to the Liouville transformation in investigating the Sturm-Liouville problem. The condition $\operatorname{ess\,inf} \eta_\rho(\cdot) > 0$ can make sure that the kernel space of the variable coefficient wave operator has finite dimensions, then the free oscillations can be easily controlled. On the other hand, it is well known that the solvability of the nonlinear problems depend on the properties of nonlinear terms. For the nonlinearity with power-law growth, Rudakov in [33] constructed the periodic solutions by the variational method. Ji and his collaborators in [5, 6, 8, 9, 34] obtained some interesting results on periodic solutions for several classes of nonlinear problems under various homogeneous boundary conditions via variational methods. With the help of a global inverse function theorem, Chen in [35] got an existence and uniqueness theorem for a system with a variable coefficient. Ji et al. in [7] found that there is at least two periodic solutions on some subspaces of the L^2 space by using the topological degree theory. These works are all focused on the case $\operatorname{ess\,inf} \eta_\rho(\cdot) > 0$. However, for the case of $\eta_\rho(\cdot) = 0$, the kernel space becomes infinite dimensional, which, together with the effects of a variable coefficient, provokes further difficulties; hence, this problem was posed by Barbu and Pavel in [2] as an open problem. Recently, Ji and his collaborators considered this problem and obtained some interesting results in [36–38], where $\eta_\rho(\cdot)$ could be equal to zero or even be of sign-changing.

In addition to the effect of the sign of $\eta_\rho(\cdot)$ on the dimension of kernel space, the spectrum of the variable coefficient wave operator has the accumulation points (e.g., [39]); thus, the existing compactness conditions are not sufficient to deal with super-linear problems. However, for the Dirichlet-Neumann boundary condition and the Dirichlet-Robin boundary condition, when T satisfies (1.3), the compactness can be improved to be good enough. Recently, when the sign of $\eta_\rho(\cdot)$ can change, Rudakov in [40] assumed $\eta_\rho(\cdot) \neq \frac{\rho'(\pi)}{\rho(\pi)}$ to guarantee that the dimension of the kernel is finite, then infinitely many periodic solutions are constructed for the super-linear problem under the Dirichlet-Neumann boundary condition.

This paper aims to establish the multiplicity of large periodic solutions to the problems (1.1), (1.2), and (1.4) with general variable coefficients. The word "general" means that we do not impose any restrictions on $\eta_\rho(\cdot)$. This results in the kernel space being infinitely dimensional. To overcome this difficulty, we make use of the monotonicity method and approximation argument to estimate the component of solutions in the kernel space. To get the discrete spectrum, we assume that the period T satisfies (1.3), which guarantees that the subspace $E^+ \oplus E^-$ of function space E defined in Section 2 is compactly embedded in L^p space for $p > 1$. This compactness condition is sufficient for dealing with the super-linear problem. Compared with the results in [40], we remove the restriction on $\eta_\rho(\cdot)$ and consider the Dirichlet-Robin boundary condition. Since the sign of $\eta_\rho(\cdot)$ can change, our results can be applied to the classical wave equation.

In this paper, we make the following assumptions:

(H1) $\rho \in C^2[0,\pi]$ and $\rho(x) > 0$, $\forall x \in [0,\pi]$;

(H2) $-\tilde{f}(\cdot,\cdot,\xi) = \tilde{f}(\cdot,\cdot,-\xi)$ for all $(\cdot,\cdot,\xi) \in \Omega\times\mathbb{R}$, and $\tilde{f}(\cdot,\cdot,\xi)$ is nondecreasing in ξ and $\tilde{f}(\cdot,\cdot,\xi) = 0$ if, and only if, $\xi = 0$, where

$$\tilde{f}(t,x,\xi) = \frac{f(t,x,\xi)}{\rho(x)}.$$

(H3) There are $M > 0$, $\mu > 2$, and $a_1, a_2 > 0$ such that

$$\mu\widetilde{F}(\cdot,\cdot,\xi) \leq \tilde{f}(\cdot,\cdot,\xi)\xi, \quad \forall|\xi| \geq M, \tag{1.7}$$

and

$$|\tilde{f}(\cdot,\cdot,\xi)| \leq a_1|\xi|^{\mu-1} + a_2, \quad \forall(\cdot,\cdot,\xi) \in \Omega\times\mathbb{R}, \tag{1.8}$$

where

$$\widetilde{F}(\cdot,\cdot,\xi) = \int_0^\xi \tilde{f}(\cdot,\cdot,s)\mathrm{d}s.$$

At the end of this section, we show the outline of this paper. Section 2 gives the main result and some preliminaries and proves that the periodic solutions of problems (1.1), (1.2), and (1.4) are equal to the critical points of the variational problem. We study the restricted functional on a sequence of subspaces with increasing dimension and construct approximate solutions in Section 3. Finally, we obtain the main result by combining uniform boundedness and an approximation argument in Section 4, and we present our conclusions in Section 5.

2. Notations, definitions and results

Let

$$\Psi = \{\psi \in C^2(\Omega) : \psi(0,x) = \psi(T,x), \psi_t(0,x) = \psi_t(T,x),$$
$$\alpha_1\psi(t,0) + \beta_1\psi_x(t,0) = 0,\ \alpha_2\psi(t,\pi) + \beta_2\psi_x(t,\pi) = 0\}$$

and

$$L^p(\Omega) = \Big\{u : \|u\|^p_{L^p(\Omega)} = \int_\Omega |u(t,x)|^p \rho(x)\mathrm{d}t\mathrm{d}x < \infty\Big\},\quad p \geq 1.$$

The inner product on the Hilbert space $L^2(\Omega)$ is defined as

$$\langle v, w\rangle = \int_\Omega v(t,x)\overline{w(t,x)}\rho(x)\mathrm{d}t\mathrm{d}x,\ \forall v, w \in L^2(\Omega).$$

Definition 2.1. *A function $u \in L^p(\Omega)$ is a weak solution of the problems* (1.1)*,* (1.2)*, and* (1.4) *if*

$$\int_\Omega u(\rho\psi_{tt} - (\rho\psi_x)_x)\mathrm{d}t\mathrm{d}x - \int_\Omega \tilde{f}(t,x,u)\psi\rho\mathrm{d}t\mathrm{d}x = 0,\ \forall\psi \in \Psi.$$

The main results are given as follows.

Theorem 2.1. *Let α_i,β_i satisfy* (1.5) *for $i = 1, 2$, the period T satisfy* (1.3)*, and let ρ and f satisfy (H1)–(H3), then, there are infinitely many periodic solutions u_n for the problems* (1.1)*,* (1.2)*, and* (1.4)*, satisfying*

$$\|u_n\|_{L^\mu(\Omega)} \to \infty,\ \ as\ \ n \to \infty.$$

Furthermore, $u_n \in C(\Omega) \cap H^1(\Omega)$ for the Dirichlet-Neumann boundary condition and $u_n \in C(\Omega)$ for the Dirichlet-Robin boundary condition.

The sequence of eigenfunctions $\{\phi_j(t)\varphi_k(x) : j \in \mathbb{Z}, k \in \mathbb{N}\}$ forms a completely orthonormal basis of $L^2(\Omega)$ ([41]), where

$$\phi_j(t) = \frac{e^{i\nu_j t}}{\sqrt{T}}\ \text{ with }\ \nu_j = \frac{2j\pi}{T},\ j \in \mathbb{Z},$$

and λ_k, $\varphi_k(x)$ are determined by the following Sturm-Liouville problem

$$(\rho(x)\varphi_k'(x))' = -\lambda_k\rho(x)\varphi_k(x),\ k \in \mathbb{N},$$
$$\alpha_1\varphi_k(0) + \beta_1\varphi_k'(0) = 0,\ \alpha_2\varphi_k(\pi) + \beta_2\varphi_k'(\pi) = 0.$$

According to Section 4 in [42], a direct calculation shows that the eigenvalues λ_k have the following asymptotic formula

$$\lambda_k = (k + \frac{1}{2})^2 + \frac{\kappa}{\pi} + O(\frac{1}{k^2}), \tag{2.1}$$

where

$$\kappa = \begin{cases} -\frac{\rho'(\pi)}{\rho(\pi)} + \int_0^\pi \eta_\rho(x)\mathrm{d}x, & \text{for Dirichlet} - \text{Neumann boundary condition},\\ \frac{2\alpha_2}{\beta_2} - \frac{\rho'(\pi)}{\rho(\pi)} + \int_0^\pi \eta_\rho(x)\mathrm{d}x, & \text{for Dirichlet} - \text{Robin boundary condition}, \end{cases}$$

with

$$\eta_\rho(x) = \frac{1}{2}\frac{\rho''}{\rho} - \frac{1}{4}\left(\frac{\rho'}{\rho}\right)^2.$$

The linear operator L_0 is defined as

$$L_0\psi = \rho^{-1}\left(\rho\psi_{tt} - (\rho\psi_x)_x\right), \ \forall\psi \in \Psi,$$

and its extension L is a self-adjoint operator in $L^2(\Omega)$. Furthermore, we have $\lambda_k - \nu_j^2$ as the eigenvalues of L. Clearly, $u \in L^2(\Omega)$ is a weak solution of problems (1.1), (1.2), and (1.4) if, and only if,

$$Lu = \tilde{f}(t, x, u).$$

For any $u, v \in L^2(\Omega)$, we rewrite it as $u(t, x) = \sum_{j,k} u_{jk}\phi_j(t)\varphi_k(x)$ and $v(t, x) = \sum_{j,k} v_{jk}\phi_j(t)\varphi_k(x)$, where u_{jk} and v_{jk} are the Fourier coefficients. Set

$$E^+ = \overline{\text{span}}\left\{\phi_j(t)\varphi_k(x) : \lambda_k > \nu_j^2\right\},$$

$$E^0 = \overline{\text{span}}\left\{\phi_j(t)\varphi_k(x) : \lambda_k = \nu_j^2\right\},$$

$$E^- = \overline{\text{span}}\left\{\phi_j(t)\varphi_k(x) : \lambda_k < \nu_j^2\right\},$$

then their direct sum $E := E^+ \oplus E^- \oplus E^0$ is a Hilbert space with the inner product

$$(u, v) = \sum_{\lambda_k \neq \nu_j^2} |\lambda_k - \nu_j^2| u_{jk}\bar{v}_{jk} + \sum_{\lambda_k = \nu_j^2} u_{jk}\bar{v}_{jk},$$

and its norm is denoted by $\|u\|_E$. For any $u \in E$, split it into $u = u^+ + u^0 + u^-$ with $u^+ \in E^+$, $u^0 \in E^0$, $u^- \in E^-$. In particular,

$$\|u^0\|_E = \|u^0\|_{L^2(\Omega)}.$$

For any $m, n \in \mathbb{N}^+$, denote the finite dimensional spaces by

$$W_m = \text{span}\left\{\phi_j\varphi_k \mid -m \leq j \leq m, \ 0 \leq k \leq m\right\}$$

Define the "upper direct sum" spaces and the "under direct sum" spaces by

$$E^m = \left(W_m \cap (E^- \oplus E^0)\right) \oplus E^+, \ \ E_n = E^- \oplus E^0 \oplus \left(W_n \cap E^+\right).$$

Let

$$E_n^m = E^m \cap E_n.$$

Obviously, $E^m \subset E^{m+1}$ and E_n^m is a finite dimensional space. Furthermore,

$$E = \bigcup_{m \in \mathbb{N}^+} E^m.$$

Define the energy functional corresponding to problems (1.1), (1.2), and (1.4) as

$$\Phi(u) = \frac{1}{2}(\|u^+\|_E^2 - \|u^-\|_E^2) - \int_\Omega \widetilde{F}(t, x, u)\rho \mathrm{d}t\mathrm{d}x, \ \ \forall u \in E. \tag{2.2}$$

Since $\tilde{f}$ is odd, Φ is an even C^1 functional on E. In addition,

$$\langle \Phi'(u), v\rangle = (u^+, v^+) - (u^-, v^-) - \int_\Omega \tilde{f}(t, x, u)v\rho \mathrm{d}t\mathrm{d}x, \quad \forall u,\ v \in E. \tag{2.3}$$

Therefore, u is a weak solution of problems (1.1), (1.2), and (1.4) if, and only if, u is a critical point of Φ, namely,

$$\Phi'(u) = 0.$$

In what follows, we first consider the restricted functional $\Phi_m = \Phi|_{E^m}$ and construct its critical points.

3. Constructing approximate solutions

Proposition 3.1. *For any $q > 1$, the embedding*

$$E^- \oplus E^+ \hookrightarrow L^q(\Omega) \tag{3.1}$$

is compact.

Proof. A similar proof as Lemma 2.1 in [36] shows that the series

$$\sum_{\lambda_k \neq \nu_j^2} \frac{1}{(\lambda_k - \nu_j^2)^2}$$

is convergent. Thus,

$$\lim_{j,k\to\infty} \left|\lambda_k - \nu_j^2\right| = \infty, \quad \text{for } \lambda_k \neq \nu_j^2.$$

In consequence, by using the method of Lemma 1 in [40], we obtain the result.

Lemma 3.1. *For any $m \in \mathbb{N}^+$, let $\{u_i\} \subset E^m$ satisfy $\Phi_m(u_i) \leq \tilde{d}$ (a constant) and $\Phi'_m(u_i) \to 0$ as $i \to \infty$, then $\{u_i\}$ has a convergent subsequence, i.e., Φ_m satisfies the Palais-Smale (PS) condition.*

Proof. Split $u_i = u_i^+ + \hat{u}_i$ with $u_i^+ \in E^+$, $\hat{u}_i \in W_m \cap (E^- \oplus E^0)$. Obviously, $\hat{u}_i = u_i^- + u_i^0$ with $u_i^- \in W_m \cap E^-$, $u_i^0 \in W_m \cap E^0$.

Since $\Phi_m(u_i) \leq \tilde{d}$ and $\langle \Phi'_m(u_i), u_i\rangle \to 0$, by (1.7), we have,

$$o(1)\|u_i\|_E + (\frac{1}{2} - \frac{1}{\mu}) \int_\Omega \tilde{f}(t, x, u_i)u_i\rho \mathrm{d}t\mathrm{d}x \leq \tilde{d} + M_1,$$

for some constant $M_1 > 0$. Thus,

$$\int_\Omega \tilde{f}(t, x, u_i)u_i\rho \mathrm{d}t\mathrm{d}x \leq M_2,$$

for some constant $M_2 > 0$. Taking advantage of (1.7) again and the above estimate, we have

$$\int_\Omega \widetilde{F}(t, x, u_i)\rho \mathrm{d}t\mathrm{d}x \leq M_3, \tag{3.2}$$

for some constant $M_3 > 0$.

According to (1.7), it follows that

$$\widetilde{F}(t,x,u_i) \geq a_3|u_i|^{\mu} - a_4, \tag{3.3}$$

for some constants $a_3, a_4 > 0$. By (3.2) and (3.3), we have

$$\|u_i\|_{L^{\mu}(\Omega)} \leq M_4, \tag{3.4}$$

for some constant $M_4 > 0$. Moreover, from (1.8), we have

$$\|\tilde{f}(t,x,u_i)\|_{L^{\mu'}(\Omega)} \leq M_5, \tag{3.5}$$

where $\mu' = \mu/(\mu - 1)$.

Noting $\Phi'_m(u_i) \to 0$, by (3.1), (3.4), and (3.5), it follows that

$$\|u_i^+\|_E^2 \leq o(1)\|u_i^+\|_E + M_6\|u_i^+\|_E.$$

Therefore, $\{u_i^+\}$ is bounded in E.

By $\dim\left(W_m \cap (E^- \oplus E^0)\right) < \infty$, from (3.4), we have that $\{\hat{u}_i\}$ is bounded in E.

Consequently, $\{u_i\}$ is bounded in E, thus $u_i \rightharpoonup u$ in E as $i \to \infty$ for some $u \in E$. Let u^+, $\hat{u}$ denote weak limits of $\{u_i^+\}$, $\{\hat{u}_i\}$, respectively, where $u_i^+, u^+ \in E^+$, $\hat{u}_i, \hat{u} \in W_m \cap (E^- \oplus E^0)$.

Since $\dim\left(W_m \cap (E^- \oplus E^0)\right) < \infty$, then

$$\|\hat{u}_i - \hat{u}\|_E \to 0, \quad \text{as } i \to \infty.$$

For $u_i^+ \in E^+$, from (2.3), it follows that

$$\|u_i^+ - u^+\|_E^2 \leq o(1)\|u_i^+ - u^+\|_E + \|\tilde{f}(t,x,u_i)\|_{L^{\mu'}(\Omega)}\|u_i^+ - u^+\|_{L^{\mu}(\Omega)} + o(1).$$

By (3.1), u_i^+ weakly converges to u^+, and it follows that

$$\|u_i^+ - u^+\|_{L^{\mu}(\Omega)} \to 0, \quad \text{as } i \to \infty.$$

Therefore,

$$\|u_i^+ - u^+\|_E \to 0, \quad \text{as } i \to \infty,$$

which completes the proof.

Proposition 3.2. *Set*

$$\zeta_n = \sup_{u \in (E_{n-1})^{\perp} \setminus \{0\}} \frac{\|u\|_{L^{\mu}(\Omega)}}{\|u\|_E}, \tag{3.6}$$

then $\zeta_n \to 0$ as $n \to \infty$.

Proof. Noting that $(E_{n-1})^{\perp} \subset E^+$ and the embedding $E^+ \hookrightarrow L^{\mu}(\Omega)$ is compact, a similar proof as [43] yields the result.

Lemma 3.2. *For any $n \in \mathbb{N}^+$, there exist the constants σ_n, $r_n > 0$ satisfying*

$$\Phi(u) \geq \sigma_n, \quad \forall u \in (E_{n-1})^{\perp} \cap S_{r_n} := \{u \in E : \|u\|_E = r_n\},$$

and

$$\sigma_n \to \infty, \quad as \;\; n \to \infty.$$

Proof. From (3.6), (1.8), and (2.2), for $u \in (E_{n-1})^{\perp} \subset E^{+}$, it follows that

$$\Phi(u) \geq \frac{1}{2}\|u\|_E^2 - a_1\|u\|_E^{\mu}\zeta_n^{\mu} - a_2 T\pi.$$

Taking $r_n = (\mu a_1 \zeta_n^{\mu})^{\frac{1}{2-\mu}}$ in the above estimate, with the help of $\mu > 2$ and $\zeta_n \to 0$ as $n \to \infty$, for n large enough, it follows that

$$\Phi(u) \geq (\frac{1}{2} - \frac{1}{\mu})r_n^2 - a_2 T\pi > 0.$$

For n large enough, $\sigma_n := (\frac{1}{2} - \frac{1}{\mu})r_n^2 - a_2 T\pi > 0$ and $\sigma_n \to \infty$ as $n \to \infty$, and the proof is complete.

Lemma 3.3. *For any $n \in \mathbb{N}^{+}$, there exist the constants R_n, $\varrho_n > 0$ satisfying*

$$\Phi(u) \leq 0, \quad \forall u \in E_n, \quad \|u\|_E \geq R_n,$$

$$\Phi(u) \leq \varrho_n, \quad \forall u \in E_n, \quad \|u\|_E \leq R_n.$$

Proof. By (3.3), it follows that

$$\Phi(u) \leq \frac{1}{2}(\|u^{+}\|_E^2 - \|u^{-}\|_E^2) - M_8(\|u^0\|_{L^{\mu}(\Omega)}^{\mu} + \|u^{+}\|_{L^{\mu}(\Omega)}^{\mu}) + M_7,$$

for some positive constants M_7, M_8.

Since $u^{+} \in W_n \cap E^{+}$ and $\dim(W_n \cap E^{+}) < \infty$, then $\|u^{+}\|_{L^{\mu}(\Omega)}^{\mu} \geq C_1\|u^{+}\|_E^{\mu}$. Moreover, since $\|u^0\|_E = \|u^0\|_{L^2(\Omega)}$ and $L^{\mu}(\Omega) \hookrightarrow L^2(\Omega)$, then $C_2\|u^0\|_E \leq \|u^0\|_{L^{\mu}(\Omega)}$. Therefore,

$$\Phi(u) \leq \frac{1}{2}(\|u^{+}\|_E^2 - \|u^{-}\|_E^2) - M_8(C_2\|u^0\|_E^{\mu} + C_1\|u^{+}\|_E^{\mu}) + M_7.$$

Thus, noting $\mu > 2$, we arrive at the result.

Let

$$\mathscr{F}_{mn} = \{\gamma \in C(B_n^m, E^m) \mid \gamma \text{ is odd and } \gamma|_{\partial B_n^m} = id\},$$

where $B_n^m = \{u \in E_n^m \mid \|u\|_E \leq R_n\}$, ∂B_n^m denotes the boundary of B_n^m, the constant R_n is given in Lemma 3.3, and id is the identity map.

Define $A_c = \{u \in E^m \mid \Phi_m(u) \leq c\}$, $\forall c \in \mathbb{R}$, and

$$K = \{u \in E^m \mid \Phi_m'(u) = 0\}.$$

Define

$$c_{mn} = \inf_{\gamma \in \mathscr{F}_{mn}} \max_{u \in B_n^m} \Phi_m(\gamma(u)). \tag{3.7}$$

Lemma 3.4. *For n large, c_{mn} are the critical values of Φ_m and satisfy*

$$0 < \sigma_n \leq c_{mn} \leq \varrho_n.$$

Proof. First, it is proved by contradiction. In virtue of Φ_m satisfying the (PS) condition, suppose that c_{mn} are not the critical values of Φ_m and there is $\bar{\varepsilon} > 0$ satisfying $\Phi_m^{-1}[c_{mn} - \bar{\varepsilon}, c_{mn} + \bar{\varepsilon}] \cap K = \emptyset$. By the definition of c_{mn} and taking $\gamma_0 \in \mathscr{F}_{mn}$ such that $\max_{u \in B_n^m} \Phi_m(\gamma_0(u)) \leq c_{mn} + \bar{\varepsilon}$, we have

$$\gamma_0(B_n^m) \subset A_{c_{mn}+\bar{\varepsilon}}.$$

By the standard deformation lemma, there is an odd mapping $\eta_t(\cdot) := \eta(t, \cdot) \in C([0, 1] \times E^m, E^m)$ such that $\eta_1(A_{c_{mn}+\bar{\varepsilon}}) \subset A_{c_{mn}-\bar{\varepsilon}}$, which implies

$$\eta_1(\gamma_0(B_n^m)) \subset A_{c_{mn}-\bar{\varepsilon}}.$$

Consequently, $\eta_1 \circ \gamma_0 \mid_{\partial B_n^m} = id$ and $\eta_1 \circ \gamma_0$ is odd, i.e., $\eta_1 \circ \gamma_0 \in \mathscr{F}_{mn}$. Therefore,

$$c_{mn} \leq \max_{u \in B_n^m} \Phi_m(\eta_1(\gamma_0(u))) \leq c_{mn} - \bar{\varepsilon}.$$

This is a contradiction.

Now, we prove $\sigma_n \leq c_{mn} \leq \varrho_n$. According to Lemma 3.3 and the definitions of c_{mn}, we have $c_{mn} \leq \varrho_n$.

On the other hand, for each $\gamma \in \mathscr{F}_{mn}$, let

$$B_{r_n} = \{u \in B_n^m \mid \|\gamma(u)\|_E < r_n\},$$

where the constant r_n is present in Lemma 3.2. Since γ is odd continuous, then B_{r_n} is a symmetrically bounded open ball and $0 \in B_{r_n}$. Moreover, from Lemmas 3.2 and 3.3, it is easy to see $R_n > r_n$. The combining of $R_n > r_n$ and $\gamma|_{\partial B_n^m} = id$ yields $B_{r_n} \cap \partial B_n^m = \emptyset$. Let $P : E^m \to E_{n-1}^m$ be the natural projection. Therefore, by the Borsuk-Ulam theorem [44], there exists $u_0 \in \partial B_{r_n}$ satisfying $P\gamma(u_0) = 0$, then we have $\|\gamma(u_0)\|_E = r_n$ and $\gamma(u_0) \in (E_{n-1})^{\perp}$. Thus, by Lemma 3.2, we have

$$\max_{u \in B_n^m} \Phi_m(\gamma(u)) \geq \sigma_n.$$

We arrive at the conclusion.

For each n large, suppose that u_{mn} are the critical points of Φ_m corresponding to c_{mn}. In what follows, to obtain Theorem 2.1, we are going to prove the uniform boundedness of $\{u_{mn}^{\pm}\}$ for any $n \in \mathbb{N}^+$, then we use the approximation argument to get the desired results.

4. Proof of Theorem 2.1

Let $M_i > 0$ $(i = 9, 10, 11, 12)$ denote the constants that are independent of m.

We have

$$\langle Lu, v\rangle = (u^+, v^+) - (u^-, v^-), \quad \forall u, v \in E.$$

Moreover, since u_{mn} are the critical points of Φ_m, then

$$(u_{mn}^+, v^+) - (u_{mn}^-, v^-) = \int_{\Omega} \tilde{f}(t, x, u_{mn}) v\rho \mathrm{d}t\mathrm{d}x, \quad \forall v \in E^m.$$

Thus, for any $v \in E^m$, it follows that

$$\langle Lu_{mn}, v\rangle = \langle \tilde{f}(t, x, u_{mn}), v\rangle. \tag{4.1}$$

Lemma 4.1. *The sequence* $\{u^{\pm}_{mn}\}$ *is uniformly bounded for any* $n \in \mathbb{N}^+$.

Proof. Since

$$\langle Lu_{mn}, u_{mn}\rangle = \int_{\Omega} \tilde{f}(t, x, u_{mn})u_{mn}\rho \mathrm{d}t\mathrm{d}x, \tag{4.2}$$

from (1.7), (2.2), (4.2), and Lemma 3.4, there exists $M_9 > 0$ such that

$$(\frac{1}{2} - \frac{1}{\mu}) \int_{\Omega} \tilde{f}(t, x, u_{mn})u_{mn}\rho \mathrm{d}t\mathrm{d}x \leq \varrho_n + M_9.$$

Thus, by (1.7), we have that $\int_{\Omega} \widetilde{F}(t, x, u_{mn})\rho \mathrm{d}t\mathrm{d}x$ is uniformly bounded.

From (3.3), it follows that

$$\|u_{mn}\|_{L^{\mu}(\Omega)} \leq M_{10}. \tag{4.3}$$

From (1.8), it follows that

$$\|\tilde{f}(t, x, u_{mn})\|_{L^{\mu'}(\Omega)} \leq M_{11}, \tag{4.4}$$

where $\mu' = \mu/(\mu - 1)$. Since

$$\begin{aligned} \|u^+_{mn}\|^2_E &\leq \|\tilde{f}(t, x, u_{mn})\|_{L^{\mu'}(\Omega)}\|u^+_{mn}\|_{L^{\mu}(\Omega)} \\ &\leq M_{12}\|u^+_{mn}\|_E, \end{aligned}$$

we have that $\{u^+_{mn}\}$ is uniformly bounded for any $n \in \mathbb{N}^+$. The similar conclusion holds for $\{u^-_{mn}\}$. We arrive at the conclusion.

Since $L^p(\Omega)$ and E are reflexive and the embedding $E^- \oplus E^+ \hookrightarrow L^q(\Omega)$ is compact for $q > 1$, then by the above lemma and (4.3), without loss of generality, we have

$$\begin{aligned} &u_{mn} \rightharpoonup u_n \text{ in } L^{\mu}(\Omega), \text{ as } m \to \infty, \\ &u^{\pm}_{mn} \rightharpoonup u^{\pm}_n \text{ in } E, \quad \text{as } m \to \infty, && (4.5) \\ &u^{\pm}_{mn} \to u^{\pm}_n \text{ in } L^{\mu}(\Omega), \text{ as } m \to \infty. && (4.6) \end{aligned}$$

Thanks to the above lemmas, now let's prove Theorem 2.1.

Proof. Let $P_m : E \to E^m$ be the natural projection. According to $u^{\pm}_{mn} \in E^m$ and $u^{\pm}_n \in E = \bigcup_{m\in\mathbb{N}^+} E^m$, it follows that

$$\|u^+_{mn}\|^2_E = (u^+_{mn}, u^+_{mn}) = (u^+_{mn}, u^+_{mn} - P_m u^+_n) + (u^+_{mn}, u^+_n).$$

In virtue of (3.1) and $\|(P_m - id)u^+_n\|_E \to 0$ as $m \to \infty$, we have

$$\|(P_m - id)u^+_n\|_{L^{\mu}(\Omega)} \to 0, \text{ as } m \to \infty. \tag{4.7}$$

Replacing v with $u^+_{mn} - P_m u^+_n$ in (4.1), with the aid of (4.4)–(4.7), we have

$$\begin{aligned} (u^+_{mn}, u^+_{mn} - P_m u^+_n) &= \int_{\Omega} \tilde{f}(t, x, u_{mn})(u^+_{mn} - P_m u^+_n)\rho \mathrm{d}t\mathrm{d}x \\ &\leq M_{11}\|u^+_{mn} - P_m u^+_n\|_{L^{\mu}(\Omega)} \\ &\leq M_{11}\|u^+_{mn} - u^+_n\|_{L^{\mu}(\Omega)} + M_{11}\|(id - P_m)u^+_n\|_{L^{\mu}(\Omega)} \to 0, \end{aligned}$$

as $m \to \infty$. Therefore, by (4.5), we have

$$\|u_{mn}^+\|_E \to \|u_n^+\|_E, \text{ as } m \to \infty.$$

By using (4.5) again, a similar proof shows

$$\|u_{mn}^-\|_E \to \|u_n^-\|_E, \text{ as } m \to \infty.$$

Consequently,

$$\|u_{mn}^\pm - u_n^\pm\|_E \to 0, \text{ as } m \to \infty.$$

To continue discussion, for any $v \in E$ and since $(id - P_m)v \in (E^m)^\perp$ and $u_{mn} - P_m u_n \in E^m$, u_{mn} is the critical point of Φ_m, then from (4.1), we have

$$\begin{aligned} \langle Lu_{mn}, u_{mn} - v\rangle &= \langle Lu_{mn}, u_{mn} - P_m v\rangle + \langle Lu_{mn}, P_m v - v\rangle \\ &= \langle \tilde{f}(t, x, u_{mn}), u_{mn} - P_m v\rangle. \end{aligned}$$

Since $\tilde{f}$ is monotone in u, a simple calculation yields

$$\langle Lu_{mn}, u_{mn} - v\rangle - \langle \tilde{f}(t, x, v), u_{mn} - v\rangle \geq \langle \tilde{f}(t, x, u_{mn}), v - P_m v\rangle. \tag{4.8}$$

Moreover, according to $\|(id - P_m)v\|_{L^\mu(\Omega)} \to 0$ and (4.4), we have

$$|\langle \tilde{f}(t, x, u_{mn}), (id - P_m)v\rangle| \leq \|\tilde{f}(t, x, u_{mn})\|_{L^{\mu'}(\Omega)} \|(id - P_m)v\|_{L^\mu(\Omega)} \longrightarrow 0, \tag{4.9}$$

as $m \to \infty$. In virtue of the embedding $L^\mu(\Omega) \hookrightarrow L^2(\Omega)$ and $u_{mn} \rightharpoonup u_n$ in $L^\mu(\Omega)$, it follows that $u_{mn} \rightharpoonup u_n$ in $L^2(\Omega)$ as $m \to \infty$. From (4.8) and (4.9), with the help of $u_{mn}^\pm \to u_n^\pm$ in E, we have

$$\begin{aligned} 0 &= \lim_{m\to\infty} \langle \tilde{f}(t, x, u_{mn}), v - P_m v\rangle \\ &\leq \lim_{m\to\infty} \langle Lu_{mn}, u_{mn} - v\rangle - \lim_{m\to\infty} \langle \tilde{f}(t, x, v), u_{mn} - v\rangle \\ &= \langle Lu_n, u_n - v\rangle - \langle \tilde{f}(t, x, v), u_n - v\rangle. \end{aligned} \tag{4.10}$$

For $s > 0$ and $\psi \in E$, taking $v = u_n - s\psi$ and dividing by s in (4.10) shows

$$\langle Lu_n, \psi\rangle - \langle \tilde{f}(t, x, u_n - s\psi), \psi\rangle \geq 0,$$

then letting $s \to 0$ gets

$$\langle Lu_n, \psi\rangle - \langle \tilde{f}(t, x, u_n), \psi\rangle \geq 0.$$

By using the arbitrariness of ψ, it follows that

$$\langle Lu_n, \psi\rangle - \langle \tilde{f}(t, x, u_n), \psi\rangle = 0.$$

Therefore, u_n is the critical point of Φ for n large enough.

Moreover, since $\tilde{f}$ is a nondecreasing function with respect to u, then its primitive function $\widetilde{F}$ is convex with respect to u. Thus, according to $u_{mn}^\pm \to u_n^\pm$ in E, we have

$$\Phi_m(u_{mn}) \to \Phi(u_n), \text{ as } m \to \infty.$$

In consequence, by (1.8), Lemma 3.2 and the embedding $L^\mu(\Omega) \hookrightarrow L^1(\Omega)$, we have

$$\sigma_n \le \Phi(u_n) \le \frac{a_1}{2}\|u_n\|^\mu_{L^\mu(\Omega)} + C_3\|u_n\|_{L^\mu(\Omega)},$$

where the constant C_3 is independent of n. Thus, taking into account $\sigma_n \to \infty$ as $n \to \infty$, we have

$$\|u_n\|_{L^\mu(\Omega)} \to \infty, \quad \text{as} \quad n \to \infty.$$

Moreover, we have

$$\langle Lu_n, v\rangle - \langle \tilde{f}(t, x, u_n), v\rangle = 0, \quad \forall v \in E.$$

Replacing v with $\phi_j(t)\varphi_k(x)$ in the above equation, we obtain

$$(\lambda_k - \nu_j^2)u_{jk} = \tilde{f}_{jk},$$

where u_{jk}, $\tilde{f}_{jk}$, respectively, denote the Fourier coefficients of u_n, $\tilde{f}$. Noting that the series $\sum_{\lambda_k \ne \nu_j^2} \frac{1}{|\lambda_k - \nu_j^2|^\mu}$ is convergent and the combination of the Hausdorff-Young and Hölder inequalities yields

$$\sum_{\lambda_k \ne \nu_j^2} |u_{jk}| \le \Big(\sum_{\lambda_k \ne \nu_j^2} \frac{1}{|\lambda_k - \nu_j^2|^\mu}\Big)^{\frac{1}{\mu}}\Big(\sum_{\lambda_k \ne \nu_j^2} |\tilde{f}_{jk}|^{\mu'}\Big)^{\frac{1}{\mu'}} \le C_4\|\tilde{f}\|_{L^{\mu'}(\Omega)}.$$

Therefore, recalling $\dim E^0 < \infty$ (see [36]), we have $u_n \in C(\Omega)$.

Furthermore, under the Dirichlet-Neumann boundary condition, the system

$$\Big\{\frac{\varphi_k'}{\sqrt{\lambda_k}}\Big\}$$

forms an orthonormal basis of $L^2(0, \pi)$. By the methods in [40], we have

$$|\lambda_k - \nu_j^2| \ge C_0(k + |j|), \quad \text{for} \quad \lambda_k \ne \nu_j^2.$$

By the above estimate and (2.1), the sequences $\{\frac{|j|}{|\lambda_k - \nu_j^2|}\}$ and $\{\frac{\sqrt{\lambda_k}}{|\lambda_k - \nu_j^2|}\}$ are bounded. Therefore, we have $u_n \in H^1(\Omega)$, and the proof is complete.

5. Conclusion

In this paper, we established the multiplicity of large periodic solutions for the super-linear problem under the Dirichlet-Neumann boundary condition and the Dirichlet-Robin boundary condition. We remove the only restrict condition $\eta_\rho(\cdot) \ne \frac{\rho'(\pi)}{\rho(\pi)}$ on $\eta_\rho(\cdot)$ in [40]; thus, we do not impose any restrictions on $\eta_\rho(\cdot)$. To get better compactness conditions, we assume the period T satisfies (1.3). Finally, since the sign of $\eta_\rho(\cdot)$ can change, our results can be applied to the classical wave equation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

References

1. V. Barbu, N. H. Pavel, Periodic solutions to one-dimensional wave equation with piece-wise constant coefficients, *J. Differential Equations*, **132** (1996), 319–337. https://doi.org/10.1006/jdeq.1996.0182
2. V. Barbu, N. H. Pavel, Periodic solutions to nonlinear one dimensional wave equation with x-dependent coefficients, *Trans. Amer. Math. Soc.*, **349** (1997), 2035–2048. https://doi.org/10.1090/S0002-9947-97-01714-5
3. V. Barbu, N. H. Pavel, Determining the acoustic impedance in the 1-D wave equation via an optimal control problem, *SIAM J. Control Optim.*, **35** (1997), 1544–1556. https://doi.org/10.1137/S0363012995283698
4. J. Casado-Díaz, J. Couce-Calvo, F. Maestre, J. D. Martín Gómez, Homogenization and correctors for the wave equation with periodic coefficients, *Math. Models Methods Appl. Sci.*, **24** (2014), 1343–1388. https://doi.org/10.1142/S0218202514500031
5. S. Ji, Time periodic solutions to a nonlinear wave equation with x-dependent coefficients, *Calc. Var. Partial Differential Equations*, **32** (2008), 137–153. https://doi.org/10.1007/s00526-007-0132-7
6. S. Ji, Time-periodic solutions to a nonlinear wave equation with periodic or anti-periodic boundary conditions, *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, **465** (2009), 895–913. https://doi.org/10.1098/rspa.2008.0272
7. S. Ji, Y. Gao, W. Zhu, Existence and multiplicity of periodic solutions for Dirichlet-Neumann boundary value problem of a variable coefficient wave equation, *Adv. Nonlinear Stud.*, **16** (2016), 765–773. https://doi.org/10.1515/ans-2015-5058
8. S. Ji, Y. Li, Periodic solutions to one-dimensional wave equation with x-dependent coefficients, *J. Differential Equations*, **229** (2006), 466–493. https://doi.org/10.1016/j.jde.2006.03.020
9. S. Ji, Y. Li, Time-periodic solutions to the one-dimensional wave equation with periodic or anti-periodic boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. A*, **137** (2007), 349–371. https://doi.org/10.1017/S0308210505001174
10. M. Berti, P. Bolle, Periodic solutions of nonlinear wave equations with general nonlinearities, *Comm. Math. Phys.*, **243** (2003), 315–328. https://doi.org/10.1007/s00220-003-0972-8
11. H. Brézis, L. Nirenberg, Forced vibrations for a nonlinear wave equation, *Comm. Pure Appl. Math.*, **31** (1978), 1–30. https://doi.org/10.1002/cpa.3160310102

12. K. Chang, Solutions of asymptotically linear operator equations via Morse theory, *Comm. Pure Appl. Math.*, **34** (1981), 693–712. https://doi.org/10.1002/cpa.3160340503
13. W. Craig, A bifurcation theory for periodic solutions of nonlinear dissipative hyperbolic equations, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.*, **10** (1983), 125–167.
14. P. H. Rabinowitz, Periodic solutions of nonlinear hyperbolic partial differential equations, *Comm. Pure Appl. Math.*, **20** (1967), 145–205. https://doi.org/10.1002/cpa.3160200105
15. P. H. Rabinowitz, Free vibrations for a semilinear wave equation, *Comm. Pure Appl. Math.*, **31** (1978), 31–68. https://doi.org/10.1002/cpa.3160310103
16. C. E. Wayne, Periodic and quasi-periodic solutions of nonlinear wave equations via KAM theory, *Comm. Math. Phys.*, **127** (1990), 479–528. https://doi.org/10.1007/BF02104499
17. G. Arioli, H. Koch, Families of periodic solutions for some Hamiltonian PDEs, *SIAM J. Appl. Dyn. Syst.*, **16** (2017), 1–15. https://doi.org/10.1137/16M1070177
18. M. Timoumi, periodic and subharmonic solutions for a class of sublinear first-order Hamiltonian systems, *Mediterr. J. Math.*, **17** (2020), 1. https://doi.org/10.1007/s00009-019-1430-y
19. M. Berti, P. Bolle, Sobolev periodic solutions of nonlinear wave equations in higher spatial dimensions, *Arch. Ration. Mech. Anal.*, **195** (2010), 609–642. https://doi.org/10.1007/s00205-008-0211-8
20. J. Chen, Z. Zhang, Infinitely many periodic solutions for a semilinear wave equation in a ball in $\mathbb{R}^n$, *J. Differential Equations*, **256** (2014), 1718–1734. https://doi.org/10.1016/j.jde.2013.12.004
21. J. Chen, Z. Zhang, Existence of infinitely many periodic solutions for the radially symmetric wave equation with resonance, *J. Differential Equations*, **260** (2016), 6017–6037. https://doi.org/10.1016/j.jde.2015.12.026
22. S. Ma, J. Sun, H. Yu, Global existence and stability of temporal periodic solution to non-isentropic compressible Euler equations with a source term, *Commun. Anal. Mech.*, **15** (2023), 245–266. https://doi.org/10.3934/cam.2023013
23. M. Dilmi, S. Otmani, Existence and asymptotic stability for generalized elasticity equation with variable exponent, *Opuscula Math.*, **43** (2023), 409–428. https://doi.org/10.7494/OpMath.2023.43.3.409
24. K. P. Jin, L. Wang, Uniform decay estimates for the semi-linear wave equation with locally distributed mixed-type damping via arbitrary local viscoelastic versus frictional dissipative effects, *Adv. Nonlinear Anal.*, **12** (2023), 20220285. https://doi.org/10.1515/anona-2022-0285
25. Q. Lin, Y. Luo, Blowup phenomena for some fourth-order strain wave equations at arbitrary positive initial energy level, *Opuscula Math.*, **42** (2022), 219–238. https://doi.org/10.7494/opmath.2022.42.2.219
26. J. Pan, J. Zhang, Blow-up solutions with minimal mass for nonlinear Schrödinger equation with variable potential, *Adv. Nonlinear Anal.*, **11** (2022), 58–71. https://doi.org/10.1515/anona-2020-0185
27. H. Brézis, Periodic solutions of nonlinear vibrating strings and duality principles, *Bull. Amer. Math. Soc. (N.S.)*, **8** (1983), 409–426. https://doi.org/10.1090/S0273-0979-1983-15105-4
28. H. Brézis, J. M. Coron, Periodic solutions of nonlinear wave equations and Hamiltonian systems, *Amer. J. Math.*, **103** (1981), 559–570. https://doi.org/10.2307/2374104

29. K. Chang, S. Wu, S. Li, Multiple periodic solutions for an asymptotically linear wave equation, *Indiana Univ. Math. J.*, **31** (1982), 721–731. https://doi.org/10.1512/iumj.1982.31.31051

30. Y. Ding, S. Li, M. Willem, Periodic solutions of symmetric wave equations, *J. Differential Equations*, **145** (1998), 217–241. https://doi.org/10.1006/jdeq.1997.3380

31. J. Mawhin, Periodic solutions of some semilinear wave equations and systems: a survey, *Chaos Solitons Fractals*, **5** (1995), 1651–1669. https://doi.org/10.1016/0960-0779(94)00169-Q

32. W. Craig, C. E. Wayne, Newton's method and periodic solutions of nonlinear wave equations, *Comm. Pure Appl. Math.*, **46** (1993), 1409–1498. https://doi.org/10.1002/cpa.3160461102

33. I. A. Rudakov, Periodic solutions of a nonlinear wave equation with nonconstant coefficients, *Math. Notes*, **76** (2004), 395–406. https://doi.org/10.1023/B:MATN.0000043467.04680.1d

34. H. Wei, S. Ji, Existence of multiple periodic solutions to a semilinear wave equation with x-dependent coefficients, *Proc. Roy. Soc. Edinburgh Sect. A*, **150** (2020), 2586–2606. https://doi.org/10.1017/prm.2019.25

35. J. Chen, Periodic solutions to nonlinear wave equation with spatially dependent coefficients, *Z. Angew. Math. Phys.*, **66** (2015), 2095–2107. https://doi.org/10.1007/s00033-015-0497-y

36. S. Ji, Periodic solutions for one dimensional wave equation with bounded nonlinearity, *J. Differential Equations*, **264** (2018), 5527–5540. https://doi.org/10.1016/j.jde.2018.02.0010022

37. S. Ji, Y. Li, Time periodic solutions to the one-dimensional nonlinear wave equation, *Arch. Ration. Mech. Anal.*, **199** (2011), 435–451. https://doi.org/10.1007/s00205-010-0328-4

38. H. Wei, S. Ji, Periodic solutions of a semilinear variable coefficient wave equation under asymptotic nonresonance conditions, *Sci. China Math.* **66** (2023), 79–90. https://doi.org/10.1007/s11425-020-1900-5

39. H. Wei, S. Ji, Infinitely many periodic solutions for a semilinear wave equation with x-dependent coefficients, *ESAIM Control Optim Calc Var*, **26** (2020). https://doi.org/10.1051/cocv/2019007

40. I. A. Rudakov, Periodic solutions of the quasilinear equation of forced vibrations of an inhomogeneous string, *Math. Notes*, **101** (2017), 137–148. https://doi.org/10.1134/S000143461701014X

41. K. Yosida, *Functional Analysis*, 6^{th} edition, Springer-Verlag, Berlin, 1980. https://link.springer.com/book/10.1007/978-3-642-61859-8

42. C. T. Fulton, S. A. Pruess, Eigenvalue and eigenfunction asymptotics for regular Sturm-Liouville problems, *J. Math. Anal. Appl.*, **188** (1994), 297–340. https://doi.org/10.1006/jmaa.1994.1429

43. T. Bartsch, M. Willem, Periodic solutions of nonautonomous Hamiltonian systems with symmetries, *J. Reine Angew. Math.*, **451** (1994), 149–159. https://doi.org/10.1515/crll.1994.451.149

44. K. Chang, *Methods in Nonlinear Analysis*, Springer-Verlag, Berlin, 2005. https://link.springer.com/book/10.1007/3-540-29232-2