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Research article

A generalized time fractional Schrödinger equation with signed potential

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Abstract: In this work, by stochastic analyses, we study stochastic representation, well-posedness, and regularity of generalized time fractional Schrödinger equation

$$\begin{cases} \partial_t^w u(t,x) = \mathcal{L}u(t,x) - \kappa(x)u(t,x), \ t \in (0,\infty), \ x \in \mathcal{X}, \\ u(0,x) = g(x), \ x \in \mathcal{X}, \end{cases}$$

where the potential κ is signed, $\mathcal{X}$ is a Lusin space, ∂_t^w is a generalized time fractional derivative, and $\mathcal{L}$ is infinitesimal generator in terms of semigroup induced by a symmetric Markov process X. Our results are applicable to some typical physical models.

Keywords: fractional Schrödinger equation; well-posedness; regularity
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1. Introduction

Anomalous diffusions are ubiquitous in the natural world, the probability distributions of which are usually governed by equations with fractional operators. It is well known that the diffusion equation $\partial_t u(t,x) = \Delta u(t,x)$ with $u(0,x) = f(x)$ allows the stochastic solution $u(t,x) = \mathbb{E}^x[f(B_t)]$, where B_t is Brownian motion started at $x \in \mathbb{R}^d$ with infinitesimal generator Δ, describing the normal diffusion that shows, e.g., heat propagation in homogeneous medium. Owing to particle sticking and/or trapping phenomena, the following equation

$$\partial_t^\beta u(t,x) = \Delta u(t,x) \text{ with } u(0,x) = f(x), \tag{1.1}$$

has been used to simulate the anomalous diffusions displaying subdiffusive behavior widely such as thermal diffusion in fractal media, protein diffusion within cells, and contaminant transport in groundwater. The Caputo derivative ∂_t^β, with fractional order $\beta \in (0,1)$, can be defined by

$$\partial_t^\beta f(t) := \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t (t-s)^{-\beta}(f(s) - f(0))ds, \tag{1.2}$$

where the Gamma function $\Gamma(\lambda) := \int_0^\infty t^{\lambda-1}e^{-t}dt$. In particular, Scheffer and Meerschaert [1, Theorem 5.1] recognized, based on [2], that the stochastic representation $u(t,x) = \mathbb{E}^x[f(B_{E_t})]$ satisfies the equation (1.1), where E_t is an inverse β-stable subordinator that is independent of B_t.

Solutions of the linear as well as nonlinear fractional partial differential equations have attracted a lot of attention and have been extensively discussed; see, e.g., [3–9] and the references therein. There are also some theoretical results and numerical methods for nonlinear time-fractional Schrödinger equations [10–13]. It can be noted that most of the quoted papers are concentrated on the Caputo derivative of fractional order. This attention has been gained by anomalous diffusion phenomena emerging in diverse fields containing mathematics, physics, engineering, biology, chemistry, hydrology, and geophysics, etc [14–18]. There are also some discussions on the semilinear parabolic equations with singular potentials [19].

The present paper investigates the generalized time fractional Schrödinger equation

$$\begin{cases} \partial_t^w u(t,x) = \mathcal{L}u(t,x) - \kappa(x)u(t,x), \ t \in (0,\infty), \ x \in \mathcal{X}, \\ \ u(0,x) = g(x), \ x \in \mathcal{X}, \end{cases} \tag{1.3}$$

where κ is bounded in $\mathcal{X}$ with $\|\kappa\|_\infty \le \mathcal{K}$, $\mathcal{X}$ is a Lusin space, being a topological space homeomorphic to a Borel subset of a compact metric space. Denote $X = \{X_t, t \in [0,\infty); \mathbb{P}^x, x \in \mathcal{X}\}$ as a time-homogeneous strong Markov process on $\mathcal{X}$ whose sample paths are right continuous and have left limits on $\mathcal{X} \cup \{\partial\}$, where ∂ is an isolated cemetery point outside $\mathcal{X}$ and $X_t = \partial$ for every $t \ge \zeta := \inf\{t \ge 0 : X_t = \partial\}$. The transition semigroup $\{P_t\}_{t\ge 0}$ of X is defined as

$$P_t f(x) := \mathbb{E}^x[f(X_t)], \ x \in \mathcal{X}, \ t \ge 0,$$

for any bounded or nonnegative function f on $\mathcal{X}$ that is extended to $\mathcal{X} \cup \{\partial\}$ by setting $f(\partial) = 0$. Here $\mathbb{P}^x$ denotes the probability law of X starting from position x, and $\mathbb{E}^x$ is the mathematical expectation taken under probability law $\mathbb{P}^x$. We assume in addition that the strong Markov process X on $\mathcal{X}$ is ν-symmetric, i.e., for any nonnegative functions f and g on $\mathcal{X}$ and $t > 0$,

$$\int_X f(x)P_t g(x)\nu(dx) = \int_X g(x)P_t f(x)\nu(dx),$$

where ν is a σ-finite measure on $\mathcal{X}$ with full support. The transition semigroup is strongly continuous on $L^2(\mathcal{X};\nu)$ with $\|P_t\| \le 1$ for every $t \ge 0$; denote its infinitesimal generator by $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$. Then $\mathcal{L}$ is a nonpositive definite self-adjoint operator in $L^2(\mathcal{X};\nu)$ [20, 21]. The notations $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ are, respectively, the norm and inner product of $L^2(\mathcal{X};\nu)$. Besides, the generalized time fractional derivative is defined by

$$\partial_t^w f(t) := \frac{d}{dt} \int_0^t w(t-s)(f(s) - f(0))ds,$$

where the given function $w : (0,\infty) \to [0,\infty)$ is unbounded, non-increasing, and having $\int_0^\infty \min\{1,s\}(-dw(s)) < \infty$. Such a function w is in one-to-one correspondence with an infinite Lévy measure μ on $(0,\infty)$ so that $w(x) = \mu(x,\infty)$. This Lévy measure μ in turn is in one-to-one correspondence with a driftless subordinator $\{S_t\}_{t\geq 0}$ having

$$\phi(\lambda) := \int_0^\infty (1-e^{-\lambda x})\mu(dx)$$

as its Laplace exponent; that is

$$\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)} \text{ for } \lambda > 0. \tag{1.4}$$

In particular, when $w(s) = \frac{1}{\Gamma(1-\beta)}s^{-\beta}$ for $\beta \in (0,1)$, $\partial_t^w f$ is just the Caputo derivative of order β in (1.2). Through out this paper, $\{S_t\}_{t\geq 0}$ is a driftless subordinator with $S_0 = 0$ that has a density $p(t,r)$ for every $t > 0$ and $\phi(\lambda)$ is the Laplace exponent of the driftless subordinator $\{S_t\}_{t\geq 0}$ having Lévy measure μ. Define $E_t = \inf\{s > 0 : S_s > t\}$ for $t > 0$, the inverse subordinator. The assumption that the Lévy measure μ is infinite, which is equivalent to $w(x) = \mu(x,\infty)$ being unbounded, shows that $t \mapsto S_t$ is strictly increasing and hence $t \mapsto E_t$ is continuous.

When $\kappa = 0$ in equation (1.3), Chen [22, Theorem 2.1] shows that the existence and uniqueness of stochastic strong solution take a form:

$$u(t,x) = \mathbb{E}[P_{E_t}g(x)] = \mathbb{E}^x[g(X_{E_t})],$$

where the infinitesimal generator $\mathcal{L}$ generates a uniformly bounded and strongly continuous semigroup in continuous function space $C_\infty(\mathcal{X})$ or L^p space. The result in [22] for strong solution was extended to that of weak solutions in [23] when the infinitesimal generator $\mathcal{L}$ is a symmetric operator in Hilbert L^2 space. The main characteristic of the method in [22, 23] is a detailed analysis of the subordinator associated with the function w together with a stochastic representation of the solution. Recently, the authors [24, Page 6022] and [25, Remark 3.3] point out that the stochastic representation of equation (1.3) takes a different form. When $\kappa \geq 0$, it follows directly from [22, Theorem 2.1] that the unique solution to (1.3) is given by

$$u(t,x) = \mathbb{E}^x\left[e^{-\int_0^{E_t}\kappa(X_s)ds}g(X_{E_t})\right]. \tag{1.5}$$

The reason is that $\mathcal{L}^\kappa = \mathcal{L} - \kappa(x)$ with $\mathcal{D}(\mathcal{L}^\kappa) = \mathcal{D}(\mathcal{L})$ is the infinitesimal generator in terms of strongly continuous contraction semigroup $\{P_t^\kappa\}_{t\geq 0}$ induced by the subprocess X^κ of X killed at rate $\kappa(x)$, that is,

$$P_t^\kappa f(x) := \mathbb{E}^x[f(X_t^\kappa)] = \mathbb{E}^x\left[e^{-\int_0^t \kappa(X_s)ds}f(X_t)\right], x \in \mathcal{X}, t \geq 0.$$

The stochastic representation (1.5) seems to be the solution to equation (1.3) in the case that the potential κ is signed. However, this conclusion has not been proved. To overcome this difficulty, we first study the existence and uniqueness of the weak solution for (1.3) by utilizing the contraction mapping principle. Then we establish regularity of this solution to obtain the strong solution of (1.3). To the best of our knowledge, the regularity investigation of equation (1.3) is largely missing in the literature, apart from the case $\kappa \geq 0$ [25]. We will try to make some contributions to this research field. The current research can be viewed as a sequel to [22, 23, 25].

2. Preliminaries and main results

For a real-valued function f defined on $[0,\infty)$, we use $\widehat{f}$ to denote its Laplace transform,

$$\widehat{f}(\lambda) := \mathscr{L}\{f\}(\lambda) := \int_0^\infty e^{-\lambda t}f(t)dt, \ \lambda > 0,$$

whenever the integral is absolutely convergent. Recall from [22] that $\widehat{w}(\lambda) = \frac{\phi(\lambda)}{\lambda}$ for $\lambda > 0$.

Let $G^S(s) := \int_0^\infty p(r, s)dr$ be the potential density of the subordinator S. Since S is transient (see [26, Theorem 35.4]),

$$\int_0^t G^S(s)ds < \infty \text{ for every } t > 0, \tag{2.1}$$

i.e., $(r, s) \mapsto p(r, s)$ is an integrable function on $[0, \infty)\times[0, t]$ for every $t > 0$. In fact, by [27, Proposition III.1], there exist constants $C_1,\ C_2 > 0$ such that

$$\frac{C_1}{\phi(1/t)} \le \int_0^t G^S(s)ds \le \frac{C_2}{\phi(1/t)} \text{ for } t > 0. \tag{2.2}$$

We will need the following estimates from Chen [22].

Lemma 2.1. *[22, Lemma 2.1 and Corollary 2.1] It holds that:*

(i) Define $G(0) = 0$ and $G(t) = \int_0^t w(x)dx$ for all $t > 0$. We have that for every $t, r > 0$, $0 \le \int_0^t w(t-s)\mathbb{P}(S_r \ge s)ds = G(t) - \mathbb{E}[G(t - S_r)\mathbf{1}_{\{t\ge S_r\}}]$.

(ii) $\mathbb{P}(S_r \ge t) = \int_0^r \mathbb{E}[w(t - S_s)\mathbf{1}_{\{t\ge S_s\}}]ds$ for every $r > 0$ and $t \in (0,\infty) \setminus \mathcal{N}$, where the Borel set $\mathcal{N} \subset (0,\infty)$ has zero Lebesgue measure.

(iii) $\int_0^\infty \mathbb{E}[w(t - S_s)\mathbf{1}_{\{t\ge S_s\}}]ds = 1$ for $t \in (0,\infty) \setminus \mathcal{N}$.

Theorem 2.2. *Set $X := \left\{u \in C([0,\infty); L^2(\mathcal{X}; \nu));\ \sup_{t\ge0} e^{-kt}\|u(t,\cdot)\| < \infty\right\}$ with norm $\|u\|_X = \sup_{t\ge0} e^{-kt}\|u(t,\cdot)\|$. Let $g \in L^2(\mathcal{X}; \nu)$. Then equation* (1.3) *has a unique weak solution $u \in X$ with stochastic representation* (1.5) *in the sense that for every $t \in (0,\infty)$ and $\varphi \in \mathcal{D}(\mathcal{L})$,*

$$\left\langle \int_0^t w(t-s)(u(s,\cdot) - g(\cdot))ds, \varphi \right\rangle = \int_0^t \langle u(s,\cdot), \mathcal{L}\varphi\rangle ds - \int_0^t \langle \kappa(\cdot)u(s,\cdot), \varphi\rangle ds. \tag{2.3}$$

Moreover,

$$u \in C((0,\infty); \mathcal{D}((-\mathcal{L})^\alpha)) \textit{ for } \alpha \in (0,1), \tag{2.4}$$

and

$$\int_0^T \|(-\mathcal{L})^\alpha u(t,\cdot)\|dt < \infty \textit{ for every } T > 0.$$

Proof of Theorem 2.2. **(Existence)** We prove that equation (1.3) has a weak solution $u \in X$ using the contraction mapping principle. For $u \in X$, we define

$$\begin{aligned} T(u)(t,x) :&= \mathbb{E}[P_{E_t}g(x)] - \int_{s=0}^t \int_{r=0}^\infty P_r(\kappa(\cdot)u(s,\cdot))(x)p(r, t-s)drds \\ :&= u_1(t,x) - u_2(t,x). \end{aligned}$$

First, it is needed to show that $T(u) \in X$. Clearly, for $g \in L^2(\mathcal{X}; \nu)$, $\sup_{t\ge0} \|u_1(t,\cdot)\| \le \|g\|$. Since $t \mapsto E_t$ is continuous a.s. and $\{P_t\}_{t\ge0}$ is a strongly continuous contraction semigroup in $L^2(\mathcal{X}; \nu)$, we have by the bounded convergence theorem that $t \mapsto u_1(t,\cdot)$ is continuous in $L^2(\mathcal{X}; \nu)$. For $u_2(t,\cdot)$, we have for every given $t \ge 0$,

$$\begin{aligned} \|u_2(t,\cdot)\| &\le \int_{s=0}^t \int_{r=0}^\infty \|P_r(\kappa(\cdot)u(s,\cdot))\|p(r, t-s)drds \\ &\le \mathcal{K} \sup_{s\in[0,t]} \|u(s,\cdot)\| \int_{s=0}^t \int_{r=0}^\infty p(r, t-s)drds \le C \int_0^t G^S(s)ds. \end{aligned}$$

Thus, $u_2(t,\cdot)$ is well defined as an element in $L^2(\mathcal{X};\nu)$. We now show the continuity of $t \mapsto u_2(t,\cdot)$ in $L^2(\mathcal{X};\nu)$. For $t \geq 0$, $\Delta t > 0$,

$$\begin{aligned}
&\|u_2(t+\Delta t,\cdot) - u_2(t,\cdot)\| \\
\leq &\int_{s=t}^{t+\Delta t}\int_{r=0}^{\infty} \|P_r(\kappa(\cdot)u(s,\cdot))\| p(r,t+\Delta t - s)drds \\
&+ \int_{s=0}^{t}\int_{r=0}^{\infty} \|P_r(\kappa(\cdot)u(s,\cdot))\| |p(r,t+\Delta t-s) - p(r,t-s)|drds \\
\leq &\mathcal{K} \sup_{s\in[t,t+\Delta t]} \|u(s,\cdot)\| \int_{s=t}^{t+\Delta t}\int_{r=0}^{\infty} p(r,t+\Delta t-s)drds \\
&+ \mathcal{K} \sup_{s\in[0,t]} \|u(s,\cdot)\| \int_{s=0}^{t}\int_{r=0}^{\infty} |p(r,t+\Delta t-s) - p(r,t-s)|drds \\
\leq &C\left[\int_0^{\Delta t} G^S(s)ds + \int_{s=0}^{t}\int_{r=0}^{\infty} |p(r,\Delta t+s) - p(r,s)|drds\right].
\end{aligned}$$

Using (2.2) and the fact $\lim_{r\to\infty}\phi(r) = \infty$ in the first term, L^1-continuity of the integrable function $(r,s) \mapsto p(r,s)$ on $[0,\infty)\times[0,t+1]$ in the second term, one can see that as $\Delta t \to 0$ both terms go to zero.

Now we show that u_2 belongs to X. By (1.4), we have

$$\begin{aligned}
\|u_2\|_X &\leq \sup_{t\geq 0} e^{-kt} \int_{s=0}^{t}\int_{r=0}^{\infty} \|P_r(\kappa(\cdot)u(s,\cdot))\| p(r,t-s)drds \\
&\leq \mathcal{K} \sup_{t\geq 0} e^{-kt} \int_{s=0}^{t}\int_{r=0}^{\infty} \|u(s,\cdot)\| p(r,t-s)drds \\
&\leq \mathcal{K}\|u\|_X \sup_{t\geq 0} e^{-kt} \int_{s=0}^{t}\int_{r=0}^{\infty} e^{ks} p(r,t-s)drds \\
&\leq \mathcal{K}\|u\|_X \int_{s=0}^{\infty}\int_{r=0}^{\infty} e^{-ks} p(r,s)drds = \frac{\mathcal{K}}{\phi(k)}\|u\|_X.
\end{aligned}$$

Next, one can see that T is a contraction mapping. Indeed, for $v_1, v_2 \in X$,

$$\begin{aligned}
\|Tv_1 - Tv_2\|_X &\leq \sup_{t\geq 0} e^{-kt} \int_{s=0}^{t}\int_{r=0}^{\infty} \|P_r(\kappa(\cdot)v_1(s,\cdot)) - P_r(\kappa(\cdot)v_2(s,\cdot))\| p(r,t-s)drds \\
&\leq \sup_{t\geq 0} e^{-kt}\mathcal{K} \int_{s=0}^{t}\int_{r=0}^{\infty} \|v_1(s,\cdot) - v_2(s,\cdot)\| p(r,t-s)drds \\
&\leq \mathcal{K}\|v_1 - v_2\|_X \sup_{t\geq 0} e^{-kt} \int_{s=0}^{t}\int_{r=0}^{\infty} e^{ks} p(r,t-s)drds \\
&\leq \mathcal{K}\|v_1 - v_2\|_X \int_{s=0}^{\infty}\int_{r=0}^{\infty} e^{-ks} p(r,s)drds = \frac{\mathcal{K}}{\phi(k)}\|v_1 - v_2\|_X,
\end{aligned}$$

which implies that $\|Tv_1 - Tv_2\|_X < \|v_1 - v_2\|_X$ for large enough k. Hence, there exists a unique fixed point $u \in X$ such that

$$u(t,x) = \mathbb{E}[P_{E_t} g(x)] - \int_{s=0}^{t}\int_{r=0}^{\infty} P_r(\kappa(\cdot)u(s,\cdot))(x) p(r,t-s)drds. \tag{2.5}$$

Now we show that u appearing in (2.5) satisfies the equation (2.3). Denote $u = u_1 - u_2$. For u_1, we have for every $t \geq 0$ and $\varphi \in \mathcal{D}(\mathcal{L})$,

$$\begin{aligned}
&\left\langle \int_0^t w(t-s)(u_1(s,\cdot)-g)ds, \varphi \right\rangle \\
&= \int_{s=0}^t w(t-s) \int_{r=0}^\infty (\langle P_r g, \varphi\rangle - \langle g, \varphi\rangle) d_r \mathbb{P}(S_r \geq s) ds \\
&= \int_{r=0}^\infty (\langle P_r g, \varphi\rangle - \langle g, \varphi\rangle) d_r \left(\int_{s=0}^t w(t-s)\mathbb{P}(S_r \geq s) ds \right) \\
&= -\int_0^\infty (\langle g, P_r\varphi\rangle - \langle g, \varphi\rangle) d_r \mathbb{E}[G(t-S_r)\mathbf{1}_{\{t\geq S_r\}}] \\
&= \int_0^\infty \mathbb{E}[G(t-S_r)\mathbf{1}_{\{t\geq S_r\}}]\langle P_r g, \mathcal{L}\varphi\rangle dr.
\end{aligned} \tag{2.6}$$

The first equality follows from $\mathbb{P}(E_t \leq r) = \mathbb{P}(S_r \geq t)$; the second is justified by the Riemann sum approximation of Stieltjes integrals; the third follows from self-adjointness of $\{P_r\}_{r\geq 0}$ and Lemma 2.1 (i); and the last follows the integration by parts. On the other hand, we find that for every $t \geq 0$ and $\varphi \in \mathcal{D}(\mathcal{L})$,

$$\begin{aligned}
&\int_0^t \langle u_1(s,\cdot), \mathcal{L}\varphi\rangle ds = \int_0^t \left\langle \int_0^\infty P_r g d_r \mathbb{P}(E_s \leq r), \mathcal{L}\varphi \right\rangle ds \\
&= \int_{s=0}^t \int_{r=0}^\infty \langle P_r g, \mathcal{L}\varphi\rangle \mathbb{E}[w(s-S_r)\mathbf{1}_{\{s\geq S_r\}}] dr ds \\
&= \int_0^\infty \langle P_r g, \mathcal{L}\varphi\rangle \mathbb{E}[G(t-S_r)\mathbf{1}_{\{t\geq S_r\}}] dr,
\end{aligned} \tag{2.7}$$

the second equality of which follows from Lemma 2.1 (ii); the third one is an application of Tonelli's Theorem and a simple change of variables. Thus, by (2.6) and (2.7) we conclude that for every $t \geq 0$ and $\varphi \in \mathcal{D}(\mathcal{L})$,

$$\left\langle \int_0^t w(t-s)(u_1(s,\cdot)-g)ds, \varphi \right\rangle = \int_0^t \langle u_1(s,\cdot), \mathcal{L}\varphi\rangle ds. \tag{2.8}$$

Next, we consider u_2, we have for every $t \geq 0$,

$$\begin{aligned}
\int_0^t w(t-s)u_2(s,x)ds &= \int_{s=0}^t w(t-s) \left(\int_{\tau=0}^s \int_{r=0}^\infty P_r(\kappa(\cdot)u(\tau,\cdot))(x) p(r, s-\tau) dr d\tau \right) ds \\
&= \int_{\tau=0}^t \int_{r=0}^\infty P_r(\kappa(\cdot)u(\tau,\cdot))(x) \left(\int_{s=\tau}^t w(t-s)p(r,s-\tau)ds \right) dr d\tau \\
&= \int_{\tau=0}^t \int_{r=0}^\infty P_r(\kappa(\cdot)u(\tau,\cdot))(x) \left(\int_{s=0}^{t-\tau} w(t-\tau-s)p(r,s)ds \right) dr d\tau \\
&= \int_{\tau=0}^t \int_{r=0}^\infty P_r(\kappa(\cdot)u(\tau,\cdot))(x) d_r \mathbb{P}(E_{t-\tau} \leq r) d\tau \\
&= \int_0^t \mathbb{E}[P_{E_{t-s}}(\kappa(\cdot)u(s,\cdot))(x)] ds,
\end{aligned}$$

the first equality of which follows the definition of u_2; the second and the third ones are application of Tonelli's Theorem and a simple change of variables; the fourth one is due to Lemma 2.1 (i). Thus, for every $\varphi \in \mathcal{D}(\mathcal{L})$ and $t \geq 0$, by the self-adjointness of $\{P_r\}_{r\geq 0}$ and Fubini theorem, we have

$$\begin{aligned}
&\left\langle \int_0^t w(t-s)u_2(s,\cdot)ds, \varphi \right\rangle = \left\langle \int_0^t \mathbb{E}P_{E_{t-s}}(\kappa(\cdot)u(s,\cdot))ds, \varphi \right\rangle \\
&= \int_0^t \langle \kappa(\cdot)u(s,\cdot), \mathbb{E}P_{E_{t-s}}\varphi \rangle ds \\
&= \int_0^t \langle \kappa(\cdot)u(s,\cdot), \varphi \rangle ds + \int_0^t \left\langle \kappa(\cdot)u(\tau,\cdot), \mathbb{E}\int_0^{E_{t-\tau}} P_s\mathcal{L}\varphi ds \right\rangle d\tau \\
&= \int_0^t \langle \kappa(\cdot)u(s,\cdot), \varphi \rangle ds + \int_0^t \left\langle \mathbb{E}\int_0^{E_{t-\tau}} P_s(\kappa(\cdot)u(\tau,\cdot))ds, \mathcal{L}\varphi \right\rangle d\tau.
\end{aligned} \tag{2.9}$$

Then, by a direct computation, we have

$$\begin{aligned}
&\int_0^t \left(\mathbb{E}\int_0^{E_{t-\tau}} P_s(\kappa(\cdot)u(\tau,\cdot))ds \right) d\tau \\
&= \int_0^t \left(\int_0^\infty \mathbb{P}(E_{t-\tau} > s)P_s(\kappa(\cdot)u(\tau,\cdot))ds \right) d\tau \\
&= \int_0^t \left(\int_0^\infty \mathbb{P}(S_s \leq t-\tau)P_s(\kappa(\cdot)u(\tau,\cdot))ds \right) d\tau \\
&= \int_0^t \left(\int_{s=0}^\infty \left(\int_{r=0}^{t-\tau} p(s,r)dr \right) P_s(\kappa(\cdot)u(\tau,\cdot))ds \right) d\tau \\
&= \int_{\tau=0}^t \int_{r=0}^{t-\tau} \int_{s=0}^\infty P_s(\kappa(\cdot)u(\tau,\cdot))p(s,t-\tau-r)dsdrd\tau \\
&= \int_{r=0}^t \int_{\tau=0}^{t-r} \int_{s=0}^\infty P_s(\kappa(\cdot)u(\tau,\cdot))p(s,t-r-\tau)dsd\tau dr \\
&= \int_0^t u_2(t-r,\cdot)dr = \int_0^t u_2(s,\cdot)ds.
\end{aligned} \tag{2.10}$$

Thus by (2.9) and (2.10) one can conclude that for every $t \geq 0$ and $\varphi \in \mathcal{D}(\mathcal{L})$,

$$\left\langle \int_0^t w(t-s)u_2(s,\cdot)ds, \varphi \right\rangle = \int_0^t \langle \kappa(\cdot)u(s,\cdot), \varphi \rangle ds + \int_0^t \langle u_2(s,\cdot), \mathcal{L}\varphi \rangle ds. \tag{2.11}$$

Hence by (2.8) and (2.11), u appearing in (2.5) satisfies the equation (2.3).

(Uniqueness) For the uniqueness of weak solution of equation (1.3), it is enough to show that u solves (2.3) if and only if u satisfies (2.5). The 'if' direction has been proved by the above discussion. For the 'only if' direction, let $u \in X$ satisfy (2.3). Combining (2.3), (2.8), and (2.11), we have for every $t \geq 0$ and $\varphi \in \mathcal{D}(\mathcal{L})$,

$$\left\langle \int_0^t w(t-s)(u(s,\cdot) - u_1(s,\cdot) + u_2(s,\cdot))ds, \varphi \right\rangle = \int_0^t \langle u(s,\cdot) - u_1(s,\cdot) + u_2(s,\cdot), \mathcal{L}\varphi \rangle ds.$$

Let $h(t,x) := u(t,x) - u_1(t,x) + u_2(t,x)$. We have $\|h(t,\cdot)\| \leq Ce^{kt}$ for large enough k. Therefore for every $\lambda > k$, $\widehat{h}(\lambda,\cdot) \in L^2(\mathcal{X};\nu)$. Taking Laplace transform w.r.t. t on both sides yields that for every $\lambda > k$ and $\varphi \in \mathcal{D}(\mathcal{L})$,

$$\frac{\phi(\lambda)}{\lambda}\langle \widehat{h}(\lambda,\cdot), \varphi \rangle = \frac{1}{\lambda}\langle \widehat{h}(\lambda,\cdot), \mathcal{L}\varphi \rangle.$$

That is, for every $\lambda > k$ and $\varphi \in \mathcal{D}(\mathcal{L})$,

$$\int_{\mathcal{X}} \widehat{h}(\lambda, x)(\phi(\lambda) - \mathcal{L})\varphi(x)\nu(dx) = 0.$$

Since $\mathcal{L}$ is the infinitesimal generator of strongly continuous contraction semigroup $\{P_t\}_{t\geq 0}$ on Banach space $L^2(\mathcal{X}; \nu)$, the resolvent $G_{\phi(\lambda)} = \int_0^\infty e^{-\phi(\lambda)t} P_t dt$ is well defined and is the inverse to $\phi(\lambda) - \mathcal{L}$. Taking $\varphi = G_{\phi(\lambda)}\psi$ yields that,

$$\int_{\mathcal{X}} \widehat{h}(\lambda, x)\psi(x)\nu(dx) = 0 \text{ for every } \lambda > k, \psi \in L^2(\mathcal{X}; \nu).$$

Thus, we have $\widehat{h}(\lambda, \cdot) = 0$ for every $\lambda > k$. By the uniqueness of Laplace transform, we have $h(t, \cdot) = 0$ for a.e. $t \geq 0$. Since $t \mapsto h(t, \cdot)$ is continuous for $t \geq 0$, we conclude that $h(t, \cdot) = 0$ for $t \geq 0$, and hence $u = u_1 - u_2$ satisfies (2.5).

Remark 2.3. Similar to the above discussions, when the nonlinear function f satisfies Lipschitz condition, i.e., there exists a positive constant $\mathcal{K}$ such that for $t \geq 0$ and $\phi_1, \phi_2 \in L^2(\mathcal{X}; \nu)$,

$$\|f(t, \cdot, \phi_1) - f(t, \cdot, \phi_2)\| \leq \mathcal{K}\|\phi_1 - \phi_2\|,$$

then the nonlinear time fractional equation

$$\begin{cases} \partial_t^w u(t, x) = \mathcal{L}u(t, x) + f(t, x, u(t, x)), \ t \in (0, \infty), \ x \in \mathcal{X}, \\ \quad u(0, x) = g(x), \ x \in \mathcal{X}, \end{cases}$$

has an unique weak solution in the sense that for every $t \in (0, \infty)$ and $\varphi \in \mathcal{D}(\mathcal{L})$,

$$\left\langle \int_0^t w(t - s)(u(s, \cdot) - g(\cdot))ds, \varphi \right\rangle = \int_0^t \langle u(s, \cdot), \mathcal{L}\varphi\rangle ds + \int_0^t \langle f(s, \cdot, u(s, \cdot)), \varphi\rangle ds.$$

Proof of Theorem 2.2*-continued.* **(Stochastic representation)** Denote

$$v(t, x) = \mathbb{E}^x \left[e^{-\int_0^t \kappa(X_s)ds} g(X_t) \right].$$

Then we have

$$\mathbb{E}^x \left[e^{-\int_0^{E_t} \kappa(X_s)ds} g(X_{E_t}) \right] = \mathbb{E}v(E_t, x).$$

It remains to establish for $t > 0$, the stochastic representation defined by (1.5) satisfies (2.5), i.e.,

$$\mathbb{E}^x \left[e^{-\int_0^{E_t} \kappa(X_s)ds} g(X_{E_t}) \right] = \mathbb{E}^x[g(X_{E_t})] - \int_{s=0}^t \int_{r=0}^\infty P_r(\kappa(\cdot)\mathbb{E}v(E_s, \cdot))(x)p(r, t - s)drds. \tag{2.12}$$

On the one hand, denote by $\mathcal{F}^S$ the σ-field generated by the subordinator S. By the independence between X and S (and hence its inverse E) and the Markov property of X,

$$
\begin{aligned}
&\mathbb{E}^x[g(X_{E_t})] - \mathbb{E}^x\left[e^{-\int_0^{E_t}\kappa(X_s)ds}g(X_{E_t})\right] \\
&= -\mathbb{E}^x\left[\left(e^{-\int_0^{E_t}\kappa(X_s)ds}-1\right)g(X_{E_t})\right] \\
&= \mathbb{E}\left[\mathbb{E}^x\left[\int_0^{E_t}\kappa(X_r)e^{-\int_r^{E_t}\kappa(X_s)ds}g(X_{E_t})dr\right\|\mathcal{F}^S\right] \\
&= \mathbb{E}\left[\mathbb{E}^x\left[\int_0^{E_t}\kappa(X_r)\mathbb{E}^{X_r}\left[e^{-\int_r^{E_t}\kappa(X_{s-r})ds}g(X_{E_t-r})\right]dr\right\|\mathcal{F}^S\right] \\
&= \mathbb{E}\left[\mathbb{E}^x\left[\int_0^{E_t}\kappa(X_r)v(E_t-r,X_r)dr\right\|\mathcal{F}^S\right] \\
&= \mathbb{E}\left[\int_0^{E_t}P_r\left(\kappa(\cdot)v(E_t-r,\cdot)\right)(x)dr\right].
\end{aligned}
\tag{2.13}
$$

By Lemma 2.1 (ii), taking Laplace transform w.r.t. t yields that for $\lambda > k$,

$$
\begin{aligned}
&\mathscr{L}\left\{\mathbb{E}\left[\int_0^{E_t}P_r\left(\kappa(\cdot)v(E_t-r,\cdot)\right)(x)dr\right]\right\}(\lambda) \\
&= \mathscr{L}\left\{\int_{\tau=0}^{\infty}\int_{r=0}^{\tau}P_r\left(\kappa(\cdot)v(\tau-r,\cdot)\right)(x)dr d_\tau\mathbb{P}(E_t\le\tau)\right\}(\lambda) \\
&= \mathscr{L}\left\{\int_{\tau=0}^{\infty}\int_{r=0}^{\tau}P_r\left(\kappa(\cdot)v(\tau-r,\cdot)\right)(x)dr\mathbb{E}[w(t-S_\tau)\mathbf{1}_{\{t\ge S_\tau\}}]d\tau\right\}(\lambda) \\
&= \frac{\phi(\lambda)}{\lambda}\int_{\tau=0}^{\infty}\int_{r=0}^{\tau}P_r\left(\kappa(\cdot)v(\tau-r,\cdot)\right)(x)dre^{-\tau\phi(\lambda)}d\tau \\
&= \frac{\phi(\lambda)}{\lambda}\int_{\tau=0}^{\infty}\int_{r=0}^{\infty}e^{-(\tau+r)\phi(\lambda)}P_r\left(\kappa(\cdot)v(\tau,\cdot)\right)(x)drd\tau.
\end{aligned}
\tag{2.14}
$$

On the other hand, taking Laplace transform w.r.t. t yields that for $\lambda > k$,

$$
\begin{aligned}
&\mathscr{L}\left\{\int_{s=0}^{t}\int_{r=0}^{\infty}P_r(\kappa(\cdot)\mathbb{E}v(E_s,\cdot))(x)p(r,t-s)drds\right\}(\lambda) \\
&= \mathscr{L}\left\{\int_{\tau=0}^{\infty}\int_{s=0}^{t}\int_{r=0}^{\infty}P_r(\kappa(\cdot)v(\tau,\cdot))(x)p(r,t-s)drdsd_\tau\mathbb{P}(E_s\le\tau)\right\}(\lambda) \\
&= \mathscr{L}\left\{\int_{\tau=0}^{\infty}\int_{r=0}^{\infty}\int_{s=0}^{t}P_r(\kappa(\cdot)v(\tau,\cdot))(x)p(r,t-s)\mathbb{E}[w(s-S_\tau)\mathbf{1}_{\{s\ge S_\tau\}}]dsdrd\tau\right\}(\lambda) \\
&= \frac{\phi(\lambda)}{\lambda}\int_{\tau=0}^{\infty}\int_{r=0}^{\infty}e^{-(\tau+r)\phi(\lambda)}P_r\left(\kappa(\cdot)v(\tau,\cdot)\right)(x)drd\tau.
\end{aligned}
\tag{2.15}
$$

Combining (2.13), (2.14), and (2.15), Eq. (2.12) can be proved by the uniqueness of Laplace transform.

(Regularity) One has the estimate $\|(-\mathcal{L})^\alpha P_t\| \le C_\alpha t^{-\alpha}$ for $t > 0$ by utilizing spectral representation of the self-adjoint operator $\mathcal{L}$. By Lemma 2.1 (ii), for $t > 0$,

$$\begin{aligned}
&\int_0^\infty \|(-\mathcal{L})^\alpha P_r g\| d_r \mathbb{P}(E_t \le r) \\
\le& C_\alpha \|g\| \int_0^\infty \frac{1}{r^\alpha} d_r \mathbb{P}(E_t \le r) = C_\alpha \|g\| \mathbb{E}\left[\frac{1}{E_t^\alpha}\right] \\
=& C_\alpha \|g\| \int_0^\infty \left(\frac{1}{r^\alpha} \mathbb{E}[w(t - S_r)\mathbf{1}_{\{t \ge S_r\}}]\right) dr.
\end{aligned} \tag{2.16}$$

For given $t > 0$, by (2.1),

$$\begin{aligned}
&\int_{s=0}^t \int_{r=0}^\infty \frac{p(r,s)}{r^\alpha} dr ds = \int_0^\infty \frac{\mathbb{P}(S_r \le t)}{r^\alpha} dr \\
=& \int_0^1 \frac{\mathbb{P}(S_r \le t)}{r^\alpha} dr + \int_1^\infty \frac{\mathbb{P}(S_r \le t)}{r^\alpha} dr \\
\le& \int_0^1 \frac{1}{r^\alpha} dr + \int_0^\infty \mathbb{P}(S_r \le t) dr \\
\le& \frac{1}{1-\alpha} + \int_0^t G^S(s) ds < \infty.
\end{aligned} \tag{2.17}$$

Then, for a.e. $t > 0$, $\int_0^\infty \frac{p(r,t)}{r^\alpha} dr < \infty$. By Lemma 2.1 (iii), for a.e. $t > 0$,

$$\int_1^\infty \frac{1}{r^\alpha} \mathbb{E}[w(t - S_r)\mathbf{1}_{\{t \ge S_r\}}] dr \le \int_0^\infty \mathbb{E}[w(t - S_r)\mathbf{1}_{\{t \ge S_r\}}] dr = 1.$$

For each $t > 0$, using Lemma 2.1 (i) and the Fubini theorem,

$$\begin{aligned}
&\int_{s=0}^t \int_{r=0}^1 \frac{1}{r^\alpha} \mathbb{E}[w(s - S_r)\mathbf{1}_{\{s \ge S_r\}}] dr ds \\
=& \int_0^1 \frac{1}{r^\alpha} \mathbb{E}\left[\int_0^t w(s - S_r)\mathbf{1}_{\{s \ge S_r\}} ds\right] dr \\
=& \int_0^1 \frac{1}{r^\alpha} \mathbb{E}[G(t - S_r)\mathbf{1}_{\{t \ge S_r\}}] dr \\
\le& G(t) \int_0^1 \frac{1}{r^\alpha} dr < \infty.
\end{aligned}$$

Thus, for a.e. $t > 0$, $\int_0^\infty \frac{1}{r^\alpha} \mathbb{E}[w(t - S_r)\mathbf{1}_{\{t \ge S_r\}}] dr < \infty$. It follows that $u_1(t, \cdot) \in \mathcal{D}((-\mathcal{L})^\alpha)$ for a.e. $t > 0$. By the monotonicity of $t \mapsto \mathbb{E}[\frac{1}{E_t^\alpha}]$, $u_1(t, \cdot) \in \mathcal{D}((-\mathcal{L})^\alpha)$ for $t > 0$. Moreover, we have for each $t > 0$,

$$(-\mathcal{L})^\alpha u_1(t, x) = \int_0^\infty (-\mathcal{L})^\alpha P_r g(x) d_r \mathbb{P}(E_t \le r)$$

and

$$\int_0^T \|(-\mathcal{L})^\alpha u_1(t, \cdot)\| dt < \infty \text{ for every } T > 0. \tag{2.18}$$

In addition, by Lemma 2.1 (ii) and the Fubini theorem, for $t > 0$,

$$\begin{aligned}\int_0^\infty \frac{\mathbb{P}(E_t \le r)}{r^{\alpha+1}}dr &= \int_{r=0}^\infty \frac{1}{r^{\alpha+1}}\int_{s=0}^r (\mathbb{E}[w(t-S_s)\mathbf{1}_{\{t\ge S_s\}}])dsdr \\ &= \int_{s=0}^\infty (\mathbb{E}[w(t-S_s)\mathbf{1}_{\{t\ge S_s\}}])\int_{r=s}^\infty \frac{1}{r^{\alpha+1}}drds \\ &= \int_0^\infty \frac{1}{\alpha s^\alpha}(\mathbb{E}[w(t-S_s)\mathbf{1}_{\{t\ge S_s\}}])ds < \infty.\end{aligned}$$

Thus we have for $t > 0$,

$$\int_0^\infty \frac{1}{r^\alpha}d_r\mathbb{P}(E_t \le r) = \frac{\mathbb{P}(E_t \le r)}{r^\alpha}\Big|_{r=0}^\infty + \alpha\int_0^\infty \frac{\mathbb{P}(E_t \le r)}{r^{\alpha+1}}dr.$$

It follows that for $t > 0$,

$$\lim_{r\to 0^+}\frac{\mathbb{P}(E_t \le r)}{r^\alpha} = 0. \tag{2.19}$$

We now show the continuity of $t \mapsto (-\mathcal{L})^\alpha u_1(t,\cdot)$ in $L^2(\mathcal{X};\nu)$. By (2.19) and the integration by parts, for $t > 0$, $\Delta t > 0$,

$$\begin{aligned}(-\mathcal{L})^\alpha u_1(t+\Delta t,x) - (-\mathcal{L})^\alpha u_1(t,x) &= \int_0^\infty (-\mathcal{L})^\alpha P_r g(x) d_r[\mathbb{P}(E_{t+\Delta t} \le r) - \mathbb{P}(E_t \le r)] \\ &= \int_0^\infty [\mathbb{P}(E_t \le r) - \mathbb{P}(E_{t+\Delta t} \le r)](-\mathcal{L})^{\alpha+1}P_r g(x)dr.\end{aligned}$$

Hence, by Levi monotone convergence theorem, as $\Delta t \to 0$,

$$\begin{aligned}\|(-\mathcal{L})^\alpha u_1(t+\Delta t,\cdot) - (-\mathcal{L})^\alpha u_1(t,\cdot)\| &\le \int_0^\infty |\mathbb{P}(E_t \le r) - \mathbb{P}(E_{t+\Delta t} \le r)|\|(-\mathcal{L})^{\alpha+1}P_r g\|dr \\ &\le C_{\alpha+1}\|g\|\int_0^\infty |\mathbb{P}(E_t \le r) - \mathbb{P}(E_{t+\Delta t} \le r)|\frac{1}{r^{\alpha+1}}dr \to 0.\end{aligned} \tag{2.20}$$

We next consider u_2, for given $t \ge 0$, by (2.17),

$$\begin{aligned}&\int_{s=0}^t\int_{r=0}^\infty \|(-\mathcal{L})^\alpha P_r(\kappa(\cdot)u(s,\cdot))\|p(r,t-s)drds \\ \le& C_\alpha \mathcal{K}\sup_{s\in[0,t]}\|u(s,\cdot)\|\int_{s=0}^t\int_{r=0}^\infty \frac{1}{r^\alpha}p(r,t-s)drds \\ \le& C\left(\frac{1}{1-\alpha} + \int_0^t G^S(s)ds\right) < \infty.\end{aligned}$$

It follows that $u_2(t,\cdot) \in \mathcal{D}((-\mathcal{L})^\alpha)$ for $t \ge 0$. Moreover, we have for each $t \ge 0$,

$$(-\mathcal{L})^\alpha u_2(t,x) = \int_{s=0}^t\int_{r=0}^\infty (-\mathcal{L})^\alpha P_r(\kappa(\cdot)u(s,\cdot))(x)p(r,t-s)drds$$

and

$$\int_0^T \|(-\mathcal{L})^\alpha u_2(t,\cdot)\|dt < \infty \text{ for every } T > 0. \tag{2.21}$$

We now show the continuity of $t \mapsto (-\mathcal{L})^\alpha u_2(t,\cdot)$ in $L^2(\mathcal{X};\nu)$. For $t \geq 0$, $\Delta t > 0$,

$$\begin{aligned}
&\|(-\mathcal{L})^\alpha u_2(t+\Delta t,\cdot) - (-\mathcal{L})^\alpha u_2(t,\cdot)\| \\
\leq &\int_{s=t}^{t+\Delta t}\int_{r=0}^{\infty} \|(-\mathcal{L})^\alpha P_r(\kappa(\cdot)u(s,\cdot))\| p(r,t+\Delta t-s)drds \\
&+ \int_{s=0}^{t}\int_{r=0}^{\infty} \|(-\mathcal{L})^\alpha P_r(\kappa(\cdot)u(s,\cdot))\| |p(r,t+\Delta t-s)-p(r,t-s)|drds \\
\leq & C_\alpha \mathcal{K} \sup_{s\in[t,t+\Delta t]} \|u(s,\cdot)\| \int_{s=t}^{t+\Delta t}\int_{r=0}^{\infty} \frac{1}{r^\alpha} p(r,t+\Delta t-s)drds \\
&+ C_\alpha \mathcal{K} \sup_{s\in[0,t]} \|u(s,\cdot)\| \int_{s=0}^{t}\int_{r=0}^{\infty} \frac{1}{r^\alpha}|p(r,t+\Delta t-s)-p(r,t-s)|drds \\
\leq & C\left[\int_0^\infty \frac{1}{r^\alpha}\mathbb{P}(S_r \leq \Delta t)dr + \int_{s=0}^{t}\int_{r=0}^{\infty}\frac{1}{r^\alpha}|p(r,\Delta t+s)-p(r,s)|drds\right].
\end{aligned} \tag{2.22}$$

Using Levi monotone convergence theorem in the first term, L^1-continuity of the integrable function $(r,s) \mapsto p(r,s)/r^\alpha$ on $[0,\infty)\times[0,t+1]$ in the second term, one can see that as $\Delta t \to 0$ both terms go to zero.

When κ and g have some regularity, we show that u appearing in Theorem 2.2 is a unique strong solution of equation (1.3).

Theorem 2.4. *Suppose that $g \in \mathcal{D}((-\mathcal{L})^\epsilon)$ with $\epsilon > 0$ small enough, and $(\kappa(\cdot)u(t,\cdot)) \in \mathcal{D}((-\mathcal{L})^\epsilon)$ for $t \geq 0$ with $\|(-\mathcal{L})^\epsilon(\kappa(\cdot)u(t,\cdot))\| \leq C_\kappa \|(-\mathcal{L})^\epsilon u(t,\cdot)\|$ for some constant C_κ. Then u is the unique strong solution of equation* (1.3) *satisfying $u(t,\cdot) \in \mathcal{D}(\mathcal{L})$ for a.e. $t > 0$ and $\int_0^T \|\mathcal{L}u(t,\cdot)\|dt < \infty$ for $T > 0$ in the sense that for every $t > 0$,*

$$\int_0^t w(t-s)(u(s,\cdot)-g(\cdot))ds = \int_0^t \mathcal{L}u(s,\cdot)ds - \int_0^t \kappa(\cdot)u(s,\cdot)ds \text{ in } L^2(\mathcal{X};\nu). \tag{2.23}$$

Moreover,

$$u \in C([0,\infty);\mathcal{D}((-\mathcal{L})^\epsilon)) \cap C((0,\infty);\mathcal{D}(\mathcal{L})). \tag{2.24}$$

Proof. By $g \in \mathcal{D}((-\mathcal{L})^\epsilon)$ and the similar calculations as the proof of (2.4), we have $u \in C([0,\infty);\mathcal{D}((-\mathcal{L})^\epsilon))$. It remains to establish for a.e. $t > 0$, $u(t,\cdot) \in \mathcal{D}(\mathcal{L})$ and $\int_0^T \|\mathcal{L}u(t,\cdot)\|dt < \infty$ for $T > 0$, and hence (2.3) can be strengthened to (2.23) by the fact that $\mathcal{D}(\mathcal{L})$ is dense in $L^2(\mathcal{X};\nu)$.

In fact, by the same arguments as those in (2.16), we have for $t > 0$,

$$\begin{aligned}
&\int_0^\infty \|\mathcal{L}P_r g\| d_r\mathbb{P}(E_t \leq r) \\
=&\int_0^\infty \|(-\mathcal{L})^{1-\epsilon}P_r(-\mathcal{L})^\epsilon g\| d_r\mathbb{P}(E_t \leq r) \\
\leq & C_{1-\epsilon}\|(-\mathcal{L})^\epsilon g\| \int_0^\infty \frac{1}{r^{1-\epsilon}} d_r\mathbb{P}(E_t \leq r) \\
=& C_{1-\epsilon}\|(-\mathcal{L})^\epsilon g\| \mathbb{E}\left[\frac{1}{E_t^{1-\epsilon}}\right] < \infty.
\end{aligned}$$

Hence $u_1(t,\cdot)\in\mathcal{D}(\mathcal{L})$ and for $t>0$,

$$\mathcal{L}u_1(t,x)=-\int_0^\infty(-\mathcal{L})^{1-\epsilon}P_r(-\mathcal{L})^\epsilon g(x)d_r\mathbb{P}(E_t\le r).$$

Let $\widetilde{u_1}(t,x)=\mathbb{E}[P_{E_t}(-\mathcal{L})^\epsilon g(x)]$, by (2.18),

$$\int_0^T\|\mathcal{L}u_1(t,\cdot)\|dt=\int_0^T\|(-\mathcal{L})^{1-\epsilon}\widetilde{u_1}(t,\cdot)\|dt<\infty\text{ for every }T>0. \tag{2.25}$$

For u_2, by (2.17) and $\|(-\mathcal{L})^\epsilon(\kappa(\cdot)u(t,\cdot))\|\le C_\kappa\|(-\mathcal{L})^\epsilon u(t,\cdot)\|$, we have for given $t\ge0$,

$$\begin{aligned}&\int_{s=0}^t\int_{r=0}^\infty\|\mathcal{L}P_r(\kappa(\cdot)u(s,\cdot))\|p(r,t-s)drds\\
=&\int_{s=0}^t\int_{r=0}^\infty\|(-\mathcal{L})^{1-\epsilon}P_r(-\mathcal{L})^\epsilon(\kappa(\cdot)u(s,\cdot))\|p(r,t-s)drds\\
\le&C_{1-\epsilon}C_\kappa\sup_{s\in[0,t]}\|(-\mathcal{L})^\epsilon u(s,\cdot)\|\int_{s=0}^t\int_{r=0}^\infty\frac{1}{r^{1-\epsilon}}p(r,t-s)drds\\
\le&C_{1-\epsilon}C_\kappa\sup_{s\in[0,t]}\|(-\mathcal{L})^\epsilon u(s,\cdot)\|\left(\frac{1}{\epsilon}+\int_0^t G^S(s)ds\right)<\infty.\end{aligned}$$

It follow that $u_2(t,\cdot)\in\mathcal{D}(\mathcal{L})$ for $t\ge0$, and

$$\mathcal{L}u_2(t,x)=-\int_{s=0}^t\int_{r=0}^\infty(-\mathcal{L})^{1-\epsilon}P_r(-\mathcal{L})^\epsilon(\kappa(\cdot)u(s,\cdot))(x)p(r,t-s)drds.$$

Let $\widetilde{u_2}(t,x)=\int_{s=0}^t\int_{r=0}^\infty P_r(-\mathcal{L})^\epsilon(\kappa(\cdot)u(s,\cdot))(x)p(r,t-s)drds$, by (2.21),

$$\int_0^T\|\mathcal{L}u_2(t,\cdot)\|dt=\int_0^T\|(-\mathcal{L})^{1-\epsilon}\widetilde{u_2}(t,\cdot)\|dt<\infty\text{ for every }T>0. \tag{2.26}$$

Therefore $\int_0^T\|\mathcal{L}u(t,\cdot)\|dt<\infty$ for every $T>0$ following from (2.25) and (2.26).

The continuity of $t\mapsto\mathcal{L}u(t,\cdot)$ in $L^2(\mathcal{X};\nu)$ is as follows. By the same arguments as those in (2.20), for any $t>0$, as $\Delta t\to0$,

$$\begin{aligned}\|\mathcal{L}u_1(t+\Delta t,\cdot)-\mathcal{L}u_1(t,\cdot)\|=&\int_0^\infty|\mathbb{P}(E_t\le r)-\mathbb{P}(E_{t+\Delta t}\le r)|\|(-\mathcal{L})^{2-\epsilon}P_r(-\mathcal{L})^\epsilon g\|dr\\
\le&C_{2-\epsilon}\|(-\mathcal{L})^\epsilon g\|\int_0^\infty|\mathbb{P}(E_t\le r)-\mathbb{P}(E_{t+\Delta t}\le r)|\frac{1}{r^{2-\epsilon}}dr\to0.\end{aligned}$$

By the same arguments as those in (2.22), for $t\ge0$, as $\Delta t\to0$,

$$\begin{aligned}&\|\mathcal{L}u_2(t+\Delta t,\cdot)-\mathcal{L}u_2(t,\cdot)\|\\
\le&\int_{s=t}^{t+\Delta t}\int_{r=0}^\infty\|(-\mathcal{L})^{1-\epsilon}P_r(-\mathcal{L})^\epsilon(\kappa(\cdot)u(s,\cdot))\|p(r,t+\Delta t-s)drds\\
&+\int_{s=0}^t\int_{r=0}^\infty\|(-\mathcal{L})^{1-\epsilon}P_r(-\mathcal{L})^\epsilon(\kappa(\cdot)u(s,\cdot))\||p(r,t+\Delta t-s)-p(r,t-s)|drds\\
\le&C_{1-\epsilon}C_\kappa\sup_{s\in[0,t+1]}\|(-\mathcal{L})^\epsilon u(s,\cdot)\|\int_0^\infty\frac{1}{r^{1-\epsilon}}\mathbb{P}(S_r\le\Delta t)dr\\
&+C_{1-\epsilon}C_\kappa\sup_{s\in[0,t]}\|(-\mathcal{L})^\epsilon u(s,\cdot)\|\int_{s=0}^t\int_{r=0}^\infty\frac{1}{r^{1-\epsilon}}|p(r,\Delta t+s)-p(r,s)|drds\to0.\end{aligned}$$

Then the proof of (2.24) is completed.

Remark 2.5. As for the rationality of the assumptions in Theorem 2.4, one can note that $\|(-\Delta)^{\epsilon}(\kappa\varphi)\| \leq C_{\kappa}\|(-\Delta)^{\epsilon}\varphi\|$ for the potential function $\kappa \in C_c^{\infty}(\mathbb{R}^d)$ and every $\varphi \in \mathcal{D}((-\Delta)^{\epsilon}) = H^{2\epsilon}(\mathbb{R}^d)$, in which C_{κ} is a constant, and we take the Lusin space $\mathcal{X} = \mathbb{R}^d$, $L^2(\mathcal{X}; \nu) = L^2(\mathbb{R}^d)$, $(\mathcal{L}, \mathcal{D}(\mathcal{L})) = (\Delta, H^2(\mathbb{R}^d))$, and the symmetric Markov process as Brownian motion B_t.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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