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*Theory article*

# Existence of infinitely many solutions for critical sub-elliptic systems via genus theory

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**Abstract:** We are devoted to the study of the following sub-Laplacian system with Hardy-type potentials and critical nonlinearities

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu_1 \dfrac{\psi^2 u}{\mathrm{d}(z)^2} = \lambda_1 \dfrac{\psi^\alpha |u|^{2^*(\alpha)-2} u}{\mathrm{d}(z)^\alpha} + \beta p_1 f(z) \dfrac{\psi^\gamma |u|^{p_1-2} u |v|^{p_2}}{\mathrm{d}(z)^\gamma} & \text{in } \mathbb{G}, \\ -\Delta_{\mathbb{G}} v - \mu_2 \dfrac{\psi^2 v}{\mathrm{d}(z)^2} = \lambda_2 \dfrac{\psi^\alpha |v|^{2^*(\alpha)-2} v}{\mathrm{d}(z)^\alpha} + \beta p_2 f(z) \dfrac{\psi^\gamma |u|^{p_1} |v|^{p_2-2} v}{\mathrm{d}(z)^\gamma} & \text{in } \mathbb{G}, \end{cases}$$

where $-\Delta_{\mathbb{G}}$ is the sub-Laplacian on Carnot group $\mathbb{G}$, $\mu_1, \mu_2 \in [0, \mu_{\mathbb{G}})$, $\alpha, \gamma \in (0,2)$, $\lambda_1, \lambda_2, \beta, p_1, p_2 > 0$ with $1 < p_1 + p_2 < 2$, $\mathrm{d}(z)$ is the $\Delta_{\mathbb{G}}$-gauge, $\psi = |\nabla_{\mathbb{G}} \mathrm{d}(z)|$, $2^*(\alpha) := \frac{2(Q-\alpha)}{Q-2}$ is the critical Sobolev-Hardy exponents, and $\mu_{\mathbb{G}} = (\frac{Q-2}{2})^2$ is the best Hardy constant on $\mathbb{G}$. By combining a variant of the symmetric mountain pass theorem with the genus theory, we prove the existence of infinitely many weak solutions whose energy tends to zero when $\beta$ or $\lambda_1, \lambda_2$ belong to a suitable range.

**Keywords:** sub-Laplacian problem; critical exponents; genus theory; carnot groups
**Mathematics Subject Classification:** 35R03, 35J70, 35B33

## 1. Introduction

In this paper, we are concerned with the following sub-Laplacian system with Sobolev-Hardy critical nonlinearities on Carnot group $\mathbb{G}$:

$$\begin{cases} -\Delta_{\mathbb{G}}u-\mu_1\dfrac{\psi^2 u}{\mathrm{d}(z)^2}=\lambda_1\dfrac{\psi^\alpha|u|^{2^*(\alpha)-2}u}{\mathrm{d}(z)^\alpha}+\beta p_1 f(z)\dfrac{\psi^\gamma|u|^{p_1-2}u|v|^{p_2}}{\mathrm{d}(z)^\gamma} & \text{in } \mathbb{G},\\ -\Delta_{\mathbb{G}}v-\mu_2\dfrac{\psi^2 v}{\mathrm{d}(z)^2}=\lambda_2\dfrac{\psi^\alpha|v|^{2^*(\alpha)-2}v}{\mathrm{d}(z)^\alpha}+\beta p_2 f(z)\dfrac{\psi^\gamma|u|^{p_1}|v|^{p_2-2}v}{\mathrm{d}(z)^\gamma} & \text{in } \mathbb{G},\end{cases}\tag{1.1}$$

where $\Delta_{\mathbb{G}}$ stands for the sub-Laplacian operator on Carnot group $\mathbb{G}$, $\mu_1,\ \mu_2\in[0,\mu_{\mathbb{G}})$, $\alpha,\ \gamma\in(0,2)$, $\lambda_1,\ \lambda_2,\ \beta$ are positive parameters, $p_1,\ p_2>0$ with $1<p_1+p_2<2$, $\psi=|\nabla_{\mathbb{G}}\mathrm{d}(z)|$, $\nabla_{\mathbb{G}}$ denotes the horizontal gradient and d is the natural gauge associated with the fundamental solution of $-\Delta_{\mathbb{G}}$ on $\mathbb{G}$. Here, $\mu_{\mathbb{G}}=(\frac{Q-2}{2})^2$ is the best Hardy constant and $2^*(\alpha):=\frac{2(Q-\alpha)}{Q-2}$ is the Sobolev-Hardy critical exponents, $Q\geq 3$ being the homogeneous dimension of the space $\mathbb{G}$ with respect to the dilation $\delta_\gamma$. Moreover, the function $f(z)$ satisfies the following assumption:

($f$) $f(z)\in L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{d(z)^\gamma}dz)$ and the Lebesgue measure of set $\{z\in\mathbb{G}:\ f(z)>0\}$ is positive, where $p_*=\frac{2^*(\gamma)}{2^*(\gamma)-(p_1+p_2)}$, $0<\gamma<2$.

Our goal is to prove, by means of variational methods, the existence of weak solutions to (1.1). We define the energy functional $I_{\lambda_1,\lambda_2,\beta}$ associated to (1.1) as follows

$$\begin{aligned} I_{\lambda_1,\lambda_2,\beta}(u,v)=&\frac{1}{2}\int_{\mathbb{G}}\Big(|\nabla_{\mathbb{G}}u|^2+|\nabla_{\mathbb{G}}v|^2-\mu_1\frac{\psi^2|u|^2}{\mathrm{d}(z)^2}-\mu_2\frac{\psi^2|v|^2}{\mathrm{d}(z)^2}\Big)dz-\frac{\lambda_1}{2^*(\alpha)}\int_{\mathbb{G}}\frac{\psi^\alpha|u|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz\\ &-\frac{\lambda_2}{2^*(\alpha)}\int_{\mathbb{G}}\frac{\psi^\alpha|v|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz-\beta\int_{\mathbb{G}}f(z)\frac{\psi^\gamma|u|^{p_1}|v|^{p_2}}{\mathrm{d}(z)^\gamma}dz\end{aligned}$$

defined on the product space $\mathcal{H}:=S_0^1(\mathbb{G})\times S_0^1(\mathbb{G})$, where the Folland-Stein space $S_0^1(\mathbb{G})=\{u\in L^{2^*}(\mathbb{G}):\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u|^2dz<+\infty\}$ is the closure of $C_0^\infty(\mathbb{G})$ with respect to the norm

$$\|u\|_{S_0^1(\mathbb{G})}=\Big(\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u|^2dz\Big)^{\frac{1}{2}}.$$

Here, $2^*=\frac{2Q}{Q-2}$ is the Sobolev critical exponent. Further, we endow the product space $\mathcal{H}$ with the following norm

$$\|(u,v)\|_{\mathcal{H}}=(\|u\|_{\mu_1}^2+\|v\|_{\mu_2}^2)^{\frac{1}{2}},$$

where

$$\|u\|_{\mu_i}^2=\int_{\mathbb{G}}(|\nabla_{\mathbb{G}}u|^2-\mu_i\frac{\psi^2|u|^2}{\mathrm{d}(z)^2})dz,\quad \forall i=1,\ 2.$$

The above norm is well-defined due to the following Hardy-type inequality on Carnot group

$$\mu_{\mathbb{G}}\int_{\mathbb{G}}\frac{\psi^2|u|^2}{\mathrm{d}(z)^2}dz\leq\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u|^2dz,\quad \forall u\in C_0^\infty(\mathbb{G}),\tag{1.2}$$

where $\mu_{\mathbb{G}}=(\frac{Q-2}{2})^2$ is the optimal constant for (1.2). We can note that the norms $\|\cdot\|_{\mu_i}$ and $\|\cdot\|_{S_0^1(\mathbb{G})}$ for any $\mu_i<\mu_{\mathbb{G}}$ with $i=1,2$ are equivalent due to the Hardy's inequality (1.2).

The inequality (1.2) was first proved by Garofalo and Lanconelli in [1] for the Heisenberg group (see also [2]), and extended it to Carnot groups by D' Ambrosio, see [3]. In the Euclidean space setting, the weight function $\psi$ appearing in the l.h.s. of (1.2) is constant, i.e., $\psi \equiv 1$. So, (1.2) becomes the well-known Hardy inequality:

$$\bar{\mu} \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} dx \leq \int_{\mathbb{R}^N} |\nabla u|^2 dx, \quad \forall u \in C_0^\infty(\mathbb{R}^N),$$

where $\bar{\mu} = (\frac{N-2}{2})^2$ is the best constant and it is never attained. In the Euclidean space, the existence and non-existence, as well as qualitative properties, of nontrivial weak solutions for $p$-Laplacian equations with singular potentials and critical exponents were recently studied by several authors, we refer, e.g., in bounded domains and for $p = 2$ to [4–8], and for general $p > 1$ to [9–12]; while in $\mathbb{R}^n$ and for $p = 2$ to [13–15], and for general $p > 1$ to [16–18], and for fractional $(p, q)$-Laplacian to [19], and the references therein. Moreover, a more interesting result can be found in [20], which studies the critical $p$-Laplace equation on the Heisenberg group with a Hardy-type term.

In recent years, people have paid much attention to the following singular sub-elliptic problem:

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu \dfrac{\psi^2 u}{\mathrm{d}(z)^2} = f(z, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $\Omega$ is a smooth bounded domain in Carnot group $\mathbb{G}$, $0 \in \Omega$. It should be mentioned that [21], by using Moser-type iteration, the author studied the asymptotic behavior of weak solutions to (1.3) when the function $f$ satisfies the following condition:

$$|f(z, t)| \leq C(|t| + |t|^{2^*-1}) \text{ for all } (z, t) \in \Omega \times \mathbb{R},$$

and obtained the following asymptotic behavior at origin:

$$u(z) \sim \mathrm{d}(z)^{-(\sqrt{\mu_{\mathbb{G}}} - \sqrt{\mu_{\mathbb{G}} - \mu})} \quad \text{as} \quad d(z) \to 0.$$

Subsequently, in [22] also the behavior at infinity has been determined for the purely critical problem

$$-\Delta_{\mathbb{G}} u - \mu \frac{\psi^2 u}{\mathrm{d}(z)^2} = |u|^{2^*-2} u \quad \text{on} \quad \mathbb{G}$$

for which the asymptotic estimates at the origin and at infinity are then, respectively:

$$u(z) \sim \frac{1}{\mathrm{d}(z)^{a(\mu)}} \quad \text{as} \quad d(z) \to 0,$$
$$u(z) \sim \frac{1}{\mathrm{d}(z)^{b(\mu)}} \quad \text{as} \quad d(z) \to \infty,$$

where $a(\mu) = \sqrt{\mu_{\mathbb{G}}} - \sqrt{\mu_{\mathbb{G}} - \mu}$, $b(\mu) = \sqrt{\mu_{\mathbb{G}}} + \sqrt{\mu_{\mathbb{G}} - \mu}$ and the notation $f \sim g$ means that there exists a constant $C > 0$ such that $\frac{1}{C} g(z) \leq f(z) \leq C g(z)$. From a technical point of view, these asymptotic estimates have a fundamental role in the study of the associated Brezis-Nirenberg type sub-elliptic problems on Carnot group. For more details on this topic, please refer to [22], which provides a detailed analysis of the Brezis-Nirenberg problem on Carnot group.

Motivated by the aforementioned articles and their results, we are interested in finding existence and multiplicity results for a system with critical Sobolev-Hardy critical terms. While dealing with the system (1.1), if we suppose $\mu_1 = \mu_2 = \mu$, $\lambda_1 = \lambda_2 = 1$ and $\beta = 0$, problem (1.1) reduces to a sub-elliptic critical problem

$$-\Delta_{\mathbb{G}} u - \mu\psi^2 \frac{u}{\mathrm{d}(z)^2} = \psi^\alpha \frac{|u|^{2^*(\alpha)-2}u}{\mathrm{d}(z)^\alpha} \quad \text{in } \mathbb{G}. \tag{1.4}$$

In 2015, Loiudice in the paper [23] proved the existence of ground state solutions of (1.4) using variational approach for $\mu = 0$ and $0 < \alpha < 2$, and obtained the asymptotic behavior of this solution at infinity. Recently, Zhang [24] proved the existence of ground state solutions of (1.4) $0 < \mu < \mu_{\mathbb{G}}$ and $0 < \alpha < 2$ and considered the following sub-elliptic system with critical Sobolev-Hardy nonlinearities on Carnot group

$$\begin{cases} -\Delta_{\mathbb{G}} u - \mu\psi^2 \dfrac{u}{\mathrm{d}(z)^2} = \psi^\alpha \dfrac{|u|^{2^*(\alpha)-2}u}{\mathrm{d}(z)^\alpha} + \lambda \dfrac{\eta}{\eta+\theta}\psi^\alpha \dfrac{|u|^{\eta-2}u|v|^\theta}{\mathrm{d}(z)^\alpha} & \text{in } \mathbb{G}, \\ -\Delta_{\mathbb{G}} v - \mu\psi^2 \dfrac{v}{\mathrm{d}(z)^2} = \psi^\alpha \dfrac{|v|^{2^*(\alpha)-2}v}{\mathrm{d}(z)^\alpha} + \lambda \dfrac{\theta}{\eta+\theta}\psi^\alpha \dfrac{|u|^{\eta}|v|^{\theta-2}v}{\mathrm{d}(z)^\alpha} & \text{in } \mathbb{G}, \end{cases}$$

where $\alpha \in (0, 2)$, $\lambda > 0$ and $\eta, \theta > 1$. The existence of nontrivial solutions of the above sub-Laplacian system through variational methods was obtained for the critical case, i.e., $\eta + \theta = 2^*(\alpha)$. Other subelliptic problems with multiple critical exponents can be found in [25] and the references therein.

Let us recall that solutions of (1.4) arise as minimizers $u \in S^1_0(\mathbb{G})$ of the following Rayleigh quotient:

$$S_{\alpha,\mu} = \inf_{u\in S^1_0(\mathbb{G})\setminus\{0\}} \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 dz - \mu \int_{\mathbb{G}} \frac{\psi^2|u|^2}{\mathrm{d}(z)^2} dz}{\left(\int_{\mathbb{G}} \frac{\psi^\alpha |u|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha} dz\right)^{\frac{2}{2^*(\alpha)}}}.$$

Actually, up to a normalization, it holds that

$$\int_{\mathbb{G}} |\nabla_{\mathbb{G}} u|^2 dz - \mu \int_{\mathbb{G}} \frac{\psi^2|u|^2}{\mathrm{d}(z)^2} dz = \int_{\mathbb{G}} \frac{\psi^\alpha |u|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha} dz = (S_{\alpha,\mu})^{\frac{Q-\alpha}{2-\alpha}}. \tag{1.5}$$

Moreover, for any $\varepsilon > 0$, rescaled functions $u_\varepsilon(z) = \varepsilon^{-\frac{Q-2}{2}} u(\delta_{\frac{1}{\varepsilon}}(z))$ are solutions, up to multiplicative constants, of the equation (1.4) and satisfy (1.5) too. However, the explicit form of ground state solutions is unknown, which is also the focus of our future work.

As a natural extension of the above papers, we are mainly interested in searching infinitely many solutions of singular sub-elliptic problem (1.1). Our point is here a combination of sub-Laplace operator and critical Sobolev-Hardy terms on the Carnot group. In the Euclidean elliptic setting, i.e., when $\mathbb{G}$ is the ordinary Euclidean space $(\mathbb{R}^N, +)$, starting with the pioneering work of Kajikiya [26], established a critical point theorem related to the symmetric mountain pass lemma and applied it to find the existence of infinitely many solutions to elliptic equation. A large number of scholars have investigated the application of this method and achieved rich results, such as He-Zou [27], Baldelli-Filippucci [28], Liang-Zhang [29, 30], Ambrosio-Isernia [19] and Liang-Shi [31] in this direction.

Motivated by the above results, our aim of this paper is to show the existence of infinitely many solutions of sub-elliptic problem (1.1), and that there exists a sequence of infinitely many arbitrarily small solutions converging to zero using the symmetric mountain-pass lemma due to Kajikiya [26]. To the best of our knowledge, there are only some results that deal with the sub-Laplacian problem with Sobolev-Hardy critical exponents and Hardy-type terms on the Carnot group.

Before stating our main result, let us recall the definition of weak solutions to (1.1).

**Definition 1.1.** *We say that* $(u, v) \in \mathcal{H}$ *is a weak solutions of* (1.1)*, if* $(u, v)$ *satisfies*

$$\begin{aligned}&\int_{\mathbb{G}} \nabla_{\mathbb{G}} u \cdot \nabla_{\mathbb{G}} \phi_1 dz + \int_{\mathbb{G}} \nabla_{\mathbb{G}} v \cdot \nabla_{\mathbb{G}} \phi_2 dz - \mu_1 \int_{\mathbb{G}} \frac{\psi^2 u \phi_1}{d(z)^2} dz \\ &- \mu_2 \int_{\mathbb{G}} \frac{\psi^2 v \phi_2}{d(z)^2} dz - \lambda_1 \int_{\mathbb{G}} \frac{\psi^\alpha |u|^{2^*(\alpha)-2} u \phi_1}{d(z)^\alpha} dz - \lambda_2 \int_{\mathbb{G}} \frac{\psi^\alpha |v|^{2^*(\alpha)-2} v \phi_2}{d(z)^\alpha} dz \\ &- \beta p_1 \int_{\mathbb{G}} f(z) \frac{\psi^\gamma |u|^{p_1-2} |v|^{p_2} u \phi_1}{d(z)^\gamma} dz - \beta p_2 \int_{\mathbb{G}} f(z) \frac{\psi^\gamma |u|^{p_1} |v|^{p_2-2} v \phi_2}{d(z)^\gamma} dz = 0\end{aligned}$$

*for all* $(\phi_1, \phi_2) \in \mathcal{H}$.

By Hardy-Sobolev inequality, it is clear that $I_{\lambda_1,\lambda_2,\beta}$ is well-defined on $\mathcal{H}$ and belongs to $C^1(\mathcal{H}, \mathbb{R})$. Then, from Definition 1.1 we see that any weak solution of (1.1) is just a critical point of $I_{\lambda_1,\lambda_2,\beta}$. Therefore, we are now in position to state our main result as follows.

**Theorem 1.1.** *Assume that* $(f)$ *holds, and* $1 < p_1 + p_2 < 2$, $0 \le \alpha < 2$, $0 \le \gamma < 2$. *Then*

*(i) for any* $\beta > 0$*, there exists* $\widetilde{\lambda} > 0$ *such that if* $0 < \lambda_1 < \widetilde{\lambda}$, $0 < \lambda_2 < \widetilde{\lambda}$, *problem* (1.1) *has a sequence of solutions* $\{(u_n, v_n)\} \subset \mathcal{H}$ *with* $I_{\lambda_1,\lambda_2,\beta}(u_n, v_n) < 0$ *and* $I_{\lambda_1,\lambda_2,\beta}(u_n, v_n) \to 0$ *as* $n \to \infty$.

*(ii) for any* $\lambda_1$, $\lambda_2 > 0$ *, there exists* $\widetilde{\beta} > 0$ *such that if* $0 < \beta < \widetilde{\beta}$, *problem* (1.1) *has a sequence of solutions* $\{(u_n, v_n)\} \subset \mathcal{H}$ *with* $I_{\lambda_1,\lambda_2,\beta}(u_n, v_n) < 0$ *and* $I_{\lambda_1,\lambda_2,\beta}(u_n, v_n) \to 0$ *as* $n \to \infty$.

**Remark 1.1.** *Using the symmetric mountain pass lemma (see Theorem 2.1) we can conclude that the solutions obtained from Theorem 1.1 satisfy* $(u_n, v_n) \to (0, 0)$ *as* $n \to \infty$.

The main idea to prove Theorem 1.1 is based on concentration-compactness result on the Carnot group and the symmetric mountain pass lemma [26]. One of the main difficulties to prove the existence and multiplicity of solutions of equation (1.1) using variational methods is that the energy functional does not satisfy the Palais-Smale condition for large energy levels, since the embedding $S_0^1(\mathbb{G}) \hookrightarrow L^{2^*(\alpha)}(\mathbb{G}, \frac{\psi^\alpha}{d(z)^\alpha} dz)$ is not compact. Another difficulty is that every nontrivial solution of (1.1) is singular at $\{z = 0\}$ due to the presence of the Hardy terms. Thus, different techniques are needed to deal with the singular case.

The rest of this paper is organized as follows. In Section 2, the variational setting and some preliminary are recalled. Finally, Section 3 contains several preliminary lemmas, including the crucial concentration-compactness lemma, as well as the proof of Theorem 1.1.

## 2. Preliminary

We devote this section to state some useful facts on the Carnot groups. For more details, we refer the reader to [32–36] and references therein.

A Carnot group (or Stratified group) $(\mathbb{G}, \circ)$ is a connected, simply connected nilpotent Lie group, whose Lie algebra $\mathfrak{g}$ admits a stratification, namely a decomposition $\mathfrak{g} = \oplus_{k=1}^r V_k$ with

$$[V_1, V_k] = V_{k+1} \text{ for } 1 \le k \le r-1 \quad \text{and} \quad [V_1, V_r] = \{0\}.$$

Here, the integer $r$ is called the step of $\mathbb{G}$, $\dim(V_k) = N_k$ and the symbol $[V_1, V_k]$ denotes the subspace of $\mathfrak{g}$ generated by the commutators $[X, Y]$, where $X \in V_1$ and $Y \in V_k$.

By means of the natural identification of $\mathbb{G}$ with its Lie algebra via the exponential map, it is not restrictive to suppose that $\mathbb{G}$ is a homogeneous group, i.e., Lie group equipped with a family $\{\delta_\gamma\}_{\gamma>0}$ of dilations, acting on $z \in \mathbb{R}^N$ as follows

$$\delta_\gamma(z^{(1)}, \cdots, z^{(r)}) = (\gamma^1 z^{(1)}, \gamma^2 z^{(2)}, \cdots, \gamma^r z^{(r)}),$$

where $z^{(k)} \in \mathbb{R}^{N_k}$ for every $k \in \{1, \cdots, r\}$ and $N = \sum_{k=1}^r N_k$. Then, the structure $\mathbb{G} := (\mathbb{R}^N, \circ, \{\delta_\gamma\}_{\gamma>0})$ is called a homogeneous group with homogeneous dimension $Q := \sum_{k=1}^r k \cdot N_k$. Note that the number $Q$ is naturally associated to the family $\{\delta_\gamma\}_{\gamma>0}$ since, for every $\gamma > 0$, the Jacobian of the map $z \mapsto \delta_\gamma(z)$ equals $\gamma^Q$. Moreover, the number $N := \sum_{k=1}^r N_k$ is called the topological dimension of $\mathbb{G}$.

Now, let $\{X_1, \cdots, X_{N_1}\}$ be any basis of $V_1$, the sub-Laplacian on $\mathbb{G}$ is define as the second order differential operator

$$\Delta_{\mathbb{G}} := X_1^2 + X_2^2 + \cdots + X_{N_1}^2.$$

The horizontal gradient on $\mathbb{G}$ is define as

$$\nabla_{\mathbb{G}} := (X_1, X_2 \cdots, X_{N_1}).$$

The horizontal divergence on $\mathbb{G}$ is define by

$$\text{div}_{\mathbb{G}} u = \nabla_{\mathbb{G}} \cdot u.$$

It is easy to check that $\nabla_{\mathbb{G}}$ and $\Delta_{\mathbb{G}}$ are left-translation invariant with respect to the group action $\tau_z$ and $\delta_\gamma$-homogeneous, respectively, of degree one and two, that is, $\nabla_{\mathbb{G}}(u \circ \tau_z) = \nabla_{\mathbb{G}} u \circ \tau_z$, $\nabla_{\mathbb{G}}(u \circ \delta_\gamma) = \gamma \nabla_{\mathbb{G}} u \circ \delta_\gamma$; $\Delta_{\mathbb{G}}(u \circ \tau_z) = \Delta_{\mathbb{G}} u \circ \tau_z$ and $\Delta_{\mathbb{G}}(u \circ \delta_\gamma) = \gamma^2 \Delta_{\mathbb{G}} u \circ \delta_\gamma$, where the left translation $\tau_z : \mathbb{G} \to \mathbb{G}$ is defined by

$$\tau_z(z') = z \circ z', \quad \forall z, z' \in \mathbb{G}.$$

Let us now define the homogeneous norm Carnot group $\mathbb{G}$.

**Definition 2.1.** *A continuous function $d : \mathbb{G} \to [0, +\infty)$ is said to be a homogeneous norm on $\mathbb{G}$ if it satisfies the following condition:*

*(i) $d(z) = 0$ if and only if $z = 0$;*
*(ii) $d(z^{-1}) = d(z)$ for all $z \in \mathbb{G}$;*
*(iii) $d(\delta_\gamma(z)) = \gamma d(z)$ for every $\gamma > 0$ and $z \in \mathbb{G}$.*

Throughout this paper, we almost exclusively work with the homogeneous norm, which is related to the fundamental solution of the sub-Laplace operator $-\Delta_{\mathbb{G}}$, that is the function d such that

$$\Gamma(z) = \frac{C}{\mathrm{d}(z)^{Q-2}}, \quad \forall z \in \mathbb{G}$$

is the fundamental solution of $-\Delta_{\mathbb{G}}$ with pole at 0, for a suitable constant $C > 0$, see [22, 33]. Moreover, if we define $\mathrm{d}(z_1, z_2) := \mathrm{d}(z_2^{-1} \circ z_1)$, then d is a pseudo-distence on $\mathbb{G}$. In particular, d satisfies the pseudo-triangular inequality:

$$\mathrm{d}(z_1, z_2) \le c(\mathrm{d}(z_1, z_3) + \mathrm{d}(z_3, z_2)), \quad \forall z_1, z_2, z_3 \in \mathbb{G}$$

for a suitable positive constant $c$. The ball of radius $R > 0$ centered at $z \in \mathbb{G}$ with respect to the norm d, calling them d-balls, defined as

$$B_{\mathrm{d}}(z, R) = \{y \in \mathbb{G} : \ \mathrm{d}(z, y) < R\}.$$

In fact, the norm on $\mathbb{G}$ can be induced by the Euclidean distance $|\cdot|$ on $\mathfrak{g}$ through the exponential mapping, which also induces the homogeneous pseudo-norm $|\cdot|_{\mathfrak{g}}$ on $\mathfrak{g}$, namely, for $\xi \in \mathfrak{g}$ with $\xi = \xi_1 + \cdots + \xi_k$, where $\xi_i \in V_i$, define a pseudo-norm on $\mathfrak{g}$ as follows

$$|\xi|_{\mathfrak{g}} = |(\xi_1, \cdots, \xi_k)|_{\mathfrak{g}} := (\sum_{i=1}^{k} |\xi_i|^{\frac{2k!}{i}})^{\frac{1}{2k!}}.$$

The induced norm on $\mathbb{G}$ has the form

$$|g|_{\mathbb{G}} = |\exp_{\mathbb{G}}^{-1}(g)|_{\mathfrak{g}}, \quad \forall g \in \mathbb{G}.$$

The function $|\cdot|_{\mathbb{G}}$ is usually known as the non-isotropic gauge. It defines a pseudo-distence on $\mathbb{G}$ given by

$$\mathrm{d}(g, h) := |h^{-1} \circ g|_{\mathbb{G}}, \qquad \forall g, \ h \in \mathbb{G}.$$

The simplest example of a stratified Lie group is the Heisenberg group $\mathbb{H}^N := (\mathbb{R}^{2N+1}, \circ)$ with the composition law as

$$(x, y, t) \circ (x', y', t') := \left(x_1 + x_1', \cdots, x_n + x_n', \ y_1 + y_1', \cdots, y_n + y_n', \ t + t' + 2(\langle x', y\rangle - \langle x, y'\rangle)\right),$$

where $(x, y, t), (x', y,' t') \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^1$ and $\langle \cdot, \cdot \rangle$ represents the inner product on $\mathbb{R}^N$. The sub-Laplacian on $\mathbb{H}^N$ is given by

$$\Delta_{\mathbb{H}^N} = \sum_{i=1}^{N} (X_i^2 + Y_i^2),$$

where

$$X_i = \frac{\partial}{\partial x_i} + 2y_i \frac{\partial}{\partial t}, \quad Y_i = \frac{\partial}{\partial y_i} - 2x_i \frac{\partial}{\partial t} \ \text{ for } i = 1, 2, \cdots, N.$$

In order to prove Theorem 1.1, we will recall some basic facts involved in the so-called Krasnoselskii genus, which can be found in [37, 38].

For a symmetric group $\mathbb{Z}_2 = \{\mathrm{id}, -\mathrm{id}\}$ and let $E$ be a Banach space we set

$$\Sigma = \{A \subset E \setminus \{0\} : \ A \text{ is closed and } \ A = -A\}.$$

For any $A \in \Sigma$, the Krasnoselskii's genus of $A$ is defined by

$$\gamma(A) = \inf\{k : \ \exists \phi \in C(A, \mathbb{R}^k) \ \ \phi \text{ is odd and } \ \phi(z) \neq 0\}.$$

If $k$ does not exist, we set $\gamma(A) = \infty$. By above definition , it is obvious that $\gamma(\emptyset) = 0$.

Let $\Sigma_k$ denote the family of closed symmetric subsets $A$ of $E$ such that $0 \notin E$ and $\gamma(A) \geq k$, that is,

$$\Sigma_k = \{A : A \subset E \text{ is closed symmetric, } \ 0 \notin E \ \text{ and } \ \gamma(A) \geq k\}.$$

Then we have the following result, see [26, 37].

**Proposition 2.1.** *Let $A$ and $B$ be closed symmetric subsets of $E$ which do not contain the origin. Then the following statements hold:*

*(1) If there exists an odd continuous mapping from $A$ to $B$, then $\gamma(A) \leq \gamma(B)$.*
*(2) If $A \subset B$, then $\gamma(A) \leq \gamma(B)$.*
*(3) If there is an odd homeomorphism from $A$ to $B$, then $\gamma(A) = \gamma(B)$.*
*(4) If $\gamma(B) < \infty$, then $\gamma(\overline{A \setminus B}) \geq \gamma(A) - \gamma(B)$.*
*(5) If $S^n$ is a n-dimensional sphere, then $\gamma(S^n) = n + 1$ .*
*(6) If $A$ is compact, then $\gamma(A) < +\infty$ and there exists a $\delta$-closed symmetric neighborhood of $A$, i.e., $N_\delta(A) = \{u \in E : \mathrm{dist}(u, A) \leq \delta\}$ such that $N_\delta(A) \subset \Sigma_k$ and $\gamma(N_\delta(A)) = \gamma(A)$.*

Now, we state the following variant of symmetric mountain-pass lemma due to Kajikiya [26].

**Theorem 2.1.** *Let $E$ be an infinite-dimensional Banach space, and let $J \in C^1(E, \mathbb{R})$ be a functional satisfying the conditions below:*

*(1) $J(u)$ is even, bounded from below, $J(0) = 0$ and $J(u)$ satisfies the local Palais-Smale condition, i.e. for some $\bar{c} > 0$, every sequence $\{u_n\}$ in $E$ satisfying $\lim\limits_{n\to\infty} J(u_n) = c < \bar{c}$ and $\lim\limits_{n\to\infty} \|J'(u_n)\|_{E'} = 0$ has a convergent subsequence;*
*(2) For each $k \in \mathbb{N}$, there exists $A_k \in \Sigma_k$ such that $\sup\limits_{u\in A_k} J(u) < 0$.*

*Then either (i) or (ii) below holds.*

*(i) There exists a sequence $\{u_n\}$ such that $J'(u_n) = 0$, $J(u_n) < 0$ and $\{u_n\}$ converges to zero as $n \to \infty$.*
*(ii) There exist two sequences $\{u_n\}$ and $\{v_n\}$ such that $J'(u_n) = 0$, $J(u_n) = 0$, $u_n \neq 0$, $\lim\limits_{n\to\infty} u_n = 0$; $J'(v_n) = 0$, $J(v_n) < 0$, $\lim\limits_{n\to\infty} J(v_n) = 0$, and $\{v_n\}$ converges to a non-zero limit.*

## 3. Proof of Theorem 1.1

In this section, we first discuss a compactness property for the energy functional $I_{\lambda_1,\lambda_2,\beta}$, given by the Palais-Smale condition.

Let $c \in \mathbb{R}$, $\mathcal{H}$ be a Banach space and $I_{\lambda_1,\lambda_2,\beta} \in C^1(\mathcal{H}, \mathbb{R})$. $\{(u_n, v_n)\} \subset \mathcal{H}$ is a Palais-Smale sequence for $I_{\lambda_1,\lambda_2,\beta}$ in $\mathcal{H}$ at level $c$, $(PS)_c$-sequence for short, if

$$I_{\lambda_1,\lambda_2,\beta}(u_n, v_n) \to c \;\text{ and }\; I'_{\lambda_1,\lambda_2,\beta}(u_n, v_n) \to 0 \;\text{ in }\; \mathcal{H}^{-1} \;\text{ as }\; n \to \infty.$$

We say that $I_{\lambda_1,\lambda_2,\beta}$ satisfies $(PS)_c$-condition at level $c$ if for any $(PS)_c$-sequence $\{(u_n, v_n)\} \subset \mathcal{H}$ for $I_{\lambda_1,\lambda_2,\beta}$ has a convergent subsequence in $\mathcal{H}$.

In order to apply Theorem 2.1, we need the following preliminary results for $(PS)_c$-sequence of $I_{\lambda_1,\lambda_2,\beta}$.

**Lemma 3.1.** *Suppose that $1 < p := p_1 + p_2 < 2$ and $\alpha, \gamma \in (0, 2)$. Let $\{(u_n, v_n)\} \subset \mathcal{H}$ be a $(PS)_c$-sequence for $I_{\lambda_1,\lambda_2,\beta}$. Then, $\{(u_n, v_n)\}$ is bounded in $\mathcal{H}$.*

*Proof.* Let $\{(u_n, v_n)\} \subset \mathcal{H}$ be a $(PS)_c$-sequence for $I_{\lambda_1,\lambda_2,\beta}$, then

$$I_{\lambda_1,\lambda_2,\beta}(u_n, v_n) = c + o_n(1) \quad \text{and} \quad I'_{\lambda_1,\lambda_2,\beta}(u_n, v_n) = o_n(1) \;\text{ in } \mathcal{H}^{-1} \;\text{ as } n \to \infty.$$

By Young inequality and Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{G}} f(z)\frac{\psi^\gamma |u_n|^{p_1}|v_n|^{p_2}}{\mathrm{d}(z)^\gamma}dz &\le \frac{p_1}{p}\int_{\mathbb{G}} f(z)\frac{\psi^\gamma |u_n|^{p}}{\mathrm{d}(z)^\gamma}dz + \frac{p_2}{p}\int_{\mathbb{G}} f(z)\frac{\psi^\gamma |v_n|^{p}}{\mathrm{d}(z)^\gamma}dz \\ &\le \frac{p_1}{p}\left(\int_{\mathbb{G}} |f(z)|^{\frac{2^*(\gamma)}{2^*(\gamma)-p}}\frac{\psi(z)^\gamma}{\mathrm{d}(z)^\gamma}dz\right)^{\frac{2^*(\gamma)-p}{2^*(\gamma)}}\left(\int_{\mathbb{G}}\frac{\psi^\gamma|u_n|^{2^*(\gamma)}}{\mathrm{d}(z)^\gamma}dz\right)^{\frac{p}{2^*(\gamma)}} \\ &+ \frac{p_2}{p}\left(\int_{\mathbb{G}} |f(z)|^{\frac{2^*(\gamma)}{2^*(\gamma)-p}}\frac{\psi(z)^\gamma}{\mathrm{d}(z)^\gamma}dz\right)^{\frac{2^*(\gamma)-p}{2^*(\gamma)}}\left(\int_{\mathbb{G}}\frac{\psi^\gamma|v_n|^{2^*(\gamma)}}{\mathrm{d}(z)^\gamma}dz\right)^{\frac{p}{2^*(\gamma)}} \\ &\le \|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\Big(\frac{p_1}{p}S_{\gamma,\mu_1}^{-\frac{p}{2}}\|u_n\|_{\mu_1}^p + \frac{p_2}{p}S_{\gamma,\mu_2}^{-\frac{p}{2}}\|v_n\|_{\mu_2}^p\Big) \\ &\le \|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}(S_{\gamma,\mu_1}^{-\frac{p}{2}} + S_{\gamma,\mu_2}^{-\frac{p}{2}})\|(u_n,v_n)\|_{\mathcal{H}}^p.\end{aligned}$$

Then,

$$\begin{aligned}&o_n(1)+|c|+o_n(\|(u_n,v_n)\|_{\mathcal{H}}) \\ &\quad\ge I_{\lambda_1,\lambda_2,\beta}(u_n) - \frac{1}{2^*(\alpha)}\langle I'_{\lambda_1,\lambda_2,\beta}(u_n,v_n),(u_n,v_n)\rangle \\ &\quad= \Big(\frac{1}{2}-\frac{1}{2^*(\alpha)}\Big)\|(u_n,v_n)\|_{\mathcal{H}}^2 - \beta\Big(1-\frac{p}{2^*(\alpha)}\Big)\int_{\mathbb{G}} f(z)\frac{\psi^\gamma|u_n|^{p_1}|v_n|^{p_2}}{\mathrm{d}(z)^\gamma}dz \\ &\quad\ge \frac{2-\alpha}{2(Q-\alpha)}\|(u_n,v_n)\|_{\mathcal{H}}^2 - \beta\frac{2^*(\alpha)-p}{2^*(\alpha)}\|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}(S_{\gamma,\mu_1}^{-\frac{p}{2}}+S_{\gamma,\mu_2}^{-\frac{p}{2}})\|u_n,v_n\|_{\mathcal{H}}^p,\end{aligned}$$

which implies that $\{(u_n,v_n)\}$ is bounded in $\mathcal{H}$ since $p<2<2^*(\alpha)$ and $\beta>0$.

**Proposition 3.1.** *Let* $1<p<2$, $\alpha,\ \gamma\in(0,2)$ *and let* $\{(u_n,v_n)\}\subset\mathcal{H}$ *be a* $(PS)_c$*-sequence of* $I_{\lambda_1,\lambda_2,\beta}$ *with* $c<0$. *Then,*

*(i) for any* $\lambda_1,\ \lambda_2>0$, *there exists* $\beta_*>0$ *such that if* $0<\beta<\beta_*$, $I_{\lambda_1,\lambda_2,\beta}$ *satisfies* $(PS)_c$ *condition, where* $\beta_*$ *is independent on the sequence* $\{(u_n,v_n)\}$ *;*

*(ii) for any* $\beta>0$, *there exists* $\lambda_*>0$ *such that is* $0<\lambda_1<\lambda_*$, $0<\lambda_2<\lambda_*$, $I_{\lambda_1,\lambda_2,\beta}$ *satisfies* $(PS)_c$ *condition, where* $\lambda_*$ *is independent on the sequence* $\{(u_n,v_n)\}$.

*Proof.* Since the sequence $\{(u_n,v_n)\}$ is bounded in $\mathcal{H}$, thanks to Lemma 3.1, then there exists $(u_0,v_0)\in\mathcal{H}$ such that, up to a subsequence, it follows that

$$\begin{aligned}&(u_n,v_n)\rightharpoonup(u_0,v_0)\ \text{weakly in}\ \ \mathcal{H}, \\ &(u_n,v_n)\rightharpoonup(u_0,v_0)\ \text{weakly in}\ \ [L^{2^*(\alpha)}(\mathbb{G},\frac{\psi^\alpha}{\mathrm{d}(z)^\alpha}dz)]^2, \\ &(u_n,v_n)\to(u_0,v_0)\ \text{strongly in}\ \ [L^t_{loc}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)]^2,\ \ \forall t\in[1,2^*(\gamma)), \\ &(u_n(z),v_n(z))\to(u_0(z),v_0(z))\ \text{a.e. in}\ \ \mathbb{G}.\end{aligned}$$

Then, by the concentration-compactness principle [39–41] and up to a subsequence if necessary, there exist positive finite Radon measure $\hat{\mu}$, $\hat{\nu}$, $\hat{\rho}$, $\bar{\mu}$, $\bar{\nu}$, $\bar{\rho}\in\mathcal{R}(\mathbb{G}\cup\{\infty\})$; at most countable set $J$ and $\bar{J}$; real numbers $\hat{\mu}_j$, $\hat{\nu}_j$ $(j \in J)$, $\bar{\mu}_k$, $\bar{\nu}_k\,(k \in \bar{J})$, $\hat{\mu}_0$, $\hat{\nu}_0$, $\hat{\rho}_0$, $\bar{\mu}_0$, $\bar{\nu}_0$, $\bar{\rho}_0$ and different points $z_j \in \mathbb{G}\backslash\{0\}$ $(j \in J)$, $\bar{z}_k \in \mathbb{G}\backslash\{0\}$ $(k \in \bar{J})$ such that

$$|\nabla_{\mathbb{G}}u_n|^2dz \rightharpoonup \hat{\mu} \geq |\nabla_{\mathbb{G}}u_0|^2dz + \sum_{j\in J}\delta_{z_j}\hat{\mu}_j + \delta_0\hat{\mu}_0, \tag{3.1}$$

$$|\nabla_{\mathbb{G}}v_n|^2dz \rightharpoonup \bar{\mu} \geq |\nabla_{\mathbb{G}}v_0|^2dz + \sum_{k\in \bar{J}}\delta_{\bar{z}_k}\bar{\mu}_k + \delta_0\bar{\mu}_0, \tag{3.2}$$

$$\frac{\psi^\alpha|u_n|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz \rightharpoonup \hat{\nu} = \frac{\psi^\alpha|u_0|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz + \sum_{j\in J}\delta_{z_j}\hat{\nu}_j + \delta_0\hat{\nu}_0, \tag{3.3}$$

$$\frac{\psi^\alpha|v_n|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz \rightharpoonup \bar{\nu} = \frac{\psi^\alpha|v_0|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz + \sum_{k\in \bar{J}}\delta_{\bar{z}_k}\bar{\nu}_k + \delta_0\bar{\nu}_0, \tag{3.4}$$

$$\frac{\psi^2|u_n|^2}{\mathrm{d}(z)^2}dz \rightharpoonup \hat{\rho} = \frac{\psi^2|u_0|^2}{\mathrm{d}(z)^2}dz + \delta_0\hat{\rho}_0, \tag{3.5}$$

$$\frac{\psi^2|v_n|^2}{\mathrm{d}(z)^2}dz \rightharpoonup \bar{\rho} = \frac{\psi^2|v_0|^2}{\mathrm{d}(z)^2}dz + \delta_0\bar{\rho}_0, \tag{3.6}$$

where $\delta_z$ is the Dirac mass at $z$. Moreover, by the Sobolev-Hardy and the Hardy inequalities, we get

$$\hat{\mu}_j \geq S(\alpha, \mathbb{G}) \cdot \hat{\nu}_j^{\frac{2}{2^*(\alpha)}} \quad \text{for all } j \in J \cup \{0\}, \;\; \text{and } \hat{\mu}_0 \geq \mu_{\mathbb{G}} \cdot \hat{\rho}_0, \tag{3.7}$$

$$\bar{\mu}_k \geq S(\alpha, \mathbb{G}) \cdot \bar{\nu}_k^{\frac{2}{2^*(\alpha)}} \quad \text{for all } k \in \bar{J} \cup \{0\}, \;\; \text{and } \bar{\mu}_0 \geq \mu_{\mathbb{G}} \cdot \bar{\rho}_0, \tag{3.8}$$

where $S(\alpha, \mathbb{G})$ is the best Hardy-Sobolev constant, i.e.,

$$S(\alpha, \mathbb{G}) = \inf_{u \in S_0^1(\mathbb{G})\backslash\{0\}} \frac{\int_{\mathbb{G}} |\nabla_{\mathbb{G}}u|^2dz}{\left(\int_{\mathbb{G}} \frac{\psi^\alpha|u|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz\right)^{\frac{2}{2^*(\alpha)}}}.$$

In order to study the concentration at infinity of $\{u_n\}$ and $\{v_n\}$, we use a method of concentration-compactness principle at infinity, which was first established by Chabrowski [42]. We set

$$\mu_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{G}\cap\{\mathrm{d}(z)>R\}} |\nabla_{\mathbb{G}}u_n|^2dz, \tag{3.9}$$

$$\nu_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{G}\cap\{\mathrm{d}(z)>R\}} \frac{\psi^\alpha|u_n|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz, \tag{3.10}$$

$$\rho_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{G}\cap\{\mathrm{d}(z)>R\}} \frac{\psi^2|u_n|^2}{\mathrm{d}(z)^2}dz, \tag{3.11}$$

and

$$\bar{\mu}_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{G}\cap\{\mathrm{d}(z)>R\}} |\nabla_{\mathbb{G}}v_n|^2dz,$$

$$\bar{\nu}_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{G}\cap\{\mathrm{d}(z)>R\}} \frac{\psi^\alpha|v_n|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz,$$

$$\bar{\rho}_\infty := \lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{G}\cap\{\mathrm{d}(z)>R\}}\frac{\psi^2|v_n|^2}{\mathrm{d}(z)^2}dz.$$

For the sequence $\{u_n\}$, let $\phi_j(z)\in C_0^\infty(\mathbb{G},[0,1])$ be a cut-off function centered at $z_j\in\mathbb{G}\backslash\{0\}$ with $\phi_j=1$ on $B_d(z_j,1)$, $\phi_j=0$ on $\mathbb{G}\backslash B_d(z_j,2)$. Let $\phi_{j,\varepsilon}(z)=\phi_j(\delta_{\frac{1}{\varepsilon}}(z))$. Then $|\nabla_{\mathbb{G}}\phi_{j,\varepsilon}|\le\frac{C}{\varepsilon}$ and $\{\phi_{j,\varepsilon}\,u_n\}$ is bounded in $S_0^1(\mathbb{G})$. Testing $I'_{\lambda_1,\lambda_2,\beta}(u_n,v_n)$ with $(\phi_{j,\varepsilon}u_n,0)$, we obtain $\lim_{n\to\infty}\langle I'_{\lambda_1,\lambda_2,\beta}(u_n,v_n),(\phi_{j,\varepsilon}u_n,0)\rangle=0$, that is,

$$\begin{aligned}\lim_{n\to\infty}\Big(\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u_n|^2\phi_{j,\varepsilon}dz-\mu_1\int_{\mathbb{G}}\frac{\psi^2|u_n|^2\phi_{j,\varepsilon}}{\mathrm{d}(z)^2}dz-\lambda_1\int_{\mathbb{G}}\frac{\psi^\alpha|u_n|^{2^*(\alpha)}\phi_{j,\varepsilon}}{\mathrm{d}(z)^\alpha}dz\\ -\beta p_1\int_{\mathbb{G}}f(z)\frac{\psi^\gamma|u_n|^{p_1}\phi_{j,\varepsilon}|v_n|^{p_2}}{\mathrm{d}(z)^\gamma}dz\Big)=\lim_{n\to\infty}\int_{\mathbb{G}}u_n\nabla_{\mathbb{G}}u_n\nabla_{\mathbb{G}}\phi_{j,\varepsilon}dz.\end{aligned}\tag{3.12}$$

Now, we estimate each term in (3.12). From (3.1)–(3.6), we get

$$\lim_{n\to\infty}\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u_n|^2\phi_{j,\varepsilon}dz=\int_{\mathbb{G}}\phi_{j,\varepsilon}d\hat{\mu}\ge\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u_0|^2\phi_{j,\varepsilon}dz+\hat{\mu}_j,\tag{3.13}$$

$$\lim_{n\to\infty}\int_{\mathbb{G}}\frac{\psi^\alpha|u_n|^{2^*(\alpha)}\phi_{j,\varepsilon}}{\mathrm{d}(z)^\alpha}dz=\int_{\mathbb{G}}\phi_{j,\varepsilon}d\hat{\nu}=\int_{\mathbb{G}}\frac{\psi^\alpha|u_0|^{2^*(\alpha)}\phi_{j,\varepsilon}}{\mathrm{d}(z)^\alpha}dz+\hat{\nu}_j,\tag{3.14}$$

$$\lim_{\varepsilon\to0}\lim_{n\to\infty}\Big|\int_{\mathbb{G}}\frac{\psi^2|u_n|^2\phi_{j,\varepsilon}}{\mathrm{d}(z)^2}dz\Big|\le\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{B_{\mathrm{d}}(z_j,2\varepsilon)}\frac{\psi^2|u_n|^2}{\mathrm{d}(z)^2}dz=0,\tag{3.15}$$

and

$$\begin{aligned}&\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{\mathbb{G}}f(z)\frac{\psi^\gamma|u_n|^{p_1}\phi_{j,\varepsilon}|v_n|^{p_2}}{\mathrm{d}(z)^\gamma}dz\\ &\le\lim_{\varepsilon\to0}\lim_{n\to\infty}\int_{B_{\mathrm{d}}(z_j,2\varepsilon)}f(z)\frac{\psi^\gamma|u_n|^{p_1}\phi_{j,\varepsilon}|v_n|^{p_2}}{\mathrm{d}(z)^\gamma}dz\\ &\le\lim_{\varepsilon\to0}\lim_{n\to\infty}\|f\|_{L^{p*}(B_{\mathrm{d}}(z_j,2\varepsilon),\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\Bigg[\Big(\int_{B_{\mathrm{d}}(z_j,2\varepsilon)}\frac{\psi^\gamma|u_n|^{2^*(\gamma)}}{\mathrm{d}(z)^\gamma}dz\Big)^{\frac{p}{2^*(\gamma)}}\\ &\qquad+\Big(\int_{B_{\mathrm{d}}(z_j,2\varepsilon)}\frac{\psi^\gamma|v_n|^{2^*(\gamma)}}{\mathrm{d}(z)^\gamma}dz\Big)^{\frac{p}{2^*(\gamma)}}\Bigg]=0.\end{aligned}\tag{3.16}$$

From Hölder inequality, it follows that

$$\begin{aligned}0&\le\lim_{\varepsilon\to0}\lim_{n\to\infty}\Big|\int_{\mathbb{G}}u_n\nabla_{\mathbb{G}}u_n\nabla_{\mathbb{G}}\phi_{j,\varepsilon}dz\Big|\\ &\le\lim_{\varepsilon\to0}\lim_{n\to\infty}\Big(\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u_n|^2dz\Big)^{\frac12}\Big(\int_{\mathbb{G}}|\nabla_{\mathbb{G}}\phi_{j,\varepsilon}|^2|u_n|^2dz\Big)^{\frac12}\\ &\le C\lim_{\varepsilon\to0}\Big(\int_{\mathbb{G}}|\nabla_{\mathbb{G}}\phi_{j,\varepsilon}|^2|u_0|^2dz\Big)^{\frac12}\\ &\le C\lim_{\varepsilon\to0}\Big(\int_{B_{\mathrm{d}}(z_j,2\varepsilon)}|\nabla_{\mathbb{G}}\phi_{j,\varepsilon}|^Qdz\Big)^{\frac1Q}\Big(\int_{B_{\mathrm{d}}(z_j,2\varepsilon)}|u_0|^{2^*}dz\Big)^{\frac{1}{2^*}}\\ &\le C\lim_{\varepsilon\to0}\Big(\int_{B_{\mathrm{d}}(z_j,2\varepsilon)}|u_0|^{2^*}dz\Big)^{\frac{1}{2^*}}=0.\end{aligned}\tag{3.17}$$

Consequently, from the above arguments (3.13)–(3.17), we get

$$0 = \lim_{\varepsilon\to 0}\langle I'_{\lambda_1,\lambda_2,\beta}(u_n, v_n), (\phi_\varepsilon u_n, 0)\rangle \geq \hat{\mu}_j - \lambda_1\hat{\nu}_j, \quad \forall j \in J.$$

Combining with (3.7), we have

$$\text{either } (1)\ \hat{\nu}_j = 0, \text{ or } (2)\ \hat{\nu}_j \geq \Big(\frac{S(\alpha,\mathbb{G})}{\lambda_1}\Big)^{\frac{Q-\alpha}{2-\alpha}},$$

which implies that the set $J$ is finite.

Similarly, for $\bar{\nu}_k$ and $\bar{J}$, the following conclusion holds:

$$\bar{J} \text{ is finite, and either } (1)'\ \bar{\nu}_k = 0, \text{ or } (2)'\ \bar{\nu}_k \geq \Big(\frac{S(\alpha,\mathbb{G})}{\lambda_2}\Big)^{\frac{Q-\alpha}{2-\alpha}} \text{ for } k \in \bar{J}.$$

On the other hand, choosing a suitable cutoff function centered at the origin, by the analogous argument we can prove that

$$\hat{\mu}_0 - \mu_1\hat{\rho}_0 \leq \lambda_1\hat{\nu}_0 \text{ and } \bar{\mu}_0 - \mu_1\bar{\rho}_0 \leq \lambda_1\bar{\nu}_0. \tag{3.18}$$

It follows from the definition of $S_{\alpha,\mu_1}$ and $S_{\alpha,\mu_2}$ that

$$\hat{\mu}_0 - \mu_1\hat{\rho}_0 \geq S_{\alpha,\mu_1}\cdot \hat{\nu}_0^{\frac{2}{2^*(\alpha)}} \tag{3.19}$$

$$\bar{\mu}_0 - \mu_2\bar{\rho}_0 \geq S_{\alpha,\mu_2}\cdot \bar{\nu}_0^{\frac{2}{2^*(\alpha)}}. \tag{3.20}$$

Thus, by combining (3.18) and (3.19), (3.20) we get

$$\text{either } (3)\ \hat{\nu}_0 = 0, \text{ or } (4)\ \hat{\nu}_0 \geq \Big(\frac{S_{\alpha,\mu_1}}{\lambda_1}\Big)^{\frac{Q-\alpha}{2-\alpha}} \tag{3.21}$$

and

$$\text{either } (3)'\ \bar{\nu}_0 = 0, \text{ or } (4)'\ \bar{\nu}_0 \geq \Big(\frac{S_{\alpha,\mu_2}}{\lambda_2}\Big)^{\frac{Q-\alpha}{2-\alpha}}. \tag{3.22}$$

Furthermore, the Hardy inequality (1.2) implies that

$$0 \leq \mu_{\mathbb{G}}\hat{\rho}_0 \leq \hat{\mu}_0, \qquad 0 \leq \Big(1 - \frac{\mu_1}{\mu_{\mathbb{G}}}\Big)\hat{\mu}_0 \leq \hat{\mu}_0 - \mu_1\hat{\rho}_0, \tag{3.23}$$

and

$$0 \leq \mu_{\mathbb{G}}\bar{\rho}_0 \leq \bar{\mu}_0, \qquad 0 \leq \Big(1 - \frac{\mu_2}{\mu_{\mathbb{G}}}\Big)\bar{\mu}_0 \leq \bar{\mu}_0 - \mu_2\bar{\rho}_0. \tag{3.24}$$

If $\hat{\nu}_0 = 0$, from (3.18) and (3.23), it follows that $\hat{\mu}_0 = \hat{\rho}_0 = 0$. Similarly, if $\bar{\nu}_0 = 0$, by (3.18) and (3.24), we conclude $\bar{\mu}_0 = \bar{\rho}_0 = 0$.

To analyze the concentration at infinity, for $R > 0$, we choose the function $\phi \in C_1^\infty(\mathbb{G})$ such that $0 \leq \phi \leq 1$, $\phi(z) = 0$ on $B_d(0,1)$, $\phi(z) = 1$ on $\mathbb{G}\backslash B_d(0,2)$ and $|\nabla_{\mathbb{G}}\phi| \leq \frac{c}{R}$. Set $\phi_R(z) = \phi(\delta_{\frac{1}{R}}(z))$, then $\{\phi_R u_n\} \subset S_0^1(\mathbb{G})$ is bounded. Testing $I'_{\lambda_1,\lambda_2,\beta}(u_n,v_n)$ with $(\phi_R u_n, 0)$ we obtain $\lim\limits_{n\to\infty}\langle I'_{\lambda_1,\lambda_2,\beta}(u_n,v_n), (\phi_R u_n,0)\rangle = 0$, i.e.,

$$\begin{aligned}-\lim_{n\to\infty}\int_{\mathbb{G}}\langle\nabla_{\mathbb{G}}u_n,\nabla_{\mathbb{G}}\phi_R\rangle u_n dz = \lim_{n\to\infty}\Bigg[\int_{\mathbb{G}}\Big(|\nabla_{\mathbb{G}}u_n|^2\phi_R - \mu_1\frac{\psi^2|u_n|^2}{\mathrm{d}(z)^2}\phi_R\Big)dz \\ -\lambda_1\int_{\mathbb{G}}\frac{\psi^\alpha|u_n|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}\phi_R dz - \beta p_1\int_{\mathbb{G}} f(z)\frac{\psi^\gamma|u_n|^{p_1}|v_n|^{p_2}}{\mathrm{d}(z)^\gamma}\phi_R dz\Bigg].\end{aligned} \tag{3.25}$$

Since

$$S_{\alpha,\mu_1}\Big(\int_{\mathbb{G}}\frac{\psi^\alpha|u_n\phi_R|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz\Big)^{\frac{2}{2^*(\alpha)}} \leq \int_{\mathbb{G}}\Big(|\nabla_{\mathbb{G}}(u_n\phi_R)|^2 - \mu_1\frac{\psi^2|u_n\phi_R|^2}{\mathrm{d}(z)^2}\Big)dz,$$

we conclude that

$$\begin{aligned}&\mu_1\int_{\mathbb{G}}\frac{\psi^2|u_n\phi_R|^2}{\mathrm{d}(z)^2}dz + S_{\alpha,\mu_1}\Big(\int_{\mathbb{G}}\frac{\psi^\alpha|u_n\phi_R|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz\Big)^{\frac{2}{2^*(\alpha)}} \\ &\leq \int_{\mathbb{G}}|\nabla_{\mathbb{G}}(u_n\phi_R)|^2 dz \\ &\leq \int_{\mathbb{G}}|\nabla_{\mathbb{G}}u_n|^2|\phi_R|^2 dz + \int_{\mathbb{G}}|\nabla_{\mathbb{G}}\phi_R|^2|u_n|^2 dz + 2\int_{\mathbb{G}}|\nabla_{\mathbb{G}}u_n\phi_R u_n\nabla_{\mathbb{G}}\phi_R|dz.\end{aligned} \tag{3.26}$$

By Hölder inequality, it is easy to get that

$$\begin{aligned}&\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{G}}|\phi_R\nabla_{\mathbb{G}}u_n||u_n\nabla_{\mathbb{G}}\phi_R|dz \\ &\leq \lim_{R\to\infty}\limsup_{n\to\infty}\Big(\int_{B_{\mathrm{d}}(0,2R)\backslash B_{\mathrm{d}}(0,R)}|\nabla_{\mathbb{G}}u_n|^2 dz\Big)^{\frac{1}{2}}\Big(\int_{B_{\mathrm{d}}(0,2R)\backslash B_{\mathrm{d}}(0,R)}|u_n\nabla_{\mathbb{G}}\phi_R|^2 dz\Big)^{\frac{1}{2}} \\ &\leq C\lim_{R\to\infty}\Big(\int_{B_{\mathrm{d}}(0,2R)\backslash B_{\mathrm{d}}(0,R)}|\nabla_{\mathbb{G}}\phi_R|^2|u_0|^2 dz\Big)^{\frac{1}{2}} \\ &\leq C\lim_{R\to\infty}\Big(\int_{B_{\mathrm{d}}(0,2R)\backslash B_d(0,R)}|\nabla_{\mathbb{G}}\phi_\varepsilon|^Q dz\Big)^{\frac{1}{Q}}\Big(\int_{B_{\mathrm{d}}(0,2)\backslash B_{\mathrm{d}}(0,R)}|u_0|^{2^*}dz\Big)^{\frac{1}{2^*}} \\ &\leq C\lim_{R\to\infty}\Big(\int_{B_{\mathrm{d}}(0,2R)\backslash B_d(0,R)}|u_0|^{2^*}dz\Big)^{\frac{1}{2^*}} = 0.\end{aligned} \tag{3.27}$$

Similarly,

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\mathbb{G}}|\nabla_{\mathbb{G}}\phi_R|^2|u_n|^2 dz = 0. \tag{3.28}$$

Thus, we see from(3.27), (3.28) and (3.26), we have

$$\mu_\infty - \mu_1\rho_\infty \geq S_{\alpha,\mu_1}\cdot\nu_\infty^{\frac{2}{2^*(\alpha)}}. \tag{3.29}$$

On the other hand, from Hölder inequality and the definition of $\phi_R$ we have

$$\Big|\int_{\mathbb{G}} f(z)\frac{\psi^\gamma|u_n|^{p_1}|v_n|^{p_2}}{\mathrm{d}(z)^\gamma}\phi_R dz\Big|$$

$$\leq \left|\int_{\mathbb{G}\backslash B_{\mathrm{d}}(0,R)} f(z)\frac{\psi^{\gamma}|u_n|^p}{\mathrm{d}(z)^{\gamma}}\phi_R dz\right| + \left|\int_{\mathbb{G}\backslash B_{\mathrm{d}}(0,R)} f(z)\frac{\psi^{\gamma}|v_n|^p}{\mathrm{d}(z)^{\gamma}}\phi_R dz\right|$$

$$\leq \left(\int_{\mathbb{G}\backslash B_{\mathrm{d}}(0,R)} \frac{\psi^{\gamma}|f(z)|^{\frac{2^*(\gamma)}{2^*(\gamma)-p}}}{\mathrm{d}(z)^{\gamma}}dz\right)^{\frac{2^*(\gamma)-p}{2^*(\gamma)}} \left[\left(\int_{\mathbb{G}\backslash B_{\mathrm{d}}(0,R)} \frac{\psi^{\gamma}|u_n|^{2^*(\gamma)}}{\mathrm{d}(z)^{\gamma}}\phi_R dz\right)^{\frac{p}{2^*(\gamma)}} + \left(\int_{\mathbb{G}\backslash B_{\mathrm{d}}(0,R)} \frac{\psi^{\gamma}|v_n|^{2^*(\gamma)}}{\mathrm{d}(z)^{\gamma}}\phi_R dz\right)^{\frac{p}{2^*(\gamma)}}\right]$$

$$\leq \left(\int_{\mathbb{G}\backslash B_{\mathrm{d}}(0,R)} \frac{\psi^{\gamma}}{\mathrm{d}(z)^{\gamma}}|f(z)|^{\frac{2^*(\gamma)}{2^*(\gamma)-p}}dz\right)^{\frac{2^*(\gamma)-p}{2^*(\gamma)}} \left[S_{\gamma,\mu_1}^{-\frac{p}{2}}\|u_n\|_{\mu_2}^p + S_{\gamma,\mu_1}^{-\frac{p}{2}}\|u_n\|_{\mu_2}^p\right].$$

Since $f \in L^{p_*}(\mathbb{G}, \frac{\psi^{\gamma}}{d(z)^{\gamma}}dz)$, it follows that

$$\lim_{R\to\infty}\limsup_{n\to\infty}\left|\int_{\mathbb{G}} f(z)\frac{\psi^{\gamma}|u_n|^{p_1}|v_n|^{p_2}}{\mathrm{d}(z)^{\gamma}}\phi_R dz\right| \leq \lim_{R\to\infty} C\left(\int_{\mathbb{G}\backslash B_{\mathrm{d}}(0,R)} \frac{\psi^{\gamma}|f(z)|^{\frac{2^*(\gamma)}{2^*(\gamma)-p}}}{\mathrm{d}(z)^{\gamma}}dz\right)^{\frac{2^*(\gamma)-p}{2^*(\gamma)}} = 0.$$

Thus, taking limits by letting $n \to \infty$ in (3.25), we have

$$\mu_\infty - \mu_1\rho_\infty \leq \lambda_1\nu_\infty. \tag{3.30}$$

Hence, it follows from (3.29) and (3.30) that

$$\text{either } (5)\ \nu_\infty = 0, \text{ or } (6)\ \nu_\infty \geq \Big(\frac{S_{\alpha,\mu_1}}{\lambda_1}\Big)^{\frac{Q-\alpha}{2-\alpha}}.$$

In contrast, the Hardy inequality implies that

$$0 \leq \mu_{\mathbb{G}}\rho_\infty \leq \mu_\infty, \qquad 0 \leq \Big(1 - \frac{\mu_1}{\mu_{\mathbb{G}}}\Big)\mu_\infty \leq \mu_\infty - \mu_1\rho_\infty. \tag{3.31}$$

If $\nu_\infty = 0$, by combining (3.30) and (3.31), we get $\mu_\infty = \rho_\infty = 0$.

From above argument the same conclusion holds for $\bar{\nu}_\infty$, namely,

$$\bar{\mu}_\infty - \mu_2\bar{\rho}_\infty \geq S_{\alpha,\mu_2} \cdot \bar{\nu}_\infty^{\frac{2}{2^*(\alpha)}},$$

$$\bar{\mu}_\infty - \mu_1\bar{\rho}_\infty \leq \lambda_2\bar{\nu}_\infty,$$

and

$$\text{either } (5)'\ \bar{\nu}_\infty = 0, \text{ or } (6)'\ \bar{\nu}_\infty \geq \Big(\frac{S_{\alpha,\mu_2}}{\lambda_2}\Big)^{\frac{Q-\alpha}{2-\alpha}}.$$

If $\bar{\nu}_\infty = 0$, we have that $\bar{\mu}_\infty = \bar{\rho}_\infty = 0$.

Now we claim that (2), (2)′, (4), (4)′ and (6), (6)′ cannot occur if $\lambda_1$, $\lambda_2$ and $\beta$ are chosen properly. In fact, applying $(f)$ and Hölder inequality, we have

$$\begin{aligned}
0 > c &= \lim_{n\to\infty}(I_{\lambda_1,\lambda_2,\beta}(u_n,v_n) - \frac{1}{2^*(\alpha)}\langle I'_{\lambda_1,\lambda_2,\beta}(u_n,v_n),(u_n,v_n)\rangle) \\
&= \lim_{n\to\infty}\Big(\Big(\frac{1}{2}-\frac{1}{2^*(\alpha)}\Big)\|(u_n,v_n)\|^2_{\mathcal{H}} - \beta\Big(1-\frac{p}{2^*(\alpha)}\Big)\int_{\mathbb{G}} f(z)\frac{\psi^\gamma|u_n|^{p_1}|v_n|^{p_2}}{\mathrm{d}(z)^\gamma}dz\Big) \\
&\geq \frac{2^*(\alpha)-2}{2\cdot 2^*(\alpha)}\|(u_0,v_0)\|^2_{\mathcal{H}} \\
&\quad - \frac{\beta(2^*(\alpha)-p)}{2^*(\alpha)}\|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\Big(\|u_0\|^p_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} + \|v_0\|^p_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\Big) \\
&\geq \frac{2^*(\alpha)-2}{2\cdot 2^*(\alpha)}\Big(S_{\gamma,\mu_1}\|u_0\|^2_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} + S_{\gamma,\mu_2}\|v_0\|^2_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\Big) \\
&\quad - \frac{\beta(2^*(\alpha)-p)}{2^*(\alpha)}\|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\Big(\|u_0\|^p_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} + \|v_0\|^p_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\Big).
\end{aligned} \tag{3.32}$$

Since

$$\|u_0\|^p_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} + \|v_0\|^p_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} \leq 2\left(\|u_0\|_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} + \|v_0\|_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\right)^p,$$

$$\|u_0\|^2_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} + \|v_0\|^2_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} \geq \frac{1}{2}\left(\|u_0\|_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} + \|v_0\|_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\right)^2,$$

which and (3.32) yield that

$$\begin{aligned}
&\frac{2\beta(2^*(\alpha)-p)}{2^*(\alpha)}\|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\Big(\|u_0\|_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} + \|v_0\|_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\Big)^p \\
&\quad \geq \frac{2^*(\alpha)-2}{4\cdot 2^*(\alpha)}\min\{S_{\gamma,\mu_1},S_{\gamma,\mu_2}\}\Big(\|u_0\|_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} + \|v_0\|_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\Big)^2,
\end{aligned}$$

namely,

$$\|u_0\|_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} + \|v_0\|_{L^{2^*(\gamma)}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)} \leq \left(\frac{8(2^*(\alpha)-p)\|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}}{(2^*(\alpha)-2)\min\{S_{\gamma,\mu_2},S_{\gamma,\mu_2}\}}\right)^{\frac{1}{2-p}}\beta^{\frac{1}{2-p}}. \tag{3.33}$$

If (6) or (6)′ occurs, we obtain by (3.32) and (3.33) that

$$\begin{aligned}
0 > c &= \lim_{n\to\infty}\Big(I_{\lambda_1,\lambda_2,\beta}(u_n,v_n) - \frac{1}{2^*(\alpha)}\langle I'_{\lambda_1,\lambda_2,\beta}(u_n,v_n),(u_n,v_n)\rangle\Big) \\
&\geq \frac{2^*(\alpha)-2}{2\cdot 2^*(\alpha)}\Big(\mu_\infty - \mu_1\rho_\infty + \bar{\mu}_\infty - \mu_2\bar{\rho}_\infty\Big) \\
&\quad - \frac{2}{2^*(\alpha)}\left(\frac{8}{(2^*(\alpha)-2)\min\{S_{\gamma,\mu_1},S_{\gamma,\mu_2}\}}\right)^{\frac{p}{2-p}}\left((2^*(\alpha)-p)\|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\right)^{\frac{2}{2-p}}\cdot\beta^{\frac{2}{2-p}} \\
&\geq \frac{2^*(\alpha)-2}{2\cdot 2^*(\alpha)}\Big(S_{\alpha,\mu_1}\nu_\infty^{\frac{2}{2^*(\alpha)}} + S_{\alpha,\mu_2}\bar{\nu}_\infty^{\frac{2}{2^*(\alpha)}}\Big) \\
&\quad - \frac{2}{2^*(\alpha)}\left(\frac{8}{(2^*(\alpha)-2)\min\{S_{\gamma,\mu_1},S_{\gamma,\mu_2}\}}\right)^{\frac{p}{2-p}}\left((2^*(\alpha)-p)\|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\right)^{\frac{2}{2-p}}\cdot\beta^{\frac{2}{2-p}}
\end{aligned}$$

$$\geq \frac{2^*(\alpha)-2}{2\cdot 2^*(\alpha)}\left(S_{\alpha,\mu_1}\Big[\Big(\frac{S_{\alpha,\mu_1}}{\lambda_1}\Big)^{\frac{Q-\alpha}{2-\alpha}}\Big]^{\frac{2}{2^*(\alpha)}}+S_{\alpha,\mu_2}\Big[\Big(\frac{S_{\alpha,\mu_2}}{\lambda_2}\Big)^{\frac{Q-\alpha}{2-\alpha}}\Big]^{\frac{2}{2^*(\alpha)}}\right)$$
$$-\frac{2}{2^*(\alpha)}\left(\frac{8}{(2^*(\alpha)-2)\min\{S_{\gamma,\mu_1},S_{\gamma,\mu_2}\}}\right)^{\frac{p}{2-p}}\left((2^*(\alpha)-p)\|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\right)^{\frac{2}{2-p}}\cdot\beta^{\frac{2}{2-p}}$$
$$=\frac{2^*(\alpha)-2}{2\cdot 2^*(\alpha)}\left((S_{\alpha,\mu_1})^{\frac{Q-\alpha}{2-\alpha}}\lambda_1^{-\frac{Q-2}{2-\alpha}}+(S_{\alpha,\mu_2})^{\frac{Q-\alpha}{2-\alpha}}\lambda_2^{-\frac{Q-2}{2-\alpha}}\right)$$
$$-\frac{2}{2^*(\alpha)}\left(\frac{8}{(2^*(\alpha)-2)\min\{S_{\gamma,\mu_1},S_{\gamma,\mu_2}\}}\right)^{\frac{p}{2-p}}\left((2^*(\alpha)-p)\|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\right)^{\frac{2}{2-p}}\cdot\beta^{\frac{2}{2-p}},$$

that is,

$$\begin{aligned}0>&\frac{2^*(\alpha)-2}{2\cdot 2^*(\alpha)}\left((S_{\alpha,\mu_1})^{\frac{Q-\alpha}{2-\alpha}}\lambda_1^{-\frac{Q-2}{2-\alpha}}+(S_{\alpha,\mu_2})^{\frac{Q-\alpha}{2-\alpha}}\lambda_2^{-\frac{Q-2}{2-\alpha}}\right)\\&-\frac{2}{2^*(\alpha)}\left(\frac{8}{(2^*(\alpha)-2)\min\{S_{\gamma,\mu_1},S_{\gamma,\mu_2}\}}\right)^{\frac{p}{2-p}}\left((2^*(\alpha)-p)\|f\|_{L^{p_*}p_*(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\right)^{\frac{2}{2-p}}\cdot\beta^{\frac{2}{2-p}}.\end{aligned}\tag{3.34}$$

From the above inequality, we can find that if $\beta>0$ is given, there exists $\lambda_*>0$ small enough such that for $\lambda_1,\ \lambda_2\in(0,\lambda_*)$, the right-hand side of (3.34) is greater than 0, which is a contradiction. Similarly, if $\lambda_1,\ \lambda_2>0$ is given, we can take $\beta_*>0$ so small that for $\beta\in(0,\beta_*)$, right-hand side of (3.34) is greater than 0.

Similarly we can prove that (2) , (2)′ and (4), (4)′ cannot occur. So

$$\lim_{n\to\infty}\int_{\mathbb{G}}\frac{\psi^\alpha|u_n|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz=\int_{\mathbb{G}}\frac{\psi^\alpha|u_0|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz$$

and

$$\lim_{n\to\infty}\int_{\mathbb{G}}\frac{\psi^\alpha|v_n|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz=\int_{\mathbb{G}}\frac{\psi^\alpha|v_0|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz.$$

In view of $(u_n,v_n)\rightharpoonup(u_0,v_0)$ weakly in $\mathcal{H}$ and the Brezis-Lieb lemma [38], we have

$$\lim_{n\to\infty}\int_{\mathbb{G}}\frac{\psi^\alpha|u_n-u_0|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz=0,\qquad \lim_{n\to\infty}\int_{\mathbb{G}}\frac{\psi^\alpha|v_n-v_0|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz=0.$$

We are now going to prove that $(u_n,v_n)\to(u_0,v_0)$ strongly in $\mathcal{H}$. First, we have

$$\begin{aligned}\|(u_n-u_0,v_n-v_0)\|_{\mathcal{H}}^2=&\langle(I'_{\lambda_1,\lambda_2,\beta}(u_n,v_n)-I'_{\lambda_1,\lambda_2,\beta}(u_0,v_0)),(u_n-u_0,v_n-v_0)\rangle\\&+\lambda_1\int_{\mathbb{G}}\frac{\psi^\alpha(|u_n|^{2^*(\alpha)-2}u_n-|u_0|^{2^*(\alpha)-2}u_0)(u_n-u_0)}{\mathrm{d}(z)^\alpha}dz\\&+\lambda_2\int_{\mathbb{G}}\frac{\psi^\alpha(|v_n|^{2^*(\alpha)-2}v_n-|v_0|^{2^*(\alpha)-2}v_0)(v_n-v_0)}{\mathrm{d}(z)^\alpha}dz\\&+\beta p_1\int_{\mathbb{G}}f(z)\frac{\psi^\gamma[|u_n|^{p_1-2}u_n|v_n|^{p_2}-|u_0|^{p_1-2}u_0|v_0|^{p_2}](u_n-u_0)}{\mathrm{d}(z)^\gamma}dz\\&+\beta p_2\int_{\mathbb{G}}f(z)\frac{\psi^\gamma[|u_n|^{p_1}|v_n|^{p_2-2}v_n-|u_0|^{p_1}|v_0|^{p_2-2}v_0](v_n-v_0)}{\mathrm{d}(z)^\gamma}dz.\end{aligned}\tag{3.35}$$

For the first term in (3.35), by using Hölder inequality, we get that

$$\begin{aligned}
&\Big|\int_{\mathbb{G}}\frac{\psi^{\alpha}(|u_n|^{2^*(\alpha)-2}u_n-|u_0|^{2^*(\alpha)-2}u_0)(u_n-u_0)}{\mathrm{d}(z)^{\alpha}}dz\Big|\\
&\le\int_{\mathbb{G}}\frac{\psi^{\alpha}|u_n|^{2^*(\alpha)-1}|u_n-u_0|}{\mathrm{d}(z)^{\alpha}}dz+\int_{\mathbb{G}}\frac{\psi^{\alpha}|u_0|^{2^*(\alpha)-1}|u_n-u_0|}{\mathrm{d}(z)^{\alpha}}dz\\
&\le\Big(\int_{\mathbb{G}}\frac{\psi^{\alpha}|u_n|^{2^*(\alpha)}}{\mathrm{d}(z)^{\alpha}}dz\Big)^{\frac{2^*(\alpha)-1}{2^*(\alpha)}}\Big(\int_{\mathbb{G}}\frac{\psi^{\alpha}|u_n-u_0|^{2^*(\alpha)}}{\mathrm{d}(z)^{\alpha}}dz\Big)^{\frac{1}{2^*(\alpha)}}\\
&\quad+\Big(\int_{\mathbb{G}}\frac{\psi^{\alpha}|u_0|^{2^*(\alpha)}}{\mathrm{d}(z)^{\alpha}}dz\Big)^{\frac{2^*(\alpha)-1}{2^*(\alpha)}}\Big(\int_{\mathbb{G}}\frac{\psi^{\alpha}|u_n-u_0|^{2^*(\alpha)}}{\mathrm{d}(z)^{\alpha}}dz\Big)^{\frac{1}{2^*(\alpha)}}\\
&\to 0\ \text{ as } n\to\infty.
\end{aligned}\tag{3.36}$$

Similarly,

$$\Big|\int_{\mathbb{G}}\frac{\psi^{\alpha}(|v_n|^{2^*(\alpha)-2}v_n-|v_0|^{2^*(\alpha)-2}v_0)(v_n-v_0)}{\mathrm{d}(z)^{\alpha}}dz\Big|\to 0 \text{ as } n\to\infty.\tag{3.37}$$

On the other hand, using the Hölder inequality and $(u_n, v_n) \rightharpoonup (u_0, v_0)$ weakly in $\mathcal{H}$ , we get that

$$\begin{aligned}
&\Big|\int_{\mathbb{G}}f(z)\frac{\psi^{\gamma}[|u_n|^{p_1-2}u_n|v_n|^{p_2}-|u_0|^{p_1-2}u_0|v_0|^{p_2}](u_n-u_0)}{\mathrm{d}(z)^{\gamma}}dz\Big|\\
&\le\int_{\mathbb{G}}\frac{\psi^{\gamma}|f(z)||u_n|^{p-1}|u_n-u_0|}{\mathrm{d}(z)^{\gamma}}dz+\int_{\mathbb{G}}|f(z)|\frac{\psi^{\gamma}|u_0|^{p-1}|u_n-u_0|}{\mathrm{d}(z)^{\gamma}}dz\\
&\le\Big(\int_{\mathbb{G}}\frac{\psi^{\gamma}|f(z)|^{\frac{2^*(\gamma)}{2^*(\gamma)-p}}}{\mathrm{d}(z)^{\gamma}}dz\Big)^{\frac{2^*(\gamma)-p}{2^*(\gamma)}}\Big(\int_{\mathbb{G}}\frac{\psi^{\gamma}|u_n|^{2^*(\gamma)}}{\mathrm{d}(z)^{\gamma}}dz\Big)^{\frac{p-1}{2^*(\gamma)}}\Big(\int_{\mathbb{G}}\frac{\psi^{\gamma}|u_n-u_0|^{2^*(\gamma)}}{\mathrm{d}(z)^{\gamma}}dz\Big)^{\frac{1}{2^*(\gamma)}}\\
&\quad+\Big(\int_{\mathbb{G}}\frac{\psi^{\gamma}|f(z)|^{\frac{2^*(\gamma)}{2^*(\gamma)-p}}}{\mathrm{d}(z)^{\gamma}}dz\Big)^{\frac{2^*(\gamma)-p}{2^*(\gamma)}}\Big(\int_{\mathbb{G}}\frac{\psi^{\gamma}|u_0|^{2^*(\gamma)}}{\mathrm{d}(z)^{\gamma}}dz\Big)^{\frac{p-1}{2^*(\gamma)}}\Big(\int_{\mathbb{G}}\frac{\psi^{\gamma}|u_n-u_0|^{2^*(\gamma)}}{\mathrm{d}(z)^{\gamma}}dz\Big)^{\frac{1}{2^*(\gamma)}}\\
&\to 0\quad \text{as } n\to\infty,
\end{aligned}\tag{3.38}$$

and

$$\Big|\int_{\mathbb{G}}f(z)\frac{\psi^{\gamma}[|u_n|^{p_1}|v_n|^{p_2-2}v_n-|u_0|^{p_1}|v_0|^{p_2-2}v_0](v_n-v_0)}{\mathrm{d}(z)^{\gamma}}dz\Big|\to 0\ \text{ as } n\to\infty,\tag{3.39}$$

Combining (3.36), (3.37), (3.38), (3.39), (3.35) with $\lim\limits_{n\to\infty}\langle I'_{\lambda_1,\lambda_2,\beta}(u_n, v_n), (u_n - u_0, v_n - v_0)\rangle = 0$ and $\lim\limits_{n\to\infty}\langle I'_{\lambda_1,\lambda_2,\beta}(u_0, v_0), (u_n - u_0, v_n - v_0)\rangle = 0$, we deduce that

$$\lim_{n\to\infty}\|(u_n-u_0, v_n-v_0)\|_{\mathcal{H}}=0.$$

The proof is completed.

In the end of this section, we will prove the existence of infinitely many weak solutions of (1.1) which tend to zero. First, by using Hölder's inequality and Young's inequality, we get

$$\begin{aligned}
\int_{\mathbb{G}}\frac{\psi^{\alpha}|u|^{2^*(\alpha)}}{\mathrm{d}(z)^{\alpha}}dz+\int_{\mathbb{G}}\frac{\psi^{\alpha}|v|^{2^*(\alpha)}}{\mathrm{d}(z)^{\alpha}}dz&\le S_{\alpha,\mu_1}^{-\frac{2^*(\alpha)}{2}}\|u\|_{\mu_1}^{2^*(\alpha)}+S_{\alpha,\mu_2}^{-\frac{2^*(\alpha)}{2}}\|v\|_{\mu_2}^{2^*(\alpha)}\\
&\le (S_{\alpha,\mu_1}^{-\frac{2^*(\alpha)}{2}}+S_{\alpha,\mu_2}^{-\frac{2^*(\alpha)}{2}})\|(u,v)\|_{\mathcal{H}}^{2^*(\alpha)},
\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{G}} f(z)\frac{\psi^\gamma |u_n|^{p_1}|v_n|^{p_2}}{\mathrm{d}(z)^\gamma}dz &\le \frac{p_1}{p}\int_{\mathbb{G}} f(z)\frac{\psi^\gamma |u_n|^{p}}{\mathrm{d}(z)^\gamma}dz + \frac{p_2}{p}\int_{\mathbb{G}} f(z)\frac{\psi^\gamma |v_n|^{p}}{\mathrm{d}(z)^\gamma}dz\\
&\le \frac{p_1}{p}\left(\int_{\mathbb{G}} |f(z)|^{\frac{2^*(\gamma)}{2^*(\gamma)-p}}\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz\right)^{\frac{2^*(\gamma)-p}{2^*(\gamma)}}\left(\int_{\mathbb{G}}\frac{\psi^\gamma|u_n|^{2^*(\gamma)}}{\mathrm{d}(z)^\gamma}dz\right)^{\frac{p}{2^*(\gamma)}}\\
&+ \frac{p_2}{p}\left(\int_{\mathbb{G}} |f(z)|^{\frac{2^*(\gamma)}{2^*(\gamma)-p}}\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz\right)^{\frac{2^*(\gamma)-p}{2^*(\gamma)}}\left(\int_{\mathbb{G}}\frac{\psi^\gamma|v_n|^{2^*(\gamma)}}{\mathrm{d}(z)^\gamma}dz\right)^{\frac{p}{2^*(\gamma)}}\\
&\le \|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}\Big(\frac{p_1}{p}S_{\gamma,\mu_1}^{-\frac{p}{2}}\|u_n\|_{\mu_1}^p + \frac{p_2}{p}S_{\gamma,\mu_2}^{-\frac{p}{2}}\|v_n\|_{\mu_2}^p\Big)\\
&\le \|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}(S_{\gamma,\mu_1}^{-\frac{p}{2}} + S_{\gamma,\mu_2}^{-\frac{p}{2}})\|(u_n,v_n)\|_{\mathcal{H}}^p.\end{aligned} \tag{3.40}$$

Then,

$$\begin{aligned}I_{\lambda_1,\lambda_2,\beta}(u,v) &= \frac{1}{2}\|(u,v)\|_{\mathcal{H}}^2 - \frac{\lambda_1}{2^*(\alpha)}\int_{\mathbb{G}}\frac{\psi^\alpha|u|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz - \frac{\lambda_2}{2^*(\alpha)}\int_{\mathbb{G}}\frac{\psi^\alpha|v|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}dz\\
&-\beta\int_{\mathbb{G}} f(z)\frac{\psi^\gamma|u|^{p_1}|v|^{p_2}}{\mathrm{d}(z)^\gamma}dz\\
&\ge \frac{1}{2}\|(u,v)\|_{\mathcal{H}}^2 - (\lambda_1+\lambda_2)\frac{(S_{\alpha,\mu_1}^{-\frac{2^*(\alpha)}{2}} + S_{\alpha,\mu_2}^{-\frac{2^*(\alpha)}{2}})}{2^*(\alpha)}\|(u,v)\|_{\mathcal{H}}^{2^*(\alpha)}\\
&-\beta\|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}(S_{\gamma,\mu_1}^{-\frac{p}{2}} + S_{\gamma,\mu_2}^{-\frac{p}{2}})\|(u,v)\|_{\mathcal{H}}^p.\end{aligned}$$

Define the function

$$g(t) = \frac{1}{2}t^2 - C_1(\lambda_1+\lambda_2)t^{2^*(\alpha)} - C_2\beta t^p, \quad \forall t>0,$$

where

$$C_1 := \frac{(S_{\alpha,\mu_1}^{-\frac{2^*(\alpha)}{2}} + S_{\alpha,\mu_2}^{-\frac{2^*(\alpha)}{2}})}{2^*(\alpha)}, \quad C_2 := \|f\|_{L^{p_*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}(S_{\gamma,\mu_1}^{-\frac{p}{2}} + S_{\gamma,\mu_2}^{-\frac{p}{2}}) > 0.$$

Because $1 < p < 2 < 2^*(\alpha)$, for the given $\beta > 0$, there exists $\lambda_{**} > 0$ so small that for $\lambda_1 + \lambda_2 \in (0, \lambda_{**})$, there exist $t_1$, $t_2 > 0$ with $t_1 < t_2$ such that $g(t_1) = g(t_2) = 0$, and $g(t) < 0$ for $t \in (0, t_1)$, $g(t) > 0$ for $t \in (t_1, t_2)$, $g(t) < 0$ for $t \in (t_2, +\infty)$. Similarly, given $\lambda_1$, $\lambda_2 > 0$, we can choose $\beta_{**} > 0$ small enough such that for all $\beta \in (0, \beta_{**})$, there exist $\hat{t}_1$, $\hat{t}_2 > 0$ with $\hat{t}_1 < \hat{t}_2$ such that $g(\hat{t}_1) = g(\hat{t}_2) = 0$ and $g(t) < 0$ for $t \in (0, \hat{t}_1)$, $g(t) > 0$ for $t \in (\hat{t}_1, \hat{t}_2)$, $g(t) < 0$ for $t \in (\hat{t}_2, +\infty)$.

Let us define a function $\phi \in C_0^\infty([0\,,\,\infty), \mathbb{R})$ such that $0 \le \phi(t) \le 1$, $\phi(-t) = \phi(t)$ for all $t \in [0, +\infty)$, $\phi(t) = 1$ if $t \in [0, t_1]$ and $\phi(t) = 0$ if $t \in [t_2, \infty)$. So we consider the equation

$$\begin{cases} -\Delta_{\mathbb{G}}u - \mu_1\dfrac{\psi^2 u}{\mathrm{d}(z)^2} = \lambda_1\phi(\|(u,v)\|_{\mathcal{H}})\dfrac{\psi^\alpha|u|^{2^*(\alpha)-2}u}{\mathrm{d}(z)^\alpha} + \beta p_1 f(z)\dfrac{\psi^\gamma|u|^{p_1-2}u|v|^{p_2}}{\mathrm{d}(z)^\gamma} & \text{in } \mathbb{G},\\ -\Delta_{\mathbb{G}}v - \mu_2\dfrac{\psi^2 v}{\mathrm{d}(z)^2} = \lambda_2\phi(\|(u,v)\|_{\mathcal{H}})\dfrac{\psi^\alpha|v|^{2^*(\alpha)-2}v}{\mathrm{d}(z)^\alpha} + \beta p_2 f(z)\dfrac{\psi^\gamma|u|^{p_1}|v|^{p_2-2}v}{\mathrm{d}(z)^\gamma} & \text{in } \mathbb{G},\end{cases} \tag{3.41}$$

and we observe that if $(u, v)$ is a weak solution of (3.41) such that $\|(u, v)\|_{\mathcal{H}} < t_1$, then $(u, v)$ is also a solution of (1.1). For this reason we look for critical points of the following functional $\mathcal{J}_{\lambda_1,\lambda_2,\beta} : \mathcal{H} \to \mathbb{R}$

defined as

$$\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v)=\frac{1}{2}\|(u,v)\|_{\mathcal{H}}^2-\frac{1}{2^*(\alpha)}\int_{\mathbb{G}}\phi(\|(u,v)\|_{\mathcal{H}})\Big(\lambda_1\frac{\psi^\alpha|u|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}+\lambda_2\frac{\psi^\alpha|v|^{2^*(\alpha)}}{\mathrm{d}(z)^\alpha}\Big)dz$$
$$-\beta\int_{\mathbb{G}}f(z)\frac{\psi^\gamma|u|^{p_1}|v|^{p_2}}{\mathrm{d}(z)^\gamma}dz,\quad \forall (u,v)\in\mathcal{H}.$$

In view of the definition of $\phi$ and $p<2$ we can see that $\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v)\to\infty$ as $\|(u,v)\|_{\mathcal{H}}\to\infty$, $\mathcal{J}_{\lambda_1,\lambda_2,\beta}(-u,-v)=\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v)$ and $\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v)$ is bounded from below. Moreover, $I_{\lambda_1,\lambda_2,\beta}(u,v)\le\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v)$ for all $(u,v)\in\mathcal{H}$.

Next, we show that $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$ satisfies the assumptions of Theorem 2.1.

**Lemma 3.2.** (i) *If $\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v)<0$, then $\|(u,v)\|_{\mathcal{H}}<t_1$ and $\mathcal{J}_{\lambda_1,\lambda_2,\beta}(\tilde u,\tilde v)=I_{\lambda_1,\lambda_2,\beta}(\tilde u,\tilde v)$ for all $(\tilde u,\tilde v)\in N_{(u,v)}$, where $N_{(u,v)}$ denotes the enough neighborhood of $(u,v)$.*

(ii) *For $\lambda_1$, $\lambda_2>0$, there exists $\widetilde{\beta}=\min\{\beta_*,\beta_{**}\}$ such that if $\beta\in(0,\widetilde{\beta})$ and $c\in(-\infty,0)$, then $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$ satisfies $(PS)_c$-condition;*

(iii) *For $\beta>0$, there exists $\widetilde{\lambda}=\min\{\lambda_*,\lambda_{**}\}$ such that if $\lambda_1$, $\lambda_2\in(0,\widetilde{\lambda})$ and $c\in(-\infty,0)$, then $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$ satisfies $(PS)_c$-condition.*

*Proof.* We prove (i) by contradiction, assume $\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u),v\le 0$ and $\|(u,v)\|_{\mathcal{H}}\ge t_1$. If $\|(u,v)\|_{\mathcal{H}}\ge t_2$, then we have

$$\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v)\ge\frac{1}{2}\|(u,v)\|_{\mathcal{H}}^2-\beta\|f\|_{L^{p*}(\mathbb{G},\frac{\psi^\gamma}{\mathrm{d}(z)^\gamma}dz)}(S_{\alpha,\mu_1}^{-\frac{p}{2}}+S_{\alpha,\mu_2}^{-\frac{p}{2}})\|(u,v)\|_{\mathcal{H}}^p>0.$$

This contradicts $\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v)<0$.

If $t_1\le\|(u,v)\|_{\mathcal{H}}<t_2$, since $0\le\phi(t)\le 1$, we get

$$\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v)\ge I_{\lambda_1,\lambda_2,\beta}(u,v)\ge g(\|(u,v)\|_{\mathcal{H}})>0,$$

which again contradicts $\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v)<0$. Hence, $\|(u,v)\|_{\mathcal{H}}<t_1$. Furthermore, by continuity of $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$, applying $I_{\lambda_1,\lambda_2,\beta}(u,v)=\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v)$ for all $\|(u,v)\|_{\mathcal{H}}<t_1$ there exists a small neighborhood $\mathcal{B}_{(u,v)}\subset B_{\mathrm{d}}((0,0),R)$ of $(u,v)$ such that $I_{\lambda_1,\lambda_2,\beta}(\tilde u,\tilde v)=\mathcal{J}_{\lambda_1,\lambda_2,\beta}(\tilde u,\tilde v)$ for any $(\tilde u,\tilde v)\in\mathcal{B}_{(u,v)}$, we conclude the proof of (i).

Now we prove (ii), let $\widetilde{\beta}=\min\{\beta_*,\beta_{**}\}$, and let $\{(u_n,v_n)\}\subset\mathcal{H}$ be a $(PS)_c$-sequence for $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$ with the level $c<0$, then $\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u_n,v_n)\to c$ and $\mathcal{J}'_{\lambda_1,\lambda_2,\beta}(u_n,v_n)\to 0$ in $\mathcal{H}^{-1}$. By (i), we have $\|(u_n,v_n)\|_{\mathcal{H}}<t_1$, hence $\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u_n,v_n)=I_{\lambda_1,\lambda_2,\beta}(u_n,v_n)$. By Proposition 3.1, $I_{\lambda_1,\lambda_2,\beta}$ satisfies the $(PS)_c$-condition for $c<0$. Thus, $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$ satisfies the $(PS)_c$-condition for $c<0$, (ii) holds.

The proof of (iii) goes exactly as (ii) with only minor modification, we omit it here.

Let

$$\mathcal{J}_{\lambda_1,\lambda_2,\beta}^{-\varepsilon}=\{(u,v)\in\mathcal{H}:\ \mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v)\le-\varepsilon\}.$$

**Lemma 3.3.** *Given $k\in\mathbb{N}$, there exists $\varepsilon=\varepsilon(k)>0$ such that $\gamma(\mathcal{J}_{\lambda_1,\lambda_2,\beta}^{-\varepsilon})\ge k$ for any $\lambda_1$, $\lambda_2$, $\beta>0$.*

*Proof.* Fix $\lambda_1$, $\lambda_2>0$, $k\in\mathbb{N}$ and let $E_k$ be a $k$-dimensional vectorial subspace of $\mathcal{H}$. Taking $(u,v)\in E_k\backslash\{(0,0)\}$ with $(u,v)=r_k(\omega_1,\omega_2)$, where $(\omega_1,\omega_2)\in E_k$ and $\|(\omega_1,\omega_2)\|_{\mathcal{H}}=1$. Then, by (3.40) there is a constant $C>0$ such that

$$\Big|\int_{\mathbb{G}}f(z)\frac{\psi^\gamma|\omega_1|^{p_1}|\omega_2|^{p_2}}{\mathrm{d}(z)^\gamma}dz\Big|\le C\|(\omega_1,\omega_2)\|_{\mathcal{H}}^p=C<\infty,$$

which implies that there exists $c_k \in (-\infty, +\infty)$ such that

$$\int_{\mathbb{G}} f(z)\frac{\psi^{\gamma}|\omega_1|^{p_1}|\omega_2|^{p_2}}{\mathrm{d}(z)^{\gamma}}dz \geq c_k > -\infty.$$

Thus, for each $(u, v) = r_k(\omega_1, \omega_2)$ with $r_k \in (0, t_1)$, we have

$$\begin{aligned}
\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u, v) &= \mathcal{J}_{\lambda,\beta}(r_k(\omega_1, \omega_2)) \\
&= \frac{r_k^2}{2} - \frac{r_k^{2^*(\alpha)}}{2^*(\alpha)}\phi(r_k)\int_{\mathbb{G}}(\lambda_1\frac{\psi^{\alpha}|\omega_1|^{2^*(\alpha)}}{\mathrm{d}(z)^{\alpha}} + \lambda_2\frac{\psi^{\alpha}|\omega_2|^{2^*(\alpha)}}{\mathrm{d}(z)^{\alpha}})dz \\
&\quad - \beta r_k^p\int_{\mathbb{G}} f(z)\frac{\psi^{\gamma}|\omega_1|^{p_1}|\omega_2|^{p_2}}{\mathrm{d}(z)^{\gamma}}dz \\
&\leq \frac{1}{2}r_k^2 - \beta c_k r_k^p.
\end{aligned}$$

For any $\varepsilon := \varepsilon(k) > 0$, there exists $r_k \in (0, t_1)$ small enough such that $\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u, v) \leq -\varepsilon$ for any $(u, v) \in \mathcal{H}$ with $\|(u, v)\|_{\mathcal{H}} = r_k$.

Denote $\mathcal{S}_k = \{(u, v) \in \mathcal{H} : \|(u, v)\|_{\mathcal{H}} = r_k\}$. Clearly, $\mathcal{S}_k$ is homeomorphic to the $k-1$ dimensional sphere $\mathbb{S}^{k-1}$ and $\mathcal{S}_k \cap E_k \subset \mathcal{J}^{-\varepsilon}_{\lambda_1,\lambda_2,\beta}$. By Proposition 2.1 (2) and (4) it follows that

$$\gamma(\mathcal{J}^{-\varepsilon}_{\lambda_1,\lambda_2,\beta}) \geq \gamma(\mathcal{S}_k \cap E_k) = k,$$

concluding the proof.

Let us set the number

$$c_k = \inf_{A\in\Gamma_k}\sup_{(u,v)\in A}\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u, v),$$

with

$$\Gamma_k = \{A \subset \mathcal{H} : A \text{ is closed}, A = -A \text{ and } \gamma(A) \geq k\}.$$

Clearly, $c_k \leq c_{k+1}$ for each $k \in \mathbb{N}$. Before proving our main result, we state the following technical results.

**Lemma 3.4.** $c_k < 0$ *for all* $k \in \mathbb{N}$.

*Proof.* Fix $k \in \mathbb{N}$. By Lemma 3.3, there exists $\varepsilon > 0$ such that $\gamma(\mathcal{J}^{-\varepsilon}_{\lambda_1,\lambda_2,\beta}) \geq k$. This and $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$ is a continuous even functional imply that $\mathcal{J}^{-\varepsilon}_{\lambda_1,\lambda_2,\beta} \in \Gamma_k$. Then

$$(0, 0) \notin \mathcal{J}^{-\varepsilon}_{\lambda_1,\lambda_2,\beta} \text{ and } \sup_{(u,v)\in\mathcal{J}^{-\varepsilon}_{\lambda_1,\lambda_2,\beta}}\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u, v) \leq -\varepsilon < 0.$$

Therefore, taking into account that $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$ is bounded from below, we get

$$-\infty < c_k = \inf_{A\in\Gamma_k}\sup_{(u,v)\in A}\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u, v) \leq \sup_{(u,v)\in\mathcal{J}^{-\varepsilon}_{\lambda_1,\lambda_2,\beta}}\mathcal{J}_{\lambda_1,\lambda_2,\beta}(u, v) \leq -\varepsilon < 0.$$

Let

$$K_c = \{(u, v) \in \mathcal{H} : \mathcal{J}'_{\lambda_1,\lambda_2,\beta}(u, v) = 0 \text{ and } \mathcal{J}_{\lambda_1,\lambda_2,\beta}(u, v) = c\}.$$

**Lemma 3.5.** *For any $\lambda_1$, $\lambda_2$, $\beta > 0$, the critical values $\{c_k\}_{k\in\mathbb{N}}$ of $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$ satisfy $c_k \to 0$ as $k \to \infty$.*

*Proof.* Fix $\mu_1, \mu_2 \in [0, \mu_{\mathbb{G}})$ and $\lambda_1, \lambda_2, \beta > 0$. By Lemma 3.4 it follows that $c_k < 0$. Since $c_k \le c_{k+1}$ we can assume that $\lim_{k\to\infty} c_k \to c_0 \le 0$. Moreover, by Lemma 3.2, it is easy to see that the functional $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$ satisfies the $(PS)_{c_k}$-condition at level $c_k$.

Now we prove that $c_0 = 0$. We argue by contradiction and we suppose that $c_0 < 0$. In view of Lemma 3.2, $K_{c_0}$ is compact. Furthermore, it is easy to see that

$$K_{c_0} \subset \mathcal{E} := \{A \subset \mathcal{H} \setminus \{(0,0)\} : \ A \text{ is closed and } \ A = -A\},$$

which and Proposition 2.1 (6) imply that $\gamma(K_{c_0}) = k_0 < \infty$ and there exists $\delta > 0$ such that $N_\delta(K_{c_0}) \subset \mathcal{E}$ and

$$\gamma(K_{c_0}) = \gamma(N_\delta(K_{c_0})) = k_0 < \infty, \tag{3.42}$$

where $N_\delta(K_{c_0}) = \{(u,v) \in \mathcal{H} : \operatorname{dist}((u,v), K_{c_0}) \le \delta\}$. Moreover, By [38, Theorem A.4], there exists an odd homeomorphism $\eta : \mathcal{H} \to \mathcal{H}$ such that

$$\eta(\mathcal{J}^{c_0+\varepsilon}_{\lambda_1,\lambda_2,\beta} \setminus N_\delta(K_{c_0})) \subset \mathcal{J}^{c_0-\varepsilon}_{\lambda_1,\lambda_2,\beta}, \ \text{ for some } \ \varepsilon \in (0, -c_0) \tag{3.43}$$

Taking into account that $c_{k+1} \le c_k$ and $c_k \to c_0$ as $k \to \infty$, we can find $k \in \mathbb{N}$ such that $c_k > c_0 - \varepsilon$ and $c_{k+k_0} \le c_0$, where $k_0$ given in (3.42). Take $A \in \Gamma_{k+k_0}$ such that $\sup_{(u,v)\in A} \mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v) \le c_{k+k_0} < c_0 + \varepsilon$, by using Properties 2.1 (4), we have

$$\gamma(\overline{A \setminus N_\delta(K_{c_0})}) \ge \gamma(A) - \gamma(N_\delta(K_{c_0})) \ge k \ \text{ and } \ \gamma(\eta(\overline{A \setminus N_\delta(K_{c_0})})) \ge k,$$

from which we have $\eta(\overline{A \setminus N_\delta(K_{c_0})}) \in \Gamma_k$. Hence

$$\sup_{(u,v)\in\eta(\overline{A\setminus N_\delta(K_{c_0})})} \mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v) \ge c_k > c_0 - \varepsilon. \tag{3.44}$$

On the other hand, in view of (3.43) and $A \subset \mathcal{J}^{c_0+\varepsilon}_{\lambda_1,\lambda_2,\beta}$, we see that

$$\eta(A \setminus N_\delta(K_{c_0})) \subset \eta(\mathcal{J}^{c_0+\varepsilon}_{\lambda_1,\lambda_2,\beta} \setminus N_\delta(K_{c_0})) \subset \mathcal{J}^{c_0-\varepsilon}_{\lambda_1,\lambda_2,\beta},$$

which gives a contradiction in virtue of (3.44). Hence, $c_0 = 0$ and $\lim_{k\to\infty} c_k = 0$ hold.

**Lemma 3.6.** *Let $\lambda_1$, $\lambda_2$, $\beta$ be as in (ii) or (iii) of Lemma 3.2. If $k$, $l \in \mathbb{N}$ such that $c = c_k = c_{k+1} = \cdots = c_{k+l}$, then*

$$\gamma(K_c) \ge l + 1.$$

*Proof.* From Lemma 3.4 we have that $c = c_k = c_{k+1} = \ldots = c_{k+l} < 0$. By Lemma 3.2, $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$ satisfies the $(PS)_c$-condition on the compact set $K_c$.

Suppose the result is not true, that is, $\gamma(K_c) \le l$. Then, by Proposition 2.1 (6) there is a neighborhood of $K_c$, say $N_\delta(K_c)$, such that $\gamma(N_\delta(K_c)) = \gamma(K_c) \le l$. By [38, Theorem A.4], there exists an odd homeomorphism $\eta : \mathcal{H} \to \mathcal{H}$ such that

$$\eta(\mathcal{J}^{c+\varepsilon}_{\lambda_1,\lambda_2,\beta} \setminus N_\delta(K_c)) \subset \mathcal{J}^{c-\varepsilon}_{\lambda_1,\lambda_2,\beta} \ \text{ for some } \ \varepsilon \in (0, -c). \tag{3.45}$$

From the definition of $c = c_{n+l}$, we know there exists $A \in \Gamma_{n+l}$ such that

$$\sup_{(u,v)\in A} \mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v) < c + \varepsilon,$$

that is, $A \subset \mathcal{J}^{c+\varepsilon}_{\lambda_1,\lambda_2,\beta}$, and so by (3.45) we get

$$\eta(A \setminus N_\delta(K_c)) \subset \eta(\mathcal{J}^{c+\varepsilon}_{\lambda_1,\lambda_2,\beta} \setminus N_\delta(K_c)) \subset \mathcal{J}^{c-\varepsilon}_{\lambda_1,\lambda_2,\beta}.$$

This yields

$$\sup_{u\in\eta(\overline{A\setminus N_\delta(K_c)})} \mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v) \leq c - \varepsilon, \tag{3.46}$$

On the other hand, by parts (1), (3) of Proposition 2.1 we have

$$\gamma(\eta(\overline{A \setminus N_\delta(K_c)})) \geq \gamma(\overline{A \setminus N_\delta(K_c)}) \geq \gamma(A) - \gamma(N_\delta(K_c)) \geq n.$$

Hence, we conclude that $\eta(\overline{A \setminus N_\delta(K_c)}) \in \Gamma_n$ and so

$$\sup_{u\in\eta(\overline{A\setminus N_\delta(K_c)})} \mathcal{J}_{\lambda_1,\lambda_2,\beta}(u,v) \geq c_n = c,$$

which contradicts (3.46). Thus, we conclude $\gamma(K_c) \geq l + 1$.

***Proof of Theorem 1.1*** Let $\lambda_1, \lambda_2, \beta$ be as in (ii) or (iii) of Lemma 3.2. Putting together Lemma 3.4 and Lemma 3.2 (ii) or (iii) , we can see that the functional $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$ satisfies the $(PS)_{c_k}$-condition with $c_n < 0$. That is, $c_k$ is a critical value of $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$.

We consider two situations.

If all $c_k$'s are distinct, that is, $-\infty < c_1 < c_2 < \cdots < c_k < c_{k+1} < \cdots$, then $\gamma(K_{c_k}) \geq 1$ since $K_{c_k}$ is a compact set. Thus, in this case $\mathcal{J}_{\lambda_1,\lambda_2,\beta}$ admits infinitely many critical values. By Lemma 3.2 (1) we can see that $I_{\lambda_1,\lambda_2,\beta}$ has infinitely many critical points, i.e., (1.1) has infinitely many solutions.

If for some $k \in \mathbb{N}$ there exists $l \in \mathbb{N}$ such that $c_k = c_{k+1} = \cdots = c_{k+l} = c$, then $\gamma(K_c) \geq l + 1 \geq 2$ by Lemma 3.6. Thus, the set $K_c$ has infinitely many distinct elements, (see [38, Remark 7.3]), i.e., $I_{\lambda_1,\lambda_2,\beta}$ has infinitely many distinct critical point. Thus again, system (1.1) has infinitely many distinct weak solutions. Moreover, Lemma 3.5 implies that the energy of this solutions converges to zero.

## Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Conflict of interest

The authors declare that there are no conflicts of interest.

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