Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 16(1): 217–236.
DOI: 10.3934/cam.2024010
Received: 08 September 2023
Revised: 28 January 2024
Accepted: 01 February 2024
Published: 19 February 2024

Research article

Normalized solutions for pseudo-relativistic Schrödinger equations

Xueqi Sun[1,*], **Yongqiang Fu**[1] **and Sihua Liang**[2]

[1] College of Mathematics, Harbin Institute of Technology, Harbin, 150001, P.R. China
[2] College of Mathematics, Changchun Normal University, Changchun, 130032, P.R. China

* **Correspondence:** E-mail: 23B912005@stu.hit.edu.cn, sunxueqi1@126.com.

Abstract: In this paper, we consider the existence and multiplicity of normalized solutions to the following pseudo-relativistic Schrödinger equations

$$\begin{cases} \sqrt{-\Delta + m^2}u + \lambda u = \vartheta|u|^{p-2}v + |u|^{2^\sharp-2}v, \quad x \in \mathbb{R}^N, \\ u > 0, \quad \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases}$$

where $N \geq 2$, $a, \vartheta, m > 0$, λ is a real Lagrange parameter, $2 < p < 2^\sharp = \frac{2N}{N-1}$ and $2^\sharp$ is the critical Sobolev exponent. The operator $\sqrt{-\Delta + m^2}$ is the fractional relativistic Schrödinger operator. Under appropriate assumptions, with the aid of truncation technique, concentration-compactness principle and genus theory, we show the existence and the multiplicity of normalized solutions for the above problem.

Keywords: pseudo-relativistic Schrödinger operator; normalized solutions; Sobolev critical exponent
Mathematics Subject Classification: 35J20, 35R03, 58E05

1. Introduction

This paper deals with the following pseudo-relativistic equation of the form:

$$\begin{cases} \sqrt{-\Delta + m^2}u + \lambda u = \vartheta|u|^{p-2}u + |u|^{2^\sharp-2}u, \quad x \in \mathbb{R}^N, \\ u > 0, \quad \int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{1.1}$$

where the frequency λ as a real Lagrange parameter and is part of the unknowns, $2 < p < 2^\sharp$. For $s \in (0, 1)$, the operator $(-\Delta + m^2)^s$ is defined in Fourier space as multiplication by the symbol $(|\xi|^2 + m^2)^s$ see([1, 2]) i.e., for any $u : \mathbb{R}^N \to \mathbb{R}$ belonging to the Schwartz space $S(\mathbb{R}^N)$ of rapidly decreasing functions,

$$\mathcal{F}((-\Delta + m^2)^s u)(\xi) := (|\xi|^2 + m^2)^s \mathcal{F}u(\xi), \quad \forall\, \xi \in \mathbb{R}^N,$$

where we denote by

$$\mathcal{F}u(\xi) := (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{ik\cdot x}u(x)dx, \quad \xi \in \mathbb{R}^N,$$

the Fourier transform of u. Aslo, we show an alternative definition of $(-\Delta + m^2)^s$ (see [2, 3]):

$$(-\Delta + m^2)^s u(x) := m^{2s}u(x) + C(N, s)m^{\frac{N+2s}{2}} P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{\frac{N+2s}{2}}} K_{\frac{N+2s}{2}}(m|x - y|)dy, \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $P.V.$ is the Cauchy principal value, K_ι is the modified Bessel function of the third kind of index ι (see [4, 5]) and

$$C(N, s) := 2^{-\frac{N+2s}{2}+1}\pi^{-\frac{N}{2}}2^{2s}\frac{s(1 - s)}{\Gamma(2 - s)}.$$

Once $m \to 0$, then $(-\Delta + m^2)^s$ reduces to the classical fractional Laplacian $(-\Delta)^s$ defined via Fourier transform by

$$\mathcal{F}((-\Delta)^s u)(\xi) := |\xi|^{2s}\mathcal{F}u(\xi), \quad \xi \in \mathbb{R}^N.$$

At the same time, by singular integrals, we also get

$$(-\Delta)^s u(x) := C_{N,s}P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x - y|^{N+2s}}dy, \quad x \in \mathbb{R}^N, \quad C_{N,s} := \pi^{-\frac{N}{2}}2^{2s}\frac{\Gamma(\frac{N+2s}{2})}{\Gamma(2 - s)}s(1 - 2s) \tag{1.3}$$

for $s \in (0, 1)$. We observe that the most important difference between operators $(-\Delta)^s$ and $(-\Delta + m^2)^s$ is showed in scaling: the first one is homogeneous in scaling, while the second one is inhomogeneous, which is evident from the Bessel function K_ι in (1.2). There are many scholars devoted to the exploration of fractional Schrödinger equation

$$(-\Delta)^s u + V(x)u = f(u), \quad x \in \Omega,$$

where $(-\Delta)^s$ as the fractional Laplacian, $f(u)$ represents the nonlinearity, the function $V(x) : \mathbb{R}^N \to \mathbb{R}$ is an external potential function, and Ω is a bounded domain in $\mathbb{R}^N$ or $\Omega = \mathbb{R}^N$. It was first introduced in the work of Laskin [6, 7] and originated from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. Note that the Feynman path integral produces the classical Schrödinger equation, however, the fractional Schrödinger equation is obtained by the path integral over Lévy trajectories.

When $s = \frac{1}{2}$, the operator $\sqrt{-\Delta + m^2}$ associates with the free Hamiltonian of a free relativistic particle of mass m. It is worth noting that works of Lieb and Yau [8, 9] on the stability of relativistic matter bring great inspiration to the exploration of $\sqrt{-\Delta + m^2}$. There are some results for this topic, here we just quote a few, please refer to [10–12]. In particularly, it is interesting to consider results for fractional equations involving the operator $\sqrt{-\Delta u + m^2}$ with $m > 0$. From the perspective of mathematics, many scholars focused on finding a solution to the following pseudo-relativistic equation

$$\sqrt{-\Delta u + m^2}u + \lambda u = \vartheta|u|^{p-2}u + g(u) \text{ in } \mathbb{R}^n, \tag{1.4}$$

with $g(u) = |u|^{2^\sharp-2}u$. Now, there are two different approaches to consider problem (1.4) according to the characteristics of the frequency λ :

(i) the frequency λ is a fixed given constant,

(ii) the frequency λ is part of the unknown in problem (1.4).

In case (i), we use a variant of extension method [13] to consider problem (1.4) due to the presence of the nonlocal operator $\sqrt{-\Delta u + m^2 u}$ and we shall introduce this tool in detail in Section 2. Therefore, it can be seen that the solution of problem (1.4) is a critical point connected with the energy functional $\mathcal{I}_\lambda(v)$ defined in $H^1(\mathbb{R}^{N+1}_+)$ by

$$\mathcal{I}_\lambda(v) = \frac{1}{2}\iint_{\mathbb{R}^{N+1}_+}(|\nabla v|^2 + m^2v^2)dxdy + \frac{1}{2}\lambda\int_{\mathbb{R}^N}|v(x,0)|^2dx - \frac{\vartheta}{p}\int_{\mathbb{R}^N}|v(x,0)|^pdx - \frac{1}{2^\sharp}\int_{\mathbb{R}^N}|v(x,0)|^{2^\sharp}dx.$$

In this case, we are devoted to looking for the ground state solutions because they possess many more properties, such as positivity, symmetry and stability. In particularly, the ground state solutions are regarded as minimizers of $\mathcal{I}_\lambda$ on the Nehari manifold

$$\mathcal{M}_\lambda := \Big\{v \in H^1(\mathbb{R}^{N+1}_+)\backslash\{0\} \,:\, \langle \mathcal{I}'_\lambda(v), v\rangle = 0\Big\},$$

(see [14]). In addition, by building a nonempty closed subset of the sign-changing Nehari manifold, Yang and Tang [15] obtained the existence of least energy sign-changing solutions for Schrödinger-Poisson system involving concave-convex nonlinearities in $\mathbb{R}^3$.

Alternatively, in case (ii) other papers are devoted to looking for nontrivial solutions of problem (1.4) when the frequency λ is unknown. In this situation, λ is regarded as a Lagrange multiplier. Moreover, this method from the perspective of physics seems particularly interesting because of the conservation of mass and the mass has a clear physical meaning. On the other hand, such solutions help us to better understand the dynamical properties, such as orbital stability or instability, where $\vartheta > 0$ represents the strength of attractive interactions between cold atoms. In general, the solutions with prescribed L^2-norms of solutions is called normalized solutions, i.e., the solutions satisfy $|u|_2 = c > 0$ for a priori given c. Here, in order to look for normalized solutions of problem (1.1), we shall take advantage of a variant of extension method [13] and transform problem (1.1) into a local problem in a upper half-space $\mathbb{R}^{N+1}_+$ with Neumann boundary condition. In addition, we look for the critical point of the functional on the constraint manifold $\mathcal{S}(a)$. We shall introduce $\mathcal{S}(a)$ and the upper half-space $\mathbb{R}^{N+1}_+$ in detail in Section 2.

In recent years, many scholars have paid great attention to exploration of normalization solutions to various classes of local and non-local problems, and have obtained many results, which are not only of special significance in physics, but also closely related to nonlinear optics and Bose-Einstein condensation. In addition, more and more mathematical scholars begin to explore also solutions with prescribed L^2-norms. This kind of problems was first proposed by Jeanjean in [16], who considered the existence of normalized solutions for the Schrödinger equations

$$\begin{cases} -\Delta u = \lambda u + g(u) \quad \text{in } \mathbb{R}^N, \\ \int_{\mathbb{R}^N}|u|^2dx = a^2, \end{cases} \tag{1.5}$$

where $N \geq 1, \lambda \in \mathbb{R}$ and g satisfies suitable assumptions. Inspired by pioneering work of Jeanjean [16], with the help of variational methods, Alves et al. [17] considered the existence of normalized solutions to the nonlinear Schrödinger equation with critical growth both when $N \geq 3$ and $N = 2$. The author in [18] established existence and several properties of ground states for the following critical equation

$$\begin{cases} -\Delta u = \lambda u + \mu|u|^{q-2}u + |u|^{2^*-2}u \quad \text{in } \mathbb{R}^N, \quad N \geq 3, \\ \int_{\mathbb{R}^N}|u|^2dx = a^2, \end{cases}$$

Later, Soave [19] also was interested in existence and qualitative properties of normalized solutions of the nonlinear Schrödinger equation with combined power nonlinearities driven by two different Laplacian operators. With the aid of an approximation method, Deng and Wu [20] obtained the existence of normalized solutions for the Schrödinger equation, and the positive solution is mountain-pass type for $p = 2^*$. Li and Zou [21] were interested in the exploration of fractional Schrödinger equation, they obtained the existence of multiple normalized solutions in both the L^2-subcritical and L^2-supercritical cases by truncation technique, concentration-compactness principle, genus theory and a fiber map. Wang et al. [22] explored the existence results of normalized solutions for p-Laplacian equations in the case $(\frac{N+2}{N}p, p^*)$ by a mountain-pass argument and constrained variational methods. Yao et al. [23] considered several nonexistence and existence results of normalized solutions for the Choquard equations involving lower critical exponent by variational methods. With the aid of a perturbation method, Jeanjean et al. [24] verified the existence of two solutions involving a prescribed L^2-norm for a quasi-linear Schrödinger equation. We point out that, in [19, 25–27], several applications are discussed. However, results about the pseudo-relativistic equation are relatively few, as far as we know.

Inspired by the works above, we treat existence of the multiple normalized solutions for problem (1.1). Undoubtedly, we shall encounter some difficulties in proving the existence of the normalized solutions of problem (1.1). One is that Sobolev critical exponent $2^\sharp = \frac{2N}{N-1}$ which makes the lack of compactness occur. On the other hand, since the embedding $\mathbb{X}_{rad}(\mathbb{R}^{N+1}_+) \hookrightarrow L^2(\mathbb{R}^N)$ is not compact, we observe that the weak limit of (PS) sequence can not be established in the constraint manifold $\mathcal{S}(a)$. Therefore, we have to prove that the Lagrange multipliers λ are non-negative in case $2 < p < 2 + 2/N < 2^\sharp$, which is crucial for us to be able to obtain the compactness. Using the compactness principle, the difficulty is solved.

In the following, in case $2 < p < 2 + 2/N < 2^\sharp$, the energy functional $\mathcal{J}$ is unbounded from below on $\mathcal{S}(a)$, which results in the failure to get the existence of the solution to problem (1.1) via minimizing problem. In the case $2 < p < 2 + \frac{2}{N}$, inspired by [17, 28], we use a truncation technique that allows the truncation function to be bounded from below and coercive.

Finally, problem (1.1) is nonlocal, we shall encounter new difficulties and the study of this kind of equations becomes very meaningful. Therefore, by the extension method in [13], we transform problem (1.1) into a local problem in a upper half-space with a nonlinear Neumann boundary condition.

Our main result is stated in the following theorem:

Theorem 1.1. *Let $2 < p < 2 + \frac{2}{N} < 2^\sharp$ be satisfied. Then for given $k \in \mathbb{N}$, there exists $\beta > 0$ independent of k and $\vartheta_k := \vartheta(k)$ such that problem* (1.1) *has at least k couples $(u_j, \lambda_j) \in H^{\frac{1}{2}}(\mathbb{R}^N) \times \mathbb{R}$ of weak solutions for $\vartheta \geq \vartheta_k$ and $a \in (0, (\beta/\vartheta)^{\frac{1}{1-\theta}}]$, with $\int_{\mathbb{R}^N} |u_j|^2 dx = a^2, \lambda_j < 0$ for all $j \in [1, k]$ and $\theta = \frac{(p-2)(N-1)}{2}$.*

The organizational structure of present paper in what follows. In Section 2, we give some necessary preliminaries and outline the variational framework. In Section 3, we are devoted to the proof of Theorem 1.1.

2. Preliminaries

Let $H^{\frac{1}{2}}(\mathbb{R}^N)$ be the fractional Sobolev space defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with the following norm

$$|u|_{H^{\frac{1}{2}}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} \sqrt{|\xi|^2 + m^2} |\mathscr{F} u(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

Therefore, $H^{\frac{1}{2}}(\mathbb{R}^N)$ is continuously embedded in $L^p(\mathbb{R}^N)$ for all $p \in [2, 2^\sharp)$ and $H^{\frac{1}{2}}(\mathbb{R}^N)$ is compactly embedded in $L^p_{loc}(\mathbb{R}^N)$ for all $p \in [1, 2^\sharp)$, please refer to [2,4,29,30]. Let $H^1(\mathbb{R}^{N+1}_+)$ denote the completion of $C_c^\infty(\overline{\mathbb{R}^{N+1}_+})$ in the norm:

$$\|v\| := \|v\|_{H^1(\mathbb{R}^{N+1}_+)} = \left(\iint_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2)\, dxdy\right)^{\frac{1}{2}}.$$

According to Lemma 3.1 in [3], for $s \in (0, 1)$, we have that the continuous imbedding $H^1(\mathbb{R}^{N+1}_+) \hookrightarrow L^{2\gamma}(\mathbb{R}^{N+1}_+, y^{1-2s})$, this fact means

$$\|v\|_{L^{2\gamma}(\mathbb{R}^{N+1}_+, y^{1-2s})} \leq \hat{S}\, \|v\| \quad \text{for all } v \in H^1(\mathbb{R}^{N+1}_+), \tag{2.1}$$

for some $\hat{S} > 0$, where $\gamma := 1 + \frac{2}{N-2s}$, and $L^r(\mathbb{R}^{N+1}_+, y^{1-2s})$ is the weighted Lebesgue space for $r \in (1, \infty)$, equipped with the norm

$$\|v\|_{L^r(\mathbb{R}^{N+1}_+, y^{1-2s})} := \left(\iint_{\mathbb{R}^{N+1}_+} y^{1-2s}|v|^r\, dxdy\right)^{\frac{1}{r}}.$$

Using Lemma 3.1.2 in [31], it follows that $H^1(\mathbb{R}^{N+1}_+)$ compactly embedded in $L^2(B^+_R, y^{1-2s})$ for all $R > 0$. In the lignt of Proposition 5 in [3], there exists a (unique) linear trace operator $\mathrm{Tr} : H^1(\mathbb{R}^{N+1}_+) \to H^{\frac{1}{2}}(\mathbb{R}^N)$ such that

$$\sqrt{\sigma_s}|\mathrm{Tr}(v)|_{H^{\frac{1}{2}}(\mathbb{R}^N)} \leq \|v\|_{H^1(\mathbb{R}^{N+1}_+)} \qquad \text{for all } v \in H^1(\mathbb{R}^{N+1}_+), \tag{2.2}$$

where $\sigma_s := 2^{1-2s}\Gamma(1 - s)/\Gamma(s)$, please refer to [32, 33]. For the sake of simplicity, we will show $\mathrm{Tr}(v)$ by $v(\cdot, 0)$. It is worth noting that (2.2) implies

$$\sigma_s m^{2s} \int_{\mathbb{R}^N} v^2(x, 0)\, dx \leq \iint_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2)\, dxdy, \tag{2.3}$$

for all $v \in H^1(\mathbb{R}^{N+1}_+)$, which is equivalent to

$$\sigma_s \int_{\mathbb{R}^N} v^2(x, 0)\, dx \leq m^{-2s} \iint_{\mathbb{R}^{N+1}_+} |\nabla v|^2\, dxdy + m^{2-2s} \iint_{\mathbb{R}^{N+1}_+} v^2\, dxdy. \tag{2.4}$$

To simplify the notation, we can get rid of the constant σ_s in (2.4).

In the following, we define the work space

$$\mathbb{X} := \{v \in H^1(\mathbb{R}^{N+1}_+) : \int_{\mathbb{R}^N} |v(x, 0)|^2 dx < \infty\}$$

equipped with the norm

$$\|v\|_{\mathbb{X}} := \left(\|v\|^2 + \int_{\mathbb{R}^N} |v(x, 0)|^2 dx\right)^{\frac{1}{2}}.$$

Clearly, $\mathbb{X} \subset H^1(\mathbb{R}^{N+1}_+)$ and using (2.3), we see that

$$\|v\| \leq \|v\|_{\mathbb{X}} \ \text{ for all } \ v \in \mathbb{X}.$$

Moreover, $\mathbb{X}$ is a Hilbert space equipped with the inner product

$$\langle v, w\rangle = \iint_{\mathbb{R}^{N+1}_+} (\nabla v \cdot \nabla w + m^2 vw)dxdy + \int_{\mathbb{R}^N} v(x,0)w(x,0)dx.$$

At the same time, $\mathbb{X}^*$ is the dual space of $\mathbb{X}$.

Now, we recall some results in the case $s \in (0,1)$. Since $Tr(H^1(\mathbb{R}^{N+1}_+)) \subset H^{\frac{1}{2}}(\mathbb{R}^N)$ and the embedding $H^{\frac{1}{2}}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is continuous for any $q \in [2, 2^*_s]$ and $s \in (0,1)$. we have the following results.

Theorem 2.1. *[34] For any $u \in H^1(\mathbb{R}^{N+1}_+, y^{1-2s})$ and for any $q \in [2, 2^*_s]$*

$$\begin{aligned} C_{q,s,N}|u|^2_{L^q(\mathbb{R}^N)} &\le \kappa_s \int_{\mathbb{R}^N} (|\xi|^2 + m^2)^s |\mathcal{F}u(\xi)|^2 d\xi \\ &\le \iint_{\mathbb{R}^{N+1}_+} y^{1-2s}(|\nabla v|^2 + m^2 v^2)dxdy, \end{aligned}$$

where $\kappa_s = 2^{1-2s}\frac{\Gamma(1-s)}{\Gamma(s)}$ and $u(x) = v(x,0)$ is the trace of v on $\partial\mathbb{R}^{N+1}_+$.

Theorem 2.2. *[34] Let $H^1_{rad} = \{u \in H^1(\mathbb{R}^{N+1}_+, y^{1-2s}) : u$ is radially symmetric with respect to $x\}$. Then $H^1_{rad}(\mathbb{R}^{N+1}_+, y^{1-2s}) \hookrightarrow\hookrightarrow L^q(\mathbb{R}^N)$ for any $q \in (2, 2^*_s)$.*

We recall the trace inequality with $s = \frac{1}{2}$(see Theorem 2.1 in [32]):

$$\iint_{\mathbb{R}^{N+1}_+} |\nabla v|^2 dxdy \ge S_*(\int_{\mathbb{R}^N} |v(x,0)|^{2^\sharp} dx)^{\frac{2}{2^\sharp}} \tag{2.5}$$

for all $v \in H^1_0(\mathbb{R}^{N+1}_+)$, where $H^1_0(\mathbb{R}^{N+1}_+)$ as the completion of $C_c(\overline{\mathbb{R}^{N+1}_+})$ in the norm

$$\Big(\iint_{\mathbb{R}^{N+1}_+} |\nabla v|^2 dxdy\Big)^{\frac{1}{2}}$$

and the best constant is given by

$$S_* = \frac{2\pi^{\frac{1}{2}}\Gamma(\frac{1}{2})\Gamma(\frac{N+1}{2})\Gamma(\frac{N}{2})^{\frac{1}{N}}}{\Gamma(\frac{1}{2})\Gamma(\frac{N-1}{2})\Gamma(N)^{\frac{1}{N}}}.$$

This constant is obtained on the family of functions $\omega_\epsilon = \mathcal{E}_{1/2}(u_\epsilon)$, where $\mathcal{E}_{1/2}$ denotes the $\frac{1}{2}$-harmonic extension [13], and

$$u_\epsilon(x) := \frac{\epsilon^{\frac{N-1}{2}}}{(|x|^2 + \epsilon^2)^{\frac{N-1}{2}}}, \quad \epsilon > 0,$$

see [29, 32]. Therefore,

$$\omega_\epsilon(x,y) := (P_{1/2}(\cdot, y) * u_\epsilon)(x) = p_{N,1/2}\, y \int_{\mathbb{R}^N} \frac{u_\epsilon(\xi)}{(|x-\xi|^2 + y^2)^{\frac{N+1}{2}}} d\xi,$$

where

$$P_{1/2}(x,y) := \frac{p_{N,1/2}\, y}{(|x|^2 + y^2)^{\frac{N+1}{2}}}$$

as the Poisson kernel for the extension problem in $\mathbb{R}^{N+1}_+$. We observe that $\omega_\epsilon(x,y) = \epsilon^{\frac{1-N}{2}}\omega_1(\frac{x}{\epsilon},\frac{y}{\epsilon})$.

We are devoted to studying the existence and multiplicity of normalized solutions of problem (1.1) in present paper. To consider problem (1.1) by variational methods, we make full use of a variant of the extension method [13] given in [3, 29, 33]. To be more precise, the nonlocal operator $\sqrt{-\Delta+m^2}$ in $\mathbb{R}^N$ can be achieved by a local problem in $\mathbb{R}^N\times(0,\infty)$. In the following, we shall describe this construction in detail. For any function $u\in H^{\frac{1}{2}}(\mathbb{R}^N)$, there exists a unique function $v\in H^1(\mathbb{R}^{N+1}_+)$ (here, $\mathbb{R}^{N+1}_+=\{(x,y)\in\mathbb{R}^N\times\mathbb{R}: y>0\}$ such that

$$\begin{cases} -\Delta v+m^2v=0 & \text{in } \mathbb{R}^{N+1}_+,\\ v(x,0)=u(x) & \text{for } x\in\mathbb{R}^N=\partial\mathbb{R}^{N+1}_+. \end{cases} \tag{2.6}$$

Set

$$Tu(x)=-\frac{\partial v}{\partial y}(x,0),$$

we have the following equation

$$\begin{cases} -\Delta w+m^2w=0 & \text{in } \mathbb{R}^{N+1}_+,\\ w(x,0)=Tu(x) & \text{for } x\in\partial\mathbb{R}^{N+1}_+=\{0\}\times\mathbb{R}^N\simeq\mathbb{R}^N \end{cases}$$

with the solution $w(x,y)=-\frac{\partial v}{\partial y}(x,y)$. By (2.6), we have

$$T(Tu)(x)=-\frac{\partial w}{\partial y}(x,0)=\frac{\partial^2 v}{\partial y^2}(x,0)=(-\Delta_x v+m^2v)(x,0)$$

and hence $T^2=(-\Delta_x+m^2)$. Thus, the operator T that maps the Dirichlet-type data u to the Neumann-type data $-\frac{\partial v}{\partial y}(x,0)$ is actually $\sqrt{-\Delta+m^2}$. Therefore, for problem (1.1), we shall consider the following nonlinear boundary value problem:

$$\begin{cases} -\Delta v+m^2v=0 & \text{in } \mathbb{R}^{N+1}_+,\\ -\frac{\partial v}{\partial y}=\vartheta|v(x,0)|^{p-2}u+|v(x,0)|^{2^\sharp-2}u-\lambda v(x,0) & \text{on } \mathbb{R}^N,\\ v>0,\ \int_{\mathbb{R}^N}|v(x,0)|^2dx=a^2. \end{cases} \tag{2.7}$$

Furthermore, we shall look for the critical points of the energy functional $\mathcal{J}:\mathbb{X}_{rad}(\mathbb{R}^{N+1}_+)\to\mathbb{R}$ associated with problem (2.7):

$$\mathcal{J}(v)=\frac{1}{2}\iint_{\mathbb{R}^{N+1}_+}(|\nabla v|^2+m^2v^2)dxdy-\frac{\vartheta}{p}\int_{\mathbb{R}^N}|v(x,0)|^pdx-\frac{1}{2^\sharp}\int_{\mathbb{R}^N}|v(x,0)|^{2^\sharp}dx$$

on the constraint

$$\mathcal{S}(a):=\{v\in\mathbb{X}_{rad}: |v(x,0)|_2^2=a^2\}.$$

3. Proof of Theorem 1.1

Let us start the section by recalling the definition of genus. Let X be a Banach space and D be a subset of X. The set D is called to be *symmetric* if $-u\in D$ for all $u\in D$. Denote by Σ the family of closed symmetric subsets D of X such that $0\notin D$, that is

$$\Sigma=\{D\subset X\setminus\{0\}: D \text{ is closed and symmetric with respect to the origin}\}.$$

For $D \in \Sigma$, we define

$$\gamma(A) = \begin{cases} 0, & \text{if } D = \emptyset, \\ \inf\{k \in \mathbb{N} \,:\, \exists \text{ an odd map } \phi \in C(D, \mathbb{R}^k \setminus \{0\})\}, & \\ \infty, & \text{if such an odd map does not exist,} \end{cases}$$

and $\Sigma_k = \{D \in \Sigma \,:\, \gamma(D) \geq k\}$. Now, we are ready to give some lemmas that play important roles in proving Theorem 1.1.

Lemma 3.1. *Let $v \in H^1(\mathbb{R}^{N+1}_+)$ and $2 < t < 2^\sharp$, then*

$$\int_{\mathbb{R}^N} |v(x,0)|^t dx \leq S_*^{-\frac{2^\sharp}{2}\theta} (\int_{\mathbb{R}^N} |v(x,0)|^2 dx)^{1-\theta} (\iint_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) dxdy)^{\frac{2^\sharp \theta}{2}},$$

where $\theta = \frac{(t-2)(N-1)}{2}$.

Proof. Since $v \in H^1(\mathbb{R}^{N+1}_+)$ and $2 < t < 2^\sharp$, by Hölder inequality and (2.5), we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |v(x,0)|^t dx &= \int_{\mathbb{R}^N} |v(x,0)|^{2(1-\theta)} \cdot |v(x,0)|^{2^\sharp \theta} dx \\ &\leq (\int_{\mathbb{R}^N} |v(x,0)|^2 dx)^{1-\theta} (\int_{\mathbb{R}^N} |v|^{2^\sharp} dx)^\theta \\ &\leq (\int_{\mathbb{R}^N} |v(x,0)|^2 dx)^{1-\theta} (S_*^{-1} \int \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) dxdy)^{\frac{2^\sharp \theta}{2}} \\ &= S_*^{-\frac{2^\sharp}{2}\theta} (\int_{\mathbb{R}^N} |v(x,0)|^2 dx)^{1-\theta} (\int \int_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) dxdy)^{\frac{2^\sharp \theta}{2}}, \end{aligned}$$

where $\theta = \frac{(t-2)(N-1)}{2}$. Then we have completed the proof of Lemma 3.1.

We state the concentration-compactness principle for $s = \frac{1}{2}$ in what follows.

Lemma 3.2 (Proposition 3.1 in [35])**.** *Let $\{v_k\}$ be a bounded tight sequence in $H^1(\mathbb{R}^{N+1}_+)$, such that v_k converges weakly to v in $H^1(\mathbb{R}^{N+1}_+)$. Let μ, ν be two non-negative measures on $\mathbb{R}^{N+1}_+$ and $\mathbb{R}^N$ respectively and such that*

$$\lim_{n\to\infty} (|\nabla v_k|^2 + m^2 u_k^2) =: \mu$$

and

$$\lim_{n\to\infty} |v_k(x,0)|^{2^\sharp} =: \nu,$$

in the sense of measures. Then, there exist an at most countable set I and three families $\{x_i\}_{i\in I}$, $\{\mu_i\}_{i\in I}$, $\{\nu_i\}_{i\in I}$, with $\mu_i, \nu_i \geq 0$ for all $i \in I$, such that

$$\nu = |v(\cdot,0)|^{2^\sharp} + \sum_{i\in I} \nu_i \delta_{x_i},$$

$$\mu \geq (|\nabla v|^2 + m^2 v^2) + \sum_{i\in I} \mu_i \delta_{(x_i,0)},$$

$$\mu_i \geq S_* \nu_i^{\frac{2}{2^\sharp}} \quad \textit{for all} \quad i \in I.$$

Lemma 3.3. *Let $\{v_k\}$ in be a sequence in $H^1(\mathbb{R}_+^{N+1})$ as in Lemma 3.2 and define*

$$\mu_\infty = \lim_{n\to\infty} \limsup_{k\to\infty} \iint_{B_R^c} (|\nabla v_k|^2 + m^2 v_k^2) dxdy, \quad \nu_\infty = \lim_{n\to\infty} \limsup_{k\to\infty} \int_{B_R^c} |v_k(\cdot, 0)|^{2^\sharp} dx. \tag{3.1}$$

Then

$$\lim_{n\to\infty} \limsup_{k\to\infty} \iint_{\mathbb{R}_+^{N+1}} (|\nabla v_k|^2 + m^2 v_k^2) dxdy = \mu(\mathbb{R}_+^{N+1}) + \mu_\infty, \tag{3.2}$$

$$\lim_{n\to\infty} \limsup_{k\to\infty} \int_{\mathbb{R}^N} |v_k(\cdot, 0)|^{2^\sharp} dx = \nu(\mathbb{R}^N) + \nu_\infty, \quad \mu_\infty \geq S_* \nu_\infty^{\frac{2}{2^\sharp}}, \tag{3.3}$$

where μ, ν are the finite non-negative measures in Lemma 3.2.

Proof. Fix a sequence $\{v_k\}$ in $H^1(\mathbb{R}_+^{N+1})$, as in the statement of Lemma 3.2. Let $\eta \in C_c^\infty(\overline{\mathbb{R}_+^{N+1}})$ such that $0 \leq \eta \leq 1$, $\eta = 0$ in B_1^+ and $\eta = 1$ in $(B_2^c)^+$. Take $R > 0$ and put $\eta_R(x, y) = \eta(\frac{x}{R}, \frac{y}{R})$. We write

$$\begin{aligned} \iint_{\mathbb{R}_+^{N+1}} (|\nabla v_k|^2 + m^2 v_k^2) dxdy = & \iint_{\mathbb{R}_+^{N+1}} (|\nabla v_k|^2 + m^2 v_k^2) \eta_R^2 dxdy \\ & + \iint_{\mathbb{R}_+^{N+1}} (|\nabla v_k|^2 + m^2 v_k^2)(1 - \eta_R^2) dxdy. \end{aligned} \tag{3.4}$$

We first observe that

$$\begin{aligned} \iint_{(B_{2R}^c)^+} (|\nabla v_k|^2 + m^2 u_k^2) dxdy \leq & \iint_{\mathbb{R}_+^{N+1}} (|\nabla v_k|^2 + m^2 v_k^2) \eta_R^2 dxdy \\ \leq & \iint_{(B_R^c)^+} (|\nabla v_k|^2 + m^2 v_k^2)(1 - \eta_R^2) dxdy. \end{aligned}$$

So by (3.1),

$$\mu_\infty = \lim_{R\to\infty} \limsup_{k\to\infty} \iint_{\mathbb{R}_+^{N+1}} (|\nabla v_k|^2 + m^2 v_k^2) \eta_R^2 dxdy. \tag{3.5}$$

On the other hand, since μ is finite, $1 - \eta_R^2$ has compact support and $\eta_R \to 0$ a.e. in $\mathbb{R}_+^{N+1}$, by the definition of μ and the Dominated convergence theorem, we have

$$\begin{aligned} & \lim_{R\to\infty} \limsup_{k\to\infty} \iint_{\mathbb{R}_+^{N+1}} (|\nabla v_k|^2 + m^2 v_k^2)(1 - \eta_R^2) dxdy \\ & = \lim_{R\to\infty} \iint_{\mathbb{R}_+^{N+1}} (1 - \eta_R^2) d\mu = \mu(\mathbb{R}_+^{N+1}). \end{aligned} \tag{3.6}$$

Using (3.5)-(3.6) in (3.4), we can obtain (3.2). Arguing similarly for ν, we see that

$$\lim_{R\to\infty} \limsup_{K\to\infty} \int_{\mathbb{R}^N} (1 - \eta_R^{2^\sharp}) |v_k(\cdot, 0)|^{2^\sharp} dx = \nu(\mathbb{R}^N).$$

Thus, the first part of (3.3) is proved.

In order to verify the last part of (3.3), we consider again the function η_R. Let $K :=\text{supp}(\eta_R)$. By the fact that

$$S_* (\int_{\mathbb{R}^N} |v_k(\cdot, 0)|^{2^\sharp} dx)^{\frac{2}{2^\sharp}} \leq \iint_{\mathbb{R}_+^{N+1}} |\nabla v_k|^2 dxdy$$

$$\leq \iint_{\mathbb{R}^{N+1}_+} (|\nabla v_k|^2 + m^2 v_k^2) dxdy \tag{3.7}$$

and applying this to $\eta_R u_k$ in $H^1(\mathbb{R}^{N+1}_+)$, we get

$$S_*(\int_{\mathbb{R}^N} |v_k(\cdot,0)|^{2^\sharp} \eta_R^{2^\sharp} dx)^{\frac{2}{2^\sharp}} \leq \iint_{\mathbb{R}^{N+1}_+} (|\nabla(v_k\eta_R)|^2 + m^2(v_k\eta_R)^2) dxdy \tag{3.8}$$

for all k. On the other hand,

$$\begin{aligned}
&\iint_{\mathbb{R}^{N+1}_+} [|\nabla(\eta_R v_k)|^2 + m^2(\eta_R v_k)^2]\, dxdy \\
&= \iint_{\mathbb{R}^{N+1}_+} \eta_R^2[|\nabla v_k|^2 + m^2 v_k^2]\, dxdy + \iint_{\mathbb{R}^{N+1}_+} v_k^2 |\nabla \eta_R|^2\, dxdy \\
&\quad + 2\iint_{\mathbb{R}^{N+1}_+} v_k \eta_R \nabla \eta_R \cdot \nabla v_k\, dxdy.
\end{aligned} \tag{3.9}$$

By the definition of η_R, we know

$$\lim_{R\to\infty} \limsup_{k\to\infty} \iint_{\mathbb{R}^{N+1}_+} v_k^2 |\nabla\eta_R|^2\, dxdy \to 0. \tag{3.10}$$

Using the Hölder inequality, the boundedness of $\{v_k\}_k$ in $H^1(\mathbb{R}^{N+1}_+)$ and (3.10), we get

$$\begin{aligned}
&\left| \iint_{\mathbb{R}^{N+1}_+} v_k \eta_R \nabla\eta_R \cdot \nabla v_k\, dxdy \right| \\
&\leq \left(\iint_{\mathbb{R}^{N+1}_+} v_k^2 |\nabla\eta_R|^2\, dxdy \right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^{N+1}_+} \eta_R^2 |\nabla v_k|^2\, dxdy \right)^{\frac{1}{2}} \\
&\leq \left(\iint_{\mathbb{R}^{N+1}_+} v_k^2 |\nabla\eta_R|^2\, dxdy \right)^{\frac{1}{2}} \left(\iint_{\mathbb{R}^{N+1}_+} |\nabla v_k|^2\, dxdy \right)^{\frac{1}{2}} \\
&\leq C \left(\iint_{\mathbb{R}^{N+1}_+} v_k^2 |\nabla\eta_R|^2\, dxdy \right)^{\frac{1}{2}}.
\end{aligned} \tag{3.11}$$

Therefore, together with (3.10) and taking $R \to \infty$, $k \to \infty$ in (3.11), we obtain

$$\lim_{R\to\infty} \limsup_{k\to\infty} \iint_{\mathbb{R}^{N+1}_+} v_k \eta_R \nabla\eta_R \cdot \nabla v_k\, dxdy = 0. \tag{3.12}$$

Putting (3.10)-(3.12) into (3.8), we obtain the desired conclusion.

For $v \in \mathcal{S}(a)$, by Lemma 3.1 and (3.7), we have

$$\begin{aligned}
\mathcal{J}(v) &= \frac{1}{2} \iint_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) dxdy - \frac{\vartheta}{p} \int_{\mathbb{R}^N} |v(x,0)|^p dx - \frac{1}{2^\sharp} \int_{\mathbb{R}^N} |v(x,0)|^{2^\sharp} dx \\
&\geq \frac{1}{2} \iint_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) dxdy - \frac{\vartheta}{p} S_*^{-\frac{2^\sharp\theta}{2}} a^{1-\theta} ||v||^{2^\sharp\theta} - \frac{1}{2^\sharp} S_*^{-\frac{2^\sharp}{2}} ||v||^{2^\sharp}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\|v\|^2 - \frac{\vartheta}{p} S_*^{-\frac{2^\sharp\theta}{2}} a^{1-\theta}\|v\|^{2^\sharp} - \frac{1}{2^\sharp} S_*^{-\frac{2^\sharp}{2}}\|v\|^{2^\sharp} \\
&:= \mathcal{K}(\|v\|),
\end{aligned}
$$

where

$$
\mathcal{K}(t) = \frac{1}{2}t^2 - \frac{\vartheta}{p} S_*^{-\frac{2^\sharp\theta}{2}} a^{1-\theta} t^{2^\sharp\theta} - \frac{1}{2^\sharp} S_*^{-\frac{2^\sharp}{2}} t^{2^\sharp}
$$

and $\theta = \frac{(p-2)(N-1)}{2}$. By $2 < p < 2 + \frac{2}{N}$, we get that $0 < \theta < 1$ and there exists $\beta > 0$ such that $\vartheta a^{1-\theta} \leq \beta$. Thus, the function $\mathcal{K}$ has a positive local maximum. To be more precisely, there exist two numbers $0 < W_1 < W_2 < \infty$ such that $\mathcal{K} < 0$ in the intervals $(0, W_1)$ and (W_2, ∞), while $\mathcal{K} > 0$ in (W_1, W_2). Suppose that $\sigma \in C^\infty(\mathbb{R}^+, [0, 1])$ is a nonincreasing function such that $\sigma(t) = 1$ for $t \leq W_1$ and $\sigma(t) = 0$ for $t \geq W_2$.

We define the truncated functional by

$$
\mathcal{J}_\sigma(v) = \frac{1}{2}\iint_{\mathbb{R}^{N+1}_+} (|\nabla v|^2 + m^2 v^2) dxdy - \frac{\vartheta}{p}\int_{\mathbb{R}^N} |v(x,0)|^p dx - \frac{\sigma(\|v\|)}{2^\sharp}\int_{\mathbb{R}^N} |v(x,0)|^{2^\sharp} dx.
$$

For $v \in \mathcal{S}(a)$, by Lemma 3.1 and (3.7), we get

$$
\begin{aligned}
\mathcal{J}_\sigma(v) &\geq \frac{1}{2}\|v\|^2 - \frac{\vartheta}{p} S_*^{-\frac{2^\sharp\theta}{2}} a^{1-\theta}\|v\|^{2^\sharp\theta} - \frac{\sigma(\|v\|)}{2^\sharp S_*^{2^\sharp/2}}\|v\|^{2^\sharp} \\
&:= \widetilde{\mathcal{K}}(\|v\|),
\end{aligned}
$$

where

$$
\widetilde{\mathcal{K}}(t) = \frac{1}{2}t^2 - \frac{\vartheta}{p} S_*^{-\frac{2^\sharp\theta}{2}} a^{1-\theta} t^{2^\sharp\theta} - \frac{\sigma(t)}{2^\sharp S_*^{2^\sharp/2}} t^{2^\sharp}.
$$

Therefore, with the help of the definition of σ, we obtain $\widetilde{\mathcal{K}} < 0$ in $(0, W_1)$ and $\widetilde{\mathcal{K}} > 0$ in (W_2, ∞) when $a \in \left(0, \left(\beta/\vartheta\right)^{\frac{1}{1-\theta}}\right]$. From now on, we assume that

$$
a \in \left(0, \left(\frac{\beta}{\vartheta}\right)^{\frac{1}{1-\theta}}\right].
$$

Without loss of generality, taking $W_1 > 0$ small enough if necessary, we also assume

$$
0 < W_1^2 < S_*^N, \quad \text{so that } \frac{r^2}{2} - \frac{1}{2^\sharp S_*^{2^\sharp/2}} r^{2^\sharp} \geq 0 \quad \text{for all } r \in [0, W_1]. \tag{3.13}
$$

Lemma 3.4. *(a)* $\mathcal{J}_\sigma \in C^1(\mathbb{X}_{rad}(\mathbb{R}^{N+1}_+), \mathbb{R})$.

(b) $\mathcal{J}_\sigma$ *is coercive and bounded from below on* $\mathcal{S}(a)$. *Furtheremore, if* $\mathcal{J}_\sigma \leq 0$, *then* $\|v\| \leq W_1$ *and* $\mathcal{J}_\sigma(v) = \mathcal{J}(v)$.

(c) $\mathcal{J}_\sigma|_{\mathcal{S}(a)}$ *satisfies the* $(PS)_c$ *condition for all* $c < 0$.

Proof. (a) and (b) hold true with the aid of a standard argument.

For (a). As the proof of the Proposition B.10 in the book [36], conclusion (a) is satisfied.

For (b). Let $v \in \mathcal{S}(a)$, by the definition of σ, we obtain $\sigma(\|v\|^2) = 0$ when $\|v\| \to \infty$. Thus,

$$\mathcal{J}_\sigma(v) \geq \frac{1}{2}\|v\|^2 - \frac{\vartheta}{p} S_*^{-\frac{2^\sharp\theta}{2}} a^{1-\theta}\|v\|^{2^\sharp\theta} \to +\infty,$$

since $N(p-2) < 2$ and $\theta = \frac{(p-2)(N-1)}{2}$, that is $\mathcal{J}_\sigma$ is coercive. On the other hand, it follows from the definition of $\widetilde{\mathcal{K}}(t)$ that $\widetilde{\mathcal{K}}$ has a maximum value, and then $\mathcal{J}_\sigma(v)$ is bounded from below on $\mathcal{S}(a)$. Furthermore, if $\mathcal{J}_\sigma(v) \leq 0$, so $\widetilde{\mathcal{K}} < 0$. Also, by the definition of $\widetilde{\mathcal{K}}$, we obtain $\|v\| \leq W_1$. Therefore, from the definition of σ, we get $\sigma = 1$. This fact implies $\mathcal{J}_\sigma(v) = \mathcal{J}(v)$.

For (c). Assume that $\{v_k\}_k$ is a $(PS)_c$ sequence of $\mathcal{J}_\sigma$ restricted to $\mathcal{S}(a)$ with $c < 0$, that is,

$$\mathcal{J}_\sigma(v_k) \to c < 0 \ \text{ and } \ \|\mathcal{J}'_\sigma|_{\mathcal{S}(a)}(v_k)\| \to 0 \ \text{ as } \ k \to \infty.$$

By (b), $\|v_k\| \leq W_1$ for k large enough. Therefore, $\{v_k\}_k$ is bounded in $\mathbb{X}_{rad}(\mathbb{R}^{N+1}_+)$. Then, up to subsequence, there exists $\mathbb{X}_{rad}(\mathbb{R}^{N+1}_+)$ such that $v_k \rightharpoonup v$ in $\mathbb{X}_{rad}(\mathbb{R}^{N+1}_+)$ and $v_k \to v$ in $L^p(\mathbb{R}^N)$ for all $p \in (2, 2^\sharp)$ and $v_k \to v$ a.e. in $\mathbb{R}^N$. Due to the fact that $2 < p < 2 + \frac{2}{N} < 2^\sharp$, we get

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |v_n(x,0)|^p dx = \int_{\mathbb{R}^N} |v(x,0)|^p dx.$$

Moreover, we claim $v \neq 0$. Otherwise, $\lim\limits_{k\to\infty} \int_{\mathbb{R}^N} |v_k|^p dx = 0$. Combining this and (3.13), we see that

$$\begin{aligned} 0 > c = \lim_{k\to\infty} \mathcal{J}(v_k) &= \lim_{k\to\infty} \Big[\frac{1}{2}\iint_{\mathbb{R}^{N+1}_+} (|\nabla v_k|^2 + m^2 v_k^2) dxdy - \frac{\vartheta}{p}\int_{\mathbb{R}^N} |v_k(x,0)|^p dx - \frac{1}{2^\sharp}\int_{\mathbb{R}^N} |v_k(x,0)|^{2^\sharp} dx\Big] \\ &\geq \lim_{k\to\infty} \Big[\frac{1}{2}\iint_{\mathbb{R}^{N+1}_+} (|\nabla v_k|^2 + m^2 v_k^2) dxdy - \frac{\vartheta}{p}\int_{\mathbb{R}^N} |v_k(x,0)|^p dx - \frac{1}{2^\sharp S_*^{2^\sharp/2}}\|v\|^{2^\sharp}\Big] \\ &\geq \lim_{k\to\infty} -\frac{\vartheta}{p}\int_{\mathbb{R}^N} |v_k(x,0)|^p dx = 0 \end{aligned}$$

which is impossible and proves the claim.

Let

$$\Psi(v) := \frac{1}{2}\int_{\mathbb{R}^N} |v(x,0)|^2 dx, \quad \forall\, v \in \mathbb{X}(\mathbb{R}^{N+1}_+).$$

Thus, $\mathcal{S}(a) = \Psi^{-1}(\{\frac{a^2}{2}\})$. By the Lagrange multiplier, there exists $\lambda_a \in \mathbb{R}$ such that

$$\mathcal{J}'(v) = \lambda_a \Psi'(v)$$

in $(H^1({\mathbb{R}_+}^{N+1}))^*$. Therefore, using this fact, we have

$$\begin{cases} -\Delta v + m^2 v = 0 & \text{in } \mathbb{R}^{N+1}_+, \\ -\frac{\partial v}{\partial y} = \vartheta |v(x,0)|^{p-2} v + |v(x,0)|^{2^\sharp-2} v - \lambda_a v(x,0) & \text{on } \mathbb{R}^N, \\ v > 0, \ \int_{\mathbb{R}^N} |v(x,0)|^2 dx = a^2. \end{cases} \tag{3.14}$$

With the help of Proposition 5.12 in [14], there exists $\lambda_k \in \mathbb{R}$ such that

$$\|\mathcal{J}'(v_k) - \lambda_k \Psi'(v_k)\| \to 0$$

as $k \to \infty$. Hence, for $\varphi \in \mathbb{X}_{rad}(\mathbb{R}^{N+1}_+)$,

$$\iint_{\mathbb{R}^{N+1}_+} (\nabla v_k \cdot \nabla\varphi + m^2 v_k\varphi)dxdy - \vartheta \int_{\mathbb{R}^N} |v_k(x,0)|^{p-2} v_k(x,0)\varphi dx - \int_{\mathbb{R}^N} |v_k(x,0)|^{2^\sharp-2} v_k(x,0)\varphi dx$$
$$= \lambda_k \int_{\mathbb{R}^N} v_k\varphi dx + o(1)\|\varphi\|. \tag{3.15}$$

In particular,

$$\|v_k\|^2 - \vartheta \int_{\mathbb{R}^N} |v_k(x,0)|^p dx - \int_{\mathbb{R}^N} |v_k(x,0)|^{2^\sharp} dx = \lambda_k a^2 + o(1). \tag{3.16}$$

The boundedness of $\{\|v_k\|\}_k$ implies that $\{\lambda_k\}_k$ is also bounded in $\mathbb{R}$. Therefore, up to a subsequence, there exists $\lambda_a \in \mathbb{R}$ such that $\lambda_k \to \lambda_a$ as $k \to \infty$. Therefore, by (3.15) and a standard argument, we obtain that v satisfies problem (3.14). In fact, for any $\varphi \in \mathbb{X}_{rad}(\mathbb{R}^{N+1}_+)$, it follows from the definition of weak convergence that

$$\iint_{\mathbb{R}^{N+1}_+} (\nabla v_k \nabla\varphi + m^2 v_k\varphi)dxdy \to \iint_{\mathbb{R}^{N+1}_+} (\nabla v \nabla\varphi + m^2 v\varphi)dxdy$$

as $k \to \infty$. Since $\lambda_k \to \lambda_a$ as $k \to \infty$, we also obtain that

$$\lambda_k \int_{\mathbb{R}^N} v_k\varphi dx \to \lambda_a \int_{\mathbb{R}^N} v\varphi dx \tag{3.17}$$

as $k \to \infty$. Moreover, since $\{|v_k|^{2^\sharp-2} v_k\}_k$ is bounded in $L^{\frac{2^\sharp}{2^\sharp-1}}(\mathbb{R}^N)$ and

$$|v_k(x,0)|^{2^\sharp-2} u_k(x,0) \to |v(x,0)|^{2^\sharp-2} v(x,0) \quad \text{a.e. in } \mathbb{R}^N, \tag{3.18}$$

then

$$|v_k(x,0)|^{2^\sharp-2} v_k(x,0) \rightharpoonup |v(x,0)|^{2^\sharp-2} v(x,0) \quad \text{in } L^{\frac{2^\sharp}{2^\sharp-1}}(\mathbb{R}^N).$$

This implies that

$$\int_{\mathbb{R}^N} |v_k(x,0)|^{2^\sharp-2} v_k\varphi dx \to \int_{\mathbb{R}^N} |v(x,0)|^{2^\sharp-2} v\varphi dx$$

as $k \to \infty$. Next, we show that $\lambda_a < 0$. Indeed, thanks to $2 < p < 2 + \frac{2}{N} < 2^\sharp$, we have

$$0 > c = \liminf_{k\to\infty} \mathcal{J}(v_k) = \liminf_{k\to\infty} \Big(\mathcal{J}(v_k) - \frac{1}{2}\|\mathcal{J}'(v_k) - \lambda_k \Psi'(v_k)\|\Big)$$
$$= (\frac{1}{2} - \frac{1}{p})\vartheta \int_{\mathbb{R}^N} |v(x,0)|^p dx + (\frac{1}{2} - \frac{1}{2^\sharp}) \int_{\mathbb{R}^N} |v(x,0)|^{2^\sharp} dx + \frac{1}{2}\lambda_a \int_{\mathbb{R}^N} |v(x,0)|^2 dx.$$

Therefore,

$$\frac{1}{2}\lambda_a \int_{\mathbb{R}^N} |v(x,0)|^2 dx < -(\frac{1}{2} - \frac{1}{p})\vartheta \int_{\mathbb{R}^N} |v(x,0)|^p dx - (\frac{1}{2} - \frac{1}{2^\sharp}) \int_{\mathbb{R}^N} |v(x,0)|^{2^\sharp} dx < 0$$

which shows $\lambda_a < 0$.

In the following, we shall recover the compactness with an application of the concentration-compactness principle [35]. Indeed, since $\|v_k\| \le W_1$ for k enough large, using the Prokhorov theorem [37, Theorem 8.6.2], there exist two positive measures $\mu, \nu \in \mathcal{M}(\mathbb{R}_+^{N+1})$ such that

$$\lim_{k\to\infty}(|\nabla v_k|^2 dx + m^2 v_k^2) =: \mu \quad \text{and} \quad \lim_{k\to\infty}|v_k(x,0)|^{2^\sharp} =: \nu \quad \text{in } \mathcal{M}(\mathbb{R}_+^{N+1}). \tag{3.19}$$

Hence, Lemma 3.2-Lemma 3.3 hold. Together with Lemma 3.2, either $v_k \to v$ in $L^{2^\sharp}(\mathbb{R}^N)$ or there exists a (at most countable) set of distinct points $\{x_i\}_i \subset \mathbb{R}^N$ and positive numbers $\{\nu_i\}_i$ such that

$$\nu = |v^+(x,0)|^{2^\sharp} + \sum_{i\in I}\nu_i\delta_{x_i}.$$

If the latter holds, we can also verify $v_k \to v$ in $L^{2^\sharp}(\mathbb{R}^N)$. We shall verify the following three claims hold.

Claim 1. We verify that $\mu(x_i) \le \nu_i$ for any $i \in I$.

Assume that $x_i \in \mathbb{R}^N$ for some $i \in I$. For any $\rho > 0$, we define, $\varphi_\rho(x,y) = \varphi(\frac{x-x_i}{\rho}, \frac{y}{\rho})$, where $\varphi \in C_c(\overline{\mathbb{R}_+^{N+1}})$ such that $\varphi = 1$ in B_1^+ and $\varphi = 0$ in $(B_2^+)^c$, $\varphi \in [0,1]$ and $\|\nabla\varphi\|_{L^\infty(\mathbb{R}_+^{N+1})} \le 2$. We suppose that $\rho > 0$ such that $\mathrm{supp}(\varphi_\rho(\cdot,0)) \subset \mathbb{R}_+^{N+1}$. By the boundedness of $\{v_k\}$ in $\mathbb{X}_{rad}(\mathbb{R}_+^{N+1})$, we know that $\{\varphi_\rho v_k\}$ is also bounded in $\mathbb{X}_{rad}(\mathbb{R}_+^{N+1})$. Therefore,

$$\begin{aligned} o(1) = (\mathcal{J}'(v_k), v_k\varphi_\rho) = &\iint_{\mathbb{R}_+^{N+1}}(|\nabla v_k|^2 + m^2 v_k^2)\varphi_\rho dxdy + \iint_{\mathbb{R}_+^{N+1}} v_k\nabla v_k\cdot\nabla\varphi_\rho dxdy \\ &- \vartheta\int_{\mathbb{R}^N}\varphi_\rho|v_k(x,0)|^p dx - \int_{\mathbb{R}^N}\varphi_\rho|v_k(x,0)|^{2^\sharp}dx. \end{aligned} \tag{3.20}$$

That means

$$\begin{aligned} \iint_{\mathbb{R}_+^{N+1}}(|\nabla v_k|^2 + m^2 v_k^2)\varphi_\rho dxdy = \vartheta\int_{\mathbb{R}^N}\varphi_\rho|v_k(x,0)|^p dx - \iint_{\mathbb{R}_+^{N+1}} v_k\nabla v_k\cdot\nabla\varphi_\rho dxdy \\ + \int_{\mathbb{R}^N}\varphi_\rho|v_k(x,0)|^{2^\sharp}dx + o(1). \end{aligned}$$

Consequently,

$$\lim_{\rho\to0^+}\lim_{k\to\infty}\iint_{\mathbb{R}_+^{N+1}}(|\nabla v_k|^2 + m^2 v_k^2)\varphi_\rho dxdy = \lim_{\rho\to0^+}\lim_{k\to\infty}\iint_{\mathbb{R}_+^{N+1}}\varphi_\rho d\mu \ge \mu_j. \tag{3.21}$$

Together with the definition of φ_ρ, we obtain

$$\lim_{\rho\to0^+}\lim_{k\to\infty}\int_{\mathbb{R}^N}\varphi_\rho|v_k(x,0)|^p dx = \lim_{\rho\to0^+}\int_{\mathbb{R}^N}\varphi_\rho|v(x,0)|^p dx = \lim_{\rho\to0^+}\int_{B_{2\rho}^+}\varphi_\rho|v(x,0)|^p dx = 0. \tag{3.22}$$

Moreover, (3.19) implies

$$\lim_{\rho\to0^+}\lim_{k\to\infty}\int_{\mathbb{R}^N}\varphi_\rho|v_k(x,0)|^{2^\sharp}dx = \lim_{\rho\to0^+}\int_{\mathbb{R}^N}\varphi_\rho d\nu = \nu_i. \tag{3.23}$$

In the following, we show that

$$\lim_{\rho\to0^+}\limsup_{k\to\infty}\iint_{\mathbb{R}_+^{N+1}} v_k\nabla v_k\cdot\nabla\varphi_\rho dxdy = 0. \tag{3.24}$$

In fact, by the Hölder inequality, the boundedness of $\{v_k\}_k$ in $\mathbb{X}_{rad}(\mathbb{R}^{N+1}_+)$, the fact that $\|\nabla\varphi_\rho\|_{L^\infty(\mathbb{R}^{N+1}_+)} \le \frac{C}{\rho}$ and $\mathbb{X}(\mathbb{R}^{N+1}_+)$ is compactly embedded into $L^2(B^+_\rho(x_i,0), y^{1-2s})$ with $s = \frac{1}{2}$, we obtain

$$\begin{aligned}
&\limsup_{k\to\infty} \Big|\iint_{\mathbb{R}^{N+1}_+} v_k \nabla v_k \cdot \nabla\varphi_\rho dxdy\Big| \\
&\le \limsup_{k\to\infty} \Big(\iint_{\mathbb{R}^{N+1}_+} |\nabla v_k|^2 dxdy\Big)^{\frac{1}{2}} \Big(\iint_{B^+_\rho(x_i,0)} |v_k|^2 |\nabla\varphi_\rho|^2 dxdy\Big)^{\frac{1}{2}} \\
&\le \frac{C}{\rho}\Big(\iint_{B^+_\rho(x_i,0)} |v_k|^2 dxdy\Big)^{\frac{1}{2}}.
\end{aligned}$$

By Hölder inequality with $\frac{1}{r} + \frac{r-1}{r} = 1$ and (2.1), we have

$$\begin{aligned}
&\frac{C}{\rho}\Big(\iint_{B^+_\rho(x_i,0)} |v_k|^2 dxdy\Big)^{\frac{1}{2}} \\
&\le \frac{C}{\rho}\Big(\iint_{B^+_\rho(x_i,0)} |v_k|^{2r} dxdy\Big)^{\frac{1}{2r}} \Big(\iint_{B^+_\rho(x_i,0)} dxdy\Big)^{\frac{r-1}{2r}} \\
&\le C\Big(\iint_{B^+_\rho(x_i,0)} |v_k|^{2r} dxdy\Big)^{\frac{1}{2r}} \to 0 \ \text{ as } \ \rho \to 0^+
\end{aligned}$$

which shows that (3.24) holds. Therefore, inserting (3.21)-(3.24) into (3.20), taking $k \to \infty$ and $\rho \to 0^+$, we obtain

$$\mu(x_i) \le \nu_i$$

and the claim holds.

Claim 2. We claim that $\mu_\infty \le \nu_\infty$.

Let $\phi \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ such that $0 \le \phi \le 1$, $\phi = 0$ in B^+_1 and $\phi = 1$ in $(B^c_2)^+$. Take $R > 0$ and put $\phi_R(x) = \phi(\frac{x-x_i}{R}, \frac{y}{R})$. Again, by the boundedness of $\{v_k\}_k$ in $\mathbb{X}_{rad}(\mathbb{R}^{N+1}_+)$, we know that $\{v_k\phi_R\}_k$ is also bounded in $\mathbb{X}_{rad}(\mathbb{R}^{N+1}_+)$. Hence,

$$\begin{aligned}
o(1) = (\mathcal{J}'(v_k), v_k\phi_R) = &\iint_{\mathbb{R}^{N+1}_+} (|\nabla v_k|^2 + m^2 v_k^2)\phi_R dxdy + \iint_{\mathbb{R}^{N+1}_+} v_k \nabla v_k \nabla\phi_R dxdy \\
&- \vartheta \int_{\mathbb{R}^N} \phi_R |v_k(x,0)|^p dx - \int_{\mathbb{R}^N} \phi_R |v_k(x,0)|^{2^\sharp} dx. \qquad (3.25)
\end{aligned}$$

From the aforementioned proof, we obtain

$$\lim_{R\to\infty}\lim_{k\to\infty} \iint_{\mathbb{R}^{N+1}_+} (|\nabla v_k|^2 + m^2 v_k^2)\phi_R dxdy = \iint_{\mathbb{R}^{N+1}_+} \phi_R d\mu \ge \mu_\infty.$$

By Hölder's inequality, $0 \le \phi_R \le 1$ and $\{v_k\}$ is bounded in $\mathbb{X}_{rad}(\mathbb{R}^{N+1}_+)$, we have

$$\begin{aligned}
&\Big|\iint_{\mathbb{R}^{N+1}_+} v_k \nabla v_k \nabla\phi_R dxdy\Big| \le \frac{C}{R}\iint_{\mathbb{R}^{N+1}_+} v_k |\nabla v_k| dxdy \\
&\le \frac{C}{R}\Big(\iint_{\mathbb{R}^{N+1}_+} |v_k|^2 dxdy\Big)^{\frac{1}{2}} \Big(\iint_{\mathbb{R}^{N+1}_+} |\nabla v_k|^2 dxdy\Big)^{\frac{1}{2}}
\end{aligned}$$

$$\leq \frac{C}{R} \to 0$$

as $R \to \infty$. Therefore,

$$\lim_{R\to\infty} \limsup_{k\to\infty} \iint_{\mathbb{R}^{N+1}_+} v_k \nabla v_k \nabla \phi_R dxdy \to 0.$$

By the proof of Lemma 3.3 in [38], we obtain

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{\mathbb{R}^N} \phi_R |v_k(x,0)|^p dx = \lim_{R\to\infty} \int_{\mathbb{R}^N} \phi_R |v(x,0)|^p dx = \lim_{R\to\infty} \int_{|x|\geq R} \phi_R |v(x,0)|^p dx = 0$$

and

$$\lim_{R\to\infty} \lim_{k\to\infty} \int_{\mathbb{R}^N} \phi_R |v_k(x,0)|^{2^\sharp} dx = \nu_\infty.$$

Therefore, it follows from (3.25) that

$$\mu_\infty \leq \nu_\infty$$

and this proves Claim 2.

Claim 3. We shall veify that $\nu_i = 0$ for any $i \in I$ and $\nu_\infty = 0$.

By contradiction, we suppose that there exists $i \in I$ such that $\nu_i > 0$. Steps 1 implies that

$$\nu_i \leq (S_*^{-1} \mu(x_i))^{\frac{2^\sharp}{2}} \leq (S_*^{-1} \nu_i)^{\frac{2^\sharp}{2}}.$$

It implies that $\nu_i \geq S_*^N$. If this case is valid, we get

$$\begin{aligned} W_1^2 &\geq \lim_{\rho\to 0^+} \lim_{k\to\infty} \|v_k\|^2 \geq S_* \lim_{\rho\to 0^+} \lim_{k\to\infty} |v_k(x,0)|^2_{2^\sharp} \\ &\geq \lim_{\rho\to 0^+} \lim_{k\to\infty} S_* \Big(\int_{\mathbb{R}^N} \varphi_\rho |v_k(x,0)|^{2^\sharp} dx \Big)^{\frac{2}{2^\sharp}} = S_* \lim_{k\to\infty} \Big(\int_{\mathbb{R}^N} \phi_\rho d\nu \Big)^{\frac{2}{2^\sharp}} \\ &= S_* \cdot \nu_i^{\frac{2}{2^\sharp}} \geq S_*^N \end{aligned}$$

which is impossible by (3.13). If the latter holds, by the same discussion above, we get

$$W_1^2 \geq \lim_{R\to\infty} \lim_{k\to\infty} \|v_k\|^2 \geq \mu_\infty \geq S_* \cdot \nu_\infty^{\frac{2}{2^\sharp}} \geq S_*^N$$

which contradicts with (3.13), and together with Lemma 3.2 implies $v_k \to v$ in $L^{2^\sharp}_{loc}(\mathbb{R}^N)$ Moreover, combining with Lemma 3.3, we obtain $v_k \to v$ in $L^{2^\sharp}(\mathbb{R}^N)$. Taking into account (3.15)–(3.17), we obtain

$$\begin{aligned} &\lim_{k\to\infty} \Big[\|v_k\|^2 - \lambda_a |v_k(x,0)|_2^2 \Big] \\ &\quad = \lim_{k\to\infty} \Big[\vartheta |v_k(x,0)|_p^p + |v_k(x,0)|_{2^\sharp}^{2^\sharp} + o(1) \Big] \\ &\quad = \vartheta |v(x,0)|_p^p + |v(x,0)|_{2^\sharp}^{2^\sharp} = \|v\|^2 - \lambda_a |v(x,0)|_2^2. \end{aligned} \tag{3.26}$$

Since $\lambda_a < 0$,

$$\begin{aligned}-\lambda_a|v(x,0)|_2^2 &\le \liminf_{k\to\infty} -\lambda_a|v_k(x,0)|_2^2 \le \limsup_{k\to\infty} -\lambda_a|v_k(x,0)|_2^2\\ &\le \limsup_{k\to\infty} -\lambda_a|v_k(x,0)|_2^2 + \liminf_{k\to\infty} \|v_k\|^2 - \|v\|^2\\ &\le \limsup_{k\to\infty}\left[\|v_k\|^2 - \lambda_a|v_k(x,0)|_2^2\right] - \|v\|^2\\ &= -\lambda_a|v(x,0)|_2^2.\end{aligned}$$

Hence,

$$\lim_{k\to\infty} -\lambda_a|v_k(x,0)|_2^2 = -\lambda_a|v(x,0)|_2^2.$$

Moreover, we obtain

$$\lim_{k\to\infty} |v_k(x,0)|_2^2 = |v(x,0)|_2^2.$$

By (3.26), we get

$$\lim_{k\to\infty} \|v_k\|^2 = \|v\|^2.$$

Then $v_k \to v$ in $\mathbb{X}_{rad}(\mathbb{R}_+^{N+1})$ and $|v_k(x,0)|_2 = a$. The proof of Lemma 3.4 is completed.

Set

$$\mathcal{J}_\sigma^{-\epsilon} = \{v \in \mathbb{X}_{rad}(\mathbb{R}_+^{N+1}) \cap \mathcal{S}(a) \,:\, \mathcal{J}_\sigma(v) \le -\epsilon\} \subset \mathbb{X}_{rad}(\mathbb{R}_+^{N+1})$$

for $\varepsilon > 0$. By the fact that $\mathcal{J}_\sigma$ is even and continuous on $\mathbb{X}_{rad}(\mathbb{R}_+^{N+1})$, gives that $\mathcal{J}_\sigma^{-\epsilon}$ is closed and symmetric. Consequently, the following lemma is true and its proof is the same as that of Lemma 3.2 in [28].

Lemma 3.5. *Given $k \in \mathbb{N}$, there exist $\epsilon_k := \epsilon(k)$ and $\vartheta_k := \vartheta(k)$ such that whenever $0 < \epsilon \le \epsilon_k$ and $\vartheta \ge \vartheta_k, \gamma(\mathcal{J}_\sigma^{-\epsilon}) \ge k$.*

Set

$$\Sigma_k := \{E \subset \mathbb{X}_{rad}(\mathbb{R}_+^{N+1}) \cap \mathcal{S}(a) \,:\, E \text{ is closed and symmetric,} \quad \gamma(E) \ge k\}$$

and

$$c_k := \inf_{E\in\Sigma_k}\sup_{u\in E} \mathcal{J}_\sigma(v) > -\infty$$

for all $k \in E$ by Lemma 3.4 (b). In order to verify Theorem 1.1, we given by

$$\mathcal{K}_c = \{v \in \mathbb{X}_{rad}(\Omega) \cap \mathcal{S}(a) \,:\, \mathcal{J}_\sigma'(v) = 0, \mathcal{J}_\sigma(v) = c\}.$$

Therefore, we obtain that the following result holds.

Lemma 3.6. *If $c = c_k = c_{k+1} = \cdots = c_{k+m}$, then $\gamma(\mathcal{K}_c) \ge m + 1$. Especially, $\mathcal{J}_\sigma$ has at least $m + 1$ nontrivial critical points.*

Proof. For $\epsilon > 0$, we know that $\mathcal{J}_\sigma^{-\epsilon} \in \Sigma$. With the help of Lemma 3.5, for any $k \in \mathbb{N}$, there exists $\epsilon_k = \epsilon(k) > 0$ and $\vartheta_k = \vartheta(k)$ such that if $0 < \epsilon \le \epsilon_k$ and $\vartheta \ge \vartheta_k$, we have $\gamma(\mathcal{J}_\sigma^{-\epsilon}) \ge k$. Therefore, $\mathcal{J}_\sigma^{-\epsilon_k} \in \Sigma_k$, and

$$c_k \le \sup_{v\in\mathcal{J}_\sigma^{-\epsilon_k}} \mathcal{J}_\sigma(v) = -\epsilon_k < 0.$$

Let $0 > c = c_k = c_{k+1} = \cdots = c_{k+m}$ are satisfied. Therefore, Lemma 3.4 (c) shows that $\mathcal{J}_\sigma$ satisfies the $(PS)_c$ condition. Consequently, $\mathcal{K}_c$ is a compact set. Theorem 2.1 in [39] yields that $\mathcal{J}_\sigma|_{S(a)}$ has at least $m+1$ critical points.

Proof of Theorem 1.1. By Lemma 3.4 (b) the critical points of $\mathcal{J}_\sigma$ obtained in Lemma 3.6 are the critical points of $\mathcal{J}$. Hence, we complete the proof.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

Sihua Liang is supported by the Science and Technology Development Plan Project of Jilin Province, China (Grant No. YDZJ202201ZYTS582), the Research Foundation of Department of Education of Jilin Province (Grant No. JJKH20230902KJ) and Innovation and Entrepreneurship Talent Funding Project of Jilin Province (No.2023QN21).

Conflict of interest

The authors declare there is no conflict of interest.

References

1. L. Hörmander, The analysis of linear partial differential operators. III: Pseudo-differential operators, Reprint of the 1994 edition, Classics in Mathematics, Springer, Berlin, 2007.
2. E. Lieb, M. Loss, Analysis, *Graduate Studies in Mathematics*, vol. 14, American Mathematical Society, Providence, RI, 1997.
3. M. Fall, V. Felli, Unique continuation properties for relativistic Schrödinger operators with a singular potential, *Discrete Contin. Dyn. Syst.*, **35** (2015), 5827–5867. https://doi.org/10.1006/jfan.1999.3462
4. N. Aronszajn, K. T. Smith, Theory of Bessel potentials. *I, Ann. Inst. Fourier (Grenoble),* **11** (1961), 385–475. https://doi.org/10.5802/aif.116
5. A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher transcendental functions. Vol. II, Based on notes left by Harry Bateman, Reprint of the 1953 original, Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981.
6. N. Laskin, Fractional quantum mechanics and Lévy path integrals, *Phys. Lett. A.*, **268** (2000), 298–305. https://doi.org/10.1016/S0375-9601(00)00201-2
7. N. Laskin, Fractional Schrödinger equation, *Phys. Rev. E.*, **66** (2002), 056108. https://doi.org/10.1103/PhysRevE.66.056108
8. E. Lieb, H. Yau, The Chandrasekhar theory of stellar collapse as the limit of quantum mechanics, *Comm. Math. Phys.*, **112** (1987), 147–174. https://doi.org/10.1007/BF01217684

9. E. Lieb, H. Yau, The stability and instability of relativistic matter, *Comm. Math. Phys.*, **118** (1988), 177–213. https://doi.org/10.1007/BF01218577
10. V. Ambrosio, Existence of heteroclinic solutions for a pseudo-relativistic Allen-Cahn type equation, *Adv. Nonlinear Stud,* **15** (2015), 395–414. https://doi.org/10.1515/ans-2015-0207
11. W. Choi, J. Seok, Nonrelativistic limit of standing waves for pseudo-relativistic nonlinear Schrödinger equations, arXiv:1506.00791. https://doi.org/10.1063/1.4941037
12. V. Coti Zelati, M. Nolasco, Existence of ground states for nonlinear, pseudo-relativistic Schrödinger equations, *Rend. Lincei. Mat. Appl.*, **22** (2011), 51–72. https://doi.org/10.4171/RLM/587
13. L. Caffarelli, L. Silvestre, An extension problem related to the fractional Laplacian, *Comm. Partial Differential Equations*, **32** (2007), 1245–1260. https://doi.org/10.1080/03605300600987306
14. M. Willem, Minimax Theorems, Birkhäuser, Boston, 1996.
15. C. Yang, C. Tang, Sign-changing solutions for the Schrödinger-Poisson system with concave-convex nonlinearities in $\mathbb{R}^N$, *Commun. Anal. Mech.*, **15** (2023), 638–657. https://doi.org/10.3934/cam.2023032
16. L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations, *Nonlinear Anal*, **28** (1997), 1633–1659. https://doi.org/10.1016/S0362-546X(96)00021-1
17. C. O. Alves, C. Ji, O. H. Miyagaki, Normalized solutions for a Schrödinger equation with critical growth in $\mathbb{R}^N$, *Calc. Var. Partial Differential Equations*, **61** (2022), 18. https://doi.org/10.1007/s00526-021-02123-1
18. N. Soave, Normalized ground states for the NLS equation with combined nonlinearities: The Sobolev critical case, *J. Funct. Anal.*, **279** (2020), 108610. https://doi.org/10.1016/j.jfa.2020.108610
19. N. Soave, Normalized ground states for the NLS equation with combined nonlinearities, *J. Differential Equations*, **269** (2020), 6941–6987. https://doi.org/10.1016/j.jde.2020.05.016
20. S. Deng, Q. Wu, Existence of normalized solutions for the Schrödinger equation, *Commun. Anal. Mech.*, **15** (2023), 575–585. https://doi.org/10.3934/cam.2023028
21. Q. Li, W. Zou, The existence and multiplicity of the normalized solutions for fractional Schrödinger equations involving Sobolev critical exponent in the L^2-subcritical and L^2-supercritical cases, *Adv. Nonlinear Anal*, **11** (2022), 1531–1551. https://doi.org/10.1515/anona-2022-0252
22. W. Wang, Q. Li, J. Zhou, Y. Li, Normalized solutions for p-Laplacian equations with a L^2-supercritical growth, *Ann. Funct. Anal.*, **12** (2021), 1–19. https://doi.org/10.1007/s43034-020-00101-w
23. S. Yao, H. Chen, V.D. Rǎdulescu, J. Sun, Normalized solutions for lower critical Choquard equations with critical Sobolev perturbation, *SIAM J. Math. Anal.*, **54** (2022), 3696–3723. https://doi.org/10.1137/21M1463136
24. L. Jeanjean, T. Luo, Z. Wang, Multiple normalized solutions for quasi-linear Schrödinger equations, *J. Differential Equations*, **259** (2015), 3894–3928. https://doi.org/10.1016/j.jde.2015.05.008
25. T. Bartsch, S. de Valeriola, Normalized solutions of nonlinear Schrödinger equations, *Arch. Math.*, **100** (2013), 75–83. https://doi.org/10.48550/arXiv.1209.0950

26. J. Bellazzini, L. Jeanjean, T. Luo, Existence and instability of standing waves with prescribed norm for a class of Schrödinger-Poisson equations, *Proc. Lond. Math. Soc.*, **107** (2013), 303–339. https://doi.org/10.1112/plms/pds072
27. X. Luo, Normalized standing waves for the Hartree equations, *J. Differential Equations*, **267** (2019), 4493–4524. https://doi.org/10.1016/j.jde.2019.05.009
28. C. O. Alves, C. Ji, O. H. Miyagaki, Multiplicity of normalized solutions for a Schrödinger equation with critical in $\mathbb{R}^N$, arXiv: 2103.07940, 2021. https://doi.org/10.48550/arXiv.2103.07940
29. A. Cotsiolis, N. Tavoularis, Best constants for Sobolev inequalities for higher order fractional derivatives, *J. Math. Anal. Appl.*, **295** (2004), 225–236. https://doi.org/10.1016/j.jmaa.2004.03.034
30. E. Di Nezza, G. Palatucci, E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.*, **136** (2012), 521–573. https://doi.org/10.1016/j.bulsci.2011.12.004
31. S. Dipierro, M. Medina, E. Valdinoci, Fractional elliptic problems with critical growth in the whole of $\mathbb{R}^n$, *Scuola Normale Superiore*, 2017. https://doi.org/10.1007/978-88-7642-601-8
32. C. Brändle, E. Colorado, A. de Pablo, U. Sánchez, A concave-convex elliptic problem involving the fractional Laplacian, *Proc. Roy. Soc. Edinburgh Sect. A*, **143** (2013), 39–71. https://doi.org/10.48550/arXiv.2105.13632
33. P. Stinga, J. Torrea, Extension problem and Harnack's inequality for some fractional operators, *Comm. Partial Differential Equations*, **35** (2010), 2092–2122. https://doi.org/10.1080/03605301003735680
34. V. Ambrosio, Ground states solutions for a non-linear equation involving a pseudo-relativistic Schrödinger operator, *J. Math. Phys.*, **57** (2016), 051502. https://doi.org/10.1063/1.4949352
35. V. Ambrosio, Concentration phenomena for a fractional relativistic Schrödinger equation with critical growth, arXiv preprint arXiv:2105.13632, 2021. https://doi.org/10.48550/arXiv.2105.13632
36. P. Rabinowitz, Minimax methods in critical point theory with applications to differential equations, in: *CBME Regional Conference Series in Mathematics*, vol. 65, American Mathematical Society, Providence, RI, 1986.
37. V. Bogachev, Measure Theory, Vol. II, *Springer-Verlag, Berlin*, 2007.
38. X. Zhang, B. Zhang, D. Repovs, Existence and symmetry of solutions for critical fractional Schrödinger equations with bounded potentials, *Nonlinear Anal*, **142** (2016), 48–68. https://doi.org/10.1016/j.na.2016.04.012
39. L. Jeanjean, S. Lu, Nonradial normalized solutions for nonlinear scalar field equations, *Nonlinearity*, **32** (2019), 4942–4966. https://doi.org/10.1088/1361-6544/ab435e

AIMS Press