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Research article

Well-posedness and stability for a nonlinear Euler-Bernoulli beam equation

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Abstract: We study the well-posedness and stability for a nonlinear Euler-Bernoulli beam equation modeling railway track deflections in the framework of input-to-state stability (ISS) theory. More specifically, in the presence of both distributed in-domain and boundary disturbances, we prove first the existence and uniqueness of a classical solution by using the technique of lifting and the semigroup method, and then establish the L^r-integral input-to-state stability estimate for the solution whenever $r \in [2, +\infty]$ by constructing a suitable Lyapunov functional with the aid of Sobolev-like inequalities, which are used to deal with the boundary terms. We provide an extensive extension of relevant work presented in the existing literature.

Keywords: Euler-Bernoulli beam equation; input-to-state stability; integral input-to-state stability; boundary disturbance; Lyapunov method
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1. Introduction

The notion of input-to-state stability (ISS) was originally introduced by Sontag in 1989 during the study of nonlinear systems governed by ordinary differential equations (ODEs) [1]. It was mainly used to quantify the influence of external inputs or disturbances on the stability of control systems, and has been proved to be a powerful tool for describing the robustness of nonlinear systems in control theory and applications. In numerous physical and engineering scenarios, external inputs or disturbances often exhibit unbounded characteristics. To provide a more comprehensive understanding of the stability of nonlinear systems subjected to such external influences, Sontag introduced a variation of ISS in 1998, known as the integral input-to-state stability (iISS); see [2]. The iISS offers a description of stability in a sense weaker than the ISS, specifically permitting unbounded inputs with finite energy. In recent years, the ISS, iISS, and their variations have been developed as the ISS theory and have found extensive applications in various fields; see, e.g., [3] for comprehensive surveys.

In the past decade, ISS theory for ODE systems has been widely extended and applied to systems described by partial differential equations (PDEs). For instance, ISS-Lyapunov characterizations were provided for abstract infinite-dimensional systems including PDEs [4–10]; different ISS estimates were established for PDE systems with various types of disturbances [9, 11–22]; and the ISS was applied to PDEs arising in engineering, such as multi-agent control [23], the railway track model [24], and bridge vibrations [25], etc. We refer [9, 14, 22] for summaries on this topic.

For PDE systems, external disturbances typically manifest within the interior of the domain, on the boundary of the domain, or simultaneously within the interior and on the boundary of the domain. Regarding PDEs with in-domain disturbances, the Lyapunov method is the most common approach for establishing the ISS in various norms; see [6, 17, 20, 26–28]. However, studying PDE systems with boundary disturbances is much more challenging. This is because when a PDE system has boundary disturbances, it is not easy to handle the boundary terms without involving time derivatives of the disturbances. To overcome this obstacle, different solutions have been proposed for different PDE systems. For instance:

(i) In the context of linear PDEs with boundary disturbances, the approach of spectral decomposition and finite difference schemes can be effectively employed for the ISS analysis of systems governed by Sturm-Liouville operators [12, 13], while the Riesz-spectral approach is suitable for establishing the ISS of Riesz-spectral systems [15, 16];

(ii) Regarding nonlinear PDEs with boundary disturbances, various approaches have been proposed for assessing the ISS of PDE systems with Dirichlet type boundary disturbances, such as the monotonicity method [29], the technique of De Giorgi iteration [30], and the maximum principle-based approach [21, 31], etc., while the Lyapunov method remains the primary one for the ISS analysis in the systems with only Robin or Neumann type boundary disturbances [9, 18, 20, 28].

In this paper, we intend to investigate the well-posedness and stability in the framework of ISS theory for a nonlinear Euler-Bernoulli beam equation with both in-domain and boundary disturbances. It is worth mentioning that, as one of the representative PDEs, the Euler-Bernoulli beam equation and its variations have attracted a lot of attention in the past few decades; see, e.g., [32–35]. In particular, a railway track model governed by a class of nonlinear Euler-Bernoulli beam equations was studied in [24, 36, 37], and the ISS for the system was established when only in-domain disturbances appeared (see [24]), whereas the effect of boundary disturbances on the stability of the system has not been considered. In addition, the iISS corresponding to the integrals of in-domain or boundary disturbances has not been studied for such a system, while it is worthy of probing. Motivated by these facts and based on the model considered in [24], we focus on the situation where the nonlinear Euler-Bernoulli beam equation involves both in-domain and boundary disturbances. Within the framework of ISS theory, we will prove first the well-posedness for the system and then establish both the ISS and the iISS estimates for the solution to characterize the influence of the L^r-integral (w.r.t. t) of disturbances on the stability of the system whenever $r \in [2, +\infty]$.

It is worth noting that, as previously mentioned, the presence of boundary disturbances in PDE systems leads to significant complexity in the well-posedness and stability analysis. Consequently, the problem addressed in this paper represents more of a challenge compared to the one tackled in [24]:

(i) Regarding the well-posedness analysis, due to the fact that the nonlinear Euler-Bernoulli beam equation considered in [24] is subject to homogeneous boundary conditions and the nonlinear term solely depends on the state, classical and mild solutions can be directly obtained by using the semigroup method after transforming the specific system into an abstract system. However, in the presence of boundary disturbances, the abstract system involves an unbounded operator, making it non-trivial to prove the well-posedness. To overcome this difficulty, in this paper, we will employ the technique of lifting to transform the original system with non-homogeneous boundary conditions into an equivalent system with homogeneous boundary conditions. Nevertheless, after applying lifting, more nonlinear terms will arise in the equivalent system and depend simultaneously on both the time variable and the state, introducing complexity when verifying properties of these nonlinear terms.

(ii) For the stability analysis, it is worth noting that analyzing the stability of nonlinear systems is inherently more challenging compared to linear ones. Nevertheless, in [24], owing to the homogeneous boundary conditions of the considered nonlinear system, the authors were able to construct a suitable ISS-Lyapunov functional and employ various technical lemmas, such as Poincaré's inequality, Young's inequality, and Gronwall's inequality, to establish the ISS for the system with only in-domain disturbances. For the nonlinear equation under consideration in this paper, the presence of boundary disturbances leads to a challenge. Indeed, after applying the technique of lifting, if an equivalent system with homogeneous boundary conditions is considered and stability analysis is performed by using the Lyapunov method as in [24], the resulting stability estimates must contain the time derivatives of the boundary disturbances, which do not strictly adhere to the real ISS or iISS property as pointed out in [12, 13, 20]. To address this issue, we will directly deal with the original system with non-homogeneous boundary conditions for the stability analysis, but handling the boundary terms is a non-trivial task. Indeed, this requires more techniques than those presented in [24], thereby amplifying the complexity of the problem under consideration.

The outline of this paper is as follows: Section 2 introduces notations and auxiliary results used in the paper. Section 3 presents the problem formulation and the main result, which is divided into two propositions stated in Section 4 and Section 5, respectively. More specifically, the first proposition is concerned with the well-posedness of the considered system and is presented in Section 4, while the second one states the result on the ISS and iISS for the considered system and is presented in Section 5.

2. Notations and preliminaries

In this paper, let $\mathbb{R} := (-\infty, +\infty)$, $\mathbb{R}_{\geq 0} := [0, +\infty)$, and $\mathbb{R}_{>0} := (0, +\infty)$.

The following sets of comparison functions are defined in the standard way; see, e.g., [3, A.1]:

$$\begin{aligned}
\mathcal{K} &:= \left\{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \middle| \gamma \text{ is continuous, strictly increasing, and } \gamma(0) = 0\right\},\\
\mathcal{K}_\infty &:= \left\{\gamma \in \mathcal{K} \middle| \gamma \text{ is unbounded}\right\},\\
\mathcal{L} &:= \left\{\gamma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \middle| \gamma \text{ is continuous, strictly decreasing, and } \lim_{t\to\infty} \gamma(t) = 0\right\},\\
\mathcal{KL} &:= \left\{\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \middle| \beta \text{ is continuous, } \beta(\cdot, t) \in \mathcal{K}, \beta(r, \cdot) \in \mathcal{L}, \forall t \in \mathbb{R}_{\geq 0}, \forall r \in \mathbb{R}_{>0}\right\}.
\end{aligned}$$

For a given operator $\mathcal{A}$, its range and resolvent set are denoted by $R(\mathcal{A})$ and $\rho(\mathcal{A})$, respectively. The kernel of $\mathcal{A}$ is denoted by $\ker(\mathcal{A})$. For given subsets C and D in normed linear spaces, the set of all bounded linear operators from C to D is defined in the standard way as in, e.g., [38, Definition A.3.1], and is denoted by $\mathscr{L}(C; D)$. In particular, let $\mathscr{L}(C; C) := \mathscr{L}(C)$.

For a given function $u : [0, 1] \times \mathbb{R}_{\geq 0} \to \mathbb{R}$, we use the notation $u[t]$ to denote the profile at certain $t \in \mathbb{R}_{\geq 0}$, i.e., $u[t](x) = u(x, t)$ for all $x \in [0, 1]$.

Let AC([0, 1]) denote the set of all absolutely continuous functions defined on [0, 1]. The following Sobolev-like inequalities, which can be proved as in [20], play an important role in dealing with boundary terms when establishing the ISS and iISS for PDE systems with boundary disturbances.

Lemma 2.1. *Suppose that $\upsilon \in AC([0, 1])$. Then, the following inequalities hold true:*

(i) $\upsilon^2(c) \leq 2\|\upsilon\|^2_{L^2(0,1)} + \|\upsilon_x\|^2_{L^2(0,1)}, \forall c \in [0, 1]$;

(ii) $\|\upsilon\|^2_{L^2(0,1)} \leq \upsilon^2(c) + \frac{1}{2}\|\upsilon_x\|^2_{L^2(0,1)}$ *for* $c = 0$ *or* $c = 1$.

We provide the concept of the Fréchet derivative, which can be found in, e.g., [38, Definition A.5.25, p. 629].

Definition 2.2. Consider the mapping F from the Banach space $\mathbb{Y}$ to the Banach space $\mathbb{Z}$. Given $y_0 \in \mathbb{Y}$, if a linear bounded operator $\mathbb{d}F(y_0)$ exists such that

$$\lim_{\|h\|_{\mathbb{Y}} \to 0} = \frac{\|F(y_0 + h) - F(y_0) - \mathbb{d}F(y_0)h\|_{\mathbb{Z}}}{\|h\|_{\mathbb{Y}}} = 0,$$

then F is said to be Fréchet differentiable at y_0, and $\mathbb{d}F(y_0)$ is said to be the Fréchet derivative of F at y_0.

3. Problem formulation and main result

In this paper, we study the well-posedness and stability of the following nonlinear Euler-Bernoulli beam equation in the framework of ISS theory:

$$\begin{aligned}
w_{tt} + (aw_{xx} + bw_{txx})_{xx} + cw_t + kw + lw^3 &= f(x, t), && (x, t) \in [0, 1] \times \mathbb{R}_{\geq 0}, && (3.1a)\\
w(0, t) &= 0, && t \in \mathbb{R}_{\geq 0}, && (3.1b)\\
w_x(1, t) &= 0, && t \in \mathbb{R}_{\geq 0}, && (3.1c)\\
(aw_{xx} + bw_{txx})(0, t) &= d_1(t), && t \in \mathbb{R}_{\geq 0}, && (3.1d)\\
(aw_{xx} + bw_{txx})_x(1, t) &= d_2(t), && t \in \mathbb{R}_{\geq 0}, && (3.1e)\\
w(x, 0) &= \varphi_1(x), && x \in [0, 1], && (3.1f)\\
w_t(x, 0) &= \varphi_2(x), && x \in [0, 1], && (3.1g)
\end{aligned}$$

where

$$a, b, c, k \in \mathbb{R}_{>0} \quad \text{and} \quad k, l \in \mathbb{R}_{\geq 0},$$

the function f represents the distributed in-domain disturbance, the functions d_1, d_2 represent the boundary disturbances, and φ_1, φ_2 are given initial data. It is worth mentioning that equation (3.1a) with $c = k = l = 0$ is well-known as the one-dimensional Euler-Bernoulli beam equation [39, 40]; equation (3.1a) with $k = l = 0$ is a model of flexible aircraft wing with Kelvin-Voigt damping [19, 41]; while equation (3.1a) with $l = 0$ or the general case of equation (3.1a) can be used to model railway track deflections [24, 36, 42, 43].

Before defining a solution, and the ISS and iISS for the system (3.1), we would like to reformulate (3.1) in an abstract form. More specifically, we introduce first the Hilbert space

$$H^2_{[0]}(0,1) := \left\{ w \in H^2(0,1) \middle| w(0) = w_x(1) = 0 \right\},$$

which is endowed with the inner product

$$\langle w_1, w_2 \rangle_{H^2_{[0]}(0,1)} := \int_0^1 a w_{1xx} w_{2xx} \mathrm{d}x, \forall w_1, w_2 \in H^2_{[0]}(0,1),$$

and the norm

$$\|w\|_{H^2_{[0]}(0,1)} := \left\| \sqrt{a} w_{xx} \right\|_{L^2(0,1)}, \forall w \in H^2_{[0]}(0,1),$$

respectively. Define $\mathbb{H} := H^2_{[0]}(0,1) \times L^2(0,1)$, which is also a Hilbert space endowed with the inner product

$$\langle (w_1, v_1), (w_2, v_2) \rangle_{\mathbb{H}} := \langle w_1, w_2 \rangle_{H^2_{[0]}(0,1)} + \langle v_1, v_2 \rangle_{L^2(0,1)}, \forall (w_1, v_1), (w_2, v_2) \in \mathbb{H},$$

and the norm

$$\|(w, v)\|_{\mathbb{H}} := \left(\left\| \sqrt{a} w_{xx} \right\|^2_{L^2(0,1)} + \|v\|^2_{L^2(0,1)} \right)^{\frac{1}{2}}, \forall (w, v) \in \mathbb{H},$$

respectively.

Let the linear operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbb{H} \to \mathbb{H}$ be defined by

$$\mathcal{A}(w, v) := (v, -(a w_{xx} + b v_{xx})_{xx})$$

with the domain

$$\begin{aligned} D(\mathcal{A}) := \Big\{ (w, v) \in \mathbb{H} \Big| & v \in H^2_{[0]}(0,1), a w_{xx} + b v_{xx} \in H^2(0,1), \\ & a w_{xx} + b v_{xx} \in \mathrm{AC}([0,1]), (a w_{xx} + b v_{xx})_x \in \mathrm{AC}([0,1]) \Big\}, \end{aligned}$$

which is dense in $\mathbb{H}$. Let the nonlinear operator $\mathcal{A}_1$ be defined by

$$\mathcal{A}_1(w, v) := \left(0, -cv - kw - lw^3\right), \forall (w, v) \in D(\mathcal{A}_1) := \mathbb{H}.$$

Let the boundary operator $\mathcal{B}$ be defined by

$$\mathcal{B}(w, v) := ((a w_{xx} + b v_{xx})(0), (a w_{xx} + b v_{xx})_x (1)), \forall (w, v) \in D(\mathcal{B}) := D(\mathcal{A}).$$

Throughout this paper, for the in-domain disturbance f, the boundary disturbances d_1, d_2, and the initial data φ_1, φ_2, we always assume that

(H1) $f \in C^1([0,1] \times \mathbb{R}_{\geq 0})$ and $d_1, d_2 \in C^2(\mathbb{R}_{\geq 0})$;

(H2) $(\varphi_1, \varphi_2) \in D(\mathcal{A})$ and satisfies the compatibility condition $\mathcal{B}(\varphi_1, \varphi_2) = (d_1(0), d_2(0))$.

Now, let $\mathbf{X}(t) := (w[t], w_t[t])$ be the state of system (3.1), and $\mathbf{X}_0 := (\varphi_1, \varphi_2)$ be the corresponding initial datum. Let $F(t) := (0, f[t])$. Then, system (3.1) can be written in the following abstract form:

$$\dot{\mathbf{X}}(t) = (\mathcal{A} + \mathcal{A}_1)\mathbf{X}(t) + F(t), \tag{3.2a}$$

$$\mathcal{B}\mathbf{X}(t) = (d_1(t), d_2(t)), \tag{3.2b}$$

$$\mathbf{X}(0) = \mathbf{X}_0 \in D(\mathcal{A}) \quad \text{with} \quad \mathcal{B}\mathbf{X}_0 = (d_1(0), d_2(0)). \tag{3.2c}$$

Definition 3.1. For any $T \in \mathbb{R}_{>0}$, if $\mathbf{X} \in C([0,T]; D(\mathcal{A})) \cap C^1((0,T); \mathbb{H})$ satisfies equation (3.2a) for all $t \in (0,T)$, boundary condition (3.2b) for all $t \in [0,T]$, and intitial-value condition (3.2c), then $\mathbf{X}$ is said to be a classical solution to system (3.2).

Definition 3.2. For certain $r \in [1, +\infty]$, system (3.2) is said to be L^r-integral input-to-state stable (L^r-iISS) in the norm of $\mathbb{H}$ w.r.t. the in-domain disturbance f and the boundary disturbances d_1, d_2 if there exist functions $\mu \in \mathcal{KL}$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathcal{K}$ such that the solution to system (3.2) satisfies

$$\|\mathbf{X}(t)\|_{\mathbb{H}} \leq \mu\left(\|\mathbf{X}_0\|_{\mathbb{H}}, t\right) + \gamma_1\left(\|d_1\|_{L^r(0,t)}\right) + \gamma_2\left(\|d_2\|_{L^r(0,t)}\right) + \gamma_3\left(\|f\|_{L^r((0,t);L^2(0,1))}\right), \forall t \in \mathbb{R}_{\geq 0}. \tag{3.3}$$

In particular, system (3.2) is said to be input-to-state stable (ISS) in the norm of $\mathbb{H}$ w.r.t. the in-domain disturbance f and the boundary disturbances d_1, d_2 if inequality (3.3) is fulfilled with $r = +\infty$.

Remark 3.3. The notions of ISS and iISS provide a powerful tool of characterizing the robustness of nonlinear systems in presence of disturbances. For instance, inequality (3.3) implies that the disturbance-free system (3.2) is asymptotically stable, while the state remains bounded when bounded external disturbances are involved. In particular, the state becomes smaller when external disturbances become smaller in a certain sense.

The main result obtained in this paper is stated as follows:

Theorem 3.4. *System* (3.2) *admits a unique classical solution. Moreover, for any* $r \in [2, +\infty]$, *system* (3.2) *is* L^r*-iISS in the norm of* $\mathbb{H}$ *w.r.t. the in-domain disturbance* f *and the boundary disturbances* d_1, d_2.

In the following, we will divide Theorem 3.4 into two propositions and provide their proofs in Section 4 and Section 5, respectively.

4. Well-posedness analysis

In order to prove the well-posedness of system (3.2), we employ the technique of lifting to transform the original system into an equivalent one, which has homogeneous boundary conditions. Indeed, letting

$$g_1(x,t) := \left(x^2 - 2x\right)\left(-\frac{1}{b}\int_0^t d_1(s)\mathrm{e}^{-\frac{a}{b}(t-s)}\mathrm{d}s\right),$$

$$g_2(x,t) := \left(\frac{1}{6}x^3 - \frac{1}{2}x\right)\left(-\frac{1}{b}\int_0^t d_2(s)\mathrm{e}^{-\frac{a}{b}(t-s)}\mathrm{d}s\right),$$

and

$$\widetilde{w} := w + g_1 + g_2,$$

we transform system (3.1) into

$$\begin{aligned} \widetilde{w}_{tt} + (a\widetilde{w}_{xx} + b\widetilde{w}_{txx})_{xx} + c\widetilde{w}_t + k\widetilde{w} =& f + g_{1tt} + g_{2tt} + \big(a(g_{1xx} + g_{2xx}) + b(g_{1txx} + g_{2txx})\big)_{xx} \\ &+ c(g_{1t} + g_{2t}) + k(g_1 + g_2) - l(\widetilde{w} - g_1 - g_2)^3, && (4.1a) \\ \widetilde{w}(0,t) =& 0, && (4.1b) \\ \widetilde{w}_x(1,t) =& 0, && (4.1c) \\ (a\widetilde{w}_{xx} + b\widetilde{w}_{txx})(0,t) =& 0, && (4.1d) \\ (a\widetilde{w}_{xx} + b\widetilde{w}_{txx})_x(1,t) =& 0, && (4.1e) \\ \widetilde{w}(x,0) =& \widetilde{\varphi}_1(x), && (4.1f) \\ \widetilde{w}_t(x,0) =& \widetilde{\varphi}_2(x). && (4.1g) \end{aligned}$$

Define $\mathcal{A}_2 \in \mathscr{L}(\mathbb{H})$ by

$$\mathcal{A}_2(\widetilde{w},\widetilde{v}) := (0, -c\widetilde{v} - k\widetilde{w}), \forall(\widetilde{w},\widetilde{v}) \in \mathbb{H},$$

and $\widetilde{\mathcal{A}} : D(\widetilde{\mathcal{A}}) \subset \mathbb{H} \to \mathbb{H}$ by

$$\widetilde{\mathcal{A}} := \mathcal{A}|_{D(\widetilde{\mathcal{A}})}$$

with the domain $D(\widetilde{\mathcal{A}}) := D(\mathcal{A}) \cap \ker(\mathcal{B})$, respectively.

For the given functions f, g_1, and g_2, we define the nonlinear functional $\widetilde{F} : \mathbb{R}_{\geq 0} \times \mathbb{H} \to \mathbb{H}$ by

$$\begin{aligned} \widetilde{F}(t,\mathbf{Z})(x) := \Big(0, & f + g_{1tt} + g_{2tt} + (a(g_{1xx} + g_{2xx}) + b(g_{1txx} + g_{2txx}))_{xx} \\ &+ c(g_{1t} + g_{2t}) + k(g_1 + g_2) - l(\widetilde{w} - g_1 - g_2)^3\Big), \forall \mathbf{Z} := (\widetilde{w},\widetilde{v}) \in \mathbb{H}. \qquad (4.2) \end{aligned}$$

Let $\widetilde{\mathbf{X}}(t) := (\widetilde{w}[t], \widetilde{w}_t[t])$ and $\widetilde{\mathbf{X}}_0 := (\widetilde{\varphi}_1, \widetilde{\varphi}_2)$ be the state and the corresponding initial datum of system (4.1), respectively. Then, system (4.1) can be written in the following abstract form:

$$\begin{aligned} \dot{\widetilde{\mathbf{X}}}(t) =& (\widetilde{\mathcal{A}} + \mathcal{A}_2)\widetilde{\mathbf{X}}(t) + \widetilde{F}(t, \widetilde{\mathbf{X}}(t)), && (4.3a) \\ \mathcal{B}\widetilde{\mathbf{X}}(t) =& (0,0), && (4.3b) \\ \widetilde{\mathbf{X}}(0) =& \widetilde{\mathbf{X}}_0 \in D(\widetilde{\mathcal{A}}). && (4.3c) \end{aligned}$$

Definition 4.1. For any $T \in \mathbb{R}_{>0}$, if a function $\widetilde{\mathbf{X}} \in C(\mathbb{R}_{\geq 0}; D(\widetilde{\mathcal{A}})) \cap C^1(\mathbb{R}_{\geq 0}; \mathbb{H})$ satisfies equation (4.3a) for all $t \in (0,T)$, boundary condition (4.3b) for all $t \in [0,T]$, and initial-value condition (4.3c), then $\widetilde{\mathbf{X}}$ is said to be a classical solution to system (4.3).

In order to prove the well-posedness of system (4.1), it suffices to prove the well-posedness of system (4.3). Indeed, we have the following result:

Proposition 4.2. *System* (4.3) *admits a unique classical solution, and hence system* (3.2) *admits a unique classical solution.*

Before proving Proposition 4.2, we present first some auxiliary results.

Lemma 4.3. *The inverse of the linear operator $\widetilde{\mathcal{A}}$, denoted by $\widetilde{\mathcal{A}}^{-1}$, exists and is bounded, namely, $\widetilde{\mathcal{A}}^{-1} \in \mathscr{L}(\mathbb{H}; D(\widetilde{\mathcal{A}}))$. Thus, $0 \in \rho(\widetilde{\mathcal{A}})$ and $\widetilde{\mathcal{A}}$ is a closed operator.*

Proof. Let us show first that $\widetilde{\mathcal{A}}$ is surjective, namely, for any $(\widehat{w}, \widehat{v}) \in \mathbb{H}$, we need to find $(\widetilde{w}, \widetilde{v}) \in D(\widetilde{\mathcal{A}})$ such that $\widetilde{\mathcal{A}}(\widetilde{w}, \widetilde{v}) = (\widehat{w}, \widehat{v})$. Indeed, for any $(\widehat{w}, \widehat{v}) \in \mathbb{H}$, we consider the solution to the following equations

$$\widetilde{v} = \widehat{w}, \tag{4.4a}$$
$$-(a\widetilde{w}_{xx} + b\widetilde{v}_{xx})_{xx} = \widehat{v}, \tag{4.4b}$$
$$(a\widetilde{w}_{xx} + b\widetilde{v}_{xx})(0) = 0, \tag{4.4c}$$
$$(a\widetilde{w}_{xx} + b\widetilde{v}_{xx})_x(1) = 0. \tag{4.4d}$$

For any $y \in [0, 1]$, integrating both sides of equation (4.4b) over the interval $[y, 1]$, we obtain

$$-\int_y^1 (a\widetilde{w}_{zz}(z) + b\widetilde{v}_{zz}(z))_{zz}\,\mathrm{d}z = \int_y^1 \widehat{v}(z)\mathrm{d}z. \tag{4.5}$$

By integrating by parts and using boundary condition (4.4c), we obtain

$$-\int_y^1 (a\widetilde{w}_{zz}(z) + b\widetilde{v}_{zz}(z))_{zz}\,\mathrm{d}z = -(a\widetilde{w}_{zz}(z) + b\widetilde{v}_{zz}(z))_z\Big|_{z=y}^{z=1} = \Big(a\widetilde{w}_{yy}(y) + b\widetilde{v}_{yy}(y)\Big)_y,$$

which, along with equality (4.5), yields

$$\Big(a\widetilde{w}_{yy}(y) + b\widetilde{v}_{yy}(y)\Big)_y = \int_y^1 \widehat{v}(z)\mathrm{d}z, \forall y \in [0, 1]. \tag{4.6}$$

Analogously, integrating both sides of equation (4.6) and using boundary condition (4.4d), we obtain

$$a\widetilde{w}_{xx}(x) + b\widetilde{v}_{xx}(x) = \int_0^x \int_y^1 \widehat{v}(z)\mathrm{d}z\mathrm{d}y, \forall x \in [0, 1]. \tag{4.7}$$

In view of equation (4.4b) and $\widehat{w} \in H^2_{[0]}(0, 1)$, equation (4.7) is equivalent to

$$a\widetilde{w}_{xx}(x) = \int_0^x \int_y^1 \widehat{v}(z)\mathrm{d}z\mathrm{d}y - b\widehat{w}_{xx}(x) := M(x), \forall x \in [0, 1],$$

which implies that

$$\widetilde{w}(x) = \frac{1}{a}\int_x^0 \int_q^1 M(p)\mathrm{d}p\mathrm{d}q, \forall x \in [0, 1]. \tag{4.8}$$

It is clear that $(\widetilde{w}, \widetilde{v}) \in D(\widetilde{\mathcal{A}})$ and $\widetilde{\mathcal{A}}(\widetilde{w}, \widetilde{v}) = (\widehat{w}, \widehat{v})$. This proves that $\widetilde{\mathcal{A}}$ is surjective.

Next, we show that $\widetilde{\mathcal{A}}$ is injective. Noting that $\widetilde{\mathcal{A}}$ is linear, it suffices to prove the implication

$$\widetilde{\mathcal{A}}(\widetilde{w},\widetilde{v}) = 0 \Rightarrow (\widetilde{w},\widetilde{v}) = (0,0).$$

Indeed, setting $(\widehat{w},\widehat{v}) = (0,0)$ in (4.4), and in view of equalities (4.4a) and (4.8), we get $(\widetilde{w},\widetilde{v}) = (0,0)$ immediately.

It has been shown that $\widetilde{\mathcal{A}}$ is bijective. Thus, the inverse of $\widetilde{\mathcal{A}}$, i.e., $\widetilde{\mathcal{A}}^{-1} : \mathbb{H} \to D(\widetilde{\mathcal{A}}) \subset \mathbb{H}$, exists.

Now, we prove that $\widetilde{\mathcal{A}}^{-1} \in \mathscr{L}(\mathbb{H}; D(\widetilde{\mathcal{A}}))$. For any $(\widehat{w},\widehat{v}) \in \mathbb{H}$, due to the fact that there is $(\widetilde{w},\widetilde{v}) \in D(\widetilde{\mathcal{A}}) \subset \mathbb{H}$ satisfying $\widetilde{\mathcal{A}}^{-1}(\widehat{w},\widehat{v}) = (\widetilde{w},\widetilde{v})$, it follows that $(\widehat{w},\widehat{v}) = \widetilde{\mathcal{A}}(\widetilde{w},\widetilde{v})$, or, equivalently,

$$\widetilde{v} = \widehat{w}, \tag{4.9a}$$

$$-(a\widetilde{w}_{xx} + b\widetilde{v}_{xx})_{xx} = \widehat{v}. \tag{4.9b}$$

Applying Lemma 2.1(ii) to $\widehat{w}$ and using equality (4.9a), we have

$$\|\widetilde{v}\|^2_{L^2(0,1)} = \|\widehat{w}\|^2_{L^2(0,1)} \le \frac{1}{4}\|\widehat{w}_{xx}\|^2_{L^2(0,1)} = \frac{1}{4}\|\widetilde{v}_{xx}\|^2_{L^2(0,1)}.$$

Since $(\widetilde{w},\widetilde{v}) \in D(\widetilde{\mathcal{A}})$, we also have

$$\|a\widetilde{w}_{xx} + b\widetilde{v}_{xx}\|^2_{L^2(0,1)} \le \frac{1}{4}\,\|(a\widetilde{w}_{xx} + b\widetilde{v}_{xx})_{xx}\|^2_{L^2(0,1)}\,.$$

Then, we deduce by equation (4.9b) that

$$\begin{aligned}
\left\|\widetilde{\mathcal{A}}^{-1}(\widehat{w},\widehat{v})\right\|_{\mathbb{H}} &= \|(\widetilde{w},\widetilde{v})\|_{\mathbb{H}} \\
&= \left(\int_0^1 \left(a\widetilde{w}_{xx}^2 + \widetilde{v}^2\right) \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le \left(\int_0^1 \left(a\widetilde{w}_{xx}^2 + \frac{1}{4}\widetilde{v}_{xx}^2\right) \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C_1 \left(\int_0^1 \left((a\widetilde{w}_{xx} + b\widetilde{v}_{xx})^2 + a\widetilde{v}_{xx}^2\right) \mathrm{d}x\right)^{\frac{1}{2}} \\
&\le C_2 \left(\int_0^1 \left(a\widetilde{v}_{xx}^2 + ((a\widetilde{w}_{xx} + b\widetilde{v}_{xx})_{xx})^2\right) \mathrm{d}x\right)^{\frac{1}{2}} \\
&= C_2 \left(\int_0^1 \left(a\widehat{w}_{xx}^2 + \widehat{v}_{xx}^2\right) \mathrm{d}x\right)^{\frac{1}{2}} \\
&= C_2 \left\|(\widehat{w},\widehat{v})\right\|_{\mathbb{H}},
\end{aligned}$$

where C_1 and C_2 are positive constants depending only on a and b. Therefore, $\widetilde{\mathcal{A}}^{-1} \in \mathscr{L}(\mathbb{H}; D(\widetilde{\mathcal{A}}))$, and hence $0 \in \rho(\widetilde{\mathcal{A}})$ and $\widetilde{\mathcal{A}}$ is a closed operator (see [38, Theorem A.3.46, p. 596]).

Lemma 4.4. *The operator* $\widetilde{\mathcal{A}} : D(\widetilde{\mathcal{A}}) \to \mathbb{H}$ *is dissipative w.r.t.* $\langle\cdot,\cdot\rangle_{\mathbb{H}}$.

Proof. Indeed, in view of the definitions of $\widetilde{\mathcal{A}}$ and $\langle\cdot,\cdot\rangle_{\mathbb{H}}$, and by integrating by parts, we have

$$\begin{aligned}\langle\widetilde{\mathcal{A}}(\widetilde{w},\widetilde{v}),(\widetilde{w},\widetilde{v})\rangle_{\mathbb{H}} &= \int_0^1 (a\widetilde{w}_{xx}\widetilde{v}_{xx} - \widetilde{v}(a\widetilde{w}_{xx} + b\widetilde{v}_{xx})_{xx})\,\mathrm{d}x \\ &= \int_0^1 a\widetilde{w}_{xx}\widetilde{v}_{xx}\mathrm{d}x - \int_0^1 \widetilde{v}_{xx}(a\widetilde{w}_{xx} + b\widetilde{v}_{xx})\mathrm{d}x \\ &= -b\int_0^1 \widetilde{v}_{xx}^2\mathrm{d}x \\ &\leq 0, \forall(\widetilde{w},\widetilde{v}) \in D(\widetilde{\mathcal{A}}),\end{aligned}$$

which indicates the dissipativity of $\widetilde{\mathcal{A}}$.

Lemma 4.5. *The operator $\widetilde{\mathcal{A}} + \mathcal{A}_2$ generates a C_0-semigroup of contractions on $\langle\cdot,\cdot\rangle_{\mathbb{H}}$.*

Proof. We prove first that the linear operator $\widetilde{\mathcal{A}}$ generates a C_0-semigroup of contractions on $\langle\cdot,\cdot\rangle_{\mathbb{H}}$. Indeed, we see from Lemma 4.4 that $\widetilde{\mathcal{A}}$ is dissipative. In view of Lemma 4.3, $\widetilde{\mathcal{A}}$ is closed, and thus the resolvent set $\rho(\widetilde{\mathcal{A}})$ is open (see [38, Lemma A.4.8, p. 612]). Since $0 \in \rho(\widetilde{\mathcal{A}})$, there must be a positive number λ_0 such that $R(\lambda_0 I - \widetilde{\mathcal{A}}) = \mathbb{H}$, where I denotes the identity operator defined on $D(\widetilde{\mathcal{A}})$. According to the Lumer-Philips Theorem (see [44, Theorem 4.3, p. 14]), the linear operator $\widetilde{\mathcal{A}}$ generates a C_0-semigroup of contractions on $\langle\cdot,\cdot\rangle_{\mathbb{H}}$.

Next, we prove that $\mathcal{A}_2$ is bounded. Indeed, for any $(\widetilde{w},\widetilde{v}) \in \mathbb{H}$, due to the fact that $\|\widetilde{w}\|^2_{L^2(0,1)} \leq \frac{1}{4}\|\widetilde{w}_{xx}\|^2_{L^2(0,1)}$, we have

$$\begin{aligned}\|\mathcal{A}_2(\widetilde{w},\widetilde{v})\|_{\mathbb{H}} &= \left(\int_0^1 (c\widetilde{v} + k\widetilde{w})^2\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^1 2\left(c^2\widetilde{v}^2 + k^2\widetilde{w}^2\right)\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \left(\int_0^1 2\left(c^2\widetilde{v}^2 + \frac{1}{4}k^2\widetilde{w}_{xx}^2\right)\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq C_3\left(\int_0^1 \left(a\widetilde{w}_{xx}^2 + \widetilde{v}^2\right)\mathrm{d}x\right)^{\frac{1}{2}} \\ &= C_3\,\|(\widetilde{w},\widetilde{v})\|_{\mathbb{H}}\,,\end{aligned}$$

where C_3 is a positive constant depending only on a, c, k when $k > 0$, and only on c when $k = 0$, respectively. Therefore, the linear operator $\widetilde{\mathcal{A}}_2$ is bounded.

Finally, according to [38, Theorem 3.2.1, p. 110], we conclude that $\widetilde{\mathcal{A}} + \mathcal{A}_2$ generates a C_0-semigroup of contractions on $\langle\cdot,\cdot\rangle_{\mathbb{H}}$.

Lemma 4.6. *For any $T \in \mathbb{R}_{>0}$, the nonlinear functional $\widetilde{F} : [0,T] \times \mathbb{H} \to \mathbb{H}$ is Fréchet differentiable.*

Proof. Let $g(x,t){:=}g_1(x,t) + g_2(x,t)$. Recalling the definition of $\widetilde{F}$ (see (4.2)), for any $t \in [0,T]$ and $\mathbf{Z} := (\widetilde{w},\widetilde{v}) \in \mathbb{H}$, we have

$$\widetilde{F}(t,\mathbf{Z})(x){=}\left(0, f + g_{tt} + (ag_{xx} + bg_{txx})_{xx} + cg_t + kg - l(\widetilde{w} - g)^3\right).$$

By virtue of the regularity of f and g, it suffices to show that $\left(0,(\widetilde{w}-g)^3\right)$ is Fréchet differentiable on $\mathbb{Y}:=[0,T]\times\mathbb{H}\to\mathbb{H}$. Furthermore, due to the fact that

$$\left(0,(\widetilde{w}-g)^3\right)=\left(0,\widetilde{w}^3\right)-\left(0,3\widetilde{w}^2g\right)+\left(0,3\widetilde{w}g^2\right)-\left(0,g^3\right),$$

it suffices to show that $\left(0,\widetilde{w}^3\right)$, $\left(0,\widetilde{w}^2g\right)$, and $\left(0,\widetilde{w}g^2\right)$ are Fréchet differentiable on $\mathbb{Y}$. Since the proof can proceed in a standard way (see, e.g., [24]), we only show that $\widetilde{F}_1(t,\mathbf{Z})(x):=\left(0,\widetilde{w}^2g\right)$ is Fréchet differentiable on $\mathbb{Y}$ in the following. More specifically, for any $y_0:=(t_0,\mathbf{Z}_0)\in\mathbb{Y}$ with $\mathbf{Z}_0:=(\widetilde{w}_0,\widetilde{v}_0)$, we would like to prove that the Fréchet derivative of $\widetilde{F}_1$ at y_0 is given by

$$\mathbb{d}\widetilde{F}_1(y_0)h:=(0,2\widetilde{w}_0g[t_0]\widetilde{w}+\widetilde{w}_0^2g_t[t_0]t),$$

where $h:=(t,\mathbf{Z})$ with $t\in[0,T]$ and $\mathbf{Z}:=(\widetilde{w},\widetilde{v})\in\mathbb{H}$, namely, we shall prove that

$$\lim_{\|h\|_{\mathbb{Y}}\to 0}\frac{\left\|\widetilde{F}_1(y_0+h)-\widetilde{F}_1(y_0)-\mathbb{d}\widetilde{F}_1(y_0)h\right\|_{\mathbb{H}}}{\|h\|_{\mathbb{Y}}}=0,$$

or, equivalently,

$$\lim_{|t|+\|\widetilde{w}\|_{H^2_{[0]}(0,1)}+\|\widetilde{v}\|_{L^2(0,1)}\to 0}\frac{\left\|(\widetilde{w}+\widetilde{w}_0)^2g[t+t_0]-\widetilde{w}_0^2g[t_0]-\widetilde{w}_0^2g_t[t_0]t-2\widetilde{w}_0g[t_0]\widetilde{w}\right\|_{L^2(0,1)}}{|t|+\|\widetilde{w}\|_{H^2_{[0]}(0,1)}+\|\widetilde{v}\|_{L^2(0,1)}}=0. \tag{4.10}$$

Indeed, we deduce that

$$\begin{aligned}
&\lim_{|t|+\|\widetilde{w}\|_{H^2_{[0]}(0,1)}+\|\widetilde{v}\|_{L^2(0,1)}\to 0}\frac{\left\|(\widetilde{w}+\widetilde{w}_0)^2g[t+t_0]-\widetilde{w}_0^2g[t_0]-\widetilde{w}_0^2g_t[t_0]t-2\widetilde{w}_0g[t_0]\widetilde{w}\right\|_{L^2(0,1)}}{|t|+\|\widetilde{w}\|_{H^2_{[0]}(0,1)}+\|\widetilde{v}\|_{L^2(0,1)}}\\
\le&\lim_{t\to 0,\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\frac{\left\|(\widetilde{w}+\widetilde{w}_0)^2g[t+t_0]-\widetilde{w}_0^2g[t_0]-\widetilde{w}_0^2g_t[t_0]t-2\widetilde{w}_0g[t_0]\widetilde{w}\right\|_{L^2(0,1)}}{|t|+\|\widetilde{w}\|_{H^2_{[0]}(0,1)}}\\
\le&\lim_{t\to 0,\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\frac{\left\|\widetilde{w}_0^2(g[t+t_0]-g[t_0])-\widetilde{w}_0^2g_t[t_0]t\right\|_{L^2(0,1)}}{|t|+\|\widetilde{w}\|_{H^2_{[0]}(0,1)}}+\lim_{t\to 0,\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\frac{\|\widetilde{w}^2g[t+t_0]\|_{L^2(0,1)}}{|t|+\|\widetilde{w}\|_{H^2_{[0]}(0,1)}}\\
&+\lim_{t\to 0,\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\frac{\|2\widetilde{w}\widetilde{w}_0g[t+t_0]-2\widetilde{w}_0g[t_0]\widetilde{w}\|_{L^2(0,1)}}{|t|+\|\widetilde{w}\|_{H^2_{[0]}(0,1)}}\\
\le&\lim_{t\to 0}\frac{\left\|\widetilde{w}_0^2\left(g[t+t_0]-g[t_0]\right)-\widetilde{w}_0^2g_t[t_0]t\right\|_{L^2(0,1)}}{|t|}+\lim_{t\to 0,\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\frac{\|\widetilde{w}^2g[t+t_0]\|_{L^2(0,1)}}{\|\widetilde{w}\|_{H^2_{[0]}(0,1)}}\\
&+\lim_{t\to 0,\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\frac{\|2\widetilde{w}\widetilde{w}_0g[t+t_0]-2\widetilde{w}_0g[t_0]\widetilde{w}\|_{L^2(0,1)}}{|t|}.
\end{aligned} \tag{4.11}$$

Now we assess each term on the right-hand side of inequality (4.11). First, we have

$$\lim_{t\to 0}\frac{\left\|\widetilde{w}_0^2(g[t+t_0]-g[t_0])-\widetilde{w}_0^2g_t[t_0]t\right\|_{L^2(0,1)}}{|t|}=\lim_{t\to 0}\frac{\left\|\widetilde{w}_0^2\frac{g[t+t_0]-g[t_0]}{t}t-\widetilde{w}_0^2g_t[t_0]t\right\|_{L^2(0,1)}}{|t|}$$

$$
\begin{aligned}
&= \lim_{t\to 0}\left\|\widetilde{w}_0^2\frac{g[t+t_0]-g[t_0]}{t}-\widetilde{w}_0^2 g_t[t_0]\right\|_{L^2(0,1)}\\
&= \lim_{t\to 0}\left\|\widetilde{w}_0^2 g_t[t_0]-\widetilde{w}_0^2 g_t[t_0]\right\|_{L^2(0,1)}\\
&=0. \qquad (4.12)
\end{aligned}
$$

Second, noting that $\|\widetilde{w}^2\|_{L^2(0,1)}=\|\widetilde{w}\|^2_{L^4(0,1)}$ and applying the Sobolev embedding result $H^2_{[0]}(0,1)\hookrightarrow L^4(0,1)$ with $\|\widetilde{w}\|_{L^4(0,1)}\le C_4\|\widetilde{w}\|_{H^2_{[0]}(0,1)}$ and some positive constant C_4, we infer that

$$
\begin{aligned}
\lim_{t\to 0,\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\frac{\left\|\widetilde{w}^2 g[t+t_0]\right\|_{L^2(0,1)}}{\|\widetilde{w}\|_{H^2_{[0]}(0,1)}} &\le\|g\|_{L^\infty((0,1)\times(0,T))}\lim_{\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\frac{\left\|\widetilde{w}^2\right\|_{L^2(0,1)}}{\|\widetilde{w}\|_{H^2_{[0]}(0,1)}}\\
&=\|g\|_{L^\infty((0,1)\times(0,T))}\lim_{\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\frac{\|\widetilde{w}\|^2_{L^4(0,1)}}{\|\widetilde{w}\|_{H^2_{[0]}(0,1)}}\\
&\le C_4\|g\|_{L^\infty((0,1)\times(0,T))}\lim_{\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\frac{\|\widetilde{w}\|^2_{H^2_{[0]}(0,1)}}{\|\widetilde{w}\|_{H^2_{[0]}(0,1)}}\\
&=0. \qquad (4.13)
\end{aligned}
$$

Third, we have

$$
\begin{aligned}
&\lim_{t\to 0,\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\frac{\|2\widetilde{w}\widetilde{w}_0 g[t+t_0]-2\widetilde{w}_0 g[t_0]\widetilde{w}\|_{L^2(0,1)}}{|t|}\\
&=\lim_{t\to 0,\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\left\|2\widetilde{w}\widetilde{w}_0\frac{g[t+t_0]-g[t_0]}{t}\right\|_{L^2(0,1)}\\
&=\lim_{\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\|2\widetilde{w}\widetilde{w}_0 g_t[t_0]\|_{L^2(0,1)}\\
&\le 2\|g_t\|_{L^\infty((0,1)\times(0,T))}\,\|\widetilde{w}_0\|_{L^2(0,1)}\lim_{\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\|\widetilde{w}\|_{L^2(0,1)}\\
&\le 2\|g_t\|_{L^\infty((0,1)\times(0,T))}\,\|\widetilde{w}_0\|_{L^2(0,1)}\lim_{\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\to 0}\|\widetilde{w}\|_{H^2_{[0]}(0,1)}\\
&=0. \qquad (4.14)
\end{aligned}
$$

Finally, combining inequality (4.11), equality (4.12), inequality (4.13), and inequality (4.14), we obtain equality (4.10).

Proof of Proposition 4.2. For any $T\in\mathbb{R}_{>0}$, in view of Lemma 4.5 and Lemma 4.6, it is guaranteed by [45, Theorem 6.1.4 & 6.1.5, pp. 185-187] that system (4.3) admits a unique local classical solution $\widetilde{\mathbf{X}}$ on an interval $[0,T_{max}]$ with some $T_{max}\in(0,T)$. Moreover, it is guaranteed by [45, Theorem 6.1.4, pp. 185-186] that the local classical solution can be extended to the whole interval $[0,T]$ if the solution satisfies

$$
\lim_{t\to T_{max}}\|\widetilde{\mathbf{X}}[t]\|_{\mathbb{H}}<+\infty,
$$

which is included in Section 5.

5. Stability assessment

The ISS and iISS results stated in Theorem 3.4 are re-formulated as in the following proposition:

Proposition 5.1. *For any $r \in [2, +\infty]$, system* (3.2) *is L^r-iISS w.r.t the in-domain disturbance f and the boundary disturbances d_1, d_2, having the following estimate for all $t \in \mathbb{R}_{\geq 0}$:*

$$\|\mathbf{X}(t)\|_{\mathbb{H}} \leq Ce^{-\Lambda t}\left(\|\mathbf{X}_0\|_{\mathbb{H}} + \|\mathbf{X}_0\|_{\mathbb{H}}^2\right) + C\left(\|f\|_{L^r((0,t);L^2(0,1))} + \|d_1\|_{L^r(0,t)} + \|d_2\|_{L^r(0,t)}\right), \tag{5.1}$$

where C and Λ are positive constants depending only on a, b, c, k, l, and r when $r \in [2, +\infty)$, and depending only on a, b, c, k, and l when $r = +\infty$, respectively.

Lemma 5.2. *For any positive constant m satisfying $m < \min\{4a, 1\}$, there are positive constants c_l, c_u, and c_h depending only on a, k, l, and m such that*

$$c_l\left(\|w_{xx}\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2\right) \leq \frac{1}{2}\int_0^1\left(aw_{xx}^2 + kw^2 + \frac{l}{2}w^4 + v^2 + 2mwv\right)dx \tag{5.2}$$

$$\begin{aligned}&\leq c_u\left(\|w_{xx}\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2\right)\\ &\quad + c_h\left(\|w_{xx}\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2\right)^2, \forall (w, v) \in \mathbb{H}.\end{aligned} \tag{5.3}$$

Proof. For any positive constant m satisfying $m < \min\{4a, 1\}$, and $(w, v) \in \mathbb{H}$, let

$$E := \frac{1}{2}\int_0^1\left(aw_{xx}^2 + kw^2 + \frac{l}{2}w^4 + v^2 + 2mwv\right)\mathrm{d}x.$$

It is clear that

$$\left|\int_0^1 2mwv\mathrm{d}x\right| \leq m\int_0^1\left(w^2 + v^2\right)\mathrm{d}x. \tag{5.4}$$

Since $w \in H^2_{[0]}(0, 1)$, by virtue of Lemma 2.1(ii), we have

$$\|w\|_{L^2(0,1)}^2 \leq \frac{1}{4}\|w_{xx}\|_{L^2(0,1)}^2. \tag{5.5}$$

Therefore, it holds that

$$\begin{aligned}E &\geq \frac{1}{2}\int_0^1\left(aw_{xx}^2 + (k - m)w^2 + (1 - m)v^2\right)\mathrm{d}x\\ &\geq \frac{1}{2}\int_0^1\left(aw_{xx}^2 - mw^2 + (1 - m)v^2\right)\mathrm{d}x\\ &\geq \frac{1}{2}\int_0^1\left(\left(a - \frac{m}{4}\right)w_{xx}^2 + (1 - m)v^2\right)\mathrm{d}x,\end{aligned}$$

which implies that inequality (5.2) holds true with $c_l := \frac{1}{2}\min\left\{a - \frac{m}{4}, 1 - m\right\}$.

We deduce by inequalities (5.4) and (5.5) that

$$E \leq \frac{1}{2}\int_0^1\left(aw_{xx}^2 + (k + m)w^2 + \frac{l}{2}w^4 + (1 + m)v^2\right)\mathrm{d}x$$

$$\leq \frac{1}{2}\int_0^1 \left(aw_{xx}^2 + \frac{1}{4}(k+m)w_{xx}^2 + \frac{l}{2}w^4 + (1+m)\,v^2\right)\mathrm{d}x \tag{5.6}$$

$$\leq c_u \left(\|w_{xx}\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2\right) + \frac{l}{4}\int_0^1 w^4 \mathrm{d}x \tag{5.7}$$

with $c_u := \frac{1}{2} \cdot \max\left\{a + \frac{1}{4}(k+m), 1+m\right\}$.

In view of $H^2(0,1) \hookrightarrow L^4(0,1)$, there is a positive constant C_e such that $\|w\|_{L^4(0,1)} \leq C_e \|w\|_{H^2(0,1)}$. It follows that

$$\begin{aligned}
\frac{l}{4}\int_0^1 w^4 \mathrm{d}x \leq & \frac{l}{4}C_e^4 \left(\int_0^1 \left(w^2 + w_x^2 + w_{xx}^2\right)\mathrm{d}x\right)^2 \\
\leq & \frac{l}{4}C_e^4 \left(\frac{7}{4}\int_0^1 w_{xx}^2 \mathrm{d}x\right)^2 \\
\leq & \frac{49l}{64}C_e^4 \left(\|w_{xx}\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2\right)^2 \\
:= & c_h \left(\|w_{xx}\|_{L^2(0,1)}^2 + \|v\|_{L^2(0,1)}^2\right)^2 .
\end{aligned} \tag{5.8}$$

We deduce by inequalities (5.7) and (5.8) that inequality (5.3) holds true.

Proof of Proposition 5.1. Let $\mathbf{X}(t) := (w[t], v[t])$ be the state of system (3.2). We proceed with the proof in three steps.

Step 1: We prove that there are positive constants m, λ, and C_5 depending only on a, b, c, k, and l, such that

$$E(t) \leq E(0)\mathrm{e}^{-\lambda t} + C_5 \int_0^t \left(\|f[s]\|_{L^2(0,1)}^2 + d_1^2(s) + d_2^2(s)\right)\mathrm{e}^{-\lambda(t-s)}\mathrm{d}s, \forall t \in [0, T_{max}), \tag{5.9}$$

where

$$E(t) := \frac{1}{2}\int_0^1 \left(aw_{xx}^2(x,t) + kw^2(x,t) + \frac{l}{2}w^4(x,t) + v^2(x,t) + 2mw(x,t)v(x,t)\right)\mathrm{d}x,$$

and $T_{max} \in \mathbb{R}_{>0}$ is the maximal time for the existence of a solution.

Indeed, in view of inequality (5.6), we deduce first that

$$\begin{aligned}
E(t) \leq & \frac{1}{2}\int_0^1 \left(aw_{xx}^2 + \frac{1}{4}(k+m)w_{xx}^2 + \frac{l}{2}w^4 + (1+m)\,v^2\right)\mathrm{d}x \\
\leq & \frac{1}{2}\max\left\{1 + \frac{1}{4a}(k+m), 1+m\right\}\int_0^1 \left(aw_{xx}^2 + \frac{l}{2}w^4 + v^2\right)\mathrm{d}x \\
\leq & \lambda_0 \int_0^1 \left(aw_{xx}^2 + kw^2 + \frac{l}{2}w^4 + v^2 + bv_{xx}^2\right)\mathrm{d}x,
\end{aligned} \tag{5.10}$$

where $\lambda_0 := \frac{1}{2}\max\left\{1 + \frac{1}{4a}(k+m), 1+m\right\}$.

By equation (3.1a) and direct computations, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = \int_0^1 \left(aw_{xx}v_{xx} + kwv + lw^3v + vv_t + mvv + mwv_t\right)\mathrm{d}x$$

$$
\begin{aligned}
&= \int_0^1 \left(aw_{xx}v_{xx} + kwv + lw^3v + mv^2\right) \mathrm{d}x \\
&\quad - \int_0^1 v\left((aw_{xx} + bv_{xx})_{xx} + cv + kw + lw^3 - f\right) \mathrm{d}x \\
&\quad - \int_0^1 mw\left((aw_{xx} + bv_{xx})_{xx} + cv + kw + lw^3 - f\right) \mathrm{d}x \\
&= \int_0^1 \left(aw_{xx}v_{xx} + mv^2 - cv^2 - cmwv - kmw^2 - lmw^4\right) \mathrm{d}x \\
&\quad - \int_0^1 (aw_{xx} + bv_{xx})_{xx}(v + mw)\mathrm{d}x + \int_0^1 f(v + mw)\mathrm{d}x. \qquad (5.11)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&- \int_0^1 (aw_{xx} + bv_{xx})_{xx}(v + mw)\mathrm{d}x \\
= &- (aw_{xx} + bv_{xx})_x(v + mw)\Big|_{x=0}^{x=1} + (aw_{xx} + bv_{xx})(v_x + mw_x)\Big|_{x=0}^{x=1} \\
&- \int_0^1 (aw_{xx} + bv_{xx})(v_{xx} + mw_{xx})\mathrm{d}x \\
= &- (aw_{xx} + bv_{xx})_x(1,t)(v + mw)(1,t) + (aw_{xx} + bv_{xx})_x(0,t)(v + mw)(0,t) \\
&+ (aw_{xx} + bv_{xx})(1,t)(v_x + mw_x)(1,t) - (aw_{xx} + bv_{xx})(0,t)(v_x + mw_x)(0,t) \\
&- \int_0^1 (aw_{xx} + bv_{xx})(v_{xx} + mw_{xx})\mathrm{d}x \\
= &-d_1(t)\,(v_x(0,t) + mw_x(0,t)) - d_2(t)\,(v(1,t) + mw(1,t)) \\
&- \int_0^1 (aw_{xx} + bv_{xx})(v_{xx} + mw_{xx})\mathrm{d}x.
\end{aligned}
$$

Thus, equation (5.11) becomes

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) = &- d_1(t)\,(v_x(0,t) + mw_x(0,t)) - d_2(t)\,(v(1,t) + mw(1,t)) \\
&- \int_0^1 \left(amw_{xx}^2 + kmw^2 + lmw^4 + (c - m)v^2 + bv_{xx}^2\right) \mathrm{d}x \\
&- \int_0^1 (cmwv + bmw_{xx}v_{xx})\,\mathrm{d}x + \int_0^1 f(v + mw)\mathrm{d}x. \qquad (5.12)
\end{aligned}
$$

Note that Young's inequality yields

$$
\left|\int_0^1 cmwv\mathrm{d}x\right| \le cm \int_0^1 \left(\frac{\varepsilon_0}{2}w^2 + \frac{1}{2\varepsilon_0}v^2\right) \mathrm{d}x, \qquad (5.13a)
$$

$$
\left|\int_0^1 bmw_{xx}v_{xx}\mathrm{d}x\right| \le bm \int_0^1 \left(\frac{\varepsilon_1}{2}w_{xx}^2 + \frac{1}{2\varepsilon_1}v_{xx}^2\right) \mathrm{d}x, \qquad (5.13b)
$$

$$
\left|\int_0^1 f(v + mw)\mathrm{d}x\right| \le \int_0^1 \left(\frac{1}{2\varepsilon_2}f^2 + \varepsilon_2 v^2 + \varepsilon_2 m^2 w^2\right) \mathrm{d}x, \qquad (5.13c)
$$

where $\varepsilon_0, \varepsilon_1, \varepsilon_2 \in \mathbb{R}_{>0}$ will be determined later.

In addition, by using Young's inequality and Lemma 2.1(i) and (ii), we have

$$\begin{aligned}
&|d_1(t)(v_x(0,t) + mw_x(0,t)) + d_2(t)(v(1,t) + mw(1,t))| \\
\le&\frac{1}{2\varepsilon_3}d_1^2(t) + \varepsilon_3 v_x^2(0,t) + m^2\varepsilon_3 w_x^2(0,t) + \frac{1}{2\varepsilon_4}d_2^2(t) + \varepsilon_4 v^2(1,t) + m^2\varepsilon_3 w^2(1,t) \\
\le&\frac{1}{2\varepsilon_3}d_1^2(t) + \varepsilon_3\left(2\|v_x\|^2_{L^2(0,1)} + \|v_{xx}\|^2_{L^2(0,1)}\right) + m^2\varepsilon_3\left(2\|w_x\|^2_{L^2(0,1)} + \|w_{xx}\|^2_{L^2(0,1)}\right) \\
&+ \frac{1}{2\varepsilon_4}d_2^2(t) + \varepsilon_4\left(2\|v\|^2_{L^2(0,1)} + \|v_x\|^2_{L^2(0,1)}\right) + m^2\varepsilon_4\left(2\|w\|^2_{L^2(0,1)} + \|w_x\|^2_{L^2(0,1)}\right) \\
\le&\frac{1}{2\varepsilon_3}d_1^2(t) + 2\varepsilon_3\|v_{xx}\|^2_{L^2(0,1)} + 2m^2\varepsilon_3\|w_{xx}\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon_4}d_2^2(t) + \varepsilon_4\|v_{xx}\|^2_{L^2(0,1)} \\
&+ m^2\varepsilon_4\|w_{xx}\|^2_{L^2(0,1)} \\
=&\frac{1}{2\varepsilon_3}d_1^2(t) + \frac{1}{2\varepsilon_4}d_2^2(t) + m^2(2\varepsilon_3 + \varepsilon_4)\|w_{xx}\|^2_{L^2(0,1)} + (2\varepsilon_3 + \varepsilon_4)\|v_{xx}\|^2_{L^2(0,1)},
\end{aligned} \tag{5.14}$$

where $\varepsilon_3, \varepsilon_4 \in \mathbb{R}_{>0}$ will be determined later.

Now we discuss the two cases.

Case 1: $k > 0$. Combining equation (5.12), inequality (5.13), and inequality (5.14), we deduce that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E(t) \le&\frac{1}{2\varepsilon_2}\|f[t]\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon_3}d_1^2(t) + \frac{1}{2\varepsilon_4}d_2^2(t) - \int_0^1\left(a - \frac{b\varepsilon_1}{2} - 2m\varepsilon_3 - m\varepsilon_4\right)mw_{xx}^2\mathrm{d}x \\
&- \int_0^1\left(k - \frac{c\varepsilon_0}{2} - m\varepsilon_2\right)mw^2\mathrm{d}x - \int_0^1 lmw^4\mathrm{d}x - \int_0^1\left(c - m - \frac{cm}{2\varepsilon_0} - \varepsilon_2\right)v^2\mathrm{d}x \\
&- \int_0^1\left(1 - \frac{m}{2\varepsilon_1} - \frac{2\varepsilon_3}{b} - \frac{\varepsilon_4}{b}\right)bv_{xx}^2\mathrm{d}x \\
=&\frac{1}{2\varepsilon_2}\|f[t]\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon_3}d_1^2(t) + \frac{1}{2\varepsilon_4}d_2^2(t) - m\left(1 - \frac{b\varepsilon_1}{2a} - \frac{2m\varepsilon_3}{a} - \frac{m\varepsilon_4}{a}\right)\int_0^1 aw_{xx}^2\mathrm{d}x \\
&- m\left(1 - \frac{c\varepsilon_0}{2k} - \frac{m\varepsilon_2}{k}\right)\int_0^1 kw^2\mathrm{d}x - m\int_0^1 lw^4\mathrm{d}x - m\left(\frac{c}{m} - 1 - \frac{c}{2\varepsilon_0} - \frac{\varepsilon_2}{m}\right)\int_0^1 v^2\mathrm{d}x \\
&- m\left(\frac{1}{m} - \frac{1}{2\varepsilon_1} - \frac{2\varepsilon_3}{mb} - \frac{\varepsilon_4}{mb}\right)\int_0^1 bv_{xx}^2\mathrm{d}x.
\end{aligned} \tag{5.15}$$

Define the constant

$$\lambda_1 := m\cdot\min\left\{1 - \frac{b\varepsilon_1}{2a} - \frac{2m\varepsilon_3}{a} - \frac{m\varepsilon_4}{a}, 1 - \frac{c\varepsilon_0}{2k} - \frac{m\varepsilon_2}{k}, \frac{c}{m} - 1 - \frac{c}{2\varepsilon_0} - \frac{\varepsilon_2}{m}, \frac{1}{m} - \frac{1}{2\varepsilon_1} - \frac{2\varepsilon_3}{mb} - \frac{\varepsilon_4}{mb}\right\}.$$

To ensure that λ_1 is positive, we first let $\varepsilon_0, \varepsilon_1 \in \mathbb{R}_{>0}$ satisfy

$$1 - \frac{b\varepsilon_1}{2a} > 0 \quad \text{and} \quad 1 - \frac{c\varepsilon_0}{2k} > 0.$$

Then, we let $m \in \mathbb{R}_{>0}$ satisfy $m < \min\{4a, 1\}$ and

$$\frac{1}{m} - \frac{1}{2\varepsilon_1} > 0 \quad \text{and} \quad \frac{c}{m} - 1 - \frac{c}{2\varepsilon_0} > 0.$$

In addition, we choose sufficiently small $\varepsilon_2 \in \mathbb{R}_{>0}$ such that

$$1 - \frac{c\varepsilon_0}{2k} - \frac{m\varepsilon_2}{2k} > 0 \quad \text{and} \quad \frac{c}{m} - 1 - \frac{c}{2\varepsilon_0} - \frac{\varepsilon_2}{m} > 0.$$

Furthermore, we choose sufficiently small $\varepsilon_3, \varepsilon_4 \in \mathbb{R}_{>0}$ such that

$$1 - \frac{b\varepsilon_1}{2a} - \frac{m\varepsilon_3}{2a} - \frac{m\varepsilon_4}{a} > 0 \quad \text{and} \quad \frac{1}{m} - \frac{1}{2\varepsilon_1} - \frac{2\varepsilon_3}{mb} - \frac{\varepsilon_4}{mb} > 0.$$

It is clear that $\lambda_1 > 0$ for such choices of $\varepsilon_0, \varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$, and m. Therefore, inequality (5.15) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq & \frac{1}{2\varepsilon_2}\|f[t]\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon_3}d_1^2(t) + \frac{1}{2\varepsilon_4}d_2^2(t) \\ & - \lambda_1 \int_0^1 \left(aw_{xx}^2 + kw^2 + \frac{l}{2}w^4 + v^2 + bv_{xx}^2\right)\mathrm{d}x. \end{aligned} \tag{5.16}$$

It follows from inequalities (5.16) and (5.10) that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq -\frac{\lambda_1}{\lambda_0}E(t) + \frac{1}{2\varepsilon_2}\|f[t]\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon_3}d_1^2(t) + \frac{1}{2\varepsilon_4}d_2^2(t). \tag{5.17}$$

Applying Gronwall's inequality to equation (5.17), we deduce that inequality (5.9) holds true with $\lambda := \frac{\lambda_1}{\lambda_0}$ and $C_5 := \max\left\{\frac{1}{2\varepsilon_2}, \frac{1}{2\varepsilon_3}, \frac{1}{2\varepsilon_4}\right\}$.

Case 2: $k = 0$**.** It suffices to note that inequality (5.15) becomes

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E(t) \leq & \frac{1}{2\varepsilon_2}\|f[t]\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon_3}d_1^2(t) + \frac{1}{2\varepsilon_4}d_2^2(t) - \int_0^1 \left(a - \frac{b\varepsilon_1}{2} - 2m\varepsilon_3 - m\varepsilon_4\right)mw_{xx}^2\mathrm{d}x \\ & + \int_0^1 \left(\frac{c\varepsilon_0}{2} + m\varepsilon_2\right)mw^2\mathrm{d}x - \int_0^1 lmw^4\mathrm{d}x - \int_0^1 \left(c - m - \frac{cm}{2\varepsilon_0} - \varepsilon_2\right)v^2\mathrm{d}x \\ & - \int_0^1 \left(1 - \frac{m}{2\varepsilon_1} - \frac{2\varepsilon_3}{b} - \frac{\varepsilon_4}{b}\right)bv_{xx}^2\mathrm{d}x \\ \leq & \frac{1}{2\varepsilon_2}\|f[t]\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon_3}d_1^2(t) + \frac{1}{2\varepsilon_4}d_2^2(t) - \int_0^1 \left(a - \frac{b\varepsilon_1}{2} - 2m\varepsilon_3 - m\varepsilon_4\right)mw_{xx}^2\mathrm{d}x \\ & + \int_0^1 \frac{1}{4}\left(\frac{c\varepsilon_0}{2} + m\varepsilon_2\right)mw_{xx}^2\mathrm{d}x - \int_0^1 lmw^4\mathrm{d}x - \int_0^1 \left(c - m - \frac{cm}{2\varepsilon_0} - \varepsilon_2\right)v^2\mathrm{d}x \\ & - \int_0^1 \left(1 - \frac{m}{2\varepsilon_1} - \frac{2\varepsilon_3}{b} - \frac{\varepsilon_4}{b}\right)bv_{xx}^2\mathrm{d}x \\ = & \frac{1}{2\varepsilon_2}\|f[t]\|^2_{L^2(0,1)} + \frac{1}{2\varepsilon_3}d_1^2(t) + \frac{1}{2\varepsilon_4}d_2^2(t) \\ & - m\left(1 - \frac{c\varepsilon_0}{8a} - \frac{b\varepsilon_1}{2a} - \frac{m\varepsilon_2}{4a} - \frac{2m\varepsilon_3}{a} - \frac{m\varepsilon_4}{a}\right)\int_0^1 aw_{xx}^2\mathrm{d}x \\ & - \int_0^1 lmw^4\mathrm{d}x - m\left(\frac{c}{m} - 1 - \frac{c}{2\varepsilon_0} - \frac{\varepsilon_2}{m}\right)\int_0^1 v^2\mathrm{d}x \end{aligned}$$

$$-m\left(\frac{1}{m}-\frac{1}{2\varepsilon_1}-\frac{2\varepsilon_3}{mb}-\frac{\varepsilon_4}{mb}\right)\int_0^1 bv_{xx}^2\mathrm{d}x$$

$$\leq\frac{1}{2\varepsilon_2}\|f[t]\|_{L^2(0,1)}^2+\frac{1}{2\varepsilon_3}d_1^2(t)+\frac{1}{2\varepsilon_4}d_2^2(t)-\lambda_1\int_0^1\left(aw_{xx}^2+\frac{l}{2}w^4+v^2+bv_{xx}^2\right)\mathrm{d}x \tag{5.18}$$

with a positive constant

$$\lambda_1:=m\cdot\min\left\{1-\frac{c\varepsilon_0}{8a}-\frac{b\varepsilon_1}{2a}-\frac{m\varepsilon_2}{4a}-\frac{2m\varepsilon_3}{a}-\frac{m\varepsilon_4}{a},\frac{c}{m}-1-\frac{c}{2\varepsilon_0}-\frac{\varepsilon_2}{m},\frac{1}{m}-\frac{1}{2\varepsilon_1}-\frac{2\varepsilon_3}{mb}-\frac{\varepsilon_4}{mb}\right\},$$

which is ensured by selecting sufficiently small $\varepsilon_0,\varepsilon_1,m,\varepsilon_2,\varepsilon_3,\varepsilon_4\in\mathbb{R}_{>0}$ in order.

Indeed, we first let $\varepsilon_0,\varepsilon_1\in\mathbb{R}_{>0}$ satisfy

$$1-\frac{c\varepsilon_0}{8a}-\frac{b\varepsilon_1}{2a}>0,$$

and $m\in\mathbb{R}_{>0}$ satisfy $m<\min\{4a,1\}$ and

$$\frac{1}{m}-\frac{1}{2\varepsilon_1}>0\quad\text{and}\quad\frac{c}{m}-1-\frac{c}{2\varepsilon_0}>0,$$

respectively. Then, we choose sufficiently small $\varepsilon_2\in\mathbb{R}_{>0}$ such that

$$1-\frac{c\varepsilon_0}{8a}-\frac{b\varepsilon_1}{2a}-\frac{m\varepsilon_2}{4a}>0\quad\text{and}\quad\frac{c}{m}-1-\frac{c}{2\varepsilon_0}-\frac{\varepsilon_2}{m}>0.$$

Finally, we choose sufficiently small $\varepsilon_3,\varepsilon_4\in\mathbb{R}_{>0}$ such that

$$1-\frac{c\varepsilon_0}{8a}-\frac{b\varepsilon_1}{2a}-\frac{m\varepsilon_2}{4a}-\frac{2m\varepsilon_3}{a}-\frac{m\varepsilon_4}{a}>0\quad\text{and}\quad\frac{1}{m}-\frac{1}{2\varepsilon_1}-\frac{2\varepsilon_3}{mb}-\frac{\varepsilon_4}{mb}>0.$$

It follows from inequalities (5.18) and (5.10) with $k=0$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t)\leq-\frac{\lambda_1}{\lambda_0}E(t)+\frac{1}{2\varepsilon_2}\|f[t]\|_{L^2(0,1)}^2+\frac{1}{2\varepsilon_3}d_1^2(t)+\frac{1}{2\varepsilon_4}d_2^2(t),$$

which implies that inequality (5.9) holds true with $\lambda:=\frac{\lambda_1}{\lambda_0}$ and $C_5:=\max\left\{\frac{1}{2\varepsilon_2},\frac{1}{2\varepsilon_3},\frac{1}{2\varepsilon_4}\right\}$.

Step 2: We show that estimate (5.1) holds true for all $t\in[0,T_{max})$.

Indeed, in view of the choice of m, we infer from Lemma 5.2 and inequality (5.9) that

$$\begin{aligned}c_l\left(\|w_{xx}[t]\|_{L^2(0,1)}^2+\|v[t]\|_{L^2(0,1)}^2\right)\leq&E(t)\\ \leq&E(0)\mathrm{e}^{-\lambda t}+C_5\int_0^t\left(\|f[s]\|_{L^2(0,1)}^2+d_1^2(s)+d_2^2(s)\right)\mathrm{e}^{-\lambda(t-s)}\mathrm{d}s\\ \leq&c_u\mathrm{e}^{-\lambda t}\left(\|\varphi_{1xx}\|_{L^2(0,1)}^2+\|\varphi_2\|_{L^2(0,1)}^2\right)+c_h\mathrm{e}^{-\lambda t}\left(\|\varphi_{1xx}\|_{L^2(0,1)}^2+\|\varphi_2\|_{L^2(0,1)}^2\right)^2\\ &+C_5\int_0^t\left(\|f[s]\|_{L^2(0,1)}^2+d_1^2(s)+d_2^2(s)\right)\mathrm{e}^{-\lambda(t-s)}\mathrm{d}s,\end{aligned}$$

where c_l,c_u,c_h are positive constants depending only on $a,k,l,$ and m.

Therefore, it holds that

$$\|w_{xx}[t]\|^2_{L^2(0,1)} + \|v[t]\|^2_{L^2(0,1)} \le \frac{c_u}{c_l}\mathrm{e}^{-\lambda t}\left(\|\varphi_{1xx}\|^2_{L^2(0,1)} + \|\varphi_2\|^2_{L^2(0,1)}\right) + \frac{c_h}{c_l}\mathrm{e}^{-\lambda t}\left(\|\varphi_{1xx}\|^2_{L^2(0,1)} + \|\varphi_2\|^2_{L^2(0,1)}\right)^2 + \frac{C_5}{c_l}\int_0^t \left(\|f[s]\|^2_{L^2(0,1)} + d_1^2(s) + d_2^2(s)\right)\mathrm{e}^{-\lambda(t-s)}\mathrm{d}s. \tag{5.19}$$

In the following, we shall estimate

$$I(t) := \int_0^t \left(\|f[s]\|^2_{L^2(0,1)} + d_1^2(s) + d_2^2(s)\right)\mathrm{e}^{-\lambda(t-s)}\mathrm{d}s.$$

For simplicity, let $\mathcal{F}(s) := \|f[s]\|^2_{L^2(0,1)}$. For any $p_i, q_i \in [1, +\infty]$ satisfying $\frac{1}{p_i} + \frac{1}{q_i} = 1$, $i = 1, 2, 3$, we deduce by Hölder's inequality that

$$I_1(t) := \int_0^t \mathcal{F}(s)\mathrm{e}^{-\lambda(t-s)}\mathrm{d}s \le \left\|\mathrm{e}^{-\lambda(t-\cdot)}\right\|_{L^{p_1}(0,t)} \|\mathcal{F}\|_{L^{q_1}(0,t)},$$

$$I_2(t) := \int_0^t \mathrm{e}^{-\lambda(t-s)} d_1^2(s)\mathrm{d}s \le \left\|\mathrm{e}^{-\lambda(t-\cdot)}\right\|_{L^{p_2}(0,t)} \left\|d_1^2\right\|_{L^{q_2}(0,t)},$$

$$I_3(t) := \int_0^t \mathrm{e}^{-\lambda(t-s)} d_2^2(s)\mathrm{d}s \le \left\|\mathrm{e}^{-\lambda(t-\cdot)}\right\|_{L^{p_3}(0,t)} \left\|d_2^2\right\|_{L^{q_3}(0,t)}.$$

We discuss the three cases.

Case 1: $p_i, q_i \in (1, +\infty)$**,** $i = 1, 2, 3$**.** By direct computations, we have

$$\begin{aligned} I_1(t) &\le \left\|\mathrm{e}^{-\lambda(t-\cdot)}\right\|_{L^{p_1}(0,t)} \|\mathcal{F}\|_{L^{q_1}(0,t)} \\ &= \left(\frac{1}{\lambda p_1}\left(1 - \mathrm{e}^{-\lambda t p_1}\right)\right)^{\frac{1}{p_1}} \left(\int_0^t \|f[s]\|^{2q_1}_{L^2(0,1)}\mathrm{d}s\right)^{\frac{1}{q_1}} \\ &\le \left(\frac{1}{\lambda p_1}\right)^{\frac{1}{p_1}} \left(\int_0^t \|f[s]\|^{2q_1}_{L^2(0,1)}\mathrm{d}s\right)^{\frac{1}{q_1}} \\ &= \left(\frac{1}{\lambda p_1}\right)^{\frac{1}{p_1}} \|f\|^2_{L^{r_1}\left((0,t);L^2(0,1)\right)}, \end{aligned} \tag{5.20}$$

where $r_1 := 2q_1 \in (2, +\infty)$.

Analogously, it holds that

$$I_2(t) + I_3(t) \le \left(\frac{1}{\lambda p_2}\right)^{\frac{1}{p_2}} \|d_1\|^2_{L^{r_2}(0,t)} + \left(\frac{1}{\lambda p_3}\right)^{\frac{1}{p_3}} \|d_2\|^2_{L^{r_3}(0,t)}, \tag{5.21}$$

where $r_i := 2q_i \in (2, +\infty)$, $i = 2, 3$.

Letting $C_M := \max\left\{\left(\frac{1}{\lambda p_1}\right)^{\frac{1}{p_1}}, \left(\frac{1}{\lambda p_2}\right)^{\frac{1}{p_2}}, \left(\frac{1}{\lambda p_3}\right)^{\frac{1}{p_3}}\right\}$, and combining inequalities (5.20) and (5.21), we have

$$I(t) = I_1(t) + I_2(t) + I_3(t) \le C_M\left(\|f\|^2_{L^{r_1}\left((0,t);L^2(0,1)\right)} + \|d_1\|^2_{L^{r_2}(0,t)} + \|d_2\|^2_{L^{r_3}(0,t)}\right). \tag{5.22}$$

Setting $q_1 = q_2 = q_3$, or, equivalently, $r = r_1 = r_2 = r_3$, we infer from inequalities (5.19) and (5.22) that estimate (5.1) holds true for $t \in [0, T_{max})$ with $r = r_1 \in (1, +\infty)$, $\Lambda = \frac{\lambda}{2}$, and some positive constant C depending only on c_l, c_u, c_h, C_5, and C_M.

Case 2: $p_i = 1, q_i = +\infty$**,** $i = 1, 2, 3$**.** Similarly, we have

$$I_1(t) \leq \left\| e^{-\lambda(t-\cdot)} \right\|_{L^1(0,t)} \|\mathcal{F}\|_{L^\infty(0,t)} = \|f\|^2_{L^\infty((0,t);L^2(0,1))} \int_0^t e^{-\lambda(t-s)} \mathrm{d}s \leq \frac{1}{\lambda} \|f\|^2_{L^\infty((0,t);L^2(0,1))}, \tag{5.23}$$

and

$$I_2(t) + I_3(t) \leq \frac{1}{\lambda} \left(\|d_1\|^2_{L^\infty(0,t)} + \|d_2\|^2_{L^\infty(0,t)} \right). \tag{5.24}$$

We infer from inequalities (5.23), (5.24), and (5.19) that estimate (5.1) holds true for $t \in [0, T_{max})$ with $r = +\infty$, $\Lambda = \frac{\lambda}{2}$, and some positive constant C depending only on c_l, c_u, c_h, C_5, and $\frac{1}{\lambda}$.

Case 3: $p_i = +\infty, q_i = 1$**,** $i = 1, 2, 3$**.** By direct computations, it holds that

$$I_1(t) \leq \left\| e^{-\lambda(t-\cdot)} \right\|_{L^\infty(0,t)} \|\mathcal{F}\|_{L^1(0,t)} = \int_0^t \|f[s]\|^2_{L^2(0,1)} \mathrm{d}s = \|f\|^2_{L^2((0,t);L^2(0,1))}, \tag{5.25}$$

and

$$I_2(t) + I_3(t) \leq \|d_1\|^2_{L^2(0,t)} + \|d_2\|^2_{L^2(0,t)}. \tag{5.26}$$

We infer from inequalities (5.25), (5.26), and (5.19) that estimate (5.1) holds true for $t \in [0, T_{max})$ with $r = 2$, $\Lambda = \frac{\lambda}{2}$, and some positive constant C depending only on c_l, c_u, c_h, and C_5.

Step 3: Conclusion. In view of the regularity $f, d_1, d_2, \mathbf{X}_0$, and $\mathbf{X}$, if $T_{max} < +\infty$, then estimate (5.1) ensures that

$$\lim_{t \to T_{max}} \|\mathbf{X}[t]\|_{\mathbb{H}} < +\infty.$$

We conclude that there must be $T_{max} = +\infty$. Therefore, estimate (5.1) holds true for all $t \in \mathbb{R}_{\geq 0}$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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