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Research article

Delay differential equations with fractional differential operators: Existence, uniqueness and applications to chaos

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Abstract: In this study, we consider a chaotic model in which fractional differential operators and the delay term are added. Using the Carathéodory existence-uniqueness theorem for this chaotic model modified with the Caputo fractional derivative, we show that the solution of the associated system exists and is unique. We consider the chaotic model with a delay term with Caputo, Caputo–Fabrizio and Atangana–Baleanu fractional derivatives and present a numerical algorithm for these models. We then present the numerical solution of chaotic models with delay terms by using piecewise differential operators, where fractional, classical and stochastic processes can be used. We present the numerical solution of chaotic models with delay terms, as modified by using piecewise differential operators. The graphical representations of these models are simulated for different values of the fractional order.

Keywords: Carathéodory existence-uniqueness theorem; chaotic models; delay term; piecewise derivative
Mathematics Subject Classification: 34A12,34C28, 34Kxx

1. Introduction

It can be seen that many different behaviors are observed in nature, which is full of surprises, and mathematical modeling is important in the analysis of many events. For example, let us consider earthquakes, which are a reality in the world and also in Turkey. When the ruptures caused by the earthquake cause part of the sea floor to rise, the seawater, which is very low in compressibility, also rises. When an earthquake occurs, mathematical modeling can be used to study the spread of a tsunami after the epicenter has been determined. Thanks to modeling, the scale of the disaster can be prevented from increasing by sending warnings to the regions exposed to this situation. As another example, chaos theory emerged when a meteorologist, Edward Lorenz, discovered a pattern that repeats in the printouts while graphing the weather in its simplest form on a primitive computer, and it has been used in modeling in many fields, from the stock market to meteorology, from communication to medicine, and from chemistry to mechanics. While mathematical modeling is done with the classical derivative, in recent years, the classical derivative has been replaced by the fractional derivative [1–4]. In recent years, mathematicians have introduced some fractional differential operators that have considered different laws, such as exponential [1], Mittag–Leffler kernels, and power-law [2,3]. There is no doubt that such functions have been intensively studied since they represent the behavior of several real-world problems. For example, the exponential decay law has been observed in the decay of a dead body, chemical reactions, electrostatics, heat transfer, and many others that are not listed here. Another important kernel is the generalized–Mittag–Leffler function, which is generalized exponential function. The generalized–Mittag–Leffler function can be applied to relaxation processes in viscoelastic and dielectric materials, creep, and renewal processes. The power law can be observed in processes that have properties of self-similarity of fractals, the spread of infectious diseases, and so on. These operators have appeared in chaos theory, for which there is a need for a very deep theory in the literature. Undoubtedly, the fractional differential operators have opened new doors in modeling because researchers have shown that the associated system may exhibit crossover behaviors at different fractional orders. This led researchers to focus on the analysis of these classes of differential operators. Moreover, the newly introduced piecewise differential operators have taken into account different behaviors with a system that can be considered as the processes from power-law to exponential or the processes from classical to stochastic. It can be concluded that these operators could model different processes at different time intervals by using the piecewise concept. Many models with these operators have been considered, and some analysis has been presented as related to piecewise models. Some research about the existence and uniqueness of differential equations with singular and nonsingular kernels is presented below. In [5], a critical analysis of computational modeling of a common disease, ie., hand-foot-mouth-and-disease, has been considered. This study deals with a model with delay parameters that is combined with six different sub-populations and the authors have investigated the existence and uniqueness of the solution of this model. In [6], the authors have investigated the existence of positive solutions to the boundary value problem for a high order fractional differential equation with delay and singularities. To determine some existence results for positive solutions, the properties of the Green function, a fixed-point theorem, and Leray–Schauder's nonlinear alternative theorem have been examined in [6]. In [7], the authors constructed an extension of the reproducing kernel method that is more accurate than the classical one for a larger time interval. To show the efficacy of the improved method, they have presented some numerical examples. In [8], the authors have considered a class of fractional differential equations with a proportional delay term and anti-periodic boundary conditions, and they have presented some theoretical results, including existence and stability theory, for the considered problem. In [9], fractional delay differential equations are discussed, and the authors present some theory about existence and uniqueness based on the method of steps. Also, they have presented some numerical examples for this class of differential equations. In [10], they have introduced a numerical method for the solution of the fractional delay differential equations. The authors have shown that the method is applicable and useful to solve these equations with various examples. Moreover, they have presented a comparison between the presented method and existing methods like the fractional Adams method and the predictor-corrector method. In [11], the fractional functional differential equations with bounded delay are considered, and the existence and uniqueness theorems have been proved for these equations. The existence and uniqueness of solutions of fractional delay differential equations have been examined by using the Banach contraction principle and then Krasnoselskii's fixed point theorem to establish theoretical results.

They have applied these results to the Lotka-Volterra model which is well-known in [12]. In [13], the authors have investigated the Mittag–Leffler stability of fractional-order systems with varying time delays, and they have benefited from applying the linearization method in combination with a new weighted type norm to achieve their goals. Further, they have also illustrated the efficiency of the theoretical results with some examples. In [14], the authors constructed a theorem that takes into account the properties of Mittag–Leffler functions for the existence and uniqueness of global solutions of delay differential equations with fractional differential operators. Also, they have guaranteed these solutions to be exponentially bounded providing a sufficient condition. In [15], by using a generalized Gronwalls inequality, the authors have shown that the solution of nonlinear nabla fractional difference systems exists and is unique. In [16], the existence and uniqueness of a Cauchy problem including different differential operators have been investigated by using Carathéodory's conditions. The existence of differential-difference equations of delay type is shown by using Carathéodory's functions, and some topological results have been presented by giving a topological transversality theorem in [17]. In [18], by Carathéodory's theorem and Ascoli's lemma, it has been shown under which conditions the solutions of generalized nonlinear differential equations exist. In [19], the abstract measure delay differential equations have been considered to investigate the existence of solutions of such equations by using the Leray–Schauder nonlinear alternative under Carathéodory conditions. The local and global existence of ordinary differential equations with power-law kernels have been investigated and the feed-back control of chaotic fractional differential equations is theoretically illustrated for the fractional Lorenz system in [20]. In [21], Persson presents a generalized version of Carathéodory's existence theorem for ordinary differential equations. To obtain information about the associated subject, the studies in [22–28] can be examined.

2. Carathéodory conditions for chaotic models with delay terms

In this section, we investigate the local and global existence of the following chaotic system [29] where the delay term is added. Such system is represented by:

$$\begin{aligned} x'(t) &= a_1 y(t-\tau) - a_2 x(t-\tau) + a_3 x(t) z(t) \\ y'(t) &= -b_1 x(t) z(t) - b_2 x(t-\tau) + b_3 y(t) z(t) \\ z'(t) &= c_1 - c_2 y^2 (t-\tau). \end{aligned} \tag{2.1}$$

Note that the system without a delay term has chaotic behavior with the parameters

$$a_1 = 1, a_2 = 0.7, a_3 = 0.3, b_1 = 4, b_2 = 4.3, b_3 = 1, c_1 = 10, c_2 = 1, \tag{2.2}$$

and the initial conditions are given by

$$x(0) = 0.01, y(0) = 0.01, z(0) = 0.01. \tag{2.3}$$

The numerical simulation of the chaotic system is presented in Figure 1.

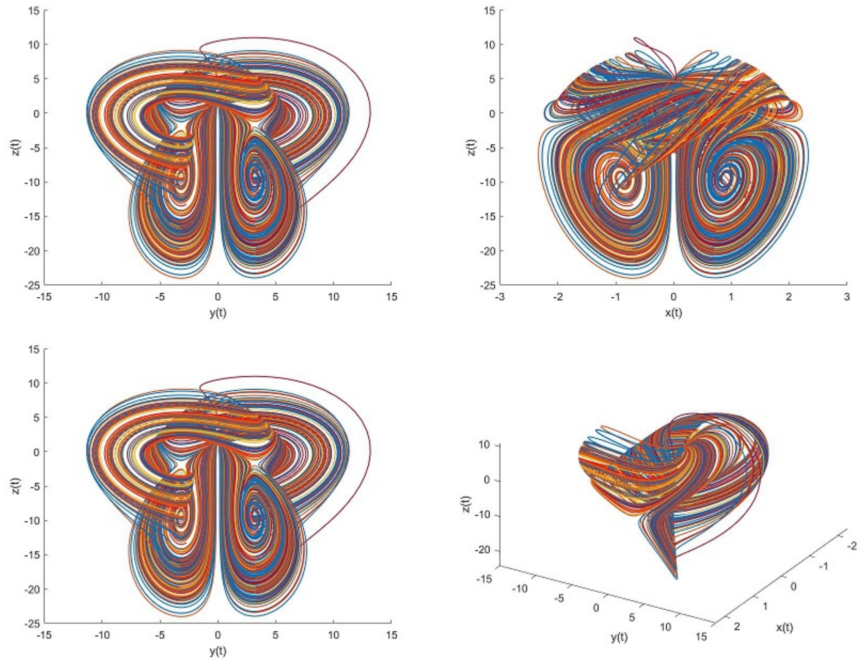

Figure 1. Numerical visualization of the considered chaotic model for $\tau = 0$.

2.1. Local existence theory for the chaotic model with Caputo fractional derivatives

In this section, the local existence of the following chaotic system [29] is proven by using the Carathéodory theorem [28]. This system is represented by:

$$\begin{aligned} {}_0^C D_t^\alpha \mathbf{x}(t) &= a_1 y(t-\tau) - a_2 x(t-\tau) + a_3 x(t) z(t) \\ {}_0^C D_t^\alpha \mathbf{y}(t) &= -b_1 x(t) z(t) - b_2 x(t-\tau) + b_3 y(t) z(t) \\ {}_0^C D_t^\alpha \mathbf{z}(t) &= c_1 - c_2 y^2(t-\tau). \end{aligned} \tag{2.4}$$

We shall define some notations for simplicity

$$\begin{aligned} \mathbf{x}(t) &= \begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix}, \\ \mathbf{F}(t, \mathbf{x}(t), \mathbf{x}(t-\tau)) &= \begin{bmatrix} a_1 y(t-\tau) - a_2 x(t-\tau) + a_3 x(t) z(t) \\ -b_1 x(t) z(t) - b_2 x(t-\tau) + b_3 y(t) z(t) \\ c_1 - c_2 y^2(t-\tau) \end{bmatrix}. \end{aligned} \tag{2.5}$$

The theory related to Carathéodory's theory for proving the existence and uniqueness of fractional differential equations has been presented in [20,27]. Here, we will discuss this theory for delay differential equations.

Theorem 1. Suppose that the function F: $\mathfrak{R} \to \mathbb{R}^3$ holds for the following conditions:

i. $F(t, \mathbf{x}(t), \mathbf{x}(t-\tau))$ is Lebesgue measurable with respect to t on $\mathfrak{I}$.
ii. $F(t, \mathbf{x}(t), \mathbf{x}(t-\tau))$ is continuous with respect to x on $\aleph$.

iii. There exists a real-valued function $C(t) \in L^2(\mathfrak{I})$ such that the following is satisfied for almost all $t \in \mathfrak{I}$ and all $x \in \aleph$:

$$\|\mathbf{F}(t, \mathbf{x}(t), \mathbf{x}(t-\tau))\| \leq C(t). \tag{2.6}$$

Note that

$$\begin{aligned} \mathfrak{R} &= \left\{(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3 \mid t \in \mathfrak{I}, \mathbf{x} \in \aleph\right\} \mathfrak{I} = [t_0 - h, t_0 + h], \\ \aleph &= \left\{\mathbf{x} \in \mathbb{R}^3 \mid \|\mathbf{x} - \mathbf{x}_0\| \leq \widetilde{M}\right\}. \end{aligned} \tag{2.7}$$

We want to check that the conditions of Carathéodory [28] hold for the considered model with a delay term. For proof of the first condition of local existence, we verify that the function $F(\mathbf{x},t)$ is Lebesgue integrable with respect to t on $\mathfrak{I}$. Here, proof will be presented for only $[t_0, t_0 + h]$; also, note that the same procedure can be performed for $[t_0 - h, t_0]$. Providing that the function $x(t)$ is Lebesgue measurable on the interval $[t_0, t_0 + h]$, the function $x(t)$ is a sequence of step functions, that is $\left\{\mathbf{x}_p(t)\right\}$ $(p = 1, 2, ..)$, such that $x_p(t) \rightarrow x(t)$ almost everywhere as $p \to \infty$. We can conclude that $F\left(\mathbf{x}_p, t\right) \rightarrow F(\mathbf{x},t)$ on the interval $[t_0, t_0 + h]$ as $p \to \infty$ from condition ii. As a result, $F(\mathbf{x},t)$ is Lebesgue measurable on the interval $[t_0, t_0 + h]$.

To show that the function $F(\mathbf{x}(s-\tau), \mathbf{x}(\tau), \tau)$ is Lebesgue integrable with respect to t on $\mathfrak{I}$, we write the following by using the third condition of Theorem 1:

$$(t-s)^{\alpha-1} \mathbf{F}(\mathbf{x}(s-\tau), \mathbf{x}(s), s) \leq (t-s)^{\alpha-1} C(s), \tag{2.8}$$

for almost every $\tau \leq t$ with $\tau, t \in \mathfrak{I}$. When $\alpha > \frac{1}{2}$, the function $(t-\tau)^{\alpha-1} \in L_2(t_0, t)$.

Integrating the above inequality on $[t_0, t]$ yields

$$\varphi(t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \mathbf{F}(\mathbf{x}(s-\tau), \mathbf{x}(s), s)\, ds \leq \varphi(t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} C(s)\, ds. \tag{2.9}$$

Using the Hölder inequality, one can obtain

$$\begin{aligned} \left|\varphi(t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \mathbf{F}(\mathbf{x}(s-\tau), \mathbf{x}(s), s)\, ds\right| &\leq \|\varphi(t)\| + \frac{1}{\Gamma(\alpha)} \sqrt{\frac{(t-t_0)^{2\alpha-1}}{2\alpha-1}} \|C(s)\|_{L_2(t_0,t)} \\ &\leq \|\varphi(t)\| + \frac{1}{\Gamma(\alpha)} \sqrt{\frac{(t-t_0)^{2\alpha-1}}{2\alpha-1}} \|C(s)\|_{L_2(t_0,t)}, \end{aligned} \tag{2.10}$$

where

$$\|C(s)\|_{L_2(t_0,t)} = \left(\int_{t_0}^{t} |C(s)|^2\, ds\right)^{1/2}. \tag{2.11}$$

Now, let us define a function sequence denoted by $\left\{\mathbf{x}_p(t)\right\}_{p=1}^{\infty}$. This sequence can be defined by

$$\mathbf{x}_p(t) = \left\{\varphi(t) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \mathbf{F}(\mathbf{x}(s-\tau), \mathbf{x}(s), s)\, ds, t_0 + \frac{h}{p} \leq t \leq t_0 + h\right.. \tag{2.12}$$

Let us prove that this set of functions is uniformly bounded and equicontinuous. Here, the function $C^2(t)$ is completely continuous, that is, for a given positive number M

$$\int_{t_0}^{t_0+h} C^2(s)\,ds \leq M \tag{2.13}$$

where $h > 0$.

Note that $F\left(\mathbf{x}_p(s-\tau), \mathbf{x}_p(s), s\right) \equiv F(\mathbf{x}_0, s)$ when $t_0 \leq s \leq t - \frac{h}{p}$. Hence, $F\left(\mathbf{x}_p(s-\tau), \mathbf{x}_p(s), s\right)$ is Lebesgue-measurable, and it is Lebesgue-integrable on the interval $\left[t_0, t - \frac{h}{p}\right]$. We next demonstrate that $x_p(s)$ is continuous on $\left[t_0, t_0 + \frac{2h}{p}\right]$ for all n.

First, we can consider the case in which $t_0 \leq t_1 \leq t_0 + \frac{h}{p} < t_2 \leq t_0 + \frac{2h}{p}$:

$$\begin{aligned}
\left\|\mathbf{x}_p(t_2) - \mathbf{x}_p(t_1)\right\| &\leq \|\varphi\| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2-\frac{h}{p}} (t_2-s)^{\alpha-1}\left\|\mathbf{F}\left(\mathbf{x}_p(s-\tau), \mathbf{x}_p(s), s\right)\right\| ds \\
&\leq \|\varphi\| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t_2-\frac{h}{p}} (t_2-s)^{\alpha-1} C(s)\,ds \\
&\leq \|\varphi\| + \frac{\sqrt{M}}{\Gamma(\alpha)}\left(\frac{(t_2-s)^{2\alpha-1}}{2\alpha-1}\Big|_{t_0}^{t_2-\frac{h}{p}}\right)^{1/2} \\
&\leq \|\varphi\| + -\frac{1}{\Gamma(\alpha)}\sqrt{\frac{M}{2\alpha-1}}\begin{pmatrix} \left(t_2 - \left(t_0 + \frac{h}{p}\right) + \frac{h}{p}\right)^{2\alpha-1} \\ -\left(\frac{h}{p}\right)^{2\alpha-1} \end{pmatrix}^{1/2} \\
&\leq \|\varphi\|(t_2-t_1) + \frac{1}{\Gamma(\alpha)}\sqrt{\frac{M}{2\alpha-1}}(t_2-t_1)^{\alpha-1/2}
\end{aligned} \tag{2.14}$$

Hence, there exists a positive number $\delta = (t_2 - t_1) > 0$:

$$\begin{aligned}
\left\|\mathbf{x}_p(t_2) - \mathbf{x}_p(t_1)\right\| &\leq \|\varphi\|\delta + \frac{1}{\Gamma(\alpha)}\sqrt{\frac{M}{2\alpha-1}}\delta^{\alpha-1/2} \\
&\leq \left(\|\varphi\|\delta + \frac{1}{\Gamma(\alpha)}\sqrt{\frac{M}{2\alpha-1}}\delta^{\alpha-1/2}\right)\|t_2 - t_1\|
\end{aligned} \tag{2.15}$$

For all p, we can have

$$\left\|\mathbf{x}_p(t_2) - \mathbf{x}_p(t_1)\right\| \leq \varepsilon\|t_2 - t_1\|. \tag{2.16}$$

such that $\varepsilon = \|\varphi\| + \frac{1}{\Gamma(\alpha)}\sqrt{\frac{M}{2\alpha-1}}\delta^{\alpha-1/2}$. Next, we have the following case in which $t_0 + \frac{h}{p} \leq t_1 < t_2 \leq t_0 + \frac{2h}{p}$:

$$\begin{aligned}
\left\|\mathbf{x}_p(t_2) - \mathbf{x}_p(t_1)\right\| &\leq \|\varphi\| + \frac{1}{\Gamma(\alpha)}\sqrt{\frac{M}{2\alpha-1}}\left(\int_{t_0}^{t_1-\frac{h}{p}}\begin{pmatrix} (t_1-s)^{\alpha-1} \\ -(t_2-s)^{\alpha-1}\end{pmatrix}^2 ds\right)^{1/2} \\
&\quad + \frac{1}{\Gamma(\alpha)}\sqrt{\frac{M}{2\alpha-1}}\left(\int_{t_1-\frac{h}{p}}^{t_2-\frac{h}{p}} (t_2-s)^{2\alpha-2}\,ds\right)^{1/2} \\
&\leq \|\varphi\| + -\frac{1}{\Gamma(\alpha)}\sqrt{\frac{2M}{2\alpha-1}}\left[-\left((t_1-s)^{2\alpha-1} + (t_2-s)^{2\alpha-1}\right)\Big|_{t_0}^{t_1-\frac{h}{p}}\right]^{1/2}
\end{aligned} \tag{2.17}$$

$$
\begin{aligned}
& +\frac{1}{\Gamma(\alpha)}\sqrt{\frac{2M}{2\alpha-1}}\left[-(t_2-s)^{2\alpha-1}\Big|_{t_1-\frac{h}{p}}^{t_2-\frac{h}{p}}\right]^{1/2} \\
\leq\ & \|\varphi\|+\frac{1}{\Gamma(\alpha)}\sqrt{\frac{2M}{2\alpha-1}}\left[\begin{array}{c}(t_1-t_0)^{2\alpha-1}+(t_2-t_0)^{2\alpha-1}-\frac{2h}{p}^{2\alpha-1} \\ -\left(t_2-t_1+\frac{h}{p}\right)^{2\alpha-1}+\frac{h}{p}^{2\alpha-1}\end{array}\right]^{1/2} \\
& +\frac{1}{\Gamma(\alpha)}\sqrt{\frac{2M}{2\alpha-1}}\left[\left(t_2-t_1+\frac{h}{p}\right)^{2\alpha-1}-\frac{h}{p}^{2\alpha-1}\right]^{1/2} \\
\leq\ & \|\varphi\|+\frac{1}{\Gamma(\alpha)}\sqrt{\frac{2M}{2\alpha-1}}(\delta_1+\delta_2)^{1/2}+\frac{1}{\Gamma(\alpha)}\sqrt{\frac{2M}{2\alpha-1}}\delta_2^{1/2} \\
<\ & \widetilde{\varepsilon}
\end{aligned}
$$

where

$$
\begin{aligned}
\delta_1 &= \left|(t_1-t_0)^{2\alpha-1}+(t_2-t_0)^{2\alpha-1}-\frac{2h}{p}^{2\alpha-1}\right| \\
\delta_2 &= \left|\left(t_2-t_1+\frac{h}{p}\right)^{2\alpha-1}-\frac{h}{p}^{2\alpha-1}\right|
\end{aligned}
\tag{2.18}
$$

This implies that

$$
\left\|\mathbf{x}_p(t_2)-\mathbf{x}_p(t_1)\right\|<\varepsilon. \tag{2.19}
$$

It is contended based on the above that the function $x_p(t)$ is continuous with respect to t on $\left[t_0, t_0+\frac{2h}{p}\right]$ for all p. On the other hand, we have that $t\in\left[t_0+\frac{h}{p}, t_0+\frac{2h}{p}\right]$

$$
\begin{aligned}
\left\|\mathbf{x}_p(t_2)-\mathbf{x}_0\right\| &\leq \|\varphi\|+\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t-\frac{h}{p}}(t-s)^{\alpha-1}C(s)\,ds \\
&\leq \|\varphi\|+\frac{1}{\Gamma(\alpha)}\sqrt{\frac{M}{2\alpha-1}}\left[(t-t_0)^{2\alpha-1}-\left(\frac{h}{p}\right)^{2\alpha-1}\right]^{1/2} \\
&\leq \|\varphi\|+\frac{1}{\Gamma(\alpha)}\sqrt{\frac{M}{2\alpha-1}}\left(\frac{h}{p}\right)^{\alpha-1/2} \\
&\leq \widetilde{M},
\end{aligned}
\tag{2.20}
$$

which yields that $\left(t, \mathbf{x}_p(t), \mathbf{x}_p(t-\tau)\right)\in D$ for all p.

Using the induction method, one can conclude that the function $x_p(t)$ is continuous with respect to t on $[t_0, t_0+h]$ such that it satisfies that $\left(t, \mathbf{x}_p(t), \mathbf{x}_p(t-\tau)\right)\in D$ for all p. We suppose that for a given integer j and all $0\leq i<j<p$, $x_p(t)$ is continuous on $\left[t_0+\frac{ih}{p}, t_0+\frac{(i+1)h}{p}\right]$ and $\left\|\mathbf{x}_p(t_2)-\mathbf{x}_0\right\|\leq\widetilde{M}$ for all p. Applying the same routine as given above, $x_p(t)$ is continuous on $\left[t_0+\frac{jh}{p}, t_0+\frac{(j+1)h}{p}\right]$ and $\left\|\mathbf{x}_p(t_2)-\mathbf{x}_0\right\|\leq\widetilde{M}$ for all p. Consequently, it can be seen that the uniform boundedness and equicontinuity of the function sequence $\left\{\mathbf{x}_p(t)\right\}_{p=1}^{\infty}$ defined on $[t_0, t_0+h]$ is verified since the estimations $\varepsilon, \widetilde{\varepsilon}, \delta, \delta_1, \delta_2, \widetilde{M}$ do not depend on the integer p.

Now, let us verify the proof of the last condition. According to the Arzela-Ascoli theorem and the results presented above, a sequence $\left\{\mathbf{x}_{n_j}(t)\right\}_{j=1}^{\infty} \triangleq \left\{\mathbf{x}_j(t)\right\}_{j=1}^{\infty}$ is involved in $\left\{\mathbf{x}_p(t)\right\}_{p=1}^{\infty}$ where $\left\{\mathbf{x}_j(t)\right\}_{j=1}^{\infty}$ is uniformly convergent to $x(t)$. So, our aim is to demonstrate that the function $x(t)$ is the solution of the chaotic system [29] considered in this chapter. There exists a positive integer ζ such that for all $j > \zeta$ since the function $F(\mathbf{x},t)$ is continuous with respect to x on $\aleph$

$$\left\|\mathbf{F}\left(\mathbf{x}_j(t-\tau), \mathbf{x}_j(t), t\right) - \mathbf{F}(\mathbf{x}(t-\tau), \mathbf{x}(t), t)\right\| \leq \frac{\Gamma(\alpha+1)}{h^\alpha}\omega, \tag{2.21}$$

for any positive ζ. We need to prove that the function $x(t)$ holds for the equation (2.4). Let us define the following mapping

$$\Lambda\mathbf{x} := \varphi(t) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\mathbf{F}(\mathbf{x}(s-\tau), \mathbf{x}(s), s)\,ds. \tag{2.22}$$

Then, one has the following

$$\begin{aligned}
\left\|\Lambda\mathbf{x}_j(t) - \Lambda\mathbf{x}(t)\right\| &= \left\|\begin{array}{c} \varphi(t) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t-\frac{h}{p}}(t-s)^{\alpha-1}\mathbf{F}\left(\mathbf{x}_j(s-\tau), \mathbf{x}_j(s), s\right)ds \\ -\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\mathbf{F}\left(\mathbf{x}(s-\tau), \mathbf{x}_j(s), s\right)ds \end{array}\right\| \\
&\leq \left\|\begin{array}{c} \varphi(t) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left(\begin{array}{c}\mathbf{F}\left(\mathbf{x}_j(s-\tau), \mathbf{x}_j(s), s\right) \\ -\mathbf{F}\left(\mathbf{x}(s-\tau), \mathbf{x}_j(s), s\right)\end{array}\right)ds \\ -\frac{1}{\Gamma(\alpha)}\int_{t-\frac{h}{p}}^{t}(t-s)^{\alpha-1}\mathbf{F}\left(\mathbf{x}_j(s-\tau), \mathbf{x}_j(s), s\right)ds \end{array}\right\| \\
&\leq \|\varphi(t)\| + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left\|\begin{array}{c}\mathbf{F}\left(\mathbf{x}_j(s-\tau), \mathbf{x}_j(s), s\right) \\ -\mathbf{F}(\mathbf{x}(s-\tau), \mathbf{x}(s), s)\end{array}\right\|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\left\|\int_{t-\frac{h}{p}}^{t}(t-s)^{\alpha-1}\mathbf{F}\left(\mathbf{x}_j(s-\tau), \mathbf{x}_j(s), s\right)ds\right\| \\
&\leq \|\varphi(t)\| + \frac{\Gamma(\alpha+1)}{2\alpha h^\alpha}\omega\frac{h^\alpha}{\Gamma(\alpha+1)} + \frac{\alpha}{\Gamma(\alpha)}\sqrt{\frac{M}{2\alpha-1}}\left(\frac{h}{p}\right)^{\alpha-1/2},
\end{aligned} \tag{2.23}$$

Using (2.23), one can obtain

$$\begin{aligned}
\left\|\Lambda\mathbf{x}_j(t) - \Lambda\mathbf{x}(t)\right\| &\leq \|\varphi(t)\| + \frac{\Gamma(\alpha+1)}{2\alpha h^\alpha}\omega\frac{h^\alpha}{\Gamma(\alpha+1)} + \frac{\alpha}{\Gamma(\alpha)}\sqrt{\frac{M}{2\alpha-1}}\left(\frac{h}{p}\right)^{\alpha-1/2} \\
&\leq \|\varphi(t)\| + \frac{\omega}{2} + \frac{1}{\Gamma(\alpha)}\sqrt{\frac{M}{2\alpha-1}}\left(\frac{h}{p}\right)^{\alpha-1/2} \\
&\leq \widetilde{\omega}.
\end{aligned} \tag{2.24}$$

Therefore, we get

$$\mathbf{x} := \varphi + \frac{\alpha}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\mathbf{F}(\mathbf{x}(s-\tau), \mathbf{x}(s), s)\,ds, \tag{2.25}$$

which implies that the function is the solution of the system on $[t_0, t_0 + h]$. Note that the same routine can be applied to obtain the solution of chaotic systems on the interval $[t_0 - h, t_0]$ which completes the proof.

2.2. Global existence theory for the chaotic model with Caputo fractional derivative

In this subsection, we present the global existence of the solution of a chaotic system [29] with Caputo fractional derivatives [20,27].

Theorem 2. Suppose that the first two conditions presented in Theorem 1 for the function $F(\mathbf{x}(t), t)$ in the global space hold [20] and

$$\|\mathbf{F}(\mathbf{x}(t-\tau), \mathbf{x}(t), t)\| \leq \kappa + \rho \|\mathbf{x}(t)\|, \tag{2.26}$$

for almost all $t \in \mathbb{R}$ and all $x \in \mathbb{R}^3$. Then, there at least exists a solution $x(t)$ of the chaotic system presented here on $(-\infty, \infty)$.

Proof. From the above theorem, the function $F(\mathbf{x}(t), t)$ is locally bounded in the domain

$$\aleph = \left\{(t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3 \mid |t - t_0| \leq a, \|\mathbf{x} - \mathbf{x}_0\| \leq b\right\}, \tag{2.27}$$

for $t_0 \in \mathbb{R}$ and $x_0 \in \mathbb{R}^3$. We suppose that the solution has a maximal existence interval which is defined as $(\sigma, s) \subset (-\infty, \infty), \sigma > -\infty, s < \infty$. From Theorem 1, it implies that there exists a solution for the chaotic system on $[t_0 - h, t_0 + h]$. We write

$$\begin{aligned} \|\mathbf{x}(\mathbf{t})\| &\leq \|\varphi\| + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \|\mathbf{F}(\mathbf{x}(s-\tau), \mathbf{x}(s), s)\| \, ds \\ &\leq \|\varphi\| + \frac{\kappa}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \, ds + \frac{\rho}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \|\mathbf{x}(s)\| \, ds \\ &\leq \|\varphi\| + \frac{\alpha\kappa}{\Gamma(\alpha+1)} (s-t_0)^{\alpha} + \frac{\alpha\rho}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \|\mathbf{x}(s)\| \, ds \\ &\leq \gamma + \frac{\alpha\rho}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \|\mathbf{x}(s)\| \, ds. \end{aligned} \tag{2.28}$$

By the Gronwall inequality, we obtain the following

$$\begin{aligned} \|\mathbf{x}(\mathbf{t})\| &\leq \gamma \exp\left(\frac{\alpha\rho}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} \, ds\right) \\ &\leq \frac{\gamma}{(1-\alpha)(1-\rho)} \exp\left(\frac{\alpha\rho}{\Gamma(\alpha+1)} (s-t_0)^{\alpha}\right) \triangleq \widehat{M} < \infty, \end{aligned} \tag{2.29}$$

where

$$\gamma = \|\varphi\| + \frac{\alpha\kappa}{\Gamma(\alpha+1)} (s-t_0)^{\alpha}. \tag{2.30}$$

This yields that $\|\mathbf{x}(\mathbf{t})\| \leq \widetilde{M}$ on $[t_0, s)$ where s is taken as bigger than $\widehat{M}$. Here, we can extend the function to the right side of s which is a contradiction that requires the assumption that (σ, s) is the maximal existence interval. It follows that $s = +\infty$. By applying the same routine, we can also get that $s = -\infty$. This also leads to the completion of the proof.

3. Numerical scheme based on the Newton polynomial for a chaotic problem with delay terms

3.1. Solution of chaotic model with exponential kernel

In this section, we present a numerical scheme based on the Newton polynomial [30] to solve the following fractional system of delay differential equation with different fractional differential operators. We start with the Caputo–Fabrizio case

$$\begin{aligned} {}_{0}^{CF}D_{t}^{\alpha}\mathbf{x}(t) &= F(t,\mathbf{x}(t),\mathbf{x}(t-\tau)), t\in[0,T] \\ \mathbf{x}(t) &= \mathbf{x}(t), t\in[-\pi,0] \end{aligned} \tag{3.1}$$

We consider a uniform grid

$$\{t_k = kh : k = -n, -n+1, ..., -1, 0, 1, ..., N\} \tag{3.2}$$

where

$$h = \frac{T}{N} = \frac{\tau}{n}. \tag{3.3}$$

We let

$$\mathbf{x}(t_k) = \mathbf{x}(t_k), k = -n, -n+1, ..., -1, 0, \tag{3.4}$$

and

$$\mathbf{x}(t_k - \tau) = \mathbf{x}(kh - nh) = \mathbf{x}(t_{k-n}), k = 0, 1, ...N. \tag{3.5}$$

Keeping these notations, we can apply the Caputo–Fabrizio integral at $t = t_n$ and $t = t_{n+1}$

$$\mathbf{x}(t_{k+1}) = \mathbf{x}(t_k) + (1-\alpha)[F(t_k,\mathbf{x}(t_k)),\mathbf{x}(t_k-\tau) - F(t_{k-1},\mathbf{x}(t_{k-1})),\mathbf{x}(t_{k-1}-\tau)] \tag{3.6}$$

$$+\alpha\int_{t_k}^{t_{k+1}} F(s,\mathbf{x}(s),\mathbf{x}(s-\tau))ds$$

Approximating the function $F(s,\mathbf{x}(s),\mathbf{x}(s-\tau))$ by the Newton polynomial yields

$$\begin{aligned} \mathbf{x}(t_{k+1}) &= \mathbf{x}(t_k) + (1-\alpha)[F(t_k,\mathbf{x}(t_k)),\mathbf{x}(t_k-\tau) - F(t_{k-1},\mathbf{x}(t_{k-1})),\mathbf{x}(t_{k-1}-\tau)] \\ &\quad +\alpha\left[\begin{array}{c} \frac{23}{12}F(t_k,\mathbf{x}(t_k)),\mathbf{x}(t_k-\tau) - \frac{4}{3}F(t_{k-1},\mathbf{x}(t_{k-1})),\mathbf{x}(t_{k-1}-\tau) \\ +\frac{5}{12}F(t_{k-2},\mathbf{x}(t_{k-2})),\mathbf{x}(t_{k-2}-\tau)] \end{array}\right] \\ &= \mathbf{x}(t_k) + (1-\alpha)\left[F(t_k,\mathbf{x}(t_k)),\mathbf{x}(kh-nh) - F(t_{k-1},\mathbf{x}(t_{k-1})),\mathbf{x}((k-1)h-nh)\right] \\ &\quad +\alpha\left[\begin{array}{c} \frac{23}{12}F(t_k,\mathbf{x}(t_k)),\mathbf{x}(kh-nh) - \frac{4}{3}F(t_{k-1},\mathbf{x}(t_{k-1})),\mathbf{x}((k-1)h-nh)) \\ +\frac{5}{12}F(t_{k-2},\mathbf{x}(t_{k-2})),\mathbf{x}((k-2)h-nh)) \end{array}\right]. \end{aligned} \tag{3.7}$$

Then, we get

$$\mathbf{x}(t_{k+1}) = \mathbf{x}(t_k) + (1-\alpha)\left[F(t_k,\mathbf{x}(t_k),\mathbf{x}(t_{k-n})) - F(t_{k-1},\mathbf{x}(t_{k-1}),\mathbf{x}(t_{k-n-1}))\right] \tag{3.8}$$

$$+\alpha h\left[\begin{array}{c}\frac{23}{12}F(t_k,\mathbf{x}(t_k)),\mathbf{x}(t_{k-n})-\frac{4}{3}F(t_{k-1},\mathbf{x}(t_{k-1})),\mathbf{x}(t_{k-n-1}))\\+\frac{5}{12}F(t_{k-2},\mathbf{x}(t_{k-2})),\mathbf{x}(t_{k-n-2}))\end{array}\right]$$

The numerical simulations have been performed for the considered chaotic model with exponential kernels, as shown in Figure 2.

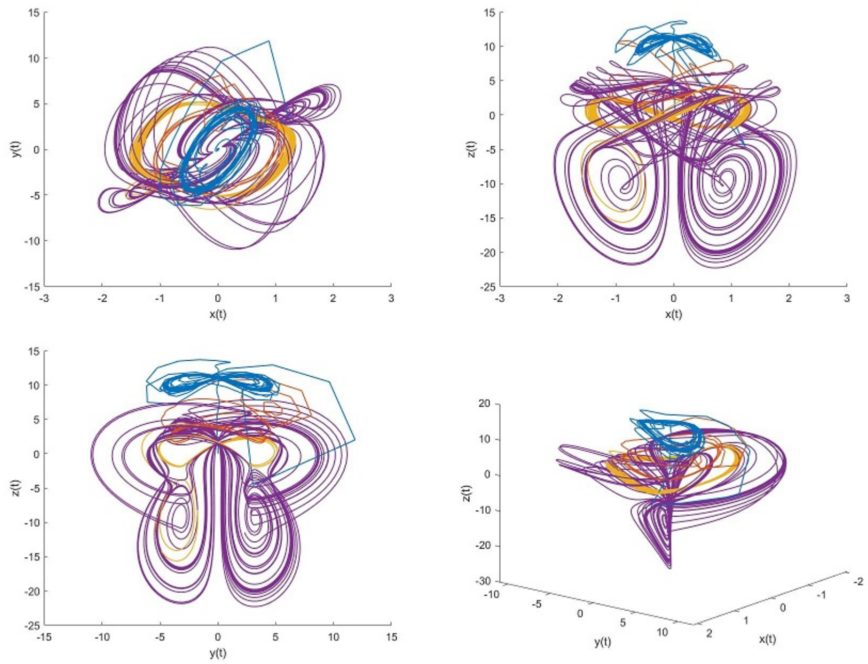

Figure 2. Numerical visualization of the considered chaotic model for $\alpha = 0.91$ and $\tau = 0.09$.

We shall note that we need to calculate the values of $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ since the Newton polynomial [30] is constructed by using three points. It can be easily seen that while the values $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ can be obtained by using the Euler and Adams–Bashforth methods. Thus, we can have

$$\begin{aligned}\mathbf{x}_1 &= \mathbf{x}_0+(1-\alpha)F(t_0,\mathbf{x}(t_0)),\mathbf{x}(t_0-\tau))+\alpha hF(t_0,\mathbf{x}(t_0)),\mathbf{x}(t_0-\tau)), \qquad (3.9)\\ \mathbf{x}_2 &= \mathbf{x}_1+(1-\alpha)\left[F(t_1,\mathbf{x}(t_1)),\mathbf{x}(t_1-\tau))-F(t_0,\mathbf{x}(t_0)),\mathbf{x}(t_0-\tau))\right]\\ &\quad+\frac{3\alpha h}{2}F(t_1,\mathbf{x}(t_1)),\mathbf{x}(t_1-\tau))-\frac{\alpha h}{2}F(t_0,\mathbf{x}(t_0)),\mathbf{x}(t_0-\tau)).\end{aligned}$$

3.2. *Solution of chaotic model with Mittag–Leffler kernel*

For the Atangana–Baleanu case [2], we consider the following system:

$$\begin{aligned}{}_0^{ABC}D_t^{\alpha}\mathbf{x}(t) &= F(t,\mathbf{x}(t),\mathbf{x}(t-\tau)),\ t\in[0,T]\\ \mathbf{x}(t) &= \widetilde{\mathbf{x}}(t),\ t\in[-\pi,0] \qquad (3.10)\end{aligned}$$

Applying the integral on both sides and considering at $t = t_{n+1}$ leads to

$$\mathbf{x}(t_{k+1}) = \mathbf{x}(0) + (1-\alpha)F(t_k, \mathbf{x}(t_k), \mathbf{x}(t_k - \tau)) + \frac{\alpha}{\Gamma(\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} F(s, \mathbf{x}(s), \mathbf{x}(s-\tau))(t_{k+1}-s)^{\alpha-1} ds. \tag{3.11}$$

By using the corresponding approach, we have

$$\begin{aligned} \mathbf{x}(t_{k+1}) = {} & \widetilde{\mathbf{x}}_0 + (1-\alpha)F(t_k, \mathbf{x}(t_k), \mathbf{x}(t_k - \tau)) \\ & + \frac{\alpha h^\alpha}{\Gamma(\alpha+1)} \sum_{j=2}^{k} F(t_{j-2}, \mathbf{x}(t_{j-2}), \mathbf{x}(t_{j-2}-\tau)) \int_{t_j}^{t_{j+1}} (t_{k+1}-s)^{\alpha-1} ds \\ & + \frac{\alpha h^\alpha}{\Gamma(\alpha+2)} \sum_{j=2}^{k} \left[\begin{array}{c} F(t_{j-1}, \mathbf{x}(t_{j-1}), \mathbf{x}(t_{j-1}-\tau)) \\ -F(t_{j-2}, \mathbf{x}(t_{j-2}), \mathbf{x}(t_{j-2}-\tau)) \end{array} \right] \\ & \times \int_{t_j}^{t_{j+1}} \left(s - t_{j-2}\right)(t_{k+1}-s)^{\alpha-1} ds \\ & + \frac{\alpha h^\alpha}{2\Gamma(\alpha+3)} \sum_{j=2}^{k} \left[\begin{array}{c} F(t_j, \mathbf{x}(t_j), \mathbf{x}(t_j-\tau)) \\ -2F(t_{j-1}, \mathbf{x}(t_{j-1}), \mathbf{x}(t_{j-1}-\tau)) \\ +F(t_{j-2}, \mathbf{x}(t_{j-2}), \mathbf{x}(t_{j-2}-\tau)) \end{array} \right] \\ & \times \int_{t_j}^{t_{j+1}} \left(s - t_{j-1}\right)\left(s - t_{j-2}\right)(t_{k+1}-s)^{\alpha-1} ds \end{aligned} \tag{3.12}$$

Calculating the above integrals yields

$$\begin{aligned} \mathbf{x}(t_{k+1}) = {} & \widetilde{\mathbf{x}}_0 + (1-\alpha)\left[F(t_k, \mathbf{x}(t_k)), \mathbf{x}(t_k - \tau) - F(t_{k-1}, \mathbf{x}(t_{k-1}), \mathbf{x}(t_{k-1}-\tau))\right] \\ & + \frac{\alpha h^\alpha}{\Gamma(\alpha+1)} \sum_{j=2}^{k} F(t_{j-2}, \mathbf{x}(t_{j-2}), \mathbf{x}(t_{j-2}-\tau))\delta_1 \\ & + \frac{\alpha h^\alpha}{\Gamma(\alpha+2)} \sum_{j=2}^{k} \left[\begin{array}{c} F(t_{j-1}, \mathbf{x}(t_{j-1}), \mathbf{x}(t_{j-1}-\tau)) \\ -F(t_{j-2}, \mathbf{x}(t_{j-2}), \mathbf{x}(t_{j-2}-\tau)) \end{array} \right] \delta_2 \\ & + \frac{\alpha h^\alpha}{2\Gamma(\alpha+3)} \sum_{j=2}^{k} \left[\begin{array}{c} F(t_j, \mathbf{x}(t_j), \mathbf{x}(t_j-\tau)) \\ -2F(t_{j-1}, \mathbf{x}(t_{j-1}), \mathbf{x}(t_{j-1}-\tau)) \\ +F(t_{j-2}, \mathbf{x}(t_{j-2}), \mathbf{x}(t_{j-2}-\tau))) \end{array} \right] \delta_3 \end{aligned}$$

where

$$\delta_1 = \left[(k-j+1)^\alpha - (k-j)^\alpha\right], \tag{3.13}$$

$$\delta_2 = \left[\begin{array}{c} (k-j+1)^\alpha (k-j+3+2\alpha) \\ -(k-j)^\alpha (k-j+3+3\alpha) \end{array} \right], \tag{3.14}$$

$$\delta_3 = \begin{bmatrix} (k-j+1)^{\alpha}\begin{bmatrix} 2(k-j)^2+(3\alpha+10)(k-j) \\ +2\alpha^2+9\alpha+12 \end{bmatrix} \\ -(k-j)^{\alpha}\begin{bmatrix} 2(k-j)^2+(5\alpha+10)(k-j) \\ +6\alpha^2+18\alpha+12 \end{bmatrix} \end{bmatrix}. \tag{3.15}$$

For the Atangana–Baleanu in the sense Caputo case, values of $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$ can be obtained by using the same procedure presented before as follows:

$$\begin{aligned} \mathbf{x}_1 &= \mathbf{x}_0 + (1-\alpha)F(t_0, \mathbf{x}(t_0)), \mathbf{x}(t_0-\tau)) \\ &+ \frac{\alpha h^{\alpha}}{\Gamma(\alpha+1)} F(t_0, \mathbf{x}(t_0)), \mathbf{x}(t_0-\tau)) \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} \mathbf{x}_2 &= \mathbf{x}_1 + (1-\alpha)\left[F(t_1, \mathbf{x}(t_1)), \mathbf{x}(t_1-\tau)) - F(t_0, \mathbf{x}(t_0)), \mathbf{x}(t_0-\tau))\right] \\ &+ \frac{\alpha h^{\alpha}}{\Gamma(\alpha+2)} F(t_1, \mathbf{x}(t_1)), \mathbf{x}(t_1-\tau)) - \frac{\alpha h^{\alpha}}{\Gamma(\alpha+1)} F(t_0, \mathbf{x}(t_0)), \mathbf{x}(t_0-\tau)). \end{aligned} \tag{3.17}$$

3.3. Solution of chaotic model with power-law kernel

We next consider a delay system with the Caputo fractional derivative:

$$\begin{aligned} {}_0^C D_t^{\alpha}\mathbf{x}(t) &= F(t, \mathbf{x}(t), \mathbf{x}(t-\tau)), t \in [0, T] \\ \mathbf{x}(t) &= \widetilde{\mathbf{x}}(t), t \in [-\pi, 0] \end{aligned} \tag{3.18}$$

We convert the above system into

$$\mathbf{x}(t_{k+1}) = \mathbf{x}(0) + \frac{1}{\Gamma(\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} F(s, \mathbf{x}(s), \mathbf{x}(s-\tau))(t_{k+1}-s)^{\alpha-1} ds. \tag{3.19}$$

Approximating the function $F(s, \mathbf{x}(s), \mathbf{x}(s-\tau))$ by the Newton polynomial [30] yields

$$\begin{aligned} \mathbf{x}(t_{k+1}) &= \widetilde{\mathbf{x}}_0 + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j=2}^{k} F(t_{j-2}, \mathbf{x}(t_{j-2}), \mathbf{x}(t_{j-2}-\tau))\delta_1 \\ &+ \frac{h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j=2}^{k} \begin{bmatrix} F(t_{j-1}, \mathbf{x}(t_{j-1}), \mathbf{x}(t_{j-1}-\tau)) \\ -F(t_{j-2}, \mathbf{x}(t_{j-2}), \mathbf{x}(t_{j-2}-\tau)) \end{bmatrix} \delta_2 \\ &+ \frac{h^{\alpha}}{2\Gamma(\alpha+3)} \sum_{j=2}^{k} \begin{bmatrix} F(t_j, \mathbf{x}(t_j), \mathbf{x}(t_j-\tau)) - 2F(t_{j-1}, \mathbf{x}(t_{j-1}), \mathbf{x}(t_{j-1}-\tau)) \\ +F(t_{j-2}, \mathbf{x}(t_{j-2}), \mathbf{x}(t_{j-2}-\tau))) \end{bmatrix} \delta_3 \end{aligned} \tag{3.20}$$

By applying the same routine as for the Caputo case, we get the following values for $\mathbf{x}(t_1)$ and $\mathbf{x}(t_2)$:

$$\mathbf{x}_1 = \mathbf{x}_0 + \frac{h^{\alpha}}{\Gamma(\alpha+1)} F(t_0, \mathbf{x}(t_0), \mathbf{x}(t_0-\tau)) \tag{3.21}$$

and

$$\mathbf{x}_2 = \mathbf{x}_1 + \frac{h^{\alpha}}{\Gamma(\alpha+2)} F\left(t_1, \mathbf{x}(t_1), \mathbf{x}(t_1-\tau)\right) - \frac{h^{\alpha}}{\Gamma(\alpha+2)} F\left(t_0, \mathbf{x}(t_0), \mathbf{x}(t_0-\tau)\right). \tag{3.22}$$

We present the numerical simulations for the considered chaotic model with power-law kernels, as shown in Figure 3.

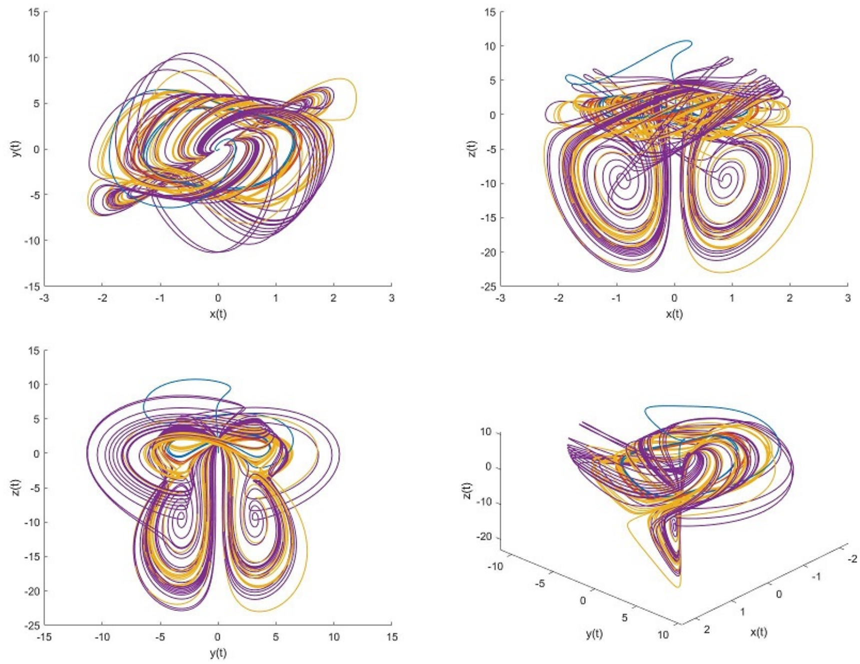

Figure 3. Numerical visualization of the considered chaotic model for $\alpha = 0.96$ and $\tau = 0.09$.

4. Chaotic model with piecewise derivative

We consider the considered model in piecewise form, where the first part is stochastic and the second part is classical derivative. The stochastic–deterministic chaotic model is represented by

$$\begin{cases} dx(t) = \left(a_1 y(t-\tau) - a_2 x(t-\tau) + a_3 x(t) z(t)\right) dt + \sigma_1 x dB_1(t) \\ dy(t) = \left(-b_1 x(t) z(t) - b_2 x(t-\tau) + b_3 y(t) z(t)\right) dt + \sigma_2 y dB_2(t) \\ dz(t) = \left(c_1 - c_2 y^2(t-\tau)\right) dt + \sigma_3 z dB_3(t) \end{cases}, \text{ if } 0 \le t \le t_0 \tag{4.1}$$

$$\begin{cases} x' = \left(a_1 y(t-\tau) - a_2 x(t-\tau) + a_3 x(t) z(t)\right) \\ y' = \left(-b_1 x(t) z(t) - b_2 x(t-\tau) + b_3 y(t) z(t)\right) \\ z' = \left(c_1 - c_2 y^2(t-\tau)\right) \end{cases}, \text{ if } t_0 \le t \le T$$

where the initial data are given by

$$x(0) = x_0, y(0) = y_0, z(0) = z_0. \tag{4.2}$$

To present a simplification numerical scheme, we shall use the following notations:

$$X=\begin{bmatrix} x\\ y\\ z\end{bmatrix}, X^0=\begin{bmatrix} x^0\\ y^0\\ z^0\end{bmatrix}, F(t,X)=\begin{bmatrix} f_1(t,X)\\ f_2(t,X)\\ f_3(t,X)\end{bmatrix} \tag{4.3}$$

where

$$\begin{array}{c} f_1(t,X)=(a_1y(t-\tau)-a_2x(t-\tau)+a_3x(t)z(t))\\ f_2(t,X)=(-b_1x(t)z(t)-b_2x(t-\tau)+b_3y(t)z(t))\\ f_3(t,X)=\left(c_1-c_2y^2(t-\tau)\right)\end{array}. \tag{4.4}$$

Using the numerical scheme presented earlier, the numerical solution of the stochastic–deterministic chaotic model is obtained by solving the following equation:

$$X(t_{n+1})=\left\{\begin{array}{l} \left\{\begin{array}{c} X(0)+\sum_{k=2}^{i}\begin{bmatrix} \frac{5}{12}F(t_{k-2},X(t_{k-2}),X(t_{k-2}-\tau))\Delta t\\ -\frac{4}{3}F(t_{k-1},X(t_{k-1}),X(t_{k-1}-\tau))\Delta t\\ +\frac{23}{12}F(t_k,X(t_k),X(t_k-\tau))\Delta t\end{bmatrix}\\ +\sum_{k=0}^{i}\sigma_lX(c_k)(B_l(t_{k+1})-B_l(t_k))\end{array}\right.\\ \left\{X(t_0)+\sum_{k=i+3}^{n}\begin{bmatrix} \frac{5}{12}F(t_{k-2},X(t_{k-2}),X(t_{k-2}-\tau))\Delta t\\ -\frac{4}{3}F(t_{k-1},X(t_{k-1}),X(t_{k-1}-\tau))\Delta t\\ +\frac{23}{12}F(t_k,X(t_k),X(t_k-\tau))\Delta t\end{bmatrix}\right.\end{array}\right., \tag{4.5}$$

such that $l=1,..,3$ and $c_k\in[t_k,t_{k+1}]$.

We now consider the considered model in piecewise form, where the first part is stochastic and the second part is the Caputo–Fabrizio derivative:

$$\left\{\begin{array}{c} dx(t)=(a_1y(t-\tau)-a_2x(t-\tau)+a_3x(t)z(t))\,dt+\sigma_1xdB_1(t)\\ dy(t)=(-b_1x(t)z(t)-b_2x(t-\tau)+b_3y(t)z(t))\,dt+\sigma_2ydB_2(t)\\ dz(t)=\left(c_1-c_2y^2(t-\tau)\right)dt+\sigma_3zdB_3(t)\end{array}\right., \text{ if } 0\le t\le t_0 \tag{4.6}$$

$$\left\{\begin{array}{c} {}_{t_0}^{CF}D_t^{\alpha}x(t)=(a_1y(t-\tau)-a_2x(t-\tau)+a_3x(t)z(t))\\ {}_{t_0}^{CF}D_t^{\alpha}y(t)=(-b_1x(t)z(t)-b_2x(t-\tau)+b_3y(t)z(t))\\ {}_{t_0}^{CF}D_t^{\alpha}z(t)=\left(c_1-c_2y^2(t-\tau)\right)\end{array}\right. \text{ if } t_0\le t\le T,$$

Using the numerical scheme presented earlier, the numerical solution of the stochastic–deterministic chaotic model is obtained by solving the following equation:

$$X(t_{n+1})=\left\{\begin{array}{l} \left\{\begin{array}{c} X(0)+\sum_{k=2}^{i}\begin{bmatrix} \frac{5}{12}F(t_{k-2},X(t_{k-2}),X(t_{k-2}-\tau))\Delta t\\ -\frac{4}{3}F(t_{k-1},X(t_{k-1}),X(t_{k-1}-\tau))\Delta t\\ +\frac{23}{12}F(t_k,X(t_k),X(t_k-\tau))\Delta t\end{bmatrix}\\ +\sum_{k=0}^{i}\sigma_lX(c_k)(B_l(t_{k+1})-B_l(t_k))\end{array}\right.\\ \left\{\begin{array}{c} X(t_0)+(1-\alpha)\begin{bmatrix} F(t_k,X(t_k),X(t_k-\tau))\\ -F(t_{k-1},X(t_{k-1}),X(t_{k-1}-\tau))\end{bmatrix}\\ +\alpha\sum_{k=i+3}^{n}\begin{bmatrix} \frac{5}{12}F(t_{k-2},X(t_{k-2}),X(t_{k-2}-\tau))\Delta t\\ -\frac{4}{3}F(t_{k-1},X(t_{k-1}),X(t_{k-1}-\tau))\Delta t\\ +\frac{23}{12}F(t_k,X(t_k),X(t_k-\tau))\Delta t\end{bmatrix}\end{array}\right.\end{array}\right., \tag{4.7}$$

such that $l=1,..,3$.

We now consider the considered model in piecewise form, where the first part is stochastic and the second part is the Atangana–Baleanu derivative:

$$\begin{cases} dx(t) = (a_1 y(t-\tau) - a_2 x(t-\tau) + a_3 x(t) z(t))\, dt + \sigma_1 x dB_1(t) \\ dy(t) = (-b_1 x(t) z(t) - b_2 x(t-\tau) + b_3 y(t) z(t))\, dt + \sigma_2 y dB_2(t) \\ dz(t) = \left(c_1 - c_2 y^2(t-\tau)\right) dt + \sigma_3 z dB_3(t) \end{cases}, \text{ if } 0 \le t \le t_0 \tag{4.8}$$

$$\begin{array}{l} {}_{t_0}^{AB}D_t^{\alpha} x(t) = (a_1 y(t-\tau) - a_2 x(t-\tau) + a_3 x(t) z(t)) \\ {}_{t_0}^{AB}D_t^{\alpha} y(t) = (-b_1 x(t) z(t) - b_2 x(t-\tau) + b_3 y(t) z(t)) \text{ if } t_0 \le t \le T, \\ {}_{t_0}^{AB}D_t^{\alpha} z(t) = \left(c_1 - c_2 y^2(t-\tau)\right) \end{array}$$

Using numerical scheme presented earlier, the numerical solution of the stochastic–deterministic chaotic model is obtained by solving the following equation:

$$X(t_{n+1}) = \begin{cases} \begin{cases} X(0) + \sum_{k=2}^{i} \begin{bmatrix} \frac{5}{12} F(t_{k-2}, X(t_{k-2}), X(t_{k-2}-\tau))\Delta t \\ -\frac{4}{3} F(t_{k-1}, X(t_{k-1}), X(t_{k-1}-\tau))\Delta t \\ +\frac{23}{12} F(t_k, X(t_k), X(t_k-\tau))\Delta t \end{bmatrix} \\ + \sum_{k=0}^{i} \sigma_l X(c_k)(B_l(t_{k+1}) - B_l(t_k)) \end{cases} \\ \begin{cases} X(t_0) + (1-\alpha) \begin{bmatrix} F(t_n, X(t_n)), X(t_n-\tau)) \\ -F(t_{n-1}, X(t_{n-1}), X(t_{n-1}-\tau)) \end{bmatrix} \\ + \frac{\alpha h^{\alpha}}{\Gamma(\alpha+1)} \sum_{k=i+3}^{n} F(t_{j-2}, X(t_{j-2}), X(t_{j-2}-\tau))\delta_1 \\ + \frac{\alpha h^{\alpha}}{\Gamma(\alpha+2)} \sum_{k=i+3}^{n} \begin{bmatrix} F(t_{j-1}, X(t_{j-1}), X(t_{j-1}-\tau)) \\ -F(t_{j-2}, X(t_{j-2}), X(t_{j-2}-\tau)) \end{bmatrix} \delta_2 \\ + \frac{\alpha h^{\alpha}}{2\Gamma(\alpha+3)} \sum_{k=i+3}^{n} \begin{bmatrix} F(t_j, X(t_j), X(t_j-\tau)) \\ -2F(t_{j-1}, X(t_{j-1}), X(t_{j-1}-\tau)) \\ +F(t_{j-2}, X(t_{j-2}), X(t_{j-2}-\tau))) \end{bmatrix} \delta_3 \end{cases} \end{cases}, \tag{4.9}$$

such that $l = 1, .., 3$.

We now consider the considered model in piecewise form, where the first part is stochastic and the second part is the Caputo derivative:

$$\begin{cases} dx(t) = (a_1 y(t-\tau) - a_2 x(t-\tau) + a_3 x(t) z(t))\, dt + \sigma_1 x dB_1(t) \\ dy(t) = (-b_1 x(t) z(t) - b_2 x(t-\tau) + b_3 y(t) z(t))\, dt + \sigma_2 y dB_2(t) \\ dz(t) = \left(c_1 - c_2 y^2(t-\tau)\right) dt + \sigma_3 z dB_3(t) \end{cases}, \text{ if } 0 \le t \le t_0 \tag{4.10}$$

$$\begin{cases} {}_{t_0}^{C}D_t^{\alpha} x(t) = (a_1 y(t-\tau) - a_2 x(t-\tau) + a_3 x(t) z(t)) \\ {}_{t_0}^{C}D_t^{\alpha} y(t) = (-b_1 x(t) z(t) - b_2 x(t-\tau) + b_3 y(t) z(t)) \text{ if } t_0 \le t \le T, \\ {}_{t_0}^{C}D_t^{\alpha} z(t) = \left(c_1 - c_2 y^2(t-\tau)\right) \end{cases}$$

Using numerical scheme presented earlier, the numerical solution of the stochastic–deterministic chaotic model is obtained by solving the following equation:

$$X\left(t_{n+1}\right)=\left\{\begin{array}{l}\left\{\begin{array}{c}X(0)+\sum_{k=2}^{i}\left[\begin{array}{c}\frac{5}{12}F\left(t_{k-2},X_{k-2},X(t_{k-2}-\tau)\right)\Delta t\\-\frac{4}{3}F\left(t_{k-1},X_{k-1},X(t_{k-1}-\tau)\right)\Delta t\\+\frac{23}{12}F\left(t_{k},X_{k},X(t_{k}-\tau)\right)\Delta t\end{array}\right]\\+\sum_{k=0}^{i}\sigma_l X\left(c_k\right)\left(B_l\left(t_{k+1}\right)-B_l\left(t_k\right)\right)\end{array}\right.\\\left\{\begin{array}{l}\widetilde{X}\left(t_0\right)+\frac{\alpha h^{\alpha}}{\Gamma(\alpha+1)}\sum_{k=i+3}^{n}F(t_{k-2},X(t_{k-2}),X(t_{k-2}-\tau))\delta_1\\+\frac{\alpha h^{\alpha}}{\Gamma(\alpha+2)}\sum_{k=i+3}^{n}\left[\begin{array}{c}F(t_{k-1},X(t_{k-1}),X(t_{k-1}-\tau))\\-F(t_{k-2},X(t_{k-2}),X(t_{k-2}-\tau))\end{array}\right]\delta_2\\+\frac{\alpha h^{\alpha}}{2\Gamma(\alpha+3)}\sum_{k=i+3}^{n}\left[\begin{array}{c}F(t_{k},X(t_{k}),X(t_{k}-\tau))\\-2F(t_{k-1},X(t_{k-1}),X(t_{k-1}-\tau))\\+F(t_{k-2},X(t_{k-2}),X(t_{k-2}-\tau)))\end{array}\right]\delta_3\end{array}\right.\end{array}\right., \tag{4.11}$$

such that $l = 1, .., 3$.

4.1. Illustrative examples

In this section, we deal with a piecewise chaotic system with delay terms, where the first part incorporates the classical derivative and the second part is chosen as the fractional derivative.

Example 1. We consider the following piecewise chaotic problem

$$\left\{\begin{array}{c}{}_{0}^{ABC}D_t^{\alpha}x=\left(a_1y\left(t-\tau\right)-a_2x\left(t-\tau\right)+a_3x\left(t\right)z\left(t\right)\right)\\{}_{0}^{ABC}D_t^{\alpha}y=\left(-b_1x\left(t\right)z\left(t\right)-b_2x\left(t-\tau\right)+b_3y\left(t\right)z\left(t\right)\right)\\{}_{0}^{ABC}D_t^{\alpha}z=\left(c_1-c_2y^2\left(t-\tau\right)\right)\end{array}\right., \text{ if } 0\leq t\leq t_0 \tag{4.12}$$

$$\left\{\begin{array}{c}{}_{t_0}^{C}D_t^{\alpha}x\left(t\right)=\left(a_1y\left(t-\tau\right)-a_2x\left(t-\tau\right)+a_3x\left(t\right)z\left(t\right)\right)\\{}_{t_0}^{C}D_t^{\alpha}y\left(t\right)=\left(-b_1x\left(t\right)z\left(t\right)-b_2x\left(t-\tau\right)+b_3y\left(t\right)z\left(t\right)\right)\\{}_{t_0}^{C}D_t^{\alpha}z\left(t\right)=\left(c_1-c_2y^2\left(t-\tau\right)\right)\end{array}\right., \text{ if } t_0\leq t\leq t_1$$

$$\left\{\begin{array}{c}x'=\left(a_1y\left(t-\tau\right)-a_2x\left(t-\tau\right)+a_3x\left(t\right)z\left(t\right)\right)\\y'=\left(-b_1x\left(t\right)z\left(t\right)-b_2x\left(t-\tau\right)+b_3y\left(t\right)z\left(t\right)\right)\\z'=\left(c_1-c_2y^2\left(t-\tau\right)\right)\end{array}\right., \text{ if } t_1\leq t$$

where $\tau = 0.04$. The piecewise system can be solved by implementing the following numerical scheme

$$X\left(t_{k+1}\right)=\left\{\begin{array}{l}\widetilde{X}\left(0\right)+\frac{\alpha h^{\alpha}}{\Gamma(\alpha+1)}\sum_{j_1=2}^{k_1}F(t_{j_1-2},X(t_{j_1-2}),X(t_{j_1-2}-\tau))\delta_1\\+\frac{\alpha h^{\alpha}}{\Gamma(\alpha+2)}\sum_{j_1=2}^{k}\left[\begin{array}{c}F(t_{j_1-1},X(t_{j_1-1}),X(t_{j_1-1}-\tau))\\-F(t_{j_1-2},X(t_{j_1-2}),X(t_{j_1-2}-\tau))\end{array}\right]\delta_2\\+\frac{\alpha h^{\alpha}}{2\Gamma(\alpha+3)}\sum_{j_1=2}^{k}\left[\begin{array}{c}F(t_{j_1},X(t_{j_1}),X(t_{j_1}-\tau))\\-2F(t_{j_1-1},X(t_{j_1-1}),X(t_{j_1-1}-\tau))\\+F(t_{j_1-2},X(t_{j_1-2}),X(t_{j_1-2}-\tau)))\end{array}\right]\delta_3\end{array}\right., \tag{4.13}$$

$$\begin{cases} \widetilde{X}(t_0) + (1-\alpha)\begin{bmatrix} F(t_k, X(t_k)), X(t_k - \tau)) \\ -F(t_{k-1}, X(t_{k-1}), X(t_{k-1} - \tau)) \end{bmatrix} \\ +\frac{\alpha h^\alpha}{\Gamma(\alpha+1)} \sum_{j_2=k_1+3}^{k_2} F(t_{j_2-2}, X(t_{j_2-2}), X(t_{j_2-2} - \tau))\delta_1 \\ +\frac{\alpha h^\alpha}{\Gamma(\alpha+2)} \sum_{j_2=k_1+3}^{k_2} \begin{bmatrix} F(t_{j_2-1}, X(t_{j_2-1}), X(t_{j_2-1} - \tau)) \\ -F(t_{j_2-2}, X(t_{j_2-2}), X(t_{j_2-2} - \tau)) \end{bmatrix} \delta_2 \\ +\frac{\alpha h^\alpha}{2\Gamma(\alpha+3)} \sum_{j_2=k_1+3}^{k_2} \begin{bmatrix} F(t_{j_2}, X(t_{j_2}), X(t_{j_2} - \tau)) \\ -2F(t_{j_2-1}, X(t_{j_2-1}), X(t_{j_2-1} - \tau)) \\ +F(t_{j_2-2}, X(t_{j_2-2}), X(t_{j_2-2} - \tau))) \end{bmatrix} \delta_3 \end{cases},$$

$$\begin{cases} \widetilde{X}(t_1) + \sum_{j_3=k_2+3}^{k_3} \begin{bmatrix} \frac{5}{12} F(t_{j_3-2}, X(t_{j_3-2}), X(t_{j_3-2} - \tau))\Delta t \\ -\frac{4}{3} F(t_{j_3-1}, X(t_{j_3-1}), X(t_{j_3-1} - \tau))\Delta t \\ +\frac{23}{12} F(t_{j_3}, X(t_{j_3}), X(t_{j_3} - \tau))\Delta t \end{bmatrix} \end{cases},$$

$$\begin{cases} \widetilde{X}(t_2) + (1-\alpha)\begin{bmatrix} F(t_k, X(t_k)), X(t_k - \tau)) \\ -F(t_{k-1}, X(t_{k-1}), X(t_{k-1} - \tau)) \end{bmatrix} \\ +\alpha \sum_{j_4=k_3+3}^{k} \begin{bmatrix} \frac{5}{12} F(t_{j_4-2}, X(t_{j_4-2}), X(t_{j_4-2} - \tau))\Delta t \\ -\frac{4}{3} F(t_{j_4-1}, X(t_{j_4-1}), X(t_{j_4-1} - \tau))\Delta t \\ +\frac{23}{12} F(t_{j_4}, X(t_{j_4}), X(t_{j_4} - \tau))\Delta t \end{bmatrix} \end{cases}.$$

The graphical representation for the piecewise system is presented for different values of fractional order in Figures 4, 5 and 6.

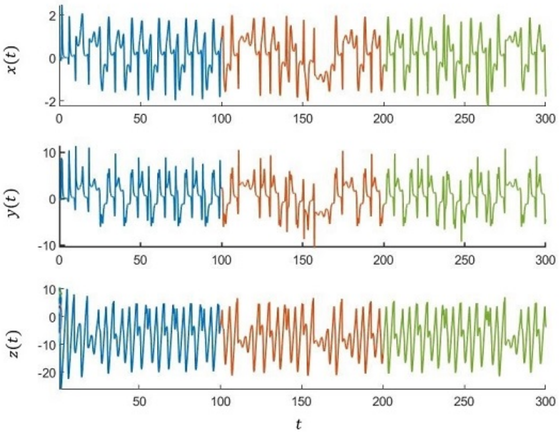

Figure 4. The graphical representation of the piecewise chaotic model for $\alpha = 1$.

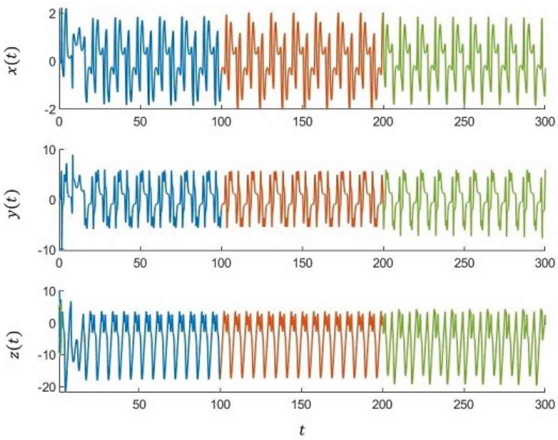

Figure 5. The graphical representation of the piecewise chaotic model for $\alpha = 0.99$.

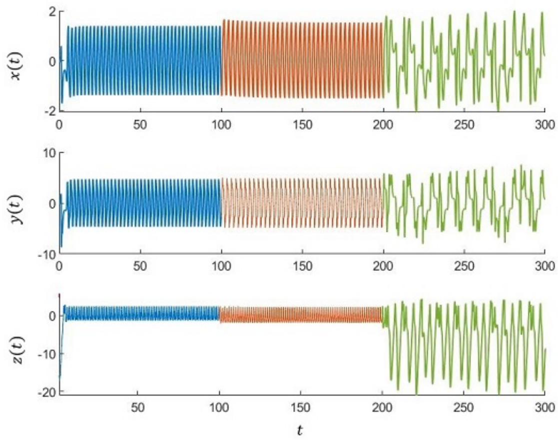

Figure 6. The graphical representation of the piecewise chaotic model for $\alpha = 0.97$.

Example 2. We deal with the following chaotic problem with piecewise derivatives:

$$\left\{\begin{array}{c} {}_0^C D_t^\alpha x = (a_1 y(t-\tau) - a_2 x(t-\tau) + a_3 x(t) z(t)) \\ {}_0^C D_t^\alpha y = (-b_1 x(t) z(t) - b_2 x(t-\tau) + b_3 y(t) z(t)) \\ {}_0^C D_t^\alpha z = \left(c_1 - c_2 y^2 (t-\tau)\right) \end{array}\right. , \text{ if } 0 \le t \le t_0 \tag{4.14}$$

$$\left\{\begin{array}{c} {}_{t_0}^{ABC} D_t^\alpha x = (a_1 y(t-\tau) - a_2 x(t-\tau) + a_3 x(t) z(t)) \\ {}_{t_0}^{ABC} D_t^\alpha y = (-b_1 x(t) z(t) - b_2 x(t-\tau) + b_3 y(t) z(t)) \\ {}_{t_0}^{ABC} D_t^\alpha z = \left(c_1 - c_2 y^2 (t-\tau)\right) \end{array}\right. , \text{ if } t_0 \le t \le t_1$$

$$\left\{\begin{array}{c} x' = (a_1 y(t-\tau) - a_2 x(t-\tau) + a_3 x(t) z(t)) \\ y' = (-b_1 x(t) z(t) - b_2 x(t-\tau) + b_3 y(t) z(t)) \\ z' = \left(c_1 - c_2 y^2 (t-\tau)\right) \end{array}\right. , \text{ if } t_1 \le t \le t_2$$

$$\begin{cases} {}^{CF}_{t_2}D_t^{\alpha}x = (a_1y(t-\tau) - a_2x(t-\tau) + a_3x(t)z(t))\,dt + \sigma_1 x dB_1(t) \\ {}^{CF}_{t_2}D_t^{\alpha}y = (-b_1x(t)z(t) - b_2x(t-\tau) + b_3y(t)z(t))\,dt + \sigma_2 y dB_2(t) \\ {}^{CF}_{t_2}D_t^{\alpha}z = \left(c_1 - c_2y^2(t-\tau)\right)dt + \sigma_3 z dB_3(t) \end{cases}, \text{ if } t_2 \le t \le T$$

where $\tau = 0.06$. The piecewise system can be solved by implementing the following numerical scheme:

$$X(t_{k+1}) = \begin{cases} \widetilde{X}(0) + \frac{\alpha h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j_1=2}^{k_1} F(t_{j_1-2}, X(t_{j_1-2}), X(t_{j_1-2}-\tau))\delta_1 \\ +\frac{\alpha h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j_1=2}^{k} \begin{bmatrix} F(t_{j_1-1}, X(t_{j_1-1}), X(t_{j_1-1}-\tau)) \\ -F(t_{j_1-2}, X(t_{j_1-2}), X(t_{j_1-2}-\tau)) \end{bmatrix} \delta_2 \\ +\frac{\alpha h^{\alpha}}{2\Gamma(\alpha+3)} \sum_{j_1=2}^{k} \begin{bmatrix} F(t_{j_1}, X(t_{j_1}), X(t_{j_1}-\tau)) \\ -2F(t_{j_1-1}, X(t_{j_1-1}), X(t_{j_1-1}-\tau)) \\ +F(t_{j_1-2}, X(t_{j_1-2}), X(t_{j_1-2}-\tau))) \end{bmatrix} \delta_3 \end{cases}, \tag{4.15}$$

$$\begin{cases} \widetilde{X}(t_0) + (1-\alpha) \begin{bmatrix} F(t_k, X(t_k)), X(t_k-\tau)) \\ -F(t_{k-1}, X(t_{k-1}), X(t_{k-1}-\tau)) \end{bmatrix} \\ +\frac{\alpha h^{\alpha}}{\Gamma(\alpha+1)} \sum_{j_2=k_1+3}^{k_2} F(t_{j_2-2}, X(t_{j_2-2}), X(t_{j_2-2}-\tau))\delta_1 \\ +\frac{\alpha h^{\alpha}}{\Gamma(\alpha+2)} \sum_{j_2=k_1+3}^{k_2} \begin{bmatrix} F(t_{j_2-1}, X(t_{j_2-1}), X(t_{j_2-1}-\tau)) \\ -F(t_{j_2-2}, X(t_{j_2-2}), X(t_{j_2-2}-\tau)) \end{bmatrix} \delta_2 \\ +\frac{\alpha h^{\alpha}}{2\Gamma(\alpha+3)} \sum_{j_2=k_1+3}^{k_2} \begin{bmatrix} F(t_{j_2}, X(t_{j_2}), X(t_{j_2}-\tau)) \\ -2F(t_{j_2-1}, X(t_{j_2-1}), X(t_{j_2-1}-\tau)) \\ +F(t_{j_2-2}, X(t_{j_2-2}), X(t_{j_2-2}-\tau))) \end{bmatrix} \delta_3 \end{cases},$$

$$\begin{cases} \widetilde{X}(t_1) + \sum_{j_3=k_2+3}^{k_3} \begin{bmatrix} \frac{5}{12}F(t_{j_3-2}, X(t_{j_3-2}), X(t_{j_3-2}-\tau))\Delta t \\ -\frac{4}{3}F(t_{j_3-1}, X(t_{j_3-1}), X(t_{j_3-1}-\tau))\Delta t \\ +\frac{23}{12}F(t_{j_3}, X(t_{j_3}), X(t_{j_3}-\tau))\Delta t \end{bmatrix} \end{cases},$$

$$\begin{cases} \widetilde{X}(t_2) + (1-\alpha) \begin{bmatrix} F(t_k, X(t_k)), X(t_k-\tau)) \\ -F(t_{k-1}, X(t_{k-1}), X(t_{k-1}-\tau)) \end{bmatrix} \\ +\alpha \sum_{j_4=k_3+3}^{k} \begin{bmatrix} \frac{5}{12}F(t_{j_4-2}, X(t_{j_4-2}), X(t_{j_4-2}-\tau))\Delta t \\ -\frac{4}{3}F(t_{j_4-1}, X(t_{j_4-1}), X(t_{j_4-1}-\tau))\Delta t \\ +\frac{23}{12}F(t_{j_4}, X(t_{j_4}), X(t_{j_4}-\tau))\Delta t \end{bmatrix} \\ +\alpha \sum_{j_4=k_3+3}^{k} \sigma_l X\left(c_{j_4}\right)\left(B_l\left(t_{j_4+1}\right) - B_l\left(t_{j_4}\right)\right) \end{cases}.$$

The graphical representation for the piecewise system is presented for different values of fractional order in Figures 7, 8 and 9.

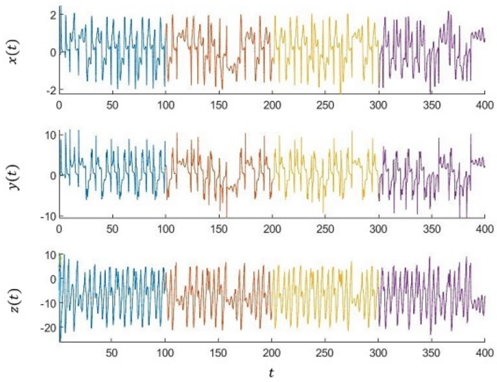

Figure 7. Numerical visualization of the considered chaotic model for $\alpha = 1, \sigma_1 = 0.5, \sigma_2 = 0.35, \sigma_3 = 0.1$.

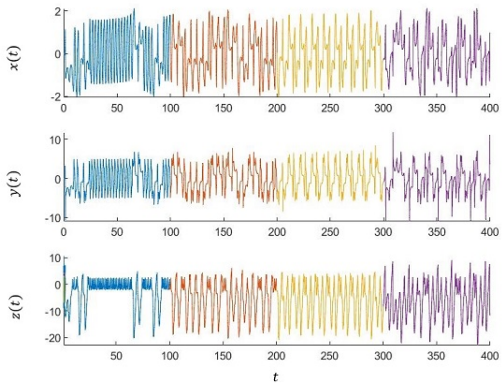

Figure 8. Numerical visualization of the considered chaotic model for $\alpha = 0.98, \sigma_1 = 0.25, \sigma_2 = 0.15, \sigma_3 = 0.14$.

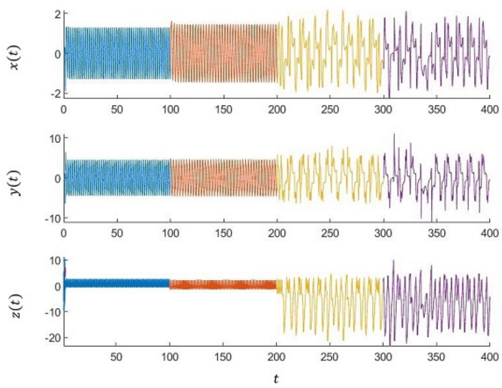

Figure 9. Numerical visualization of the considered chaotic model for $\alpha = 0.96, \sigma_1 = 0.1, \sigma_2 = 0.09, \sigma_3 = 0.08$.

5. Conclusion

In this study, we have considered a chaotic model with fractional differential operators and a delay term that has been added to this model. Moreover, the considered delay system has been modeled by using piecewise derivatives that can account for different behaviors such as stochastic, crossover and memory effects according to the kernels used. Of course, this led us to the exhibition of different behaviors of the model considered here. The existence and uniqueness of the solution of the delay chaotic model have been demonstrated by using the Carathéodory theorem. Some applications for these models are presented for different values of fractional order.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

References

1. M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Prog. Fract. Differ. Appl.*, **1** (2015), 73–85. http://dx.doi.org/10.12785/pfda/010201
2. A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel: Theory and application to heat transfer model, *Therm. Sci.*, **20** (2016), 763–769. https://doi.org/10.2298/TSCI160111018A
3. M. Caputo, Linear model of dissipation whoseQ is almost frequency independent II, *Geophys J Int.*, **13** (1967), 529–539. https://doi.org/10.1111/j.1365-246X.1967.tb02303.x
4. A. Atangana, S. Igret Araz, New concept in calculus: Piecewise differential and integral operators, *Chaos Solit. Fractals*, **145** (2021), 110638. https://doi.org/10.1016/j.chaos.2020.110638
5. B. Ghanbari, A fractional system of delay differential equation with nonsingular kernels in modeling hand-foot-mouth disease, *Adv. Differ. Equ.*, **2020** (2020), 536. https://doi.org/10.1186/s13662-020-02993-3
6. D. Liu, K. Zhang, Existence of positive solutions to a boundary value problem for a delayed singular high order fractional differential equation with sign-changing nonlinearity, *J. Appl. Anal. Comput.*, **10** (2020), 1073–1093. https://doi.org/10.11948/20190190
7. X. Li, H. Li, B. Wu, Piecewise reproducing kernel method for linear impulsive delay differential equations with piecewise constant arguments, *Appl. Math. Comput.*, **349** (2019), 304–313. https://doi.org/10.1016/j.amc.2018.12.0540
8. D. Filali, A. Ali, Z. Ali, P. Agarwal, Problem on piecewise Caputo-Fabrizio fractional delay differential equation under anti-periodic boundary conditions, *Phys. Scr.*, **98** (2023), 034001. https://doi.org/10.1088/1402-4896/acb6c4

9. M. L. Morgado, N. J. Ford, P. M. Lims, Analysis and numerical methods for fractional differential equations with delay, *J. Comput. Appl. Math.*, **252** (2013), 159–168. https://doi.org/10.1016/j.cam.2012.06.034

10. A. Jhinga, V. Daftardar-Gejji, A new numerical method for solving fractional delay differential equations, *Comp. Appl. Math.*, **38** (2019), 1–18. https://doi.org/10.1007/s40314-019-0951-0

11. D. Baleanu, F. Jarad, Existence and uniqueness theorem for a class of delay differential equations with left and right Caputo fractional derivatives, *J. Math. Phys.*, **49** (2008). https://doi.org/10.1063/1.2970709

12. S. Abbas, Existence of solutions to fractional order ordinary and delay differential equations and applications, *Electron. J. Differ. Equ.*, **2011** (2011), 72–76. https://doi.org/10.1155/2011/793023

13. H. T. Tuan, H. Trinh, A Qualitative Theory of Time Delay Nonlinear Fractional-Order Systems, *SIAM J. Control Optim.*, **58** (2020), 1491–1518. https://doi.org/10.1137/19M1299797

14. N. D. Cong, H. T. Tuan, Existence, uniqueness, and exponential boundedness of global solutions to delay fractional differential equations, *Mediterr. J. Math.*, **14** (2017), 1–12. https://doi.org/10.1007/s00009-017-0997-4

15. F. F. Wang, D. Y. Chen, X. G. Zhang, Y. Wu, The existence and uniqueness theorem of the solution to a class of nonlinear fractional order system with time delay, *Appl. Math. Lett.*, **53** (2016), 45–51. https://doi.org/10.1016/j.aml.2015.10.001

16. A. Atangana, S. Igret Araz, A modified parametrized method for ordinary differential equations with nonlocal operators, preprint, `arXiv:hal-03840759`.

17. J. W. Lee, D. O'regan, Existence results for differential delay equations, *J Differ Equ.*, **102** (1993), 342–359. https://doi.org/10.1016/0362-546X(91)90113-F

18. S. B. Hadid, Carathéodory's existence theorem of generalized order differential equations by using Ascoli's Lemma, *Antarct. J. Math.*, **11** (2014), 129–137. https://doi.org/10.7312/columbia/9780231164023.003.0011

19. B. C. Dhage, D. N. Chate, S. K. Ntouyas, A system of abstract measure delay differential equations, *Electron. J. Qual. Theory Differ. Equ.*, **8** (2003), 1–14. https://doi.org/10.14232/ejqtde.2003.1.8

20. W. Lin, Global existence theory and chaos control of fractional differential equations, *J. Math. Anal. Appl.*, **332** (2007), 709–726. https://doi.org/10.1016/j.jmaa.2006.10.040

21. J. Persson, A generalization of Carathéodory's existence theorem for ordinary differential equations, *J. Math. Anal. Appl.*, **49** (1975), 496–503. https://doi.org/10.1016/0022-247X(75)90192-4

22. J. Diblík J., G. Vážanová, Lower and upper estimates of semi-global and global solutions to mixed-type functional differential equations, *Adv. Nonlinear Anal.*, **1** (2022), 757–784. https://doi.org/10.1515/anona-2021-0218

23. H. Xiao, Z. Guo, Periodic solutions to a class of distributed delay differential equations via variational methods, *Adv. Nonlinear Anal.*, **12** (2023), 20220305. https://doi.org/10.1515/anona-2022-0305

24. M. R. Cartabia, Cucker-Smale model with time delay, *Discrete Cont. Dyn. S*, **42** (2022), 2409–2432. https://doi.org/10.3934/dcds.2021195

25. K. A. Abro, A. Siyal, A. Atangana, Q. M. Al-Mdallal, Chaos control and characterization of brushless DC motor via integral and differential fractal-fractional techniques, *Int. J. Model. Simul.*, **43** (2023), 2409–2432. https://doi.org/10.1080/02286203.2022.2086743

26. K. A. Abro, A. Siyal, A. Atangana, J. F. Gomez-Aguilar, Optimal synchronization of fractal–fractional differentials on chaotic convection for Newtonian and non-Newtonian fluids, *Eur Phys J Spec Top.*, **232** (2023), 2403–2414. https://doi.org/10.1140/epjs/s11734-023-00913-6

27. A. Atangana, S. Igret Araz, *Theory and Methods of Piecewise Defined Fractional Operators*, Academic Press, Elsevier, 2024.

28. C. Carathéodory Über den Variabilitätsbereich der Koeczienten von Potenzreihen,die gegebene Werte nicht annehmen. *Math. Ann.*, **64** (1907), 95–115. https://doi.org/10.1007/BF01449883

29. A. Akgul, I. Moroz, I. Pehlivan, S. Vaidyanathan, A new four-scroll chaotic attractor and its engineering applications, *Optik*, **127** (2016), 5491–5499. https://doi.org/10.1016/j.ijleo.2016.02.066

30. A. Atangana, S. Igret Araz, *New numerical scheme with Newton polynomial: Theory, Methods and Applications*, Academic Press, Elsevier, 2021. https://doi.org/10.1016/C2020-0-02711-8

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