Communications in Analysis and Mechanics

https://www.aimspress.com/journal/cam

Communications in Analysis and Mechanics, 16(1): 121–146.
DOI: 10.3934/cam.2024006
Received: 12 September 2023
Revised: 18 December 2023
Accepted: 12 January 2024
Published: 23 January 2024

Research article

Existence and concentration of homoclinic orbits for first order Hamiltonian systems

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Abstract: This paper is concerned with the following first-order Hamiltonian system

$$\dot{z} = \mathscr{J} H_z(t, z),$$

where the Hamiltonian function $H(t, z) = \frac{1}{2} Lz \cdot z + A(\epsilon t)G(|z|)$ and $\epsilon > 0$ is a small parameter. Under some natural conditions, we obtain a new existence result for ground state homoclinic orbits by applying variational methods. Moreover, the concentration behavior and exponential decay of these ground state homoclinic orbits are also investigated.

Keywords: Hamiltonian system; ground state homoclinic orbits; concentration
Mathematics Subject Classification: 37J45, 70H05, 58E50

1. Introduction and main result

In this paper, we are interested in the existence and some asymptotic properties of ground state homoclinic orbits for the following first-order Hamiltonian system

$$\dot{z} = \mathscr{J} H_z(t, z), \tag{1.1}$$

with the Hamiltonian function

$$H(t, z) = \frac{1}{2} Lz \cdot z + A(\epsilon t)G(|z|). \tag{1.2}$$

Here, $z = (u, v) \in \mathbb{R}^N \times \mathbb{R}^N = \mathbb{R}^{2N}$, $\epsilon > 0$ is a parameter, L is a symmetric $2N \times 2N$ matrix-valued function, and

$$\mathscr{J} = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

is the usual symplectic matrix with I being the identity matrix in $\mathbb{R}^N$. As usual, we refer to a solution z of system (1.1) as a homoclinic orbit if $z(t) \not\equiv 0$ and $z(t) \to 0$ as $|t| \to \infty$.

It is widely known that Hamiltonian systems are very important dynamical systems, which have many applications in several natural science areas such as relativistic mechanics, celestial mechanics, gas dynamics, chemical kinetics, optimization and control theory and so on. The complicated dynamical behavior of Hamiltonian systems has the attracted attention of many mathematicians and physicists ever since Newton wrote down the differential equations describing planetary motions and derived Kepler's ellipses as solutions. For more details on applications of Hamiltonian systems, one can refer to [1] and the monograph [2] of Mawhin and Willem. We also refer the readers to see [3] for the Stokes-Dirac structures of the port-Hamiltonian systems.

In the past few decades, Hamiltonian systems have attracted a considerable amount of interest due to many powerful applications in different fields; the literature related to these systems is extensive and encompasses several interesting lines of research on the topic of nonlinear analysis, including the existence, nonexistence, multiplicity and finer qualitative properties of homoclinic orbits. Here we cannot provide a complete and fully detailed list of references, but rather we limit ourselves to mentioning the works which are closely related to the content of the present paper.

A major breakthrough was the pioneering paper of Rabinowitz [4] from 1978 who, for the first time, obtained periodic solutions of system (1.1) by using variational methods. After the celebrated work of Rabinowitz [4], based on the dual action and mountain pass argument, Coti-Zelati et al. [5] obtained the existence and multiplicity of homoclinic orbits under the condition of strictly convexity. Later on, this result was further detailed by [6, 7] in which the authors established the existence result for infinitely many homoclinic orbits. Without the convexity condition, Hofer and Wysocki [8] independently investigated the existence of homoclinic orbits by combining the Fredholm operator theory and the linking argument. Tanaka [9] employed a suitable subharmonic approach to obtain one homoclinic orbit by relaxing the convexity condition. In [10], Rashkovskiy studied the quantization process of Hamiltonian and non-Hamiltonian systems.

The main unusual feature of the first-order Hamiltonian system is that the associated energy functional is strongly indefinite. Generally speaking, for the strongly indefinite functionals refined variational methods like the Nehari manifold method and mountain pass theorem still do not apply. Some general critical point theories like the generalized linking theorem and other weaker versions for strongly indefinite functionals were subsequently developed by Kryszewski and Szulkin in [11] and Bartsch and Ding in [12]. Since then, based on the critical point theorems from [11, 12] for strongly indefinite functionals, many scholars have gradually begun to investigate the existence and multiplicity of homoclinic orbits for non-autonomous Hamiltonian systems under some different conditions. More precisely, under the conditions that H depends periodically on t and has super-quadratic growth in z, Arioli and Szulkin [13], Chen and Ma [14], and Ding and Willem [15] obtained the existence result. Ding [16], Ding and Girardi [17], and Zhang et al. [18] studied the multiplicity result for homoclinic orbits. Concerning the asymptotic quadratic growth case, we refer to the work done by Szulkin and Zou [19] and Sun et al. [20]. Here we would like to emphasize that the periodicity condition is used to resolve the issue stemming from the lack of compactness since system (1.1) is set on the whole space $\mathbb{R}$.

On the other hand, without the condition of periodicity, the non-periodic problem is quite different due to the lack of compactness of Sobolev embeddings. In an early paper [21], Ding and Li utilized the coercive property of L to establish a variational framework with compactness, and they proved the existence of homoclinic orbits for the super-quadratic growth case. Also under the framework of compactness, Zhang and Liu [22] studied the sub-quadratic growth case. Regarding the asymptotically quadratic case, based on the infinite-dimensional linking argument, Ding and Jeanjean [23] established a multiplicity result for homoclinic orbits. Moreover, they imposed a control on the size of G with respect to the behavior of L to recover sufficient compactness. For the existence and exponential decay of homoclinic orbits for system (1.1) with nonperiodic super-quadratic and lack of compactness, we refer the reader to [24]. We also mention the recent paper by Zhang et al. [25] in which the existence and decay of ground state homoclinic orbits for system (1.1) with asymptotic periodicity are explored. For other results related to the Hamiltonian systems with strongly variational structure, we refer the reader to [26–32] and the references therein.

It is worth pointing out that, in all of the works mentioned above, the authors were concerned mainly with the study of the existence and multiplicity of homoclinic orbits, and there are no papers considering the asymptotic properties of homoclinic orbits. Inspired by this fact and the work done by Alves and Germano [33] in which the authors investigated the existence and concentration of ground state solutions for the Schrödinger equation; in the present paper we aim to further study the existence and some asymptotic properties of ground-state homoclinic orbits for system (1.1) with Hamiltonian function (1.2). This is a very interesting issue that has motivated the present work.

To continue the discussion, we introduce the following notation

$$S = -\left(\mathcal{J}\frac{d}{dt} + L\right),$$

then, system (1.1) takes the following form

$$Sz = A(\epsilon t)g(|z|)z, \quad t \in \mathbb{R}. \tag{1.3}$$

Before stating our results, we suppose the following conditions hold for L, A and G.

(L) L is a constant symmetric $2N \times 2N$ matrix such that $\sigma(\mathcal{J}L) \cap i\mathbb{R} = \varnothing$, where σ denotes the spectrum of operator $\mathcal{J}L$.

(A) $A \in C(\mathbb{R}, \mathbb{R})$ and $0 < \inf_{t\in\mathbb{R}} A(t) \le A_\infty := \lim_{|t|\to\infty} A(t) < A(0) = \max_{t\in\mathbb{R}} A(t)$;

(g_1) $G_z(|z|) = g(|z|)z$, $g \in C(\mathbb{R}^+, \mathbb{R}^+)$, and there exist $p > 2$ and $c_0 > 0$ such that

$$|g(s)| \le c_0(1 + |s|^{p-2}) \text{ for all } s \in \mathbb{R}^+;$$

(g_2) $g(s) = o(1)$ as $s \to 0$, and $G(s)/s^2 \to +\infty$ as $s \to +\infty$;

(g_3) $g(s)$ is strictly increasing in s on $(0, +\infty)$.

Next we state the main result of this paper as follows.

Theorem 1.1. *Assume that conditions* (L), (A), *and* (g_1)-(g_3) *hold. Then we have the following results:*

(a) *there exists* $\epsilon_0 > 0$ *such that system* (1.1) *has a ground state homoclinic orbit* z_ϵ *for each* $\epsilon \in (0, \epsilon_0)$*;*

(b) $|z_\epsilon|$ *attains its maximum at* t_ϵ*, then,*

$$\lim_{\epsilon\to 0} A(\epsilon t_\epsilon) = A(0),$$

moreover, $z_\epsilon(t + t_\epsilon) \to z$ *as* $\epsilon \to 0$*, where* z *is a ground state homoclinic orbit of the limit system*

$$Sz = A(0)g(|z|)z, \quad t \in \mathbb{R};$$

(c) *additionally, if* $A, g \in C^1$, *and* $g'(s)s = o(1)$ *as* $s \to 0$, *then there exist constants* $c, C > 0$ *such that*

$$|z(t)| \leq C \exp(-c|t - t_\epsilon|) \text{ for all } t \in \mathbb{R}.$$

We would like to emphasize that since our problem is carried out in the whole space, then the strongly indefiniteness of energy functionals and the lack of compactness are two major difficulties that we encounter in order to guarantee the existence of homoclinic orbits. More precisely, one reason is that strongly indefinite functionals are unbounded from below and from above so that the classical methods from the calculus of variations do not apply. The other reason is that the lack of compactness leads to the energy functionals not satisfying the necessary compactness property.

Let us now outline the methods involved to prove Theorem 1.1. Indeed, based on the above reasons, first, we will take advantage of the method of the generalized Nehari manifold developed by Szulkin-Weth [34] to handle system (1.1), this is because such a strategy helps to overcome the difficulty caused by strongly indefinite features. Second, we must verify that the energy functional possesses the necessary compactness property at some energy level. This target will be accomplished by applying the energy comparison argument to establish some precise comparison relationships for the ground-state energy value between the original problem and certain auxiliary problems. Finally, combining the compactness analysis technique, Kato's inequality, and the sub-solution estimate, we can obtain the concentration property and decay of homoclinic orbits. Then Theorem 1.1 follows naturally.

This paper is organized as follows. In Section 2, we establish the functional analytic setting associated with system (1.1). In Section 3, we present some technical results, and obtain the existence result for ground-state homoclinic orbits for the autonomous system. Section 4 is devoted to proofs of Theorem 1.1.

2. Functional analytic setting

Throughout the present paper, we will use the following notations:

- $\|\cdot\|_s$ denotes the norm of the Lebesgue space $L^s(\mathbb{R})$, $1 \leq s \leq +\infty$;
- $(\cdot,\cdot)_2$ denotes the usual inner product of the space $L^2(\mathbb{R})$;
- c, c_i, C_i represent various different positive constants.

In what follows, we will establish the variational framework to work for system (1.1).

Recall that $S = -(\mathcal{J}\frac{d}{dt} + L)$ is a self-adjoint operator on the space $L^2 := L^2(\mathbb{R}, \mathbb{R}^{2N})$ with the domain $\mathcal{D}(S) = H^1(\mathbb{R}, \mathbb{R}^{2N})$; according to the discussion in [13], we can know that, under the condition (L), there exists $a > 0$ such that $(-a, a) \cap \sigma(S) = \emptyset$ (see also [16, 19]). Therefore, the space L^2 has the following orthogonal decomposition

$$L^2 = L^- \oplus L^+, \quad z = z^- + z^+$$

corresponding to the spectrum decomposition of S such that S is positive definite (resp. negative definite) in L^+ (resp. L^-).

We use $|S|$ to denote the absolute value of S, and $|S|^{1/2}$ denotes the square root of $|S|$. Let $E = \mathscr{D}(|S|^{1/2})$ be the domain of the self-adjoint operator $|S|^{1/2}$, which is a Hilbert space equipped with the following inner product

$$(z, w) = (|S|^{1/2}z, |S|^{1/2}w)_2, \text{ for } z, w \in E,$$

and the norm $\|z\|^2 = (z, z)$. Evidently, E possesses the following decomposition

$$E = E^- \oplus E^+, \quad \text{where} \quad E^\pm = E \cap L^\pm,$$

which is orthogonal with respect to the two inner products $(\cdot, \cdot)_2$ and $(\cdot, \cdot)$. Moreover, by using the polar decomposition of S we can obtain that

$$Sz^- = -|S|z^-, \; Sz^+ = |S|z^+ \; \text{ for all } z = z^+ + z^- \in E.$$

Furthermore, from [16] we have the embedding theorem, that is, E embeds continuously into L^q for each $q \geq 2$ and compactly into L^q_{loc} for all $q \geq 1$. Hence, there exists a constant $\gamma_q > 0$ such that for all $z \in E$

$$\|z\|_q \leq \gamma_q \|z\| \text{ for all } q \geq 2. \tag{2.1}$$

From the assumptions (g_1) and (g_2), we can deduce that for any $\epsilon > 0$, there exists a positive constant C_ϵ such that

$$|g(s)| \leq \epsilon + C_\epsilon |s|^{p-2} \;\text{ and }\; |G(s)| \leq \epsilon |s|^2 + C_\epsilon |s|^p \;\text{ for each }\; s \in \mathbb{R}^+. \tag{2.2}$$

Next, we define the corresponding energy functional of system (1.3) on E by

$$I_\epsilon(z) = \frac{1}{2}\int_{\mathbb{R}} Sz \cdot z \mathrm{d}t - \int_{\mathbb{R}} A(\epsilon t) G(|z|) \mathrm{d}t$$

Applying the polar decomposition of S, then the energy functional I_ϵ has another representation as follows

$$I_\epsilon(z) = \frac{1}{2}(\|z^+\|^2 - \|z^-\|^2) - \int_{\mathbb{R}} A(\epsilon t) G(|z|) \mathrm{d}t.$$

Evidently, according to condition (L), we can see that I_ϵ is strongly indefinite. Furthermore, from conditions (g_1) and (g_2) we can infer that $I_\epsilon \in C^1(E, \mathbb{R})$, and we have

$$\langle I'_\epsilon(z), \psi \rangle = (z^+, \psi^+) - (z^-, \psi^-) - \int_{\mathbb{R}} A(\epsilon t) g(|z|) z \psi \mathrm{d}t, \;\forall \psi \in E.$$

Making use of a standard argument we can check that critical points of I_ϵ are homoclinic orbits of system (1.1).

3. The autonomous system

We shall make use of the techniques of the limit problem to prove the main results; in this section we introduce some related results for the autonomous system.

For any constant $\mu > 0$, in what follows we consider the autonomous system given by

$$Sz = \mu g(|z|)z, \quad t \in \mathbb{R}. \tag{3.1}$$

Similarly, following the above comments, we define the energy functional I_μ corresponding to system (3.1) as follows

$$I_\mu(z) = \frac{1}{2}\left(\|z^+\|^2 - \|z^-\|^2\right) - \mu \int_{\mathbb{R}} G(|z|) \mathrm{d}t.$$

Evidently, we have

$$\langle I'_\mu(z), \psi\rangle = (z^+, \psi^+) - (z^-, \psi^-) - \mu \int_{\mathbb{R}} g(|z|)z\psi \mathrm{d}t, \ \forall \psi \in E.$$

In order to obtain the ground state homoclinic orbits of system (3.1), we will use the method of the generalized Nehari manifold developed by Szulkin and Weth [34]. To do this, we first introduce the following generalized Nehari manifold

$$\mathcal{N}_\mu = \{z \in E\backslash E^- : \langle I'_\mu(z), z\rangle = 0 \text{ and } \langle I'_\mu(z), v\rangle = 0, \ \ \forall v \in E^-\},$$

and we define the ground state energy value c_μ of I_μ on $\mathcal{N}_\mu$

$$c_\mu = \inf_{z\in\mathcal{N}_\mu} I_\mu(z).$$

Furthermore, for every $z \in E\backslash E^-$, we also need to define the subspace

$$E(z) = E^- \oplus \mathbb{R}z = E^- \oplus \mathbb{R}z^+,$$

and the convex subset

$$\widehat{E}(z) = E^- \oplus [0, +\infty)z = E^- \oplus [0, +\infty)z^+.$$

We note that $E(z)$ and $\widehat{E}(z)$ do not depend on μ, but depend on the operator S.

The main result in this section is the following theorem:

Theorem 3.1. *Assume that condition* (L) *holds and let* (g_1)-(g_3) *be satisfied, then, problem* (3.1) *has at least a ground state homoclinic orbit* $z \in \mathcal{N}_\mu$ *such that* $I_\mu(z) = c_\mu > 0$.

3.1. Technical results

In this subsection, we are going to prove some technical results which will be used in the proof of Theorem 3.1. The following result involves the translation that will be used frequently in this paper, the proof can be found in [33, Lemma 2.1].

Lemma 3.1. *For all* $u = u^+ + u^- \in E$ *and* $y \in \mathbb{R}$, *if* $v(t) := u(t + y)$, *then* $v \in E$ *with* $v^+(t) = u^+(t + y)$ *and* $v^-(t) = u^-(t + y)$.

We give an important estimate, which plays a crucial role in the later proof.

Lemma 3.2. *Let* $z \in \mathcal{N}_\mu$, *then, for each* $v \in \mathcal{H} := \{sz + w : s \geq -1, w \in E^-\}$ *and* $v \neq 0$, *we have the following energy estimate*

$$I_\mu(z + v) < I_\mu(z).$$

Hence z *is a unique global maximum of* $I_\mu|_{\widehat{E}(z)}$.

Proof. We follow the similar ideas explored in [34, Proposition 2.3.] to prove this lemma. Observe that, for any $z \in \mathcal{N}_\mu$, we directly obtain

$$0 = \langle I'_\mu(z), \varphi\rangle = (Az, \varphi)_2 - \mu \int_{\mathbb{R}} g(|z|)z\varphi \mathrm{d}t \text{ for all } \varphi \in E(z).$$

Let $v = sz + w \in \mathcal{H}$, then, $z + v = (1 + s)z + w \in \widehat{E}(z)$. By an elemental computation, we can get

$$
\begin{aligned}
& I_\mu(z + v) - I_\mu(z) \\
&= \frac{1}{2}\Big[(A(z + v), (z + v))_2 - (Az, z)_2\Big] + \mu \int_{\mathbb{R}} \Big[G(|z|) - G(|z + v|)\Big]\mathrm{d}t \\
&= \frac{1}{2}\Big[(A((1 + s)z + w), (1 + s)z + w)_2 - (Az, z)_2\Big] + \mu \int_{\mathbb{R}} \Big[G(|z|) - G(|z + v|)\Big]\mathrm{d}t \\
&= -\frac{\|w\|^2}{2} + (Az, s(\frac{s}{2} + 1)z + (1 + s)w)_2 + \mu \int_{\mathbb{R}} \Big[G(|z|) - G(|z + v|)\Big]\mathrm{d}t \\
&= -\frac{\|w\|^2}{2} + \mu \int_{\mathbb{R}} \Big[g(|z|)z \cdot \Big(s(\frac{s}{2} + 1)z(t) + (1 + s)w(t)\Big) + G(|z|) - G(|z + v|)\Big]\mathrm{d}t \\
&= -\frac{\|w\|^2}{2} + \mu \int_{\mathbb{R}} \widetilde{g}(s, z, v)\mathrm{d}t,
\end{aligned}
$$

where

$$\widetilde{g}(s, z, v) = g(|z|)z \cdot \Big(s(\frac{s}{2} + 1)z(t) + (1 + s)w(t)\Big) + G(|z|) - G(|z + v|).$$

Using (g_2) and (g_3) and combining the arguments used in [35] (see also [36]), we can conclude that $\widetilde{g}(s, z, v) < 0$. Therefore, we have

$$I_\mu(z + v) < I_\mu(z).$$

Evidently, we know that z is a unique global maximum of $I_\mu|_{\widehat{E}(z)}$.

Lemma 3.3. *Assume that (g_1) and (g_2) are satisfied, then, we have the following two conclusions:*

(i) *there exists $\rho > 0$ such that $c_\mu = \inf_{\mathcal{N}_\mu} I_\mu \geq \inf_{S_\rho} I_\mu > 0$, where*

$$S_\rho := \{z \in E^+ : \|z\| = \rho\};$$

(ii) *for any $z \in \mathcal{N}_\mu$, then $\|z^+\|^2 \geq \max\{\|z^-\|^2, 2c_\mu\} > 0$.*

Proof. (i) Let $z \in E^+$, we can deduce from (2.1) and (2.2) that

$$I_\mu(z) = \frac{1}{2}\|z\|^2 - \mu \int_{\mathbb{R}} G(|z|)\mathrm{d}t \geq \left(\frac{1}{2} - \epsilon\mu\gamma_2^2\right)\|z\|^2 - \mu\gamma_p^p C_\epsilon \|z\|^p.$$

Evidently, we can see that there is $\rho > 0$, for $\|z\| = \rho$ small enough such that $\inf_{S_\rho} I_\mu > 0$.

On the other hand, for each $z \in \mathcal{N}_\mu$, there exists a positive constant s such that $s\|z\| = \rho$, then $sz \in \widehat{E}(z) \cap S_\rho$. From Lemma 3.2, one can derive that

$$I_\mu(z) = \max_{v \in \widehat{E}(z)} I_\mu(v) \geq I_\mu(sz).$$

Therefore, we have

$$\inf_{\mathcal{N}_\mu} I_\mu \geq \inf_{S_\rho} I_\mu > 0,$$

which shows that the conclusion (i) holds.

(ii) First we note that, from (g_3), it follows that

$$\frac{1}{2}g(s)s^2 \geq G(s) > 0 \text{ for all } s \in \mathbb{R}^+.$$

For each $z \in \mathcal{N}_\mu$, combining this with the definition of c_μ, we have

$$0 < c_\mu \leq \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2 - \mu \int_{\mathbb{R}} G(|z|)\mathrm{d}t \leq \frac{1}{2}\|z^+\|^2 - \frac{1}{2}\|z^-\|^2.$$

Hence, we can derive that $\|z^+\|^2 \geq \max\{\|z^-\|^2, 2c_\mu\} > 0$. The proof is finished.

Lemma 3.4. *Assume that $\Omega \subset E^+ \setminus \{0\}$ is a compact subset, thus, there exists $R > 0$ such that $I_\mu < 0$ on $E(z) \setminus B_R(0)$ for all $z \in \Omega$.*

Proof. The proof follows as in [34, Lemma 2.5], here, we omit the details.

Following the result in [34] (see [34, Lemma 2.6]), we can establish the uniqueness of maximum point of I_μ on the set $\widehat{E}(z)$.

Lemma 3.5. *For any $z \in E\backslash E^-$, then the set $\mathcal{N}_\mu \cap \widehat{E}(z)$ consists of precisely one point $\widetilde{m}_\mu(z) \neq 0$, which is the unique global maximum of $I_\mu|_{\widehat{E}(z)}$.*

Proof. On account of Lemma 3.2, it is sufficient to prove that $\mathcal{N}_\mu \cap \widehat{E}(z) \neq \emptyset$. Since $\widehat{E}(z) = \widehat{E}(z^+)$, without loss of generality, we can suppose that $z \in E^+$ and $\|z\| = 1$. By Lemma 3.3-(i), we obtain that $I_\mu(sz) > 0$ for $s \in (0, +\infty)$ small enough. Lemma 3.4 yields that $I_\mu(sz) < 0$ for $sz \in \widehat{E}(z)\backslash B_R(0)$. Consequently, we can deduce that $0 < \sup I_\mu(\widehat{E}(z)) < \infty$. Because $\widehat{E}(z)$ is convex and the functional I_μ is weakly supper semi-continuous on $\widehat{E}(z)$, we conclude that there exists $\widehat{z} \in \widehat{E}(z)$ such that $I_\mu(\widehat{z}) = \sup I_\mu(\widehat{E}(z))$. This shows that $\widehat{z}$ is a critical point of $I_\mu|_{\widehat{E}(z)}$; therefore,

$$\langle I'_\mu(\widehat{z}), \widehat{z}\rangle = \langle I'_\mu(\widehat{z}), \varphi\rangle = 0 \ \text{ for all } \ \varphi \in \widehat{E}(z),$$

hence, $\widehat{z} \in \mathcal{N}_\mu$. So, $\widehat{z} \in \mathcal{N}_\mu \cap \widehat{E}(z)$. The proof is finished.

Combining Lemma 3.2 with Lemma 3.5, we obtain the following conclusion.

Lemma 3.6. *For each $z \in E\backslash E^-$, then, there is a unique pair (s^*, φ^*) with $s^* \in (0, +\infty)$ and $\varphi^* \in E^-$ such that $s^*z + \varphi^* \in \mathcal{N}_\mu \cap \widehat{E}(z)$ and*

$$I_\mu(s^*z + \varphi^*) = \max_{w \in \widehat{E}(z)} I_\mu(w).$$

Moreover, if $z \in \mathcal{N}_\mu$, then we have that $s^* = 1$ and $\varphi^* = z^-$.

Lemma 3.7. *I_μ is coercive on $\mathcal{N}_\mu$, that is, $I_\mu(z) \to +\infty$ as $\|z\| \to +\infty$, $z \in \mathcal{N}_\mu$.*

Proof. Seeking for a contradiction, assume that there exists $\{z_n\} \subset \mathcal{N}_\mu$ such that

$$I_\mu(z_n) \leq \widehat{c} \ \text{ for some } \ \widehat{c} \in [c_\mu, +\infty) \ \text{ as } \ \|z_n\| \to +\infty.$$

Setting $w_n := z_n/\|z_n\|$, we obtain that $\|z_n^+\| \geq \|z_n^-\|$ from Lemma 3.3(ii), then, for every $n \in \mathbb{N}$, we get that $\|w_n^+\|^2 \geq \|w_n^-\|^2$ and $\|w_n^+\|^2 \geq \frac{1}{2}$. There exist $\{y_n\} \subset \mathbb{Z}$, $r > 0$ and $\delta > 0$ such that

$$\int_{B_r(y_n)} |w_n^+|^2 \mathrm{d}t \geq \delta, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

If this is not true, then according to Lions' concentration-compactness principle, we can conclude that $w_n^+ \to 0$ in $L^q(\mathbb{R})$ for $q > 2$. Combining (2.1) and (2.2), we know that, for every $s > 0$,

$$\mu \int_{\mathbb{R}} G(|sw_n^+|)\mathrm{d}t \leq \epsilon\mu\gamma_2^2 s^2 \|w_n^+\|^2 + \mu C_\epsilon \gamma_p^p s^p \|w_n^+\|^p \to 0,$$

then we get

$$\begin{aligned} \widehat{c} \geq I_\mu(sw_n^+) &= \frac{1}{2}s^2\|w_n^+\|^2 - \mu \int_{\mathbb{R}} G(|sw_n^+|)\mathrm{d}t \\ &\geq \frac{s^2}{4} - \mu \int_{\mathbb{R}} G(|sw_n^+|)\mathrm{d}t \to \frac{s^2}{4}. \end{aligned}$$

This yields a contradiction if $s > \sqrt{4\widehat{c}}$; hence we prove that (3.2) holds.

Let us define $\tilde{z}_n(t) := z_n(t + y_n)$, and $\tilde{w}_n(t) := w_n(t + y_n)$, then, $\tilde{w}_n^+ \rightharpoonup \tilde{w}^+$, and (3.2) yields that $\tilde{w}^+ \neq 0$. Since $\tilde{z}_n(t) = \tilde{w}_n(t)\|\tilde{z}_n\|$, it follows that $\tilde{z}_n(t) \to +\infty$ almost everywhere in $\mathbb{R}$ as $\|\tilde{z}_n\| = \|z_n\| \to +\infty$. Applying the Fatou's lemma, we can derive that

$$\int_{\mathbb{R}} \frac{G(|z_n|)}{\|z_n\|^2}\mathrm{d}t = \int_{\mathbb{R}} \frac{G(|\tilde{z}_n|)}{\|\tilde{z}_n\|^2}\mathrm{d}t = \int_{\mathbb{R}} \frac{G(|\tilde{z}_n|)}{|\tilde{z}_n|^2}|\tilde{w}_n|^2\mathrm{d}t \geq \int_{[\tilde{z}_n \neq 0]} \frac{G(|\tilde{z}_n|)}{|\tilde{z}_n|^2}|\tilde{w}_n|^2\mathrm{d}t \to +\infty,$$

where $[\tilde{z}_n \neq 0]$ is the Lebesgue measure of the set $\{t \in \mathbb{R} : \tilde{z}_n(t) \neq 0\}$. Therefore

$$\begin{aligned} 0 \leq \frac{I_\mu(z_n)}{\|z_n\|^2} &= \frac{1}{2}\|w_n^+\|^2 - \frac{1}{2}\|w_n^-\|^2 - \mu \int_{\mathbb{R}} \frac{G(|z_n|)}{\|z_n\|^2}\mathrm{d}t \\ &\leq \frac{1}{2} - \mu \int_{\mathbb{R}} \frac{G(|\tilde{z}_n|)}{|\tilde{z}_n|^2}|\tilde{w}_n|^2\mathrm{d}t \to -\infty, \end{aligned}$$

we get a contradiction. The proof is finished.

We want to utilize the method of the generalized Nehari manifold to prove the main result. To do this, we set $S^+ := \{z \in E^+ : \|z\| = 1\}$ in E^+, and we define the following mapping

$$\widetilde{m}_\mu : E^+\backslash\{0\} \to \mathcal{N}_\mu \text{ and } m_\mu = \widetilde{m}_\mu|_{S^+},$$

and the inverse of m_μ is

$$m_\mu^{-1} : \mathcal{N}_\mu \to S^+, \; m_\mu^{-1}(z) = z^+/\|z^+\|.$$

Following from the proof of [34, Lemma 2.8], we can see that $\widetilde{m}_\mu$ is continuous and m_μ is a homeomorphism.

We now consider the reduced functionals

$$\widetilde{\Phi}_\mu(z) = I_\mu(\widetilde{m}_\mu(z)) \text{ and } \Phi_\mu = \widetilde{\Phi}_\mu|_{S^+}.$$

which is continuous since $\widetilde{m}_\mu$ is continuous.

The following results establish some crucial properties involving the reduced functionals $\widetilde{\Phi}_\mu$ and Φ_μ, which play important roles in our arguments. And their proofs follow the proofs of [34, Proposition 2.9, Corollary 2.10].

Lemma 3.8. *The following conclusions are true:*

(a) $\widetilde{\Phi}_\mu \in C^1(E^+\backslash\{0\}, \mathbb{R})$ *and for* $z, v \in E^+$ *and* $z \neq 0$,

$$\langle \widetilde{\Phi}'_\mu(z), v\rangle = \frac{\|\widetilde{m}_\mu(z)^+\|}{\|z\|}\langle I'_\mu(\widetilde{m}_\mu(z)), v\rangle.$$

(b) $\Phi_\mu \in C^1(S^+, \mathbb{R})$ *and for each* $z \in S^+$ *and* $v \in T_z(S^+) = \{u \in E^+ : (z, u) = 0\}$,

$$\langle \Phi'_\mu(z), v\rangle = \|\widetilde{m}_\mu(z)^+\|\langle I'_\mu(\widetilde{m}_\mu(z)), v\rangle.$$

(c) $\{z_n\}$ *is a (PS)-sequence for* Φ_μ *if and only if* $\{\widetilde{m}_\mu(z_n)\}$ *is a (PS)-sequence for* I_μ.

(d) *We have*

$$\inf_{S^+} \Phi_\mu = \inf_{\mathcal{N}_\mu} I_\mu = c_\mu.$$

Moreover, $z \in S^+$ *is a critical point of* Φ_μ *if and only if* $\widetilde{m}_\mu(z) \in \mathcal{N}_\mu$ *is a critical point of* I_μ *and the corresponding critical values coincide.*

3.2. Proof of Theorem 3.1

Based on the above preliminaries, in this subsection we give the complete proof of Theorem 3.1, and further study the monotonicity and continuity of the ground-state energy c_μ.

Proof of Theorem 3.1: According to Lemma 3.3, it is easy to see that $c_\mu > 0$. We note that, if $z \in \mathcal{N}_\mu$ with $I_\mu(z) = c_\mu$, then $m_\mu^{-1}(z) \in S^+$ is a minimizer of Φ_μ; hence, it is a critical point of Φ_μ. Consequently, Lemma 3.8 yields that z is a critical point of I_μ. We have to prove that there exists a minimizer $\tilde{z} \in \mathcal{N}_\mu$ such that $I_\mu(\tilde{z}) = c_\mu$. Actually, Ekeland's variational principle yields that there exists a sequence $\{v_n\} \subset S^+$ such that $\Phi_\mu(v_n) \to c_\mu$ and $\Phi'_\mu(v_n) \to 0$ as $n \to \infty$. For all $n \in \mathbb{N}$, setting $z_n = \widetilde{m}_\mu(v_n) \in \mathcal{N}_\mu$, then $I_\mu(z_n) \to c_\mu$ and $I'_\mu(z_n) \to 0$ by Lemma 3.8. By virtue of Lemma 3.7, we can see that $\{z_n\}$ is bounded in E. Moreover, it satisfies

$$\varliminf_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{B_1(y)} |z_n|^2 dt > 0.$$

If this is not true, then Lions' concentration-compactness principle implies that $z_n \to 0$ in $L^q(\mathbb{R})$ for any $q > 2$. Therefore, from (2.1) and (2.2), we can derive that

$$\int_{\mathbb{R}} \left[\frac{1}{2}g(|z_n|)|z_n|^2 - G(|z_n|)\right] dt = o_n(1).$$

Then, we get

$$\begin{aligned} c_\mu + o_n(1) &= I_\mu(z_n) - \frac{1}{2}\langle I'_\mu(z_n), z_n\rangle \\ &= \mu \int_{\mathbb{R}} \left[\frac{1}{2}g(|z_n|)|z_n|^2 - G(|z_n|)\right] dt = o_n(1). \end{aligned}$$

Since $c_\mu > 0$, obviously we get a contradiction. Thus, there exist $\{y_n\} \subset \mathbb{Z}$ and $\delta > 0$ such that

$$\int_{B_2(y_n)} |z_n|^2 dt \geq \delta.$$

Let us define $\tilde{z}_n(t) = z_n(t + y_n)$, then, we have

$$\int_{B_2(0)} |\tilde{z}_n|^2 \mathrm{d}t \geq \delta. \tag{3.3}$$

Observe that I_μ is the invariant under translation since (3.1) is autonomous, then, we have $\|\tilde{z}_n\| = \|z_n\|$ and

$$I_\mu(\tilde{z}_n) \to c_\mu \ \text{ and } \ I'_\mu(\tilde{z}_n) \to 0. \tag{3.4}$$

Passing to a subsequence, we may suppose that $\tilde{z}_n \rightharpoonup \tilde{z}$ in E, $\tilde{z}_n \to \tilde{z}$ in $L^q_{loc}(\mathbb{R})$ for $q > 2$, and $\tilde{z}_n(t) \to \tilde{z}(t)$ almost everywhere on $\mathbb{R}$. According to (3.3) and (3.4), then we can derive that $\tilde{z} \neq 0$ and $I'_\mu(\tilde{z}) = 0$. This implies that $\tilde{z} \in \mathscr{N}_\mu$ and $I_\mu(\tilde{z}) \geq c_\mu$. On the other hand, applying Fatou's lemma we can obtain

$$\begin{aligned}
c_\mu &= \lim_{n\to\infty} \left(I_\mu(\tilde{z}_n) - \frac{1}{2}\langle I'_\mu(\tilde{z}_n), \tilde{z}_n \rangle \right) \\
&= \lim_{n\to\infty} \mu \int_{\mathbb{R}} \left(\frac{1}{2} g(|z_n|)|z_n|^2 - G(|z_n|) \right) \mathrm{d}t \\
&\geq \mu \int_{\mathbb{R}} \lim_{n\to\infty} \left(\frac{1}{2} g(|z_n|)|z_n|^2 - G(|z_n|) \right) \mathrm{d}t \\
&= I_\mu(\tilde{z}) - \frac{1}{2}\langle I'_\mu(\tilde{z}), \tilde{z} \rangle = I_\mu(\tilde{z}),
\end{aligned}$$

that is, $I_\mu(\tilde{z}) \leq c_\mu$. Consequently, $I_\mu(\tilde{z}) = c_\mu$ and $\tilde{z}$ is a critical point of I_μ, which implies that $\tilde{z}$ is a ground-state homoclinic orbit of problem (3.1). So, we have completed the proof of Theorem 3.1.

As a byproduct of the Theorem 3.1, we show the monotonicity and continuity of c_μ.

Lemma 3.9. *The function $\mu \mapsto c_\mu$ is decreasing and continuous on* $(0, +\infty)$.

Proof. In what follows, let z_{μ_1} and z_{μ_2} be as ground state homoclinic orbits of I_{μ_1} and I_{μ_2}, respectively. Assume that $\mu_1 > \mu_2$. First of all, we want to verify that the function $\mu \mapsto c_\mu$ is decreasing. On account of Lemma 3.6 we can find that there exist $s_1 > 0$ and $\varphi_1 \in E^-$ such that

$$I_{\mu_1}(s_1 z_{\mu_2} + \varphi_1) = \max_{z \in \widehat{E}(z_{\mu_2})} I_{\mu_1}(z),$$

then we have

$$\begin{aligned}
c_{\mu_1} &\leq I_{\mu_1}(s_1 z_{\mu_2} + \varphi_1) \\
&= I_{\mu_2}(s_1 z_{\mu_2} + \varphi_1) + (\mu_2 - \mu_1) \int_{\mathbb{R}} G(|s_1 z_{\mu_2} + \varphi_1|) \mathrm{d}t \\
&\leq I_{\mu_2}(z_{\mu_2}) + (\mu_2 - \mu_1) \int_{\mathbb{R}} G(|s_1 z_{\mu_2} + \varphi_1|) \mathrm{d}t \\
&= c_{\mu_2} + (\mu_2 - \mu_1) \int_{\mathbb{R}} G(|s_1 z_{\mu_2} + \varphi_1|) \mathrm{d}t.
\end{aligned}$$

Combining the fact that

$$\int_{\mathbb{R}} G(|s_1 z_{\mu_2} + \varphi_1|) \mathrm{d}t \geq 0,$$

with the inequality $\mu_2 - \mu_1 < 0$, we can infer that

$$c_{\mu_1} \leq c_{\mu_2}.$$

We finish the proof by demonstrating that the function $\mu \mapsto c_\mu$ is decreasing on $(0, +\infty)$.

In order to claim the continuity of c_μ, we divide the proof into two steps:

Step 1: Let $\{\mu_n\}$ be a sequence such that $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_n \leq \cdots \leq \mu$ and $\mu_n \to \mu$.

Claim 1: $c_{\mu_n} \to c_\mu$ as $n \to \infty$.

Indeed, let z_μ be the ground state homoclinic orbit of system (3.1). On account of Lemma 3.6, we can see that there exist $s_n > 0$ and $\varphi_n \in E^-$ such that

$$I_{\mu_n}(s_n z_\mu + \varphi_n) = \max_{z \in \widehat{E}(z_\mu)} I_{\mu_n}(z) \ \text{ for all } \ n \in \mathbb{N}.$$

Note that, using (g_3) and computing directly, we get

$$I_{\mu_1}(z) - I_{\mu_n}(z) = (\mu_n - \mu_1) \int_{\mathbb{R}} G(|z|) \mathrm{d}t \geq 0,$$

so, for all $n \in \mathbb{N}$ and $z \in E$, we have that $I_{\mu_1}(z) \geq I_{\mu_n}(z)$. Then by Lemma 3.4, it holds that there exists $R > 0$ such that

$$I_{\mu_n}(z) \leq I_{\mu_1}(z) \leq 0, \ \ \forall z \in \widehat{E}(z_\mu) \backslash B_R(0). \tag{3.5}$$

According to Lemma 3.3 and the monotonicity of c_μ, we can obtain

$$I_{\mu_n}(s_n z_\mu + \varphi_n) = \max_{z \in \widehat{E}(z_\mu)} I_{\mu_n}(z) \geq c_{\mu_n} \geq c_\mu > 0,$$

consequently, it follows that

$$I_{\mu_n}(s_n z_\mu + \varphi_n) > 0. \tag{3.6}$$

From (3.5) and (3.6), one can check that $\|s_n z_\mu + \varphi_n\| \leq R$; this shows that the sequence $\{s_n z_\mu + \varphi_n\}$ is bounded in E. Hence, it is easy to see that

$$\int_{\mathbb{R}} G(|s_n z_\mu + \varphi_n|) \mathrm{d}t \ \text{ is also bounded,}$$

then we get

$$\begin{aligned} c_{\mu_n} &\leq I_{\mu_n}(s_n z_\mu + \varphi_n) \\ &= I_\mu(s_n z_\mu + \varphi_n) + (\mu - \mu_n) \int_{\mathbb{R}} G(|s_n z_\mu + \varphi_n|) \mathrm{d}t \\ &\leq I_\mu(z_\mu) + (\mu - \mu_n) \int_{\mathbb{R}} G(|s_n z_\mu + \varphi_n|) \mathrm{d}t \\ &= c_\mu + o_n(1). \end{aligned}$$

On the other hand, since $c_\mu \leq c_{\mu_n}$ for all $n \in \mathbb{N}$, we can infer that

$$c_{\mu_n} \to c_\mu \text{ as } n \to \infty.$$

Step 2: Let $\{\mu_n\}$ be a sequence such that $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq \cdots \geq \mu$ and $\mu_n \to \mu$.

Claim 2: $c_{\mu_n} \to c_\mu$ as $n \to \infty$.

In fact, let z_n be the ground state homoclinic orbit of the system (3.1) with $\mu = \mu_n$, then, there exist $s_n > 0$ and $\varphi_n \in E^-$ such that

$$I_\mu(s_n z_n + \varphi_n) = \max_{z \in \widehat{E}(z_n)} I_\mu(z).$$

We can easily obtain that the sequence $\{z_n\}$ is bounded by Lemma 3.7. Moreover, we can find that there exist $\delta > 0$, $r > 0$ and $\{y_n\} \subset \mathbb{Z}$ such that for each $n \in \mathbb{N}$, we have

$$\int_{B_r(y_n)} |z_n^+|^2 \mathrm{d}t \geq \delta. \tag{3.7}$$

Otherwise, using Lions' concentration-compactness principle we can deduce that $z_n^+ \to 0$ in $L^q(\mathbb{R})$ for all $q > 2$. Combining (2.1) with (2.2), it holds that

$$\int_{\mathbb{R}} g(|z_n|) z_n z_n^+ \mathrm{d}t \to 0.$$

Therefore, we have

$$\begin{aligned} 0 = \langle I'_{\mu_n}(z_n), z_n^+ \rangle &= \|z_n^+\|^2 - \mu_n \int_{\mathbb{R}} g(|z_n|) z_n z_n^+ \mathrm{d}t \\ &= \|z_n^+\|^2 + o_n(1), \end{aligned}$$

this shows that $\|z_n^+\|^2 \to 0$, which is a contradiction to the inequality $\|z_n^+\|^2 \geq 2c_{\mu_n} > 0$ from Lemma 3.3. So, (3.7) holds.

Setting $\tilde{z}_n(t) := z_n(t + y_n)$, one can check that $\{\tilde{z}_n\}$ is bounded in E; passing to a subsequence, $\tilde{z}_n^+ \rightharpoonup \tilde{z}^+ \neq 0$ in E. Set $\mathcal{V} := \{\tilde{z}_n^+\} \subset E^+ \backslash \{0\}$; hence, $\mathcal{V}$ is bounded and the sequence does not weakly converge to zero in E. Then by Lemma 3.4, there exists $R > 0$ such that for every $z \in \mathcal{V}$, we obtain

$$I_\mu(w) < 0, \text{ for } w \in E(z) \backslash B_R(0). \tag{3.8}$$

Define $\tilde{\varphi}_n(t) := \varphi_n(t + y_n)$, we have

$$I_\mu(s_n \tilde{z}_n + \tilde{\varphi}_n) = I_\mu(s_n z_n + \varphi_n) = \max_{z \in \widehat{E}(z_n)} I_\mu(z) \geq c_\mu > 0, \quad \forall n \in \mathbb{N}. \tag{3.9}$$

In view of (3.8) and (3.9), we can conclude that $\|s_n \tilde{z}_n + \tilde{\varphi}_n\| \leq R$ for all $n \in \mathbb{N}$, then, $\|s_n z_n + \varphi_n\| \leq R$, which implies that the sequence $\{s_n z_n + \varphi_n\}$ is bounded in E, and $\int_{\mathbb{R}} G(|s_n z_n + \varphi_n|) \mathrm{d}t$ is also bounded. Thus, we obtain

$$\begin{aligned} c_\mu &\leq I_\mu(s_n z_n + \varphi_n) \\ &= I_{\mu_n}(s_n z_n + \varphi_n) + (\mu_n - \mu) \int_{\mathbb{R}} G(|s_n z_n + \varphi_n|) \mathrm{d}t \\ &\leq I_{\mu_n}(z_n) + (\mu_n - \mu) \int_{\mathbb{R}} G(|s_n z_n + \varphi_n|) \mathrm{d}t \\ &= c_{\mu_n} + o_n(1). \end{aligned}$$

Combining this with the fact that $c_\mu \geq c_{\mu_n}$ for all $n \in \mathbb{N}$, then we have

$$c_{\mu_n} \to c_\mu \text{ as } n \to \infty.$$

We have finished the proof of the lemma.

4. The proof of Theorem 1.1

4.1. Existence of ground state homoclinic orbits

In this subsection we will give the proof involving the existence result for ground state homoclinic orbits for system (1.1). As before, we define the associated generalized Nehari manifold

$$\mathscr{N}_\epsilon := \{z \in E\backslash E^- : \langle I'_\epsilon(z), z\rangle = 0 \ \text{ and } \ \langle I'_\epsilon(z), \varphi\rangle = 0, \forall \varphi \in E^-\}$$

and the ground state energy value

$$c_\epsilon = \inf_{\mathscr{N}_\epsilon} I_\epsilon.$$

Applying the same arguments explored in the Section 3, we can show that for every $z \in E\backslash E^-$ the set $\mathscr{N}_\epsilon \cap \widehat{E}(z)$ is a singleton set, and the element of this set is the unique global maximum of $I_\epsilon|_{\widehat{E}(z)}$, that is, there exists a unique pair $t > 0$ and $\varphi \in E^-$ such that

$$I_\epsilon(tz + \varphi) = \max_{w \in \widehat{E}(z)} I_\epsilon(w).$$

Therefore, the following mapping is well-defined:

$$\widetilde{m}_\epsilon : E^+\backslash\{0\} \to \mathscr{N}_\epsilon \text{ and } m_\epsilon = \widetilde{m}_\epsilon|_{S^+},$$

and the inverse of m_ϵ is

$$m_\epsilon^{-1} : \mathscr{N}_\epsilon \to S^+, \ m_\epsilon^{-1}(z) = z^+/\|z^+\|.$$

Accordingly, the reduced functional $\widetilde{\Phi}_\epsilon : E^+\backslash\{0\} \to \mathbb{R}$ and the restriction $\Phi_\epsilon : S^+ \to \mathbb{R}$ can respectively be defined by

$$\widetilde{\Phi}_\epsilon(z) = I_\epsilon(\widetilde{m}_\epsilon(z)) \text{ and } \Phi_\epsilon = \widetilde{\Phi}_\epsilon|_{S^+}.$$

Moreover, from the above discussions in Section 3, we can check that all related conclusions in Section 3 hold for I_ϵ, c_ϵ, $\mathscr{N}_\epsilon$, $\widetilde{m}_\epsilon$, m_ϵ, $\widetilde{\Phi}_\epsilon$ and Φ_ϵ, respectively.

Meanwhile, concerning the limit problem given by

$$Sz = A(0)g(|z|)z, \quad t \in \mathbb{R}, \tag{4.1}$$

for the sake of simplicity, we will use the notations I_0, c_0 and $\mathscr{N}_0$ to denote $I_{A(0)}$, $c_{A(0)}$ and $\mathscr{N}_{A(0)}$, respectively.

Next, we will state the relationship of the ground state energy value between system (1.3) and limit system (4.1), and this is very significant in our following arguments.

Lemma 4.1. *The limit* $\lim_{\epsilon \to 0} c_\epsilon = c_0$ *holds.*

Proof. Let be $\epsilon_n \to 0$ as $n \to \infty$. Evidently, using Lemma 3.9 we obtain that $c_0 \le c_{\epsilon_n}$ for all $n \in \mathbb{N}$; thus, $c_0 \le \liminf_{n\to\infty} c_{\epsilon_n}$.

On the other hand, Theorem 3.1 shows that the limit system (4.1) has a ground state homoclinic orbit z_0. Then, according to Lemma 3.6, we can find that there are $s_n \in (0, +\infty)$ and $\varphi_n \in E^-$ such that $s_n z_0^+ + \varphi_n \in \mathscr{N}_{\epsilon_n}$, and

$$I_{\epsilon_n}(s_n z_0^+ + \varphi_n) \ge c_{\epsilon_n} \ge c_0 > 0, \quad \forall n \in \mathbb{N}.$$

As in the previous section, we can see that $\{s_n z_0^+ + \varphi_n\}$ is bounded in E. Thus, without loss of generality, we assume that $s_n \to s_0$ and $\varphi_n \rightharpoonup \varphi$ in E^-. Therefore, we can deduce from the weakly lower semi-continuity of the norm and Fatou's Lemma that

$$\begin{aligned} c_0 &= \liminf_{n\to\infty} c_{\epsilon_n} \le \limsup_{n\to\infty} c_{\epsilon_n} \le \limsup_{n\to\infty} I_{\epsilon_n}(s_n z_0^+ + \varphi_n) \\ &\le \limsup_{n\to\infty} \left[\frac{1}{2}s_n^2\|z_0^+\|^2 - \frac{1}{2}\|\varphi_n\|^2 - \int_{\mathbb{R}} A(\epsilon_n t)G(|s_n z_0^+ + \varphi_n|)\mathrm{d}t\right] \\ &\le \frac{1}{2}s_0^2\|z_0^+\|^2 - \frac{1}{2}\|\varphi\|^2 - A(0)\int_{\mathbb{R}} G(|s_0 z_0^+ + \varphi|)\mathrm{d}t \\ &= I_0(s_0 z_0^+ + \varphi) \le I_0(z_0) = c_0. \end{aligned}$$

Obviously, we can get

$$\lim_{n\to\infty} c_{\epsilon_n} = c_0,$$

finishing the proof.

In view of the above discussion, we obtain that $I_0(s_0 z_0^+ + \varphi) = I_0(z_0) = c_0$; then, both $s_0 z_0^+ + \varphi$ and z_0 are the elements of $\mathscr{N}_0 \cap \widehat{E}(z_0)$. But, according to Lemma 3.6, there is only one element in $\mathscr{N}_0 \cap \widehat{E}(z_0)$, so we can conclude that $s_0 z_0^+ + \varphi = z_0$ and $s_n \to s_0 = 1$, where $\varphi_n \rightharpoonup \varphi = z_0^-$.

As a byproduct of the Lemma 4.1, we can directly obtain the following result.

Lemma 4.2. *Assume that condition* (A) *holds, then, there is* $\epsilon_0 > 0$ *such that* $c_\epsilon < c_{A_\infty}$ *for* $\epsilon \in (0, \epsilon_0)$.

Proof. From condition (A) we can see that $A(0) > A_\infty$. Then from Lemma 3.9 we have that $c_0 < c_{A_\infty}$. Observe that, Lemma 4.1 yields that there exists $\epsilon_0 > 0$ small enough such that $c_\epsilon < c_{A_\infty}$ for all $\epsilon \in (0, \epsilon_0)$. Therefore, we get that $c_\epsilon < c_{A_\infty}$ for $\epsilon \in (0, \epsilon_0)$.

Using similar arguments as for the proof of Lemma 3.7, one can easily check the following lemma.

Lemma 4.3. *The energy functional* I_ϵ *is coercive on* $\mathscr{N}_\epsilon$ *for each* $\epsilon \ge 0$.

Next we give the proof involving the existence result for ground state homoclinic orbits for system (1.1).

Lemma 4.4. *Assume that conditions* (L), (A) *and* (g_1)-(g_3) *are satisfied, then, system* (1.1) *has a ground-state homoclinic orbit for each* $\epsilon \in (0, \epsilon_0)$.

Proof. Following the proof of Theorem 3.1 and using Lemma 3.8, we must prove that there exists $z \in \mathscr{N}_\epsilon$ such that $I_\epsilon(z) = c_\epsilon$. Indeed, applying Ekeland's variational principle, there exists $\{u_n\} \subset S^+$ such that $\Phi_\epsilon(u_n) \to c_\epsilon$ and $\Phi'_\epsilon(u_n) \to 0$. Put $z_n = \widetilde{m}_\epsilon(u_n) \in \mathscr{N}_\epsilon$ for all $n \in \mathbb{N}$. Then from Lemma 3.8 we have that $I_\epsilon(z_n) \to c_\epsilon$ and $I'_\epsilon(z_n) \to 0$. Furthermore, in view of Lemma 4.3, we can prove that $\{z_n\}$ is bounded. Then, up to a subsequence, we can suppose that $z_n \rightharpoonup z$ in E. Evidently, $I'_\epsilon(z) = 0$.

In what follows we need to show that $z \neq 0$ and $I_\epsilon(z) = c_\epsilon$. Combining the fact that $z_n \in \mathscr{N}_\epsilon$ with Lemma 3.3, we have

$$\begin{aligned} o_n(1) = \langle I'_\epsilon(z_n), z_n^+\rangle &= \|z_n^+\|^2 - \int_{\mathbb{R}} A(\epsilon t)g(|z_n|)z_n z_n^+\mathrm{d}t \\ &\ge 2c_\epsilon - \int_{\mathbb{R}} A(\epsilon t)g(|z_n|)z_n z_n^+\mathrm{d}t, \end{aligned}$$

which yields that

$$\int_{\mathbb{R}} A(\epsilon t)g(|z_n|)z_n z_n^+ \mathrm{d}t \geq 2c_\epsilon > 0.$$

As in the previous section, we can check that there exists a sequence $\{y_n\} \subset \mathbb{Z}$, $r > 0$ and $\delta > 0$ such that

$$\int_{B_r(y_n)} |z_n^+|^2 \mathrm{d}t \geq \delta, \quad \forall n \in \mathbb{N}. \tag{4.2}$$

Now, we need to prove that the sequence $\{y_n\}$ is bounded in $\mathbb{R}$. Arguing by contradiction we can suppose that $\{y_n\}$ is unbounded and $|y_n| \to +\infty$ as $n \to \infty$. Setting $w_n(t) := z_n(t + y_n)$, then, $w_n \rightharpoonup w$ in E; we can obtain $w \neq 0$ from (4.2). By choosing the test function $\psi \in C_0^\infty(\mathbb{R})$, we get

$$\begin{aligned} o_n(1) =& \langle I'_\epsilon(z_n), \psi(t - y_n)\rangle \\ =& (z_n^+, \psi^+(t - y_n)) - (z_n^-, \psi^-(t - y_n)) - \int_{\mathbb{R}} A(\epsilon t)g(|z_n|)z_n\psi(t - y_n)\mathrm{d}t \\ =& (w_n^+, \psi^+) - (w_n^-, \psi^-) - \int_{\mathbb{R}} A(\epsilon t + \epsilon y_n)g(|w_n|)w_n\psi(t)\mathrm{d}t. \end{aligned} \tag{4.3}$$

Letting $n \to +\infty$, then we obtain

$$(w^+, \psi^+) - (w^-, \psi^-) - \int_{\mathbb{R}} A_\infty g(|w|)w\psi(t)\mathrm{d}t = \langle I'_\infty(w), \psi\rangle = 0. \tag{4.4}$$

This shows that w is a nontrivial solution of system (3.1) with $\mu = A_\infty$ and $w \in \mathscr{N}_{A_\infty}$.

Employing the Fatou's lemma we can derive that

$$\begin{aligned} c_{A_\infty} \leq I_{A_\infty}(w) &= I_{A_\infty}(w) - \frac{1}{2}\langle I'_{A_\infty}(w), w\rangle \\ &= \int_{\mathbb{R}} A_\infty \left[\frac{1}{2}g(|w|)|w|^2 - G(|w|)\right]\mathrm{d}t \\ &\leq \liminf_{n\to\infty} \int_{\mathbb{R}} A(\epsilon t + \epsilon y_n)\left[\frac{1}{2}g(|w_n|)|w_n|^2 - G(|w_n|)\right]\mathrm{d}t \\ &= \liminf_{n\to\infty} \int_{\mathbb{R}} A(\epsilon t)\left[\frac{1}{2}g(|z_n|)|z_n|^2 - G(|z_n|)\right]\mathrm{d}t \\ &= \liminf_{n\to\infty} \left[I_\epsilon(z_n) - \frac{1}{2}\langle I'_\epsilon(z_n), z_n\rangle\right] = c_\epsilon. \end{aligned}$$

Therefore, it follows that

$$c_{A_\infty} \leq c_\epsilon, \quad \forall \epsilon > 0.$$

However, Lemma 4.2 yields that $c_\epsilon < c_{A_\infty}$ when $\epsilon < \epsilon_0$, which leads to a contradiction. So, we can conclude that $\{y_n\}$ is bounded. Then for all $n \in \mathbb{N}$, there exists $r_0 > 0$ such that $B_r(y_n) \subset B_{r_0}(0)$, it holds that

$$\int_{B_{r_0}(0)} |z_n|^2 \mathrm{d}t \geq \int_{B_r(y_n)} |z_n|^2 \mathrm{d}t \geq \delta.$$

Therefore, we obtain that $z_n \rightharpoonup z$ in E with $z \neq 0$. By repeating the steps in (4.3) and (4.4), we know that $z \in \mathscr{N}_\epsilon$ is a nontrivial solution for system (1.1), thus, $c_\epsilon \leq I_\epsilon(z)$.

On the other hand, according to Fatou's lemma, we infer that

$$\begin{aligned} c_\epsilon &= \liminf_{n\to\infty}\left[I_\epsilon(z_n) - \frac{1}{2}\langle I'_\epsilon(z_n), z_n\rangle\right] \\ &= \liminf_{n\to\infty}\int_{\mathbb{R}} A(\epsilon t)\left[\frac{1}{2}g(|z_n|)|z_n|^2 - G(|z_n|)\right]\mathrm{d}t \\ &\geq \int_{\mathbb{R}} A(\epsilon t)\left[\frac{1}{2}g(|z|)|z|^2 - G(|z|)\right]\mathrm{d}t \\ &= I_\epsilon(z) - \frac{1}{2}\langle I'_\epsilon(z), z\rangle = I_\epsilon(z). \end{aligned}$$

Thus, $c_\epsilon = I_\epsilon(z)$. Evidently, it is easy to see that z is a ground state homoclinic orbit of system (1.1). We complete the proof.

Let

$$\mathcal{K}_\epsilon := \{z \in E\backslash\{0\} : I'_\epsilon(z) = 0\}$$

be the set of all nontrivial critical points of I_ϵ. In order to describe some important properties of ground state homoclinic orbits, next, we get the following regularity result by taking advantage of the bootstrap argument (see [37] for the iterative steps), this result can also be found in [24, Lemma 2.3].

Lemma 4.5. *If $z \in \mathcal{K}_\epsilon$ with $|I_\epsilon(z)| \leq C_1$ and $\|z\|_2 \leq C_2$; then, $z \in W^{1,q}(\mathbb{R}, \mathbb{R}^{2N})$ for any $q > 2$, and $\|z\|_{W^{1,q}} \leq C_q$, where C_q depends only on C_1, C_2 and q.*

Below, we use $\mathscr{L}$ to denote the set of all ground state homoclinic orbits of system (1.1). Let $z \in \mathscr{L}$, then, $I_\epsilon(z) = c_\mu$; applying a standard argument we can show that $\mathscr{L}$ is bounded in E; therefore, $\|z\|_2 \leq \widehat{c}$ for all $z \in \mathscr{L}$ and some $\widehat{c} > 0$. Hence, making use of Lemma 4.5, we see that, for each $q > 2$, there exists C_q such that

$$\|z\|_{W^{1,q}} \leq C_q, \quad \forall z \in \mathscr{L}. \tag{4.5}$$

Moreover, combining the Sobolev embedding theorem, we can show that there exists $C_\infty > 0$ such that

$$\|z\|_\infty \leq C_\infty, \quad \forall z \in \mathscr{L}. \tag{4.6}$$

4.2. Concentration of ground state homoclinic orbits

We now shall prove the concentration behavior of the maximum points of the ground state homoclinic orbit. Let z_ϵ be a ground state homoclinic orbit of system (1.1), which can be obtained by Lemma 4.4. Our aim is to show that if t_ϵ is a maximum point of $|z_\epsilon|$, then,

$$\lim_{\epsilon\to 0} A(\epsilon t_\epsilon) = A(0).$$

In other words, we must show that if $\epsilon_n \to 0$, up to a subsequence, $\epsilon_n t_{\epsilon_n} \to t_0$ for some $t_0 \in \mathscr{A}$, where

$$\mathscr{A} = \{t \in \mathbb{R} : A(t) = A(0)\}$$

denotes the set of the maximum points of $A(t)$.

Let $\{\epsilon_n\} \subset (0, \epsilon_0)$ with $\epsilon_n \to 0$ as $n \to \infty$ and $z_{\epsilon_n} \in \mathscr{L}$; we write $z_n := z_{\epsilon_n}$. Then, we have

$$I_{\epsilon_n}(z_n) = c_{\epsilon_n} \text{ and } I'_{\epsilon_n}(z_n) = 0$$

Evidently, in view of Lemma 4.3, we can easily check that $\{z_n\}$ is bounded in E.

Lemma 4.6. *There exist a sequence $\{y_n\} \subset \mathbb{Z}$ and two constants $r > 0$, $\delta > 0$ such that*

$$\int_{B_r(y_n)} |z_n|^2 \mathrm{d}t \geq \delta.$$

Proof. Arguing by contradiction, we suppose that for any $r_1 > 0$,

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{B_{r_1}(y)} |z_n|^2 \mathrm{d}t = 0.$$

Then, according to Lions' concentration-compactness principle, we conclude that $z_n \to 0$ in $L^q(\mathbb{R})$ for all $q > 2$. Furthermore, by (2.1) and (2.2), we obtain

$$\int_{\mathbb{R}} A(\epsilon_n t)\left[\frac{1}{2} g(|z_n|)|z_n|^2 - G(|z_n|)\right] \mathrm{d}t \to 0.$$

Therefore, it follows that

$$c_{\epsilon_n} = I_{\epsilon_n}(z_n) - \frac{1}{2}\langle I'_{\epsilon_n}(z_n), z_n\rangle = \int_{\mathbb{R}} A(\epsilon_n t)\left[\frac{1}{2} g(|z_n|)|z_n|^2 - G(|z_n|)\right] \mathrm{d}t \to 0.$$

Evidently, this is impossible because $c_{\epsilon_n} > 0$ (see Lemma 3.3). We complete the proof.

Lemma 4.7. *The sequence $\{\epsilon_n y_n\}$ is bounded, and $\lim_{n\to\infty} \epsilon_n y_n = x_0 \in \mathscr{A}$.*

Proof. Setting $v_n(t) := z_n(t + y_n)$, up to a subsequence, it is easy to see that $v_n \rightharpoonup v$ in E with $v \neq 0$ from Lemma 4.6. In what follows, we want to prove that the sequence $\{\epsilon_n y_n\}$ is bounded. If this is not true, we can suppose that there is a subsequence $\{\epsilon_n y_n\}$ such that $|\epsilon_n y_n| \to +\infty$ as $n \to +\infty$. Since z_n is the ground state homoclinic orbit of system (1.1), v_n solves the following system

$$S v_n = A(\epsilon_n t + \epsilon_n y_n) g(|v_n|) v_n, \tag{4.7}$$

and the energy

$$\begin{aligned}
\widehat{I_{\epsilon_n}}(v_n) &= \frac{1}{2}\left(\|v_n^+\|^2 - \|v_n^-\|^2\right) - \int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) G(|v_n|) \mathrm{d}t \\
&= \frac{1}{2}\left(\|z_n^+\|^2 - \|z_n^-\|^2\right) - \int_{\mathbb{R}} A(\epsilon_n t) G(|z_n|) \mathrm{d}t \\
&= \int_{\mathbb{R}} A(\epsilon_n t)\left[\frac{1}{2} g(|z_n|)|z_n|^2 - G(|z_n|)\right] \mathrm{d}t \\
&= I_{\epsilon_n}(z_n) = c_{\epsilon_n}.
\end{aligned}$$

Furthermore, for every $\phi \in E$, we have

$$(v_n^+, \phi^+) - (v_n^-, \phi^-) - \int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) g(|v_n|) v_n \phi \mathrm{d}t = 0.$$

Since $A(\epsilon_n t + \epsilon_n y_n) \to A_\infty$, given that $v_n \rightharpoonup v$ and $\phi \in C_0^\infty(\mathbb{R})$, we get

$$(v^+, \phi^+) - (v^-, \phi^-) - \int_{\mathbb{R}} A_\infty g(|v|) v \phi \mathrm{d}t = 0.$$

Thereby, v is a nontrivial homoclinic orbit of system (3.1) with $\mu = A_\infty$ and $v \in \mathscr{N}_{A_\infty}$. In view of Lemma 4.1 and Fatou's lemma, we can conclude that

$$\begin{aligned}
c_{A_\infty} &\le I_{A_\infty}(v) = I_{A_\infty}(v) - \frac{1}{2}\langle I'_{A_\infty}(v), v\rangle \\
&= A_\infty \int_{\mathbb{R}} \left[\frac{1}{2}g(|v|)|v|^2 - G(|v|)\right] \mathrm{d}t \\
&\le \liminf_{n\to\infty} \int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) \left[\frac{1}{2}g(|v_n|)|v_n|^2 - G(|v_n|)\right] \mathrm{d}t \\
&= \liminf_{n\to\infty} \int_{\mathbb{R}} A(\epsilon t) \left[\frac{1}{2}g(|z_n|)|z_n|^2 - G(|z_n|)\right] \mathrm{d}t \\
&= \liminf_{n\to\infty} \left[I_{\epsilon_n}(z_n) - \frac{1}{2}\langle I'_{\epsilon_n}(z_n), z_n\rangle\right] \\
&= \liminf_{n\to\infty} I_{\epsilon_n}(z_n) = \lim_{n\to\infty} c_{\epsilon_n} = c_0.
\end{aligned} \tag{4.8}$$

However, according to Lemma 4.2 we know that $c_0 < c_{A_\infty}$. Evidently, this is a contradiction. Therefore, $\{\epsilon_n y_n\}$ is bounded in $\mathbb{R}$, and passing to a subsequence, we can assume that $\epsilon_n y_n \to x_0$. According to the above argument, for $\forall \psi \in E$, we get

$$(v^+, \psi^+) - (v^-, \psi^-) - \int_{\mathbb{R}} A(x_0) g(|v|) v \psi \mathrm{d}t = 0,$$

Obviously, we can see that v is a ground state homoclinic orbit of the following system

$$S v = A(x_0) g(|v|) v, \quad t \in \mathbb{R}, \tag{4.9}$$

and $v \in \mathscr{N}_{A(x_0)}$. Following to the proof of (4.8), we can get tht $c_{A(x_0)} \le c_0$, then, using Lemma 3.9, it follows that $A(x_0) \ge A(0)$; together with condition (A), we can obtain that $A(x_0) = A(0)$. Hence, we show that $\lim_{n\to\infty} \epsilon_n y_n = x_0$ and $x_0 \in \mathscr{A}$. The proof is completed.

According to Lemma 4.7, we see that v is a ground state homoclinic orbit of system (4.9), then, $I_0(v) = c_0$ and $I'_0(v) = 0$. Using Lemma 4.1 and Fatou's lemma, we directly obtain

$$\begin{aligned}
c_0 &\le \int_{\mathbb{R}} A(0) \left[\frac{1}{2}g(|v|)|v|^2 - G(|v|)\right] \mathrm{d}t \\
&\le \liminf_{n\to\infty} \int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) \left[\frac{1}{2}g(|v_n|)|v_n|^2 - G(|v_n|)\right] \mathrm{d}t \\
&= \liminf_{n\to\infty} \widehat{I}_{\epsilon_n}(v_n) \le \limsup_{n\to\infty} I_{\epsilon_n}(z_n) \le c_0.
\end{aligned}$$

Hence, we have

$$\lim_{n\to\infty} \widehat{I}_{\epsilon_n}(v_n) = \lim_{n\to\infty} c_{\epsilon_n} = c_0 = I_0(v). \tag{4.10}$$

Lemma 4.8. *We have the convergence conclusion: $v_n \to v$ in E as $n \to \infty$.*

Proof. Let $\eta : [0, +\infty) \to [0, 1]$ be a smooth function satisfying that $\eta(s) = 1$ if $s \le 1$, and $\eta(s) = 0$ if $s \ge 2$. Define $\tilde{v}_n(t) = \eta(2|t|/n)v(t)$, then, for $q \in [2, +\infty)$, one has

$$\|v - \tilde{v}_n\| \to 0 \text{ and } \|v - \tilde{v}_n\|_q \to 0 \text{ as } n \to \infty. \tag{4.11}$$

Setting $\theta_n = v_n - \tilde{v}_n$, it is not difficult to verify that along a subsequence

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) \left[G(|v_n|) - G(|\theta_n|) - G(|\tilde{v}_n|) \right] \mathrm{d}t \right| = 0 \tag{4.12}$$

and

$$\lim_{n\to\infty} \left| \int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) \left[g(|v_n|)v_n - g(|\theta_n|)\theta_n - g(|\tilde{v}_n|)\tilde{v}_n \right] \varphi \mathrm{d}t \right| = 0 \tag{4.13}$$

uniformly in $\varphi \in E$ with $\|\varphi\| \le 1$. Using the fact that $A(\epsilon_n t + \epsilon_n y_n) \to A_0$ as $n \to \infty$ uniformly on any bounded set of t, and combining the decay of v and (4.11) we can easily check the following result

$$\int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) G(|\tilde{v}_n|) \mathrm{d}t \to \int_{\mathbb{R}} A_0 G(|v|) \mathrm{d}t. \tag{4.14}$$

Consequently, using (4.10), (4.11), (4.12) and (4.14) we infer that

$$\begin{aligned}
\widehat{I}_{\epsilon_n}(\theta_n) &= \widehat{I}_{\epsilon_n}(v_n) - I_0(v) \\
&\quad + \int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) \left[G(|v_n|) - G(|\theta_n|) - G(|\tilde{v}_n|) \right] \mathrm{d}t + o_n(1) \\
&= o_n(1) \text{ as } n \to \infty,
\end{aligned}$$

which implies that $\widehat{I}_{\epsilon_n}(\theta_n) \to 0$. Similarly, we also obtain

$$\begin{aligned}
\langle \widehat{I}'_{\epsilon_n}(\theta_n), \varphi \rangle &= \int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) \left[g(|v_n|)v_n - g(|\theta_n|)\theta_n - g(|\tilde{v}_n|)\tilde{v}_n \right] \varphi \mathrm{d}t + o_n(1) \\
&= o_n(1) \text{ uniformly in } \|\varphi\| \le 1 \text{ as } n \to \infty,
\end{aligned}$$

which implies that $\widehat{I}'_{\epsilon_n}(\theta_n) \to 0$. Therefore

$$o_n(1) = \widehat{I}'_{\epsilon_n}(\theta_n) - \frac{1}{2} \langle \widehat{I}'_{\epsilon_n}(\theta_n), \theta_n \rangle = \int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) \left[\frac{1}{2} g(|\theta_n|)|\theta_n|^2 - G(|\theta_n|) \right] \mathrm{d}t,$$

from which together with (g_3), we can infer that

$$\int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) g(|\theta_n|)|\theta_n|^2 \mathrm{d}t \to 0.$$

Notice that $\{\|v_n\|_\infty\}$ is bounded, thus,

$$\int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) g(|\theta_n|)|\theta_n^+ - \theta_n^-|^2 \mathrm{d}t \le C.$$

As a consequence, we obtain

$$
\begin{aligned}
\|\theta_n\|^2 = & \langle \widehat{I'_{\epsilon_n}}(\theta_n), \theta_n^+ - \theta_n^- \rangle + \int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) g(|\theta_n|)\theta_n(\theta_n^+ - \theta_n^-)\mathrm{d}t \\
= & o_n(1) + \int_{\mathbb{R}} A^{\frac{1}{2}}(\epsilon_n t + \epsilon_n y_n) g^{\frac{1}{2}}(|\theta_n|)|\theta_n| A^{\frac{1}{2}}(\epsilon_n t + \epsilon_n y_n) g^{\frac{1}{2}}(|\theta_n|)|\theta_n^+ - \theta_n^-|\mathrm{d}t \\
\leq & o_n(1) + \left(\int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) g(|\theta_n|)|\theta_n|^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
& \left(\int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) g(|\theta_n|)|\theta_n^+ - \theta_n^-|^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
\leq & o_n(1) + C\left(\int_{\mathbb{R}} A(\epsilon_n t + \epsilon_n y_n) g(|\theta_n|)|\theta_n|^2 \mathrm{d}t\right)^{\frac{1}{2}} \\
= & o_n(1),
\end{aligned}
$$

that is, $\|\theta_n\| \to 0$, which together with (4.11) leads to $v_n \to v$ in E as $n \to \infty$.

Lemma 4.9. *We have that $v_n(t) \to 0$ uniformly in $n \in \mathbb{N}$ as $t \to \infty$. Moreover, there exist $c, C > 0$ such that for all $t \in \mathbb{R}$, it holds that*

$$|v_n(t)| \leq C \exp(-c|t|).$$

Proof. Firstly, we observe that if z is a homoclinic orbit of system (1.1), then it satisfies the following relation

$$\frac{\mathrm{d}}{\mathrm{d}t} z = \mathscr{J}\Big(Lz + A(\epsilon t) g(|z|) z\Big).$$

Computing directly, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} z = (\mathscr{J} L)^2 z + Q(t, z)$$

with

$$
\begin{aligned}
Q(t,z) = \mathscr{J}\Big[& \Big(\epsilon A'(\epsilon t) + L\mathscr{J} A(\epsilon t)\Big) g(|z|) z + \Big(g_z'(|z|)|z| + g(|z|)\Big) A(\epsilon t) \mathscr{J} Lz \\
& + \Big(g_z'(|z|)|z| + g(|z|)\Big) A^2(\epsilon t) L g(|z|) z\Big].
\end{aligned}
\tag{4.15}
$$

Setting

$$\mathrm{sgn} z \begin{cases} \frac{z}{|z|} & \text{if } z \neq 0, \\ 0, & \text{if } z = 0. \end{cases}$$

Applying Kato's inequality and (4.15), and using the real positivity of $(\mathscr{J} L)^2$, we can find some $\rho > 0$ such that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}|z| \geq \frac{\mathrm{d}^2}{\mathrm{d}t^2} z(\mathrm{sgn} z) = (\mathscr{J} L)^2 z \frac{z}{|z|} + Q(t,z)\frac{z}{|z|} \geq \rho|z| - |Q(t,z)|. \tag{4.16}$$

Hence, using (2.1), (4.6), (4.15) and (4.16) we conclude that there exists $\kappa > 0$ such that

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}|z| \geq -\kappa|z| \quad \text{for all } t \in \mathbb{R}.$$

Then by the sub-solution estimate [38], there is a $\widehat{c_0}$ independent of t; we have the following estimate

$$|z(t)| \leq \widehat{c_0} \int_{B_1(t)} |z(s)| \mathrm{d}s. \tag{4.17}$$

Now we claim that $v_n(t) \to 0$ uniformly in $n \in \mathbb{N}$ as $t \to \infty$. Indeed, if it is not true, then using (4.17) we can find that there exist $c_0 > 0$ and $t_n \in \mathbb{R}$ with $|t_n| \to \infty$ such that

$$c_0 \leq |v_n(t_n)| \leq \widehat{c_0} \int_{B_1(t_n)} |v_n(t)| \mathrm{d}t,$$

this is because v_n satisfies $\widehat{I'_{\epsilon_n}}(v_n) = 0$, then, the above processes still hold for v_n. From Lemma 4.8, it follows that $v_n \to v$ in E. Therefore, we get

$$\begin{aligned} c_0 \leq |v_n(t_n)| &\leq \widehat{c_0} \int_{B_1(t_n)} |v_n(t)| \mathrm{d}t \leq \widehat{c_0} \int_{B_1(t_n)} |v_n - v| \mathrm{d}t + \widehat{c_0} \int_{B_1(t_n)} |v| \mathrm{d}t \\ &\leq \widehat{c} \Big(\int_{\mathbb{R}} |v_n - v|^2 \mathrm{d}t \Big)^{\frac{1}{2}} + \widehat{c_0} \int_{B_1(t_n)} |v| \mathrm{d}t \to 0, \end{aligned}$$

which yields a contradiction. So, the claim holds.

Note that $g(s) = o(1)$ and $g'_s(s)s = o(1)$ as $s \to 0$; then, we can find suitable constants $0 < \delta < \frac{\rho}{2}$ and $R > 0$ such that

$$|Q(t, v_n)| \leq \frac{\rho}{2} |v_n|, \quad \forall |t| \geq R.$$

Combining the above relation and (4.16), we get

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} |v_n| \geq \delta |v_n|, \quad \forall |t| \geq R.$$

Let $\Lambda(t)$ be a fundamental solution of the following equation

$$-\frac{\mathrm{d}^2}{\mathrm{d}t^2} \Lambda + \delta \Lambda = 0.$$

From the uniform boundedness, we may choose $\Lambda(t)$ such that $|v_n(t)| \leq \delta \Lambda(t)$ holds on $|t| = R$ for all $n \in \mathbb{N}$. Let $u_n = |v_n| - \delta \Lambda$; thus, we obtain

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} u_n = \frac{\mathrm{d}^2}{\mathrm{d}t^2} |v_n| - \delta \frac{\mathrm{d}^2}{\mathrm{d}t^2} \Lambda \geq \delta(|v_n| - \delta \Lambda) = \delta u_n, \quad \text{for all } |t| \geq R.$$

The maximum principle yields that $u_n(t) \leq 0$ for $|t| \geq R$, i.e., $|v_n(t)| \leq \delta \Lambda(t)$ for $|t| \geq R$. As we know that there exists $c_1 > 0$ such that

$$\Lambda(t) \leq c_1 \exp(-\sqrt{\delta} |t|) \quad \text{for all } |t| \geq 1.$$

Therefore, there are constants $C, c > 0$; we obtain

$$|v_n(t)| \leq C \exp(-c|t|) \quad \text{for all } t \in \mathbb{R}.$$

We complete the proof.

Lemma 4.10. *There exists $\nu > 0$ such that $\|v_n\|_\infty \geq \nu$ for all $n \in \mathbb{N}$.*

Proof. According to Lemma 4.6, we can see that there exist $r > 0$ and $\delta > 0$ such that

$$\int_{B_r(0)} |v_n|^2 \mathrm{d}t \geq \delta.$$

Suppose by contradiction that $\|v_n\|_\infty \to 0$ as $n \to \infty$, then, it holds that

$$0 < \delta \leq \int_{B_r(0)} |v_n|^2 \mathrm{d}t \leq |B_r| \|v_n\|_\infty^2 \to 0 \text{ as } n \to \infty,$$

which is absurd. This ends the proof.

Finally, based on the above facts, next we give the completed proof of Theorem 1.1.

Proof of Theorem 1.1 (completed). Suppose that q_n is a global maximum point of $|v_n(t)|$ for each $n \in \mathbb{N}$, then,

$$|v_n(q_n)| = \max_{t\in\mathbb{R}} |v_n(t)|.$$

Since $v_n(t) = z_n(t + y_n)$, we can see that $p_n = q_n + y_n$ is a maximum point of $|z_n(t)|$. Lemma 4.10 shows that there exists $\nu > 0$ such that

$$|v_n(q_n)| \geq \nu \text{ for all } n \in \mathbb{N},$$

then we know that $\{q_n\}$ is bounded. So, we conclude from Lemma 4.7 that

$$\epsilon_n p_n = \epsilon_n q_n + \epsilon_n y_n \to x_0 \in \mathscr{A}.$$

Consequently, it follows that

$$\lim_{n\to\infty} A(\epsilon_n p_n) = A(x_0) = A(0).$$

Furthermore, from Lemma 4.7 and Lemma 4.8, it is easy to see that $z_n(t + p_n)$ converges to a ground state homoclinic orbit v of the following limit system

$$Sz = A(0)g(|z|)z, \quad t \in \mathbb{R}.$$

From Lemma 4.9 and the boundedness of $\{q_n\}$, we derive that

$$\begin{aligned} |z_n(t)| =& |v_n(t - y_n)| \leq C \exp(-c|t - y_n|) = C \exp(-c|t - p_n + q_n|) \\ \leq& C \exp(-c|t - p_n| + c|q_n|) \leq \widetilde{C} \exp(-\tilde{c}|t - p_n|) \end{aligned}$$

for some $\tilde{c}, \widetilde{C} > 0$ and all $t \in \mathbb{R}$.

Finally, we observe that Lemma 4.2 shows that, there is $\epsilon_0 > 0$; system (1.1) has a ground state homoclinic orbit z_ϵ for each $\epsilon \in (0, \epsilon_0)$. So, the conclusion (a) of Theorem 1.1 holds. Moreover, according to the above discussions, we directly obtain the following conclusions:

(b) let t_ϵ be the maximum point of $|z_\epsilon(t)|$, then,

$$\lim_{\epsilon\to 0} A(\epsilon t_\epsilon) = A(0);$$

and $z_\epsilon(t + t_\epsilon) \to v$ in E, where v is a ground state homoclinic orbit of the limit system

$$Sz = A(0)g(|z|)z, \quad t \in \mathbb{R};$$

(c) there are two positive constants $\tilde{c}, \widetilde{C}$ such that

$$|z_\epsilon(t)| \leq \widetilde{C} \exp(-\tilde{c}|t - t_\epsilon|).$$

We have finished the proof of all conclusions of Theorem 1.1.

Use of AI tools declaration

The authors declare that no artificial intelligence tools were used in the creation of this article.

Acknowledgments

The research of Tianfang Wang was supported by the High Level Research Achievement Project-General Project of Natural Science of Baotou Teachers' College (BSYKJ2022-ZY09), the Youth Innovative Talent Project of Baotou City (2022). The research of Wen Zhang was supported by the Natural Science Foundation of Hunan Province (2022JJ30200), the Key project of Scientific Research Project of Department of Education of Hunan Province (22A0461), and Aid Program for Science and Technology Innovative Research Team in Higher Educational Institutions of Hunan Province.

Conflict of interest

The authors declare that they have no competing interests.

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