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Research article

Nontrivial p-convex solutions to singular p-Monge-Ampère problems: Existence, Multiplicity and Nonexistence

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Abstract: Our main objective of this paper is to study the singular p-Monge-Ampère problems: equations and systems of equations. New multiplicity results of nontrivial p-convex radial solutions to a single equation involving p-Monge-Ampère operator are first analyzed. Then, some new criteria of existence, nonexistence and multiplicity for nontrivial p-convex radial solutions for a singular system of p-Monge-Ampère equation are also established.

Keywords: singular p-Monge-Ampère equations and systems; nontrivial p-convex solutions; fixed point index; multiplicity
Mathematics Subject Classification: 35J96, 35J57

1. Introduction and main results

Discuss the following p-Monge-Ampère equation

$$\begin{cases} \det(D(|Du|^{p-2}Du)) = g(|x|)f(|x|, -u) \ \text{ in } B, \\ u = 0 \ \text{ on } \partial B, \end{cases} \tag{1.1}$$

and system

$$\begin{cases} \det(D(|Du_i|^{p-2}Du_i)) = g_i(|x|)f_i(|x|, -u_1, -u_2, \dots, -u_n) \ \text{ in } B, \\ u_i = 0 \ \text{ on } \partial B. \end{cases} \tag{1.2}$$

Here $p \geq 2$, $i \in I_n := \{1, 2, \cdots, n\}$, $g,\ g_i \in C[0, 1)$ are all singular at 1, $B := \{y \in \mathbb{R}^m : |y| < 1\}$, and $m,\ n \geq 2$ are integers.

A new operator proposed by Trudinger-Wang in [1] is p-Monge-Ampère operator, which is denoted by $\det(D(|Du|^{p-2}Du))$. And such operator just is Monge-Ampère operator when $p = 2$.

Let M be a $m \times m$ real symmetric array, and

$$\sigma_l(\mu(M)) = \sum_{1 \le i_1 < \cdots < i_l \le m} \mu_{i_1} \cdots \mu_{i_l}$$

denote the lth elemental symmetric function, where $\mu_1, \mu_2 \ldots, \mu_m$ are the eigenvalues of M.

If $u \in \Phi^p(\Omega)$ satisfying (1.1), then u is said to be a p-convex strong solution. Here

$$\Phi^p(\mathbb{R}^m) := \{u \in W^{2,qm}_{loc}(\mathbb{R}^m) : |Du|^{p-2}Du \in C^1(\mathbb{R}^m), \lambda(D_i(|Du|^{p-2}D_j u)) \in \Gamma_m \text{ in } \mathbb{R}^m\},$$

where $1 < q < \frac{p-1}{p-2}$, and

$$\Gamma_m := \{\lambda \in \mathbb{R}^m : \sigma_l(\lambda) > 0, l \in \{1, 2, \ldots, m\}\}.$$

Now, we review several excellent conclusions related to p-Monge-Ampère Dirichlet problem (1.1) and system (1.2).

Results of equations and systems involving p-Laplacian operator:

We refer to the following articles [2–16] for p-Laplacian equations and systems. We need to specifically mention here that Hai-Shivaji [17] investigated

$$\begin{cases} -\Delta_p v = \lambda f(v) \text{ in } D, \\ v = 0 \text{ on } \partial D. \end{cases} \tag{1.3}$$

Here $p > 1$, λ denotes a positive parameter, and D denotes the unit ball in $\mathbb{R}^n$ $(n \ge 1)$. Since a positive solution of (1.3) in the unit ball is radially symmetric, so the authors resolved the problem of ordinary differential equation

$$\begin{cases} (r^{n-1}|v'|^{p-2}v')' = -\lambda r^{n-1} f(v), \\ v'(0) = 0, \ v(1) = 0. \end{cases} \tag{1.4}$$

The authors mainly used a sub-super solution method to demonstrate the uniqueness and existence of positive solution for problem (1.4).

Recently, Feng-Zhang [18] employed the eigenvalue theory to discuss the existence of positive solution for the p-Laplacian elliptic system

$$\begin{cases} -\Delta_p z_1 = \lambda_1 h_1(|x|) z_2^{\alpha} \text{ in } D, \\ -\Delta_p z_2 = \lambda_2 h_2(|x|) z_1^{\beta} \text{ in } D, \\ z_1 = z_2 = 0 \text{ on } \partial D. \end{cases}$$

Here $\lambda_1, \ \lambda_2 \neq 0$ are parameters, $\alpha, \ \beta > 0$, and D denotes the unit ball in $\mathbb{R}^n$ $(n \ge 2)$. The authors obtained a uniqueness and approximation result by iterations of the solution.

In [19], Lan-Zhang considered the system of p-Laplace equations

$$\begin{cases} \Delta_p u_i = f_i(x, u_1, -u_2, \ldots, -u_n) \text{ in } \Omega, \\ u_i = 0 \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where $i \in \{1, 2, \cdots, n\}$. The authors obtained new existence results of nonzero positive weak solutions of (1.4) under some sublinear conditions by employing a well-known theorem of fixed point index on cones for completely continuous operators. The other recent results concerning p-Laplacian equations and systems can be found in Ju-Bisci-Zhang [20] and He-Ousbika-Allali-Zuo [21].

Results of equations and systems involving Monge-Ampère operator:

We observe that large numbers of mathematicians care about the existence of solution of equations and systems involving Monge-Ampère operator; see [22–37] and the bibliographies. Particularly, Cheng-Yau [40] considered the following Monge-Ampère problem

$$\begin{cases} \det D^2 v = v^{-(n+2)} \ \text{ in } \Omega, \\ v = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.5}$$

where $n \geq 2$. Employing the Legendre transform and the approximated method, they derived some existence results of solution of (1.5).

In [41], Feng investigated

$$\begin{cases} \det D^2 u = \mu f(-u) \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.6}$$

where μ is a positive parameter. He use sharp estimates to verify that (1.6) admits at most one nontrivial radial convex solution.

On systems involving Monge-Ampère operator, we only find a few results. In particular, Zhang-Qi [42] considered

$$\begin{cases} \det D^2 v_1 = (-v_2)^{\alpha} \ \text{ in } \Omega, \\ \det D^2 v_2 = (-v_1)^{\beta} \ \text{ in } \Omega, \\ v_1 < 0, \ \ v_2 < 0 \ \text{ in } \Omega, \\ v_1 = v_2 = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where α and β are two positive numbers. Using the index theory of fixed points on cones, they obtained several existence, nonexistence and uniqueness results of radial convex solutions when Ω denotes the unit ball in $\mathbb{R}^n$.

In [41], Feng considered a more general system

$$\begin{cases} \det D^2 v_1 = \lambda_1 f_1(-v_2) \ \text{ in } \Omega, \\ \det D^2 v_2 = \lambda_2 f_2(-v_3) \ \text{ in } \Omega, \\ \qquad\qquad \vdots \\ \det D^2 v_n = \lambda_n f_n(-v_1) \ \text{ in } \Omega, \\ v_1 = v_2 = \ldots = v_n = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.7}$$

where $\lambda_i > 0$ ($i \in \{1, 2, \ldots, n\}$) are parameters. He got some new existence and nonexistence results of (1.7) by means of the eigenvalue theory on cones.

Recently, Feng [43] derived some existence, nonexistence and multiplicity results of nontrivial radial convex solutions of

$$\begin{cases} \det D^2 v_1 = \lambda h_1(|x|) f_1(-v_2), & \text{in } \ \Omega, \\ \det D^2 v_2 = \lambda h_2(|x|) f_2(-v_1), & \text{in } \ \Omega, \\ v_1 = v_2 = 0, \ \text{ on } \ \partial\Omega \end{cases}$$

for a certain range of $\lambda > 0$.

More recently, in [44], Feng discussed the existence of nontrivial solution of

$$\begin{cases} \det(D(|Dv|^{p-2} Dv)) = \lambda h(|x|) f(|x|, -v) \ \text{ in } \Omega, \\ v = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1.8}$$

where λ denotes a positive parameter. We also refer to Xu [38] and Lian-Wang-Xu [39] for the related results of p-Monge-Ampère problems.

Remark 1. There is almost no article except [44] studying p-Monge-Ampère equation. But, in [44], the author only dealt with the existence of nontrivial solution, not the multiplicity of nontrivial solutions; and the author only studied single p-Monge-Ampère equation, not system of p-Monge-Ampère equations.

Inspired by the above, we first search the multiplicity of nontrivial p-convex radial solutions of (1.1). Our proof makes use of the fixed point index theory on cones, which is completely different from that used in [41] and [44]. Then we seek existence and multiplicity results of nontrivial solutions of (1.2) by employing the theory of fixed point on cons when f_i $(i \in I_n)$ satisfy some new growth conditions.

The article will be organized as follows. In next section, we are going to study the existence and multiplicity of nontrivial p-convex radial solutions of (1.1). In addition, many nontrivial p-convex radial solutions are also studied. The third part will search nontrivial p-convex radial solution of system (1.2).

2. Multiple nontrivial radial solutions of (1.1)

Let us seek multiple nontrivial radial solutions of (1.1) in this section. In [45], Bao-Feng pointed out that one can change (1.1) into

$$\begin{cases} r^{1-n}\left(\frac{1}{n}(u')^{(p-1)n}\right)' = g(r)f(r,-u), \quad 0 < r < 1, \\ u'(0) = u(1) = 0. \end{cases} \tag{2.1}$$

Setting $v = -u$, then one can rewrite (2.1) as follows:

$$\begin{cases} r^{1-n}\left(\frac{1}{n}(-v')^{(p-1)n}\right)' = g(r)f(r,v), \quad 0 < r < 1, \\ v'(0) = v(1) = 0. \end{cases} \tag{2.2}$$

We make the following suppositions:

($\mathbf{C_1}$) $g : [0,1) \to \mathbb{R}_+$ is continuous, $g(t) \not\equiv 0$ on any subinterval of $[0,1)$, and

$$\int_0^1 g(t)dt < +\infty,$$

where $\mathbb{R}_+ = [0,+\infty)$;

($\mathbf{C_2}$) $f : [0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.

Remark 2. It is not difficult to see that there are some elementary functions that satisfy ($\mathbf{C_1}$). For example

$$g(r) = \frac{c}{\pi\sqrt{1-r^2}},$$

where c is a positive real number.

Obviously, $g : [0,1) \to \mathbb{R}_+$ is continuous, and $g(t) \not\equiv 0$ on any subinterval of $[0,1)$.

Next, we verify that g satisfies $\int_0^1 g(t)dt < +\infty$.

In fact,

$$\begin{aligned}\int_0^1 g(r)dr &= \int_0^1 \frac{c}{\pi\sqrt{1-r^2}}dr\\ &= \lim_{b\to 0^+}\int_0^{1-b}\frac{c}{\pi\sqrt{1-r^2}}dr\\ &= \frac{c}{\pi}\lim_{b\to 0^+}[\arcsin r]_0^{1-b}\\ &= \frac{c}{\pi}\lim_{b\to 0^+}\arcsin(1-b)\\ &= \frac{c}{2},\end{aligned}$$

which indicates that $\int_0^1 g(t)dt < +\infty$.

Let $J = [0,1]$ and $E := C[0,1]$. Then E is a real Banach space (RBS for short) with the norm denoted by

$$\|x\| = \max_{t\in J}|x(t)|.$$

Lemma 2.1. If $(\mathbf{C_1})$ and $(\mathbf{C_2})$ holds, then v is a solution of (2.2) when and only when $v \in E$ is a solution of

$$v(t) = \int_t^1 \left(\int_0^\tau ns^{n-1}h(s)f(s,v(s))ds\right)^{\frac{1}{(p-1)n}}d\tau, \tag{2.3}$$

and

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} v(t) \geq \frac{1}{4}\|v\|. \tag{2.4}$$

Proof. Similar to the proof of Lemma 2.1 in [44], we can prove that Lemma 2.1 is correct.

Let $K \subset E$ be

$$K = \{u \in E : v(t) \geq 0, t \in J, \min_{\frac{1}{4}\leq t\leq\frac{3}{4}} u(t) \geq \frac{1}{4}\|u\|\}. \tag{2.5}$$

Define an operator $T : K \to E$ as

$$(Tv)(t) = \int_t^1 \left(\int_0^\tau ns^{n-1}g(s)f(s,v(s))ds\right)^{\frac{1}{(p-1)n}}d\tau,\ v \in K. \tag{2.6}$$

When $(\mathbf{C_1})$ and $(\mathbf{C_2})$ hold, one can verify that $T : K \to E$ is compact.

We shall apply the well-known fixed points index theorem to discuss problem (2.2), which can be found in Amann [46]. In addition, we use $i(A, P\cap\Omega, P)$ to denote the fixed point index over $P\cap\Omega$ with regard to P in Lemma 2.2.

Lemma 2.2. Let E be a real Banach space. Suppose that $P \subset E$ is a cone and $\Omega \subset E$ is a bounded open subset. Let $A : P\cap\bar{\Omega} \to P$ be a completely continuous operator, which admits no fixed points on $\partial\Omega$. Then the following three conclusions are correct:

(1) Suppose that there is a $v_0 > 0$ so that $v - Av \neq tv_0,\ \forall v \in P\cap\partial\Omega,\ t \geq 0$. Then

$$i(A, P\cap\Omega, P) = 0.$$

(2) Suppose that $Av \neq \mu v$ for $v \in (P\cap\partial\Omega)$ and $\mu \geq 1$. Then

$$i(A, P\cap\Omega, P) = 1.$$

(3) Suppose that U is open in P so that $\bar{U} \subset P \cap \Omega$. Then A admits a fixed point in $(P \cap \Omega) \backslash (\bar{U} \cap \Omega)$ when $i(A, P \cap \Omega, P) = 1$ and $i(A, U \cap P, P) = 0$. The same conclusion is also correct when $i(A, P \cap \Omega, P) = 0$ and $i(A, U \cap P, P) = 1$.

Remark 2.1. From Lan-Zhang [19], it is obvious to see that Lemma 2.1 is difficult to be applied to demonstrate the multiplicity of solutions of (2.2). In the present paper, we will use Lemma 2.2 to search the multiplicity of nonnegative solutions of (2.2). This needs some new ingredients in our proof.

For $\rho > 0$, we set

$$\Omega_\rho = \left\{ v \in K : \min_{t \in [\frac{1}{4}, \frac{3}{4}]} v(t) < \frac{1}{4}\rho \right\} = \left\{ v \in E : \frac{1}{4}\|v\| \le \min_{t \in [\frac{1}{4}, \frac{3}{4}]} v(t) < \frac{1}{4}\rho \right\}.$$

We also set

$$K_\rho = \{ v \in K : \|v\| < \rho \}.$$

According to a result of Lemma 2.5 in Lan [47], we can demonstrate that Ω_ρ is an open set relative to K and

(1) $K_{\frac{1}{4}\rho} \subset \Omega_\rho \subset K_\rho$;

(2) $v \in \partial\Omega_\rho$ if and only if $\min_{t \in [\frac{1}{4}, \frac{3}{4}]} v(t) = \frac{1}{4}\rho$;

(3) if $v \in \partial\Omega_\rho$, then $\frac{1}{4}\rho \le v(t) \le \rho$ for $t \in [\frac{1}{4}, \frac{3}{4}]$.

Define

$$f^\rho_{\frac{1}{4}\rho} = \min\left\{ \min_{t \in [\frac{1}{4}, \frac{3}{4}]} \frac{f(t,u)}{\rho^{(p-1)n}} : u \in \left[\frac{1}{4}\rho, \rho\right] \right\},$$

$$f^\rho_0 = \max\left\{ \max_{t \in J} \frac{f(t,u)}{\rho^{(p-1)n}} : u \in [0, \rho] \right\},$$

$$f^\infty = \limsup_{u \to \infty} \max_{t \in J} \frac{f(t,u)}{u}, \quad f_\infty = \liminf_{u \to \infty} \min_{t \in J} \frac{f(t,u)}{u},$$

$$l = \frac{1}{nd_2}, \quad L = \frac{4^{pn-1}}{nd_1},$$

where

$$d_1 = \int_{\frac{1}{4}}^{\frac{3}{4}} g(s)ds, \quad d_2 = \int_0^1 g(s)ds.$$

Theorem 2.1. Under conditions $(\mathbf{C_1})$ and $(\mathbf{C_2})$, if there are $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ satisfying $\rho_1 < \frac{1}{4}\rho_2$ and $\rho_2 < \rho_3$ so that

(C_3) $f^{\rho_1}_0 < l$ or $f^{\rho_3}_0 < l$, and

(C_4) $f^{\rho_2}_{\frac{1}{4}\rho_2} > L$,

then (2.2) admits a positive p-convex solution v with

$$v \in \Omega_{\rho_2} \backslash \bar{K}_{\rho_1} \text{ or } v \in K_{\rho_3} \backslash \bar{\Omega}_{\rho_2}.$$

Proof. For all $v \in \partial\Omega_{\rho_2}$, we assume that

$$v - Tv \neq \theta. \tag{2.7}$$

Otherwise, then there is $v \in \partial\Omega_{\rho_2}$ so that $Tv = v$.

According to (C_3), we deduce that

$$f(t,v) > L\rho_2^{(p-1)n},\ \forall t \in [\frac{1}{4},\frac{3}{4}],\ v \in [\frac{1}{4}\rho_2,\rho_2]. \tag{2.8}$$

Setting $w(t) \equiv 1,\ \forall t \in J$, then $w \in K$ with $\|w\| \equiv 1$. We claim

$$v - Tv \neq \zeta w\ \ (\forall v \in \partial\Omega_{\rho_2},\ \zeta \geq 0). \tag{2.9}$$

In reality, if there are $v_0 \in \partial\Omega_{\rho_2}$ and $\zeta_0 \geq 0$ so that $v_0 - Tv_0 = \zeta_0 w$. Then (2.7) indicates that $\zeta_0 > 0$. So, for any $t \in [\frac{1}{4}, \frac{3}{4}]$, we derive from (2.6) and (2.8) that

$$\begin{aligned}
v_0(t) &= \int_t^1 \left(\int_0^\tau ns^{n-1}g(s)f(s,v_0(s))ds \right)^{\frac{1}{(p-1)n}} d\tau + \zeta_0 w(s) \\
&\geq \int_{\frac{3}{4}}^1 \left(\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1}g(s)f(s,v_0(s))ds \right)^{\frac{1}{(p-1)n}} d\tau + \zeta_0 w(s) \\
&> \int_{\frac{3}{4}}^1 \left(\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1}g(s)L\rho_2^{(p-1)n}ds \right)^{\frac{1}{(p-1)n}} d\tau + \zeta_0 w(s) \\
&\geq \left[Ln(\tfrac{1}{4})^{n-1} \right]^{\frac{1}{(p-1)n}} \tfrac{1}{4}\rho_2 (\int_{\frac{1}{4}}^{\frac{3}{4}} g(s))ds)^{\frac{1}{(p-1)n}} + \zeta_0 w(s) \\
&= \left[Ln(\tfrac{1}{4})^{n-1} d_1 \right]^{\frac{1}{(p-1)n}} \tfrac{1}{4}\rho_2 + \zeta_0 w(s) \\
&= \rho_2 + \zeta_0 w(s).
\end{aligned}$$

This indicates that $\rho_2 > \rho_2 + \zeta_0$ by the property (3) of Ω_ρ, which leads to a conflict. So, (2.9) is correct. Hence it yields by (1) of Lemma 2.2

$$i(T,\Omega_{\rho_2},K) = 0. \tag{2.10}$$

Moreover, by the definition of $f_0^{\rho_1}$ and $f_0^{\rho_1} < l$, we derive

$$f(t,v) < l\rho_1^{(p-1)n},\ \forall t \in J,\ v \in [0,\rho_1].$$

Next, we demonstrate that

$$\forall v \in \partial K_{\rho_1},\ \mu \geq 1 \Rightarrow Tv \neq \mu v. \tag{2.11}$$

Actually, if there are $v_1 \in \partial K_{\rho_1}$ and $\mu_1 \geq 1$ so that $Tv_1 = \mu_1 v_1$, then we derive from (2.6) that

$$\begin{aligned}
\mu_1 v_1(t) &= \int_t^1 \left(\int_0^\tau ns^{n-1}g(s)f(s,v_1(s))ds \right)^{\frac{1}{(p-1)n}} d\tau \\
&\leq \int_0^1 \left(\int_0^1 ns^{n-1}g(s)f(s,v_1(s))ds \right)^{\frac{1}{(p-1)n}} d\tau \\
&< \int_0^1 \left(\int_0^1 ns^{n-1}g(s)l\rho_1^{(p-1)n}ds \right)^{\frac{1}{(p-1)n}} d\tau \\
&\leq (ln)^{\frac{1}{(p-1)n}} \rho_1 \left(\int_0^1 g(s)ds \right)^{\frac{1}{(p-1)n}} \\
&\leq (lnd_2)^{\frac{1}{(p-1)n}} \rho_1 \\
&= \rho_1,\ \forall t \in J,
\end{aligned}$$

which indicates that $\mu\|v_1\|_\infty < \rho_1$. We so derive that $\mu\rho_1 < \rho_1$. This shows that $\mu < 1$, which contradicts $\mu \geq 1$. Hence (2.11) is correct. From (2) of Lemma 2.2, we derive

$$i(T, K_{\rho_1}, K) = 1. \tag{2.13}$$

In addition, one can similarly demonstrate

$$i(T, K_{\rho_3}, K) = 1. \tag{2.14}$$

Noticing that $\rho_1 < \gamma\rho_2$, we have $\bar{K}_{\rho_1} \subset K_{\gamma\rho_2} \subset \Omega_{\rho_2}$. It so follows from (3) of Lemma 2.2 that (2.2) possesses a positive p-convex solution v satisfying $v \in \Omega_{\rho_2} \setminus \bar{K}_{\rho_1}$ or $v \in K_{\rho_3} \setminus \bar{\Omega}_{\rho_2}$. So Theorem 2.1 is correct.

From the proof of Theorem 2.1, one can obtain the following conclusions.

Theorem 2.2. Under conditions (**C₁**) and (**C₂**), if there exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\rho_1 < \frac{1}{4}\rho_2$ and $\rho_2 < \rho_3$ so that $f_0^{\rho_1} < l$ and $f_0^{\rho_3} < l$, and (C_4) holds, then problem (2.2) has two positive p-convex solutions $v_1,\ v_2$ with

$$v_1 \in \Omega_{\rho_2} \backslash \bar{K}_{\rho_1}, \quad v_2 \in K_{\rho_3} \backslash \bar{\Omega}_{\rho_2}.$$

Corollary 2.1. Under conditions (**C₁**) and (**C₂**), if there are $\rho', \rho \in (0, \infty)$ with $\rho' < \frac{1}{4}\rho$ so that $f_0^{\rho'} < l,\ f_{\frac{1}{4}\rho}^{\rho} > L$ and $0 \leq f^\infty < l$, then (2.2) admits two positive p-convex solutions in K.

Proof. We just need to verify that $0 \leq f^\infty < l$ yields that there exists a ρ_3 such that $f_0^{\rho_3} < l$.

Set $\eta \in (f^\infty, l)$. So there is $r > \eta$ so that

$$\max_{t \in J} f(t, v) \leq \eta v,\ \ \forall v \in [r, +\infty)$$

because of $0 \leq f^\infty < l$. Letting

$$\gamma = \max\{\max_{t \in J} f(t, v) : v \in [0, r]\} \text{ and } \rho_3 > \max\left\{\frac{\gamma}{l - \eta}, \rho\right\},$$

then

$$\max_{t \in J} f(t, v) \leq \eta v + \gamma \leq \eta\rho_3 + \gamma < l\rho_3,\ \ \forall v \in [0, \rho_3].$$

This indicates that $f_0^{\rho_3} < l$.

Similarly, one can derive the following result.

Theorem 2.3. Under conditions (**C₁**) and (**C₂**), suppose that there exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\rho_1 < \rho_2 < \rho_3$ so that

(C_5) $f_0^{\rho_2} < l,\ \ f_{\frac{1}{4}\rho_1}^{\rho_1} > L$ and $f_{\frac{1}{4}\rho_3}^{\rho_3} > L$,

then (2.2) admits two positive p-convex solutions $v_1,\ v_2$ with

$$v_1 \in \Omega_{\rho_2} \backslash \bar{K}_{\rho_1},\ v_2 \in K_{\rho_3} \backslash \bar{\Omega}_{\rho_2}.$$

The following conclusion is a special circumstances of Theorem 2.3.

Corollary 2.2. Under conditions (**C₁**) and (**C₂**), suppose that there are $\rho', \rho \in (0, \infty)$ with $\rho' < \frac{1}{4}\rho$ such that $f_0^{\rho} < l,\ f_{\frac{1}{4}\rho'}^{\rho'} > L$ and $L < f_\infty \leq +\infty$, then (2.2) admits two positive p-convex solutions in K.

Moreover, one can generalize Theorem 2.2 and Theorem 2.3 to derive many solutions.

Theorem 2.4. Under conditions ($\mathbf{C_1}$) and ($\mathbf{C_2}$), if there are

$$\{\rho_i\}_{i=1}^{2N_0} \subset (0, \infty) \text{ with } \rho_1 < \gamma\rho_2 < \rho_2 < \rho_3 < \gamma\rho_4 < \cdots < \rho_{2N_0}$$

so that

(C_6) $f_0^{\rho_{2N-1}} < l,\ f_{\frac{1}{4}\rho_{2N}}^{\rho_{2N}} > L,\ N \in \{1, 2, \ldots, N_0\}$,

then (2.2) admits $2N_0$ positive p-convex solutions in K.

Theorem 2.5. Under conditions ($\mathbf{C_1}$) and ($\mathbf{C_2}$), if there exist

$$\{\rho_i\}_{i=1}^{2N_0} \subset (0, \infty) \text{ with } \rho_1 < \gamma\rho_2 < \rho_2 < \rho_3 < \gamma\rho_4 < \cdots < \rho_{2N_0}$$

so that

(C_7) $f_0^{\rho_{2N-1}} < l,\ f_{\frac{1}{4}\rho_{2N}}^{\rho_{2N}} > L,\ N \in \{1, 2, \ldots, N_0\}$,

then (2.2) admits $(2N_0 - 1)$ positive p-convex solutions in K.

3. Main results of system (1.2)

In this section, we are gonging to discuss the existence and multiplicity of nontrivial p-convex solutions of system (1.2). Using a conclusion of Bao-Feng [45], we can change system (1.2) into

$$\begin{cases} r^{1-n}\Big(\frac{1}{n}(u_i')^{(p-1)n}\Big)' = g_i(r)f_i(r, -u_1, -u_2, \ldots, -u_n), \quad 0 < r < 1, \\ u_i'(0) = 0, \quad u_i(1) = 0, \quad i \in I_n. \end{cases} \tag{3.1}$$

Letting $v_i = -u_i$ for $i \in I_n$, then one can rewrite (3.1) as

$$\begin{cases} r^{1-n}\Big(\frac{1}{n}(-v_i')^{(p-1)n}\Big)' = g_i(r)f_i(r, v_1, v_2, \ldots, v_n), \quad 0 < r < 1, \\ v_i'(0) = 0, \quad v_i(1) = 0, \quad i \in I_n. \end{cases} \tag{3.2}$$

For each $i \in I_n$, we assume that g_i and f_i gratify

(**G**) $g_i : [0, 1) \to \mathbb{R}_+$ is continuous, $g_i(s) \not\equiv 0$ in any subinterval of $[0, 1)$, and

$$\int_0^1 g_i(s)ds < +\infty;$$

(**F**) $f_i : \mathbb{J} \times \mathbb{R}_+^n \to \mathbb{R}_+$ are continuous for $i \in I_n$, where

$$\mathbb{J} = [0, 1],\ \mathbb{R}_+ = [0, +\infty),\ \mathbb{R}_+^n = \overbrace{\mathbb{R}_+ \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+}^{n}.$$

Let $\mathbf{v}(t) = (v_1(t), v_2(t), \ldots, v_n(t))$. Then $\mathbf{v}(t)$ is a positive p-convex solution of system (3.2) iff $\mathbf{v}(r)$ is a solution of

$$v_i(r) = \int_r^1 \left(\int_0^t ns^{n-1}h_i(s)f_i(s, \mathbf{v}(s))ds\right)^{\frac{1}{(p-1)n}} dt,\ \forall r \in \mathbb{J},\ i \in I_n. \tag{3.3}$$

Let $|\cdot|$ denote the maximum norm in $\mathbb{R}^n$ defined by $|\mathbf{v}| = \max\{|v_i| : i \in I_n\}$, where $\mathbf{v} = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n$, and set

$$(\mathbb{R}_+^n)_J = \{\mathbf{v} \in \mathbb{R}_+^n : |\mathbf{v}| \in J\},$$

where $J = [l_1, l_2]$ if $l_1, l_2 \in [0, \infty)$ with $l_1 \le l_2$ and $J = [l_1, l_2)$ if $l_1, l_2 \in [0, \infty]$ with $l_1 < l_2$.

We also let $Y = C(\mathbb{J}; \mathbb{R}^n)$. Then Y is a RBS of continuous functions from $\mathbb{J}$ into $\mathbb{R}^n$ with norm $\|\mathbf{v}\| = \max\{\|v_i\|_0,\ i \in I_n\}$, where $\|\cdot\|_0$ denotes the supremum norm of $C[0,1]$.

Under conditions (**G**) and (**F**), if $\mathbf{v}$ is a positive solution of (3.2), then from Lemma 2.1 of Feng [43] we derive

$$\min_{r\in\mathbb{J}_0} v_i(r) \ge \frac{1}{4}\|v_i\|_0, \tag{3.4}$$

where $\mathbb{J}_0 = [\frac{1}{4}, \frac{3}{4}]$.

Hence one can define a cone P in Y as

$$P = \{\mathbf{v} = (v_1, v_2, \dots, v_n) \in Y :\ v_i(r) \ge 0,\ r \in \mathbb{J}\}. \tag{3.5}$$

Let $\mathbf{T} = (T_1, T_2, \dots, T_n)$ and (**G**) and (**F**) hold. Then we can show that $\mathbf{T} : P \to P$ is a compact operator. We understand

$$\mathbf{T}\mathbf{v} = (T_1\mathbf{v}, T_2\mathbf{v}, \dots, T_n\mathbf{v}),$$

where

$$(T_i\mathbf{v})(r) = \int_r^1 \Big(\int_0^t n s^{n-1} g_i(s) f_i(s, \mathbf{v}(s)) ds\Big)^{\frac{1}{(p-1)n}} dt,\quad i \in I_n. \tag{3.6}$$

Denote a fixed point equation by

$$\mathbf{v} = \mathbf{T}(\mathbf{v}),\ \ \mathbf{v} \in P. \tag{3.7}$$

Our main goal is to look for nonzero fixed points of operator $\mathbf{T}$ because (3.2) is equivalent to (3.7).

We will apply a well known fixed point theorem for compact maps to tackle system (3.2), which can be found in Amann [46]).

Lemma 3.1. Let E be a real Banach space. Suppose that Ω_1 and Ω_2 are two bounded open sets in E with $\theta \in \Omega_1$ and $\bar{\Omega}_1 \subset \Omega_2$. Suppose that P is a cone in E and operator $A : P \cap (\bar{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous. Let one of the following two conditions

(a) there is a $u_0 > 0$ such that $u - Au \ne tu_0, \forall u \in P \cap \partial\Omega_2, t \ge 0$; $Au \ne \mu u, \forall u \in P \cap \partial\Omega_1, \mu \ge 1$,

(b) there is a $u_0 > 0$ such that $u - Au \ne tu_0, \forall u \in P \cap \partial\Omega_1, t \ge 0$; $Au \ne \mu u, \forall u \in P \cap \partial\Omega_2, \mu \ge 1$

be satisfied. Then A admits at least one fixed point in $P \cap (\Omega_2 \backslash \bar{\Omega}_1)$.

For $i \in I_n$, let

$$(f_i)^\infty = \limsup_{|\mathbf{v}^{(p-1)n}| \to +\infty} \max_{s\in\mathbb{J}} \frac{f_i(s, \mathbf{v})}{|\mathbf{v}|^{(p-1)n}},\ \ (f_i)_\infty = \liminf_{|\mathbf{v}| \to +\infty} \min_{s\in\mathbb{J}} \frac{f_i(s, \mathbf{v})}{|\mathbf{v}|^{(p-1)n}},$$

$$(f_i)^0 = \limsup_{|\mathbf{v}| \to 0^+} \max_{s\in\mathbb{J}} \frac{f_i(s, \mathbf{v})}{|\mathbf{v}|^{(p-1)n}},\ \ (f_i)_0 = \liminf_{|\mathbf{v}| \to 0^+} \min_{s\in\mathbb{J}} \frac{f_i(s, \mathbf{v})}{|\mathbf{v}|^{(p-1)n}},$$

$$f^\infty = \max\{(f_i)^\infty,\ i \in I_n\},\ \ f_\infty = \max\{(f_i)_\infty,\ i \in I_n\},$$

$$f^0 = \max\{(f_i)^0,\ i \in I_n\},\ \ f_0 = \max\{(f_i)_0,\ i \in I_n\},$$

$$(F_i)_\infty := \lim_{|\mathbf{v}| \to +\infty} f_i(s, \mathbf{v}) \text{ uniformly for } s \in \mathbb{J},$$

and

$$D_{ni} = nd_i,\ \ D_{ni_0} = [(\frac{1}{4})^{pn-1} nd_{i_0}],$$

where

$$d_i = \int_0^1 g_i(s)ds, \ \ d_{i_0} = \int_{\frac{1}{4}}^{\frac{3}{4}} g_{i_0}(s)ds.$$

Theorem 3.1. Under conditions (**G**) and (**F**), if in addition there exists $i_0 \in I_n$ such that

$$D_{ni} f^{\infty} < 1 < D_{ni_0}(f_{i_0})_0,$$

then we derive:

(i) (3.2) has a positive p-convex solution $\mathbf{v} = (v_1, v_2, \ldots, v_n)$; and then

(ii) (1.2) has a nontrivial p-convex radial solution $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, where

$$u_i(|x|) = -v_i(r) \text{ for } i \in I_n \text{ and } r \in \mathbb{J}.$$

Proof. We assume that there is $l_1 > 0$ so that

$$\mathbf{v} - \mathbf{Tv} \neq \theta, \ \forall \mathbf{v} \in P, \ 0 < \|\mathbf{v}\| \leq l_1. \tag{3.8}$$

If not, then there is $\mathbf{v} \in P_{l_1}$ such that

$$\mathbf{Tv} = \mathbf{v}.$$

On the one hand, it yields from the definition of $(f_{i_0})_0$ and $D_{i_0}(f_{i_0})_0 > 1$ that there are $\varepsilon_1 > 0$ and $l_2 > 0$ such that

$$f_{i_0}(s, \mathbf{v}) \geq ((f_{i_0})_0 - \varepsilon_1)|\mathbf{v}|^{(p-1)n}, \ \ \forall s \in \mathbb{J}, \ \ \mathbf{v} \in \partial P_{l_2}, \tag{3.9}$$

where ε_1 gratifies that

$$D_{ni_0}((f_{i_0})_0 - \varepsilon_1) \geq 1.$$

For $i \in I_n$, letting $\mathbf{w} = \{w_1, w_2, \ldots, w_n\}$ with $w_i(s) \equiv 1$ for $s \in \mathbb{J}$, then $\mathbf{w} \in P$ with $\|w_i\|_0 \equiv 1$. Next, we demonstrate

$$\mathbf{v} - \mathbf{Tv} \neq \zeta \mathbf{w} \ \ (\forall \mathbf{v} \in \partial P_l, \ \zeta \geq 0), \tag{3.10}$$

where

$$0 < l < \min\{l_1, l_2\}.$$

In reality, if there are $\mathbf{v} \in \partial P_l$ and $\zeta \geq 0$ so that $\mathbf{v} - \mathbf{Tv} = \zeta \mathbf{w}$. Then (3.8) shows that $\zeta > 0$ and

$$v_{i_0} = \zeta w_{i_0} + T_{i_0}\mathbf{v} \geq \zeta w_{i_0}.$$

Let

$$\zeta^* = \sup\{\zeta | v_{i_0} \geq \zeta w_{i_0}\}. \tag{3.11}$$

Then

$$\zeta^* = \zeta^* \|w_{i_0}\|_0 \leq \|v_{i_0}\|_0 = l < l_2 \leq [D_{ni_0}((f_{i_0})_0 - \varepsilon_1)]^{\frac{1}{(p-1)n-1}}.$$

Therefore, for any $r \in \mathbb{J}_0$, we derive from (3.6), (3.9) and (3.11) that

$$\begin{aligned}
v_{i_0}(r) &= \int_r^1 (\int_0^t ns^{n-1} g_{i_0}(s) f_{i_0}(s, \mathbf{v}(s)) ds)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \int_{\frac{3}{4}}^1 (\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1} g_{i_0}(s) f_{i_0}(s, \mathbf{v}(s)) ds)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \int_{\frac{3}{4}}^1 (\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1} g_{i_0}(s)((f_{i_0})_0 - \varepsilon_1)|\mathbf{v}(s)|^{(p-1)n} ds)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \int_{\frac{3}{4}}^1 (\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1} g_{i_0}(s)((f_{i_0})_0 - \varepsilon_1)|v_{i_0}(s)|^{(p-1)n} ds)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \int_{\frac{3}{4}}^1 (\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1} g_{i_0}(s)((f_{i_0})_0 - \varepsilon_1)(\zeta^* w_{i_0}(s))^{(p-1)n} ds)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \frac{1}{4}\zeta^* [n(\tfrac{1}{4})^{n-1}((f_{i_0})_0 - \varepsilon_1)]^{\frac{1}{(p-1)n}} (\int_{\frac{1}{4}}^{\frac{3}{4}} g_{i_0}(s) ds)^{\frac{1}{(p-1)n}} + \zeta w_{i_0}(r) \\
&= \frac{1}{4}\zeta^* [d_{i_0} n(\tfrac{1}{4})^{n-1}((f_{i_0})_0 - \varepsilon_1)]^{\frac{1}{(p-1)n}} + \zeta w_{i_0}(r) \\
&= \zeta^* [D_{ni_0}((f_{i_0})_0 - \varepsilon_1)]^{\frac{1}{(p-1)n}} + \zeta w_{i_0}(r) \\
&\geq \zeta^* + \zeta w_{i_0}(r) \\
&= (\zeta^* + \zeta) w_{i_0}(r),
\end{aligned}$$

which conflicts with the definition of ζ^*. So, (3.10) is correct.

In addition, by the definition of f^∞ and $D_i f^\infty < 1$ we have that there are $\varepsilon_2 > 0$ and $l_3 > 0$ so that

$$f_i(s, \mathbf{v}) \leq ((f_i)^\infty + \varepsilon_2)|\mathbf{v}|^{(p-1)n}, \ \forall s \in \mathbb{J}, \ \mathbf{v} \in (\mathbb{R}_+^n)_{[l_3, \infty)}.$$

Define

$$L_i = \max_{s \in \mathbb{J}, \mathbf{v} \in (\mathbb{R}_+^n)_{[0, l_3]}} f_i(s, \mathbf{v}).$$

We so derive

$$f_i(s, \mathbf{v}) \leq ((f_i)^\infty + \varepsilon_2)|\mathbf{v}|^{(p-1)n} + L_i, \ \forall s \in \mathbb{J}, \mathbf{v} \in \mathbb{R}_+^n. \tag{3.12}$$

Set

$$R > \left\{ l_3, \left(\frac{L_i D_{ni}}{1 - D_{ni}(((f_i)^\infty + \varepsilon_2)} \right)^{(p-1)n} \right\} \tag{3.13}$$

for $i \in I_n$.

We declare

$$\forall \mathbf{v} \in \partial P_R, \ \mu \geq 1 \Rightarrow \mathbf{T}\mathbf{v} \neq \mu \mathbf{v}. \tag{3.14}$$

Actually, if there are $\mathbf{v} \in \partial P_R$ and $\mu \geq 1$ so that $\mathbf{T}\mathbf{v} = \mu \mathbf{v}$, then for each $i \in I_n$ it follows from (3.6), (3.12) and (3.13) that

$$\begin{aligned}
\mu v_i(r) &= \int_r^1 (\int_0^t ns^{n-1} g_i(s) f_i(s, \mathbf{v}(s)) ds)^{\frac{1}{(p-1)n}} dt \\
&\leq \int_0^1 (\int_0^1 ns^{n-1} g_i(s) f_i(s, \mathbf{v}(s)) ds)^{\frac{1}{(p-1)n}} dt \\
&\leq \int_0^1 (\int_0^1 n g_i(s) f_i(s, \mathbf{v}(s)) ds)^{\frac{1}{(p-1)n}} dt \\
&< \int_0^1 (\int_0^1 n g_i(s)(((f_i)^\infty + \varepsilon_2)|\mathbf{v}(s)|^{(p-1)n} + L_i) ds)^{\frac{1}{(p-1)n}} dt \\
&\leq (\int_0^1 n g_i(s)(((f_i)^\infty + \varepsilon_2)||\mathbf{v}||^{(p-1)n} + L_i) ds)^{\frac{1}{(p-1)n}} \\
&= [n(((f_i)^\infty + \varepsilon_2)||\mathbf{v}||^{(p-1)n} + L_i)]^{\frac{1}{(p-1)n}} (\int_0^1 g_i(s)) ds)^{\frac{1}{(p-1)n}} \\
&= [n d_i(((f_i)^\infty + \varepsilon_2)||\mathbf{v}||^{(p-1)n} + L_i)]^{\frac{1}{(p-1)n}} \\
&< R,
\end{aligned}$$

which shows that $\mu\|v_i\|_0 < R$, and then we have $\mu\|\mathbf{v}\| < R$. We hence derive $\mu R < R$. This indicates that $\mu < 1$, which conflicts with $\mu \geq 1$. So (3.14) is correct.

From Lemma 3.1 (b), it hence yields from (3.10) and (3.14) that $\mathbf{T}$ possesses a fixed point $\mathbf{v}$ in $P_R \backslash \bar{P}_l$ satisfying $l < \|\mathbf{v}\| < R$. So (3.2) has a positive p-convex solution $\mathbf{v}$ satisfying $l < \|\mathbf{v}\| < R$. Hence we finish the proof of Theorem 3.1.

Remark 3.1. Although the essence of Lemma 2.1 and Lemma 3.1 is the same, the specific processing technique of Theorem 3.1 is different from that of Theorem 2.1.

Theorem 3.2. Under conditions (**G**) and (**F**), if in addition there exists $i_0 \in I_n$ such that

$$D_{ni} f^0 < 1 < D_{ni_0}(f_{i_0})_\infty,$$

then we derive:

(i) (3.2) has a positive p-convex solution $\mathbf{v} = (v_1, v_2, \dots, v_n)$; and then

(ii) (1.2) has a nontrivial p-convex radial solution $\mathbf{u} = (u_1, u_2, \dots, u_n)$, where

$$u_i(|x|) = -v_i(r) \text{ for } i \in I_n \text{ and } r \in \mathbb{J}.$$

Proof. We assume that (3.8) holds. Considering the definition of $(f_{i_0})_\infty$, then there are $\varepsilon_3 > 0$ and $\hat{R} > 0$ with $\hat{R} > l_1$ so that

$$f_{i_0}(s, \mathbf{v}) \geq ((f_{i_0})_\infty - \varepsilon_3)|\mathbf{v}|^{(p-1)n} \quad (\forall s \in \mathbb{J}, \quad \mathbf{v} \in (\mathbb{R}^n_+)_{[\hat{R}, +\infty)}), \tag{3.15}$$

where ε_3 satisfies that

$$D_{ni_0}((f_{i_0})_\infty - \varepsilon_3) \geq 1.$$

For $i \in I_n$, let

$$\mathbf{w} = \{w_1, w_2, \dots, w_n\}$$

with $w_i(s) \equiv 1$ for $s \in \mathbb{J}$. Then $\mathbf{w} \in P$ with $\|w_i\|_\infty \equiv 1$. Next, we demonstrate that

$$\mathbf{v} - \mathbf{T}\mathbf{v} \neq \zeta \mathbf{w} \quad (\forall \mathbf{v} \in \partial P_R, \ \zeta \geq 0), \tag{3.16}$$

where $R = 4\hat{R}$.

In reality, if there are $\mathbf{v} \in \partial P_R$ and $\zeta \geq 0$ so that $\mathbf{v} - \mathbf{T}\mathbf{v} = \zeta \mathbf{w}$. Then (3.8) shows that $\zeta > 0$ and

$$v_{i_0} = \zeta w_{i_0} + T_{i_0}\mathbf{v} \geq \zeta w_{i_0}.$$

In addition, for $\mathbf{v} \in \partial P_R$ we derive

$$v_i(s) \geq \min_{s \in \mathbb{J}_0} v_i(s) \geq \frac{1}{4}\|v_i\|_0 \geq \frac{1}{4} \max\{\|v_i\|_0, \ i \in I_n\} = \frac{1}{4}\|\mathbf{v}\| = \hat{R}.$$

Let ζ^* be defined as in (3.11). Then, for $\mathbf{v} \in \partial P_R$ and $r \in \mathbb{J}_0$, we derive from (3.6) and (3.15) that

$$\begin{aligned}
v_{i_0}(r) &= \int_r^1 (\int_0^t ns^{n-1} g_{i_0}(s) f_{i_0}(s, \mathbf{v}(s)) ds)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \int_{\frac{3}{4}}^1 (\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1} g_{i_0}(s) f_{i_0}(s, \mathbf{v}(s)) ds)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \int_{\frac{3}{4}}^1 (\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1} g_{i_0}(s)((f_{i_0})_0 - \varepsilon_3)|\mathbf{v}(s)|^{(p-1)n} ds)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \int_{\frac{3}{4}}^1 (\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1} g_{i_0}(s)((f_{i_0})_0 - \varepsilon_3)|v_{i_0}(s)|^{(p-1)n} ds)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \int_{\frac{3}{4}}^1 (\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1} g_{i_0}(s)((f_{i_0})_0 - \varepsilon_3)(\zeta^* w_{i_0}(s))^{(p-1)n} ds)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \tfrac{1}{4}\zeta^*[n(\tfrac{1}{4})^{n-1}((f_{i_0})_0 - \varepsilon_3)]^{\frac{1}{(p-1)n}} (\int_{\frac{1}{4}}^{\frac{3}{4}} g_{i_0}(s) ds)^{\frac{1}{(p-1)n}} + \zeta w_{i_0}(r) \\
&= \tfrac{1}{4}\zeta^*[nd_{i_0}(\tfrac{1}{4})^{n-1}((f_{i_0})_0 - \varepsilon_3)]^{\frac{1}{(p-1)n}} + \zeta w_{i_0}(r) \\
&= \zeta^*[nd_{i_0}(\tfrac{1}{4})^{pn-1}((f_{i_0})_0 - \varepsilon_3)]^{\frac{1}{(p-1)n}} + \zeta w_{i_0}(r) \\
&= \zeta^*[D_{ni_0}((f_{i_0})_0 - \varepsilon_3)]^{\frac{1}{(p-1)n}} + \zeta w_{i_0}(r) \\
&\geq \zeta^* + \zeta w_{i_0}(r) \\
&= (\zeta^* + \zeta) w_{i_0}(r),
\end{aligned}$$

which conflicts with the definition of ζ^*. So, (3.16) is correct.

In addition, by the definition of f^0 and $D_{ni}f^0 < 1$ we know that there are $\varepsilon_4 > 0$ and $l > 0$ with $l < l_1$ so that

$$f_i(s, \mathbf{v}) \leq ((f_i)^\infty + \varepsilon_4)|\mathbf{v}|^{\frac{1}{(p-1)n}}, \ \forall s \in \mathbb{J}, \ \mathbf{v} \in (\mathbb{R}_+^n)_{[0,l]}, \tag{3.17}$$

where ε_4 satisfies

$$2^{(p-1)n} D_{ni}((f_i)^0 + \varepsilon_4) \leq 1.$$

We declare that

$$\forall \mathbf{v} \in \partial P_l, \ \mu \geq 1 \Rightarrow \mathbf{T}\mathbf{v} \neq \mu \mathbf{v}. \tag{3.18}$$

Actually, if there are $\mathbf{v} \in \partial P_l$ and $\mu \geq 1$ so that $\mathbf{T}\mathbf{v} = \mu\mathbf{v}$, then for each $i \in I_n$ it follows from (2.5) and (3.17) that

$$\begin{aligned}
\mu v_i(r) &= \int_r^1 \left(\int_0^t ns^{n-1} g_i(s) f_i(s, \mathbf{v}(s))) ds \right)^{\frac{1}{(p-1)n}} dt \\
&\leq \int_0^1 \left(\int_0^1 ns^{n-1} g_i(s) f_i(s, \mathbf{v}(s)) ds \right)^{\frac{1}{(p-1)n}} dt \\
&\leq \int_0^1 \left(\int_0^1 n g_i(s) f_i(\mathbf{v}(s)) ds \right)^{\frac{1}{(p-1)n}} dt \\
&< \int_0^1 \left(\int_0^1 n g_i(s)((f_i)^\infty + \varepsilon_4)|\mathbf{v}(s)|^{(p-1)n} ds \right)^{\frac{1}{(p-1)n}} dt \\
&\leq \left(\int_0^1 n g_i(s)((f_i)^\infty + \varepsilon_4)\|\mathbf{v}\|^{(p-1)n} ds \right)^{\frac{1}{(p-1)n}} \\
&= [n((f_i)^\infty + \varepsilon_4)]^{\frac{1}{(p-1)n}} \|\mathbf{v}\| (\int_0^1 g_i(s) ds)^{\frac{1}{(p-1)n}} \\
&= [nd_i((f_i)^\infty + \varepsilon_4)]^{\frac{1}{(p-1)n}} \|\mathbf{v}\| \\
&= [D_{ni}((f_i)^\infty + \varepsilon_4)]^{\frac{1}{(p-1)n}} \|\mathbf{v}\| \\
&\leq \tfrac{1}{2}\|\mathbf{v}\| = \tfrac{1}{2} l < l,
\end{aligned}$$

which shows that $\mu\|v_i\|_0 < l$, and then we have $\mu\|\mathbf{v}\| < l$. We hence derive $\mu l < l$. This indicates that $\mu < 1$, which conflicts with $\mu \geq 1$. So (3.18) is correct.

According to (b) of Lemma 3.1, it so yields from (3.16) and (3.18) that operator $\mathbf{T}$ possesses a fixed point $\mathbf{v}$ in $P_R \backslash \bar{P}_l$ satisfying $l < \|\mathbf{v}\| < R$. So (3.2) has a positive p-convex solution $\mathbf{v}$ satisfying $l < \|\mathbf{v}\| < R$. Therefore Theorem 3.2 is correct.

Theorem 3.3. Under conditions (**G**) and (**F**), if in addition there exists $i_0 \in I_n$ so that

$$D_{ni_0}(f_{i_0})_0 > 1 \text{ or } D_{ni_0}(f_{i_0})_\infty > 1$$

and there is $b_i > 0$ so that

$$\max_{s\in\mathbb{J},\ \mathbf{v}\in(\mathbb{R}^n_+)_{[0,b_i]}} f_i(s,\mathbf{v}) < \frac{1}{D_{ni}} b_i \tag{3.19}$$

for each $i \in I_n$, then we have:

(i) (3.2) possesses a positive p-convex solution $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ in P; and then

(ii) (1.2) possesses a nontrivial p-convex radial solution $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, where

$$u_i(|x|) = -v_i(r) \text{ for } i \in I_n \text{ and } r \in \mathbb{J}.$$

Proof. Here we only consider the case $D_{ni_0}(f_{i_0})_0 > 1$ and (3.19). We choose l with $0 < l < b_i$ for each $i \in I_n$.

If $D_{ni_0}(f_{i_0})_0 > 1$, then

$$\mathbf{v} - \mathbf{T}\mathbf{v} \neq \zeta\mathbf{w} \ \ (\forall \mathbf{v} \in \partial P_l,\ \zeta \geq 0). \tag{3.20}$$

The proof is similar to that of (3.10). Therefore, it is omitted.

On the other hand, by (3.19), we can define

$$L_i = \max_{s\in\mathbb{J},\mathbf{v}\in(\mathbb{R}^n_+)_{[0,b_i]}} f_i(s,\mathbf{v}) < D_{ni}^{-1} b_i^{(p-1)n}. \tag{3.21}$$

Let

$$b = \max\{b_i : i \in I_n\}. \tag{3.22}$$

Next, we prove that

$$\forall \mathbf{v} \in \partial P_b,\ \mu \geq 1 \Rightarrow \mathbf{T}\mathbf{v} \neq \mu\mathbf{v}. \tag{3.23}$$

Indeed, suppose that there are $\mathbf{v} \in \partial P_b$ and $\mu \geq 1$ such that $\mathbf{T}\mathbf{v} = \mu\mathbf{v}$, then for each $i \in I_n$ it follows from (3.6), (3.21) and (3.22) that

$$\begin{aligned}
\mu v_i(r) &= \int_r^1 \left(\int_0^t n s^{n-1} g_i(s) f_i(s,\mathbf{v}(s)) ds \right)^{\frac{1}{(p-1)n}} dt \\
&\leq \int_0^1 \left(\int_0^1 n s^{n-1} g_i(s) f_i(s,\mathbf{v}(s)) ds \right)^{\frac{1}{(p-1)n}} dt \\
&\leq \left(\int_0^1 n g_i(s) f_i(s,\mathbf{v}(s)) ds \right)^{\frac{1}{(p-1)n}} \\
&\leq \left(\int_0^1 n g_i(s) L_i ds \right)^{\frac{1}{(p-1)n}} \\
&= (nL_i)^{\frac{1}{(p-1)n}} \Big(\int_0^1 g_i(s) ds \Big)^{\frac{1}{(p-1)n}} \\
&= (nd_i L_i)^{\frac{1}{(p-1)n}} \\
&= (D_{ni} L_i)^{\frac{1}{(p-1)n}} \\
&< b_i \leq b.
\end{aligned}$$

which indicates that $\mu\|v_i\|_0 < b$, and then we get $\mu\|\mathbf{v}\| < b$. Thus we derive $\mu b < b$. So we have $\mu < 1$. This conflicts with $\mu \geq 1$. Hence (3.23) is correct.

So, by (3.20) and (3.23), it follows from (b) of Lemma 3.1 that $\mathbf{T}$ possesses a fixed point $\mathbf{v}$ in $P_b \backslash \bar{P}_l$ with $l < \|\mathbf{v}\| < b$. This follows that (3.2) has a positive p-convex solution $\mathbf{v}$ satisfying $l < \|\mathbf{v}\| < b$. So Theorem 3.3 is correct.

Similarly, one can derive the following multiplicity conclusions.

Theorem 3.4. Under conditions (**G**) and (**F**), if in addition there is $i_0 \in I_n$ so that

$$D_{ni_0}(f_{i_0})_0 > 1 \text{ and } D_{ni_0}(f_{i_0})_\infty > 1$$

and there is $b_i > 0$ so that (3.19) holds for each $i \in I_n$, then we derive:

(i) (3.2) possesses two p-convex positive solutions $\mathbf{v}^*$ and $\mathbf{v}^{**}$ in P with

$$0 < \|\mathbf{v}^*\| < \max\{b_i,\ i \in I_n\} < \|\mathbf{v}^{**}\|;$$

and then

(ii) (1.2) possesses two nontrivial p-convex radial solutions $\mathbf{u}^*$ and $\mathbf{u}^{**}$ with

$$u_i^*(|x|) = -v_i^*(r) \ \ u_i^{**}(|x|) = -v_i^{**}(r) \text{ for } i \in I_n \text{ and } r \in \mathbb{J}.$$

Next, we consider the nonexistence result on system (3.2).

Theorem 3.5. Under conditions (**G**) and (**F**), if for each $i \in I_n$ $f_i(s, \mathbf{v}) < \frac{1}{D_{ni}}|\mathbf{v}|^{(p-1)n}$ for all $s \in \mathbb{J}$ and $\mathbf{v} \in \mathbb{R}_+$ with $|\mathbf{v}| > 0$, then (3.2) possesses no positive solution.

Proof. Conversely, suppose that $\mathbf{v}$ is a positive solution of system (3.2).

So, for $r \in \mathbb{J},\ \mathbf{v} \in P$ with $\|\mathbf{v}\| > 0$ we obtain that

$$\begin{aligned}
v_i(r) &= \int_r^1 \left(\int_0^t ns^{n-1} g_i(s) f_i(s, \mathbf{v}(s)) ds \right)^{\frac{1}{(p-1)n}} dt \\
&\leq \int_0^1 \left(\int_0^1 ns^{n-1} g_i(s) f_i(s, \mathbf{v}(s)) ds \right)^{\frac{1}{(p-1)n}} dt \\
&\leq \left(\int_0^1 ng_i(s) f_i(s, \mathbf{v}(s)) ds \right)^{\frac{1}{(p-1)n}} \\
&< \left(\int_0^1 ng_i(s) (\tfrac{1}{D_{ni}} |\mathbf{v}(s)|)^{(p-1)n} ds \right)^{\frac{1}{(p-1)n}} \\
&\leq \|\mathbf{v}\| \left(\frac{1}{D_{ni}} n \right)^{\frac{1}{(p-1)n}} \left(\int_0^1 g_i(s) ds \right)^{\frac{1}{(p-1)n}} \\
&= \|\mathbf{v}\| \left(\frac{1}{D_{ni}} nd_i \right)^{\frac{1}{(p-1)n}} \\
&= \|\mathbf{v}\|.
\end{aligned}$$

This shows $\|v_i\|_0 < \|\mathbf{v}\|$, and hence we derive that $\|\mathbf{v}\| < \|\mathbf{v}\|$, a contradiction. So Theorem 3.5 is correct.

Remark 3.2. It is interesting to point out that, for $i \in I_n$, if we define

$$(f_i)^\infty = \limsup_{|\mathbf{v}| \to +\infty} \max_{s \in \mathbb{J}} \frac{f_i(s, \mathbf{v})}{|\mathbf{v}|}, \ \ (f_i)_\infty = \liminf_{|\mathbf{v}| \to +\infty} \min_{s \in \mathbb{J}} \frac{f_i(s, \mathbf{v})}{|\mathbf{v}|},$$

$$(f_i)^0 = \limsup_{|\mathbf{v}|\to 0^+} \max_{s\in\mathbb{J}} \frac{f_i(s,\mathbf{v})}{|\mathbf{v}|}, \quad (f_i)_0 = \liminf_{|\mathbf{v}|\to 0^+} \min_{s\in\mathbb{J}} \frac{f_i(s,\mathbf{v})}{|\mathbf{v}|},$$

$$f^\infty = \max\{(f_i)^\infty,\ i\in I_n\}, \quad f_\infty = \max\{(f_i)_\infty,\ i\in I_n\},$$

$$f^0 = \max\{(f_i)^0,\ i\in I_n\}, \quad f_0 = \max\{(f_i)_0,\ i\in I_n\},$$

then we derive:

Theorem 3.6. Under conditions (**H**) and (**F**), if in addition there is $i_0 \in I_n$ so that

$$D_i f^\infty < 1 < D_{i_0}(f_{i_0})_0,$$

then we derive:

(i) (3.2) has a p-convex positive solution $\mathbf{v} = (v_1, v_2, \ldots, v_n)$; and then

(ii) (1.2) has a nontrivial p-convex radial solution $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, where

$$u_i(|x|) = -v_i(r) \text{ for } i \in I_n \text{ and } r \in \mathbb{J}.$$

Proof. We assume that there is $l_1 > 0$ so that

$$\mathbf{v} - \mathbf{T}\mathbf{v} \neq \theta,\ \forall \mathbf{v} \in P,\ 0 < \|\mathbf{v}\| \leq l_1. \tag{3.24}$$

If not, then there is $\mathbf{v} \in P_{l_1}$ such that

$$\mathbf{T}\mathbf{v} = \mathbf{v}.$$

On the one hand, it yields from the definition of $(f_{i_0})_0$ and $D_{i_0}(f_{i_0})_0 > 1$ that there are $\varepsilon_1 > 0$ and $l_2 > 0$ such that

$$f_{i_0}(s,\mathbf{v}) \geq ((f_{i_0})_0 - \varepsilon_1)|\mathbf{v}|, \quad \forall s \in \mathbb{J}, \quad \mathbf{v} \in \partial P_{l_2}. \tag{3.25}$$

For $i \in I_n$, letting

$$\mathbf{w} = \{w_1, w_2, \ldots, w_n\}$$

with $w_i(s) \equiv 1$ for $s \in \mathbb{J}$, then $\mathbf{w} \in P$ with $\|w_i\|_0 \equiv 1$. Now, we clare

$$\mathbf{v} - \mathbf{T}\mathbf{v} \neq \zeta\mathbf{w} \ \ (\forall \mathbf{v} \in \partial P_l,\ \zeta \geq 0), \tag{3.26}$$

where

$$0 < l < \min\{l_1, l_2\}.$$

In reality, if there are $\mathbf{v} \in \partial P_l$ and $\zeta \geq 0$ such that

$$\mathbf{v} - \mathbf{T}\mathbf{v} = \zeta\mathbf{w}.$$

Then (3.24) indicates $\zeta > 0$ and

$$v_{i_0} = \zeta w_{i_0} + T_{i_0}\mathbf{v} \geq \zeta w_{i_0}.$$

Let

$$\zeta^* = \sup\{\zeta | v_{i_0} \geq \zeta w_{i_0}\}. \tag{3.27}$$

Then

$$\zeta^* = \zeta^*\|w_{i_0}\|_0 \leq \|v_{i_0}\|_0 = l < l_2 \leq [D_{ni_0}((f_{i_0})_0 - \varepsilon_1)]^{\frac{1}{(p-1)n-1}}.$$

Therefore, for any $r \in \mathbb{J}_0$, we derive from (3.6), (3.25) and (3.27) that

$$\begin{aligned}
v_{i_0}(r) &= \int_r^1 \Big(\int_0^t ns^{n-1} g_{i_0}(s) f_{i_0}(s, \mathbf{v}(s)) ds\Big)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \int_{\frac{3}{4}}^1 \Big(\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1} g_{i_0}(s) f_{i_0}(s, \mathbf{v}(s)) ds\Big)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \int_{\frac{3}{4}}^1 \Big(\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1} g_{i_0}(s)((f_{i_0})_0 - \varepsilon_1)|\mathbf{v}(s)| ds\Big)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \int_{\frac{3}{4}}^1 \Big(\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1} g_{i_0}(s)((f_{i_0})_0 - \varepsilon_1)|v_{i_0}(s)| ds\Big)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \int_{\frac{3}{4}}^1 \Big(\int_{\frac{1}{4}}^{\frac{3}{4}} ns^{n-1} g_{i_0}(s)((f_{i_0})_0 - \varepsilon_1)\zeta^* w_{i_0}(s) ds\Big)^{\frac{1}{(p-1)n}} dt + \zeta w_{i_0}(r) \\
&\geq \tfrac{1}{4}[n(\tfrac{1}{4})^{n-1}((f_{i_0})_0 - \varepsilon_1)\zeta^*]^{\frac{1}{(p-1)n}} \Big(\int_{\frac{1}{4}}^{\frac{3}{4}} g_{i_0}(s) ds\Big)^{\frac{1}{(p-1)n}} + \zeta w_{i_0}(r) \\
&= \tfrac{1}{4}[d_{i_0} n(\tfrac{1}{4})^{n-1}((f_{i_0})_0 - \varepsilon_1)\zeta^*]^{\frac{1}{(p-1)n}} + \zeta w_{i_0}(r) \\
&= [D_{ni_0}((f_{i_0})_0 - \varepsilon_1)\zeta^*]^{\frac{1}{(p-1)n}} + \zeta w_{i_0}(r) \\
&\geq \zeta^* + \zeta w_{i_0}(r) \\
&= (\zeta^* + \zeta) w_{i_0}(r),
\end{aligned}$$

which conflicts with the definition of ζ^*. So, (3.26) is correct.

In addition, by the definition of f^∞ and $D_i f^\infty < 1$ we know that there are $\varepsilon_2 > 0$ and $l_3 > 0$ so that

$$f_i(s, \mathbf{v}) \leq ((f_i)^\infty + \varepsilon_2)|\mathbf{v}|, \ \forall s \in \mathbb{J}, \ \mathbf{v} \in (\mathbb{R}_+^n)_{[l_3,\infty)}.$$

Define

$$L_i = \max_{s \in \mathbb{J}, \mathbf{v} \in (\mathbb{R}_+^n)_{[0,l_3]}} f_i(s, \mathbf{v}).$$

We so derive

$$f_i(s, \mathbf{v}) \leq ((f_i)^\infty + \varepsilon_2)|\mathbf{v}| + L_i, \ \forall s \in \mathbb{J}, \mathbf{v} \in \mathbb{R}_+^n. \tag{3.28}$$

Take R large enough (say $R > l_3$) such that

$$\frac{nd_i((f_i)^\infty + \varepsilon_2)|\mathbf{v}|}{R^{pn-n-1}} + \frac{nd_i L_i}{R^{pn-n}} < 1 \tag{3.29}$$

for $i \in I_n$.

We declare

$$\forall \mathbf{v} \in \partial P_R, \ \mu \geq 1 \Rightarrow \mathbf{T}\mathbf{v} \neq \mu \mathbf{v}. \tag{3.30}$$

Actually, if there are $\mathbf{v} \in \partial P_R$ and $\mu \geq 1$ so that $\mathbf{T}\mathbf{v} = \mu\mathbf{v}$, then for each $i \in I_n$ it follows from (3.6), (3.28) and (3.29) that

$$\begin{aligned}
\mu v_i(r) &= \int_r^1 \Big(\int_0^t ns^{n-1} g_i(s) f_i(s, \mathbf{v}(s)) ds\Big)^{\frac{1}{(p-1)n}} dt \\
&\leq \int_0^1 \Big(\int_0^1 ns^{n-1} g_i(s) f_i(s, \mathbf{v}(s)) ds\Big)^{\frac{1}{(p-1)n}} dt \\
&\leq \int_0^1 \Big(\int_0^1 ng_i(s) f_i(s, \mathbf{v}(s)) ds\Big)^{\frac{1}{(p-1)n}} dt \\
&< \int_0^1 \Big(\int_0^1 ng_i(s)(((f_i)^\infty + \varepsilon_2)|\mathbf{v}(s)| + L_i) ds\Big)^{\frac{1}{(p-1)n}} dt \\
&\leq \Big(\int_0^1 ng_i(s)(((f_i)^\infty + \varepsilon_2)\|\mathbf{v}\| + L_i) ds\Big)^{\frac{1}{(p-1)n}} \\
&= [n(((f_i)^\infty + \varepsilon_2)\|\mathbf{v}\| + L_i)]^{\frac{1}{(p-1)n}} \Big(\int_0^1 g_i(s)) ds\Big)^{\frac{1}{(p-1)n}} \\
&= [nd_i(((f_i)^\infty + \varepsilon_2)\|\mathbf{v}\| + L_i)]^{\frac{1}{(p-1)n}} \\
&< R.
\end{aligned}$$

This indicates that $\mu\|v_i\|_0 < R$, and then we have $\mu\|\mathbf{v}\| < R$. So we derive $\mu R < R$, which shows that $\mu < 1$. This conflicts with $\mu \geq 1$. So (3.30) is correct.

By (b) of Lemma 3.1, it so yields from (3.26) and (3.30) that operator $\mathbf{T}$ admits a fixed point $\mathbf{v}$ in $P_R \backslash \bar{P}_l$ satisfying $l < \|\mathbf{v}\| < R$. Therefore (3.2) possesses a p-convex positive solution $\mathbf{v}$ satisfying $l < \|\mathbf{v}\| < R$. Hence Theorem 3.6 is correct.

Remark 3.3. It is not difficult to see that the technique to prove Theorem 3.6 is different from that used in Theorem 3.1. However, we can not apply this technique to prove:

Theorem 3.7. Under conditions (**H**) and (**F**), if in addition there exists $i_0 \in I_n$ such that

$$D_{ni} f^0 < 1 < D_{ni_0} (f_{i_0})_\infty,$$

then we derive:

(i) (3.2) has a p-convex positive solution $\mathbf{v} = (v_1, v_2, \ldots, v_n)$; and then

(ii) (1.2) has a nontrivial p-convex radial solution $\mathbf{u} = (u_1, u_2, \ldots, u_n)$, where

$$u_i(|x|) = -v_i(r) \text{ for } i \in I_n \text{ and } r \in \mathbb{J}.$$

Theorem 3.8. Under conditions (**H**) and (**F**), if in addition there exists $i_0 \in I_n$ so that

$$D_{ni_0}(f_{i_0})_0 > 1 \text{ and } D_{ni_0}(f_{i_0})_\infty > 1$$

and there is $b_i > 0$ so that (3.19) holds for each $i \in I_n$, then we derive:

(i) (3.2) possesses two p-convex positive solutions $\mathbf{v}^*$ and $\mathbf{v}^{**}$ in P with

$$0 < \|\mathbf{v}^*\| < \max\{b_i,\ i \in I_n\} < \|\mathbf{v}^{**}\|;$$

and then

(ii) system (1.2) possesses two nontrivial p-convex radial solutions $\mathbf{u}^*$ and $\mathbf{u}^{**}$ with

$$u_i^*(|x|) = -v_i^*(r) \ \ u_i^{**}(|x|) = -v_i^{**}(r) \text{ for } i \in I_n \text{ and } r \in \mathbb{J}.$$

Remark 3. The conclusions in Theorems 3.1-3.8 can be generalized to the system of p-k-Hessian equation

$$\begin{cases} \sigma_k(\lambda(D_i(|Du_{\mathbb{I}}|^{p-2} D_j u_{\mathbb{I}}))) = h_{\mathbb{I}}(|x|) f_{\mathbb{I}}(|x|, -u_1, -u_2, \ldots, -u_n) \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega. \end{cases}$$

Here $k \in \{1, 2, \cdots, n\}$, $p \geq 2$, $h_{\mathbb{I}} \in C[0,1)$ is singular at 1 for each $\mathbb{I} \in \{1, 2, \cdots, n\}$, $f_{\mathbb{I}}$ are continuous functions, $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$. There is only a few results on problems involving p-k-Hessian operator; see Bao-Feng [45], Feng-Zhang [48], Kan-Zhang [49] and Zhang-Yang [50].

4. Conclusion

In this paper, we study the singular p-Monge-Ampère problems: equations and systems of equations. we first analyze the multiplicity of nontrivial p-convex radial solutions to a single equation involving p-Monge-Ampère . Then, we establish some new criteria of existence, nonexistence and multiplicity for nontrivial p-convex radial solutions for a singular system of p-Monge-Ampère equation.

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Use of AI tools declaration

The author declares that he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflict of interest.

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