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Research article

Characterizations of ball-covering of separable Banach space and application

Shaoqiang Shang*

College of Mathematical Sciences, Harbin Engineering University, Harbin 150001, PR China

* **Correspondence:** Email: sqshang@163.com.

Abstract: In this paper, we first prove that the space $(X, \|\cdot\|)$ is separable if and only if for every $\varepsilon \in (0,1)$, there is a dense subset G of X^* and a w^*-lower semicontinuous norm $\|\cdot\|_0$ of X^* so that (1) the norm $\|\cdot\|_0$ is Frechet differentiable at every point of G and $d_F\|x^*\|_0 \in X$ is a w^*-strongly exposed point of $B(X^{**}, \|\cdot\|_0)$ whenever $x^* \in G$; (2) $\left(1+\varepsilon^2\right)\|x^{***}\|_0 \le \|x^{***}\| \le (1+\varepsilon)\|x^{***}\|_0$ for each $x^{***} \in X^{***}$; (3) there exists $\{x_i^*\}_{i=1}^{\infty} \subset G$ such that ball-covering $\{B(x_i^*, r_i)\}_{i=1}^{\infty}$ of $(X^*, \|\cdot\|_0)$ is $(1+\varepsilon)^{-1}$-off the origin and $S(X^*, \|\cdot\|) \subset \cup_{i=1}^{\infty} B(x_i^*, r_i)$. Moreover, we also prove that if space X is weakly locally uniform convex, then the space X is separable if and only if X^* has the ball-covering property. As an application, we get that Orlicz sequence space l_M has the ball-covering property.

Keywords: Ball-covering property; strongly exposed point; separable space; Orlicz sequence space
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1. Introduction and preliminaries

Let $(X, \|\cdot\|)$ denote a real Banach space and X^* denote the dual space of X. Let $S(X)$ and $B(X)$ denote the unit sphere and unit ball of X, respectively. Let the set $B(x, r)$ denote the closed ball centered at point x and of radius $r > 0$. Let $x_n \xrightarrow{w} x$ denote that the sequence $\{x_n\}_{n=1}^{\infty}$ is weakly convergent to point x.

The geometric and topological properties of unit ball and unit sphere in Banach spaces play an important role in the geometry of Banach spaces. The geometry of Banach space can be said to be related to the unit ball and unit sphere of Banach space. Almost all geometric concepts are defined by the unit sphere, such as convexity and smoothness of Banach spaces. Not only that, many other research topics are related to the spherical representation of Banach space subsets, such as Mazur intersection property, noncompact measure and spherical topology problem. These topics have attracted the attention of many mathematicians since they were put forward. Through the tireless efforts of predecessors, many important results have been achieved in the study of these issues. These results often play an indispensable role in the in-depth study of the geometric properties of Banach spaces. It can be seen that the charm of the behavior of the ball family is amazing.

Starting with a different viewpoint, a notion of ball-covering property is introduced by Cheng[1].

Definition 1.1. (see [1]) We call that $\mathfrak{B} = \{B(x_i, r_i)\}_{i\in I}$ is a ball-covering of X if $S(X) \subset \cup_{i\in I} B(x_i, r_i)$ and $0 \notin \cup_{i\in I} B(x_i, r_i)$. Moreover, if I is a countable set, we call that X has the ball-covering property.

Definition 1.2. (see [1]) A ball-covering $\mathfrak{B} = \{B(x_i, r_i)\}_{i\in I}$ is said to be r-off the origin if $\inf_{x\in\cup\mathfrak{B}} \|x\| \geq r$.

It is easy to see that if X is separable, then X has the ball-covering property. However, if X has the ball-covering property, then X is not necessarily a separable space. In [1], Cheng proved that l^∞ has the ball-covering property, but l^∞ is not a separable spaces. In [2], Shang and Cui proved that if X is a separable space and has the Radon-Nikodym property, then X^* has the ball-covering property. As a corollary, Shang and Cui proved that if $M \in \nabla_2$, then Orlicz function space L_M has the ball-covering property. This is an example of the ball-covering property of nonseparable function space. In 2021, Shang [3] studied the ball-covering property in dual space and proved the following theorem.

Theorem 1.3. *The following statements are equivalent:*

(1) The space $(X, \|\cdot\|)$ is a separable space;

(2) for every $0 < \varepsilon < 1$, there is a norm $\|\cdot\|_1$ of X^* with $(1+\varepsilon)^{-1}\|x^*\|_1 \leq \|x^*\| \leq \|x^*\|_1$ so that $\|\cdot\|_1$ is Gâteaux differentiable on a dense subset of X^* and $(X^*, \|\cdot\|_1)$ has the ball-covering property.

In Theorem 1.3, we gave the ball-covering characteristics of separable spaces. In this paper, we further study the ball-covering characteristics of separable spaces. We first prove that a Banach space $(X, \|\cdot\|)$ is a separable space if and only if for every $\varepsilon \in (0, 1)$, there exists a dense subset G of X^* and a w^*-lower semicontinuous norm $\|\cdot\|_0$ of X^* such that

(1) the norm $\|\cdot\|_0$ is Frechet differentiable at every point of G and $d_F\|x^*\|_0 \in X$ is a w^*-strongly exposed point of $B(X^{**}, \|\cdot\|_0)$ whenever $x^* \in G$;

(2) $\left(1+\varepsilon^2\right)\|x^{***}\|_0 \leq \|x^{***}\| \leq (1+\varepsilon)\|x^{***}\|_0$ for each $x^{***} \in X^{***}$;

(3) there exists $\{x_i^*\}_{i=1}^\infty \subset G$ such that ball-covering $\{B(x_i^*, r_i)\}_{i=1}^\infty$ of $(X^*, \|\cdot\|_0)$ is $(1+\varepsilon)^{-1}$-off the origin and $S(X^*, \|\cdot\|) \subset \cup_{i=1}^\infty B(x_i^*, r_i)$.

Compared with Theorem 1.3, the result has the following two advantages

(1) The norm constructed in this result has better differentiability and better geometric properties than the norm constructed in Theorem 1.3;

(2) the closed ball sequence constructed in this result can cover the unit sphere of the original norm. Moreover, we also prove that if the space X is a weakly locally uniform convex space, then the necessary and sufficient condition that X is a separable space is that for any $\alpha \in (0, 1)$, there is a sequence $\{x_i^*\}_{i=1}^\infty \subset X^*$ so that

(1) the ball-covering $\{B(x_i^*, r_i)\}_{i=1}^\infty$ of X^* is α-off the origin;

(2) the norm of X^* is Gâteaux differentiable at every point of $\{x_i^*\}_{i=1}^\infty$;

(3) the point $d\|x_i^*\|$ belongs to X for each $i \in N$.

As an application, we obtain that Orlicz sequence space l_M has the ball-covering property. Other studies on ball covering properties can be found in [4–12]. First let us recall some definitions and lemmas that will be used in the further part of this paper.

Definition 1.4. (see [13]) Suppose that D is an open subset of Banach space X, a continuous function f is called Gâteaux (Frechet) differentiable at $x \in D$ if there exists a functional $d_G f(x) \in X^*$ ($d_F f(x) \in X^*$) such that

$$\lim_{t\to 0}\left[\frac{f(x+ty)-f(x)}{t} - \langle d_G f(x), y\rangle\right] = 0$$

$$\left(\lim_{t\to 0}\sup_{y\in B(X)}\left[\frac{f(x+ty)-f(x)}{t}-\langle d_F f(x), y\rangle\right]=0\right).$$

Definition 1.5. (see [13]) A Banach space X is said to be a Gâteaux differentiability space if for every continuous convex function f is Gâteaux differentiable on a dense subset of X.

Definition 1.6. (see [14]) A Banach space is said to be smooth if every point of $S(X)$ is Gâteaux differentiable point of norm.

Definition 1.7. (see [14]) A Banach space X is said to be locally uniform convex if for every $x \in X$ and $\{x_n\}_{n=1}^{\infty} \subset S(X)$ with $\|x_n + x\| \to 2$ as $n \to \infty$ we have $\|x_n - x\| \to 0$ as $n \to \infty$.

Definition 1.8. (see [14]) We call that X is a weakly local uniform convex space if for every $x \in X$ and $\{x_n\}_{n=1}^{\infty} \subset S(X)$ with $\|x_n + x\| \to 2$ as $n \to \infty$ we have $x_n \xrightarrow{w} x$ as $n \to \infty$.

It is easy to see that if X is a locally uniform convex space, then X is a weakly locally uniform convex space. We also know that if X is a weakly locally uniform convex space, then X is not necessarily a locally uniform convex space. Moreover, separable spaces have the norm of equivalent local uniform convexity.

Definition 1.9. (see [9]) Let C^* be a subset of X^*. A point $x_0^* \in C^*$ is called a w^*-strongly exposed point of C^* if there is a point $x_0 \in S(X)$ so that if $\{x_n^*\}_{n=1}^{\infty} \subset C^*$ and $x_n^*(x_0) \to \sup_{x^*\in C^*} x^*(x_0)$, then $\|x_n^* - x_0^*\| \to 0$.

Definition 1.10. (see [14]) Let C be a subset of X. A point $x_0 \in C$ is called a strongly exposed point of C if there is a point $x_0^* \in S(X^*)$ so that if $\{x_n\}_{n=1}^{\infty} \subset C$ and $x_0^*(x_n) \to \sup_{x\in C} x_0^*(x)$, then $\|x_n - x_0\| \to 0$.

Lemma 1.11. *(see [9]) Suppose that C^* is a bounded subset of X^*. Then σ_{C^*} is Frechet differentiable at point x_0 and $d\sigma_{C^*}(x_0) = x_0^*$ if and only if the point x_0^* is a w^*-strongly exposed point of C^* and exposed by x_0.*

Lemma 1.12. *(see [15]) Suppose that C is a bounded closed set in X and $C^{**} = \overline{C}^{w^*}$. If $x \in C$ is a strongly exposed point of C and strongly exposed by $x^* \in X^*$, then x is a w^*-strongly exposed point of C^{**} and w^*-strongly exposed by x^*.*

2. A Characterization of ball-covering on separable space

Theorem 2.1. *The space $(X, \|\cdot\|)$ is separable iff for every $\varepsilon \in (0, 1)$, there is a dense subset G of X^* and a w^*-lower semicontinuous norm $\|\cdot\|_0$ of X^* so that*

(1) the norm $\|\cdot\|_0$ is Frechet differentiable at every point of G and $d_F\|x^*\|_0 \in X$ is a w^*-strongly exposed point of $B(X^{**}, \|\cdot\|_0)$ whenever $x^* \in G$;

(2) $\left(1+\varepsilon^2\right)\|x^{***}\|_0 \le \|x^{***}\| \le (1+\varepsilon)\|x^{***}\|_0$ for each $x^{***} \in X^{***}$;

(3) there exists $\{x_i^*\}_{i=1}^{\infty} \subset G$ such that ball-covering $\{B(x_i^*, r_i)\}_{i=1}^{\infty}$ of $(X^*, \|\cdot\|_0)$ is $(1+\varepsilon)^{-1}$-off the origin and $S(X^*, \|\cdot\|) \subset \cup_{i=1}^{\infty} B(x_i^*, r_i)$.

Proof. Necessity. (a) We first prove the condition (2). Since the space $(X, \|\cdot\|)$ is a separable Banach space, there exists an equivalent norm $\|\cdot\|_1$ such that the space $(X, \|\cdot\|_1)$ is a locally uniformly convex space. This implies that $\|\cdot\|$ and $\|\cdot\|_1$ of X^* are two equivalent norms. Since the norms $\|\cdot\|$ and $\|\cdot\|_1$ of X^* are two equivalent norms, there exists a real number $a \in (1, +\infty)$ such that

$$\frac{1}{a}\|x^*\| \le \|x^*\|_1 \le a\|x^*\| \qquad \text{for each} \qquad x^* \in X^*. \tag{2.1}$$

We pick a real number $\varepsilon \in (0,1)$. Then we get that $\eta = \varepsilon/a \in (0,+\infty)$. Hence we define the symmetric bounded convex set

$$C_0^* = \{x^* \in X^* : \|x^*\| \le 1\} + \left\{x^* \in X^* : \|x^*\|_1 \le \frac{1}{2}\eta\right\}.$$

Since $\{x^* \in X^* : \|x^*\| \le 1\}$ and $\{x^* \in X^* : \|x^*\|_1 \le \eta/2\}$ are two weak* compact convex sets, we obtain that C_0^* is weak* compact. This implies that C_0^* is a weak* bounded closed convex subset of X^*. Define the norm

$$\|x^*\|_0 = \mu_{C_0^*}(x^*) = \inf\left\{\lambda \in R : \frac{1}{\lambda}x^* \in C_0^*\right\} \tag{2.2}$$

in X^*. Then $\|\cdot\|_0$ is a w^*-lower semicontinuous norm of X^* and $\|x^*\|_0 < \|x^*\|$ for every $x^* \in X^*\backslash\{0\}$. Pick a point $x_0^* \in X^*$ such that $\|x_0^*\| = 1$. Then, by the formula $\left\|x_0^*\right\|_0 < \left\|x_0^*\right\| = 1$, there exists $\lambda_0 \in (0,+\infty)$ such that $\|(1+\lambda_0)x_0^*\|_0 = 1$. We claim that $\|\lambda_0 x_0^*\|_1 \ge \eta/2$. In fact, suppose that $\|\lambda_0 x_0^*\|_1 < \eta/2$. Then we get that $\lambda_0 x_0^* \in \operatorname{int}\{x^* \in X^* : \|x^*\|_1 \le \eta/2\}$. Therefore, by the formula $\|x_0^*\| = 1$ and the definition of C_0^*, we get that $(1+\lambda_0)x_0^* \in \operatorname{int} C_0^*$. Therefore, by the formula (2.2), we have $\|(1+\lambda_0)x_0^*\|_0 < 1$, which contradicts $\|(1+\lambda_0)x_0^*\|_0 = 1$. Moreover, we can assume without loss of generality that $2\varepsilon < 1/a^2$. Then, by the formulas $\|\lambda_0 x_0^*\|_1 \ge \eta/2$ and $\|x_0^*\| = 1$, we get that

$$\lambda_0 \ge \frac{1}{2}\cdot\frac{\eta}{\|x_0^*\|_1} \ge \frac{1}{2}\cdot\frac{\eta}{a\|x_0^*\|} \ge \frac{1}{2}\cdot\frac{\eta}{a} = \frac{1}{2}\cdot\frac{\varepsilon}{a^2} > \varepsilon^2.$$

Therefore, by the above inequalities and the formula $\|(1+\lambda_0)x_0^*\|_0 = 1$, we have the following inequalities

$$\|x_0^*\|_0 = \frac{1}{1+\lambda_0} \le \frac{1}{1+\varepsilon^2} = \frac{1}{1+\varepsilon^2}\|x_0^*\|.$$

Therefore, from the above inequalities, we have the following inequalities

$$\|x^*\| \ge \left(1+\varepsilon^2\right)\|x^*\|_0 \qquad \text{for each} \qquad x^* \in X^*. \tag{2.3}$$

On the other hand, we define the two norms

$$\|x^{**}\|_0 = \sup\{x^{**}(x^*) : x^* \in C_0^*\} \qquad \text{for each} \qquad x^{**} \in X^{**}$$

and

$$\|x^{**}\| = \sup\{x^{**}(x^*) : x^* \in \{x^* \in X^* : \|x^*\| \le 1\}\} \qquad \text{for each} \qquad x^{**} \in X^{**}.$$

Then $(X^{**}, \|\cdot\|_0)$ is the dual space of $(X^*, \|\cdot\|_0)$ and $(X^{**}, \|\cdot\|)$ is the dual space of $(X^*, \|\cdot\|)$. Hence, we get that $\|x^{**}\|_0 \ge \left(1+\varepsilon^2\right)\|x^{**}\|$ for every $x^{**} \in X^{**}$. Pick a point $x_0^{**} \in X^{**}$ such that the point x_0^{**} is norm attainable on sphere $S(X^*, \|\cdot\|_0)$. Hence exists a point $x_0^* \in C_0^*$ such that $x_0^{**}(x_0^*) = \left\|x_0^{**}\right\|_0$. Then, by the definition of C_0^* and the formula $x_0^* \in C_0^*$, there exist two points

$$y_0^* \in \{x^* \in X^* : \|x^*\| \le 1\} \qquad \text{and} \qquad z_0^* \in \left\{x^* \in X^* : \|x^*\|_1 \le \frac{1}{2}\eta\right\}$$

such that $x_0^* = y_0^* + z_0^*$. Therefore, by the formula $x_0^* = y_0^* + z_0^*$ and the formula $\|x^{**}\|_1 \le a\|x^{**}\|$, we have the following inequalities

$$\left\|x_0^{**}\right\|_0 = x_0^{**}(x_0^*) = \langle x_0^{**}, y_0^* + z_0^*\rangle$$

$$\leq \sup\{x_0^{**}(x^*) : x^* \in \{x^* \in X^* : \|x^*\| \leq 1\}\}$$
$$+\sup\left\{x_0^{**}(x^*) : x^* \in \left\{x^* \in X^* : \|x^*\|_1 \leq \frac{1}{2}\eta\right\}\right\}$$
$$\leq \sup\{x_0^{**}(x^*) : x^* \in \{x^* \in X^* : \|x^*\| \leq 1\}\} + \frac{1}{2}\eta\left\|x_0^{**}\right\|_1$$
$$\leq \left\|x_0^{**}\right\| + \frac{1}{2}a\eta\left\|x_0^{**}\right\|.$$

Therefore, by formula (2.3) and the above inequalities, we get that

$$\left(1+\varepsilon^2\right)\left\|x_0^{**}\right\| \leq \left\|x_0^{**}\right\|_0 \leq \left\|x_0^{**}\right\| + \frac{1}{2}a\eta\left\|x_0^{**}\right\| = \left(1+\frac{1}{2}a\eta\right)\left\|x_0^{**}\right\|.$$

Therefore, by the Bishop-Phelps Theorem, we have the following inequalities

$$\left(1+\varepsilon^2\right)\|x^{**}\| \leq \|x^{**}\|_0 \leq \|x^{**}\| + \frac{1}{2}a\eta\|x^{**}\| = \left(1+\frac{1}{2}a\eta\right)\|x^{**}\|$$

for every $x^{**} \in X^{**}$. Therefore, by the above inequalities, we obtain that

$$\left(1+\varepsilon^2\right)\|x^{***}\|_0 \leq \|x^{***}\| \leq \left(1+\frac{1}{2}a\eta\right)\|x^{***}\|_0$$

for every $x^{***} \in X^{***}$. Therefore, by the formula $\eta = \varepsilon/a$, we obtain that

$$\left(1+\varepsilon^2\right)\|x^{***}\|_0 \leq \|x^{***}\| \leq \left(1+\frac{1}{2}\varepsilon\right)\|x^{***}\|_0 \leq (1+\varepsilon)\|x^{***}\|_0$$

for each $x^{***} \in X^{***}$. Moreover, it is easy to see that the norm $\|\cdot\|_0$ is a w^*-lower semicontinuous norm of X^{***}. Hence we get that the condition (2) is true.

(b) We will prove that there exists a dense subset G of X^* such that the norm $\|\cdot\|_0$ is Frechet differentiable on G and $d_F\|\cdot\|_0(G) \subset X$. Define the set

$$C_0 = \{x \in X : x^*(x) \leq 1,\ x^* \in C_0^*\}. \tag{2.4}$$

Then we get that C_0 is a nonempty bounded closed convex set. Since the set C_0^* is a weak* bounded closed convex subset of X^*, we get that

$$C_0^* = \{x^* \in X^* : x^*(x) \leq 1,\ x \in C_0\}.$$

Then, using the Bishop-Phelps Theorem, we get that $\overline{A_0^*} = X^*$, where

$$A_0^* = \{x^* \in X^* : \text{there exists a point } x \in C_0 \text{ such that } x^*(x) = \|x^*\|_0\}.$$

Therefore, from Theorem 2.1 of [6], we obtain the following formulas $D_0^* \neq \emptyset$ and $\overline{D_0^*} = \{x^* \in X^* : \|x^*\|_0 = 1\}$, where

$$D_0^* = \{x^* \in X^* : \text{there exists a point } x \in C_0 \text{ such that } x^*(x) = \|x^*\|_0 = 1\}.$$

Let $G = A_0^*$. Then we get that G is a dense subset of X^*. We pick a point

$$y_0^* \in \{x^* \in X^* : \text{there exists a point } x \in C_0 \text{ such that } x^*(x) = \|x^*\|_0 = 1\}.$$

Hence, we get that there exists a point $x_0 \in C_0$ such that $\langle y_0^*, x_0\rangle = \sup\{\langle z^*, x_0\rangle : z^* \in C_0^*\}$. Therefore, by the definition of y_0^*, we get that $y_0^* \in C_0^*$. Moreover, by the definition of C_0^*, there exist two point

$$u_0^* \in \{x^* \in X^* : \|x^*\| \leq 1\} \qquad \text{and} \qquad v_0^* \in \left\{x^* \in X^* : \|x^*\|_1 \leq \frac{1}{2}\eta\right\}$$

such that $y_0^* = u_0^* + v_0^*$. Hence we have the following inequalities

$$\begin{aligned}
\langle y_0^*, x_0\rangle &= \sup\{\langle z^*, x_0\rangle : z^* \in C_0^*\} \\
&= \sup\left\{\langle u^* + v^*, x_0\rangle : \|u^*\| \leq 1,\ \|v^*\|_1 \leq \frac{1}{2}\eta\right\} \\
&= \sup\left\{\langle v^*, x_0\rangle : \|v^*\|_1 \leq \frac{1}{2}\eta\right\} + \sup\{\langle u^*, x_0\rangle : \|u^*\| \leq 1\} \\
&\geq \sup\{\langle u^*, x_0\rangle : \|u^*\| \leq 1\} + \langle v_0^*, x_0\rangle \\
&= \langle u_0^*, x_0\rangle + \langle v_0^*, x_0\rangle = \langle y_0^*, x_0\rangle.
\end{aligned}$$

Therefore, by the above inequalities, we have the following formula

$$u_0^*(x_0) = \sup\{\langle u^*, x_0\rangle : \|u^*\| \leq 1\} \qquad \text{and} \qquad v_0^*(x_0) = \sup\left\{\langle v^*, x_0\rangle : \|v^*\|_1 \leq \frac{1}{2}\eta\right\}.$$

Therefore, by the formula $v_0^* \in \{x^* \in X^* : \|x^*\|_1 \leq \eta/2\}$, we obtain that the point v_0^* is norm attainable on set $\{x \in X : \|x\|_1 \leq 1\}$.

We next prove that the point y_0^* is a Frechet differentiable point of norm $\|\cdot\|_0$ in X^{***}, i.e, the point $d_F\|y_0^*\|$ is a w^*-strongly exposed point of $B(X^{**}, \|\cdot\|_0)$. Define the closed convex set

$$C_0^{**} = \{x^{**} \in X^{**} : x^{**}(x^*) \leq 1,\ x^* \in C_0^*\}.$$

It is well known that $y_0^*(x_0) = \|y_0^*\|_0 = \|x_0\|_0 = 1$. Pick a sequence $\{x_n\}_{n=1}^{\infty} \subset C_0$ such that $x_n(y_0^*) \to 1$ as $n \to \infty$. Then, from the definition of C_0, we have the following equations

$$\lim_{n\to\infty} x_n(y_0^*) = x_0(y_0^*) = 1 = \sup\{x_0(x^*) : x^* \in C_0^*\}. \tag{2.5}$$

Since the set C_0^* is a bounded set and $\text{int}C_0^* \neq \emptyset$, we obtain that C_0 is a bounded subset of X. Therefore, by the formula $\{x_n\}_{n=1}^{\infty} \subset C_0$, we obtain that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence. Hence we can assume that without loss of generality that $\{x_n(u_0^*)\}_{n=1}^{\infty}$ is a Cauchy sequence. Moreover, by the formula $y_0^* = u_0^* + v_0^*$, we get the following formula

$$y_0^* \in u_0^* + \left\{x^* \in X^* : \|x^*\|_1 \leq \frac{1}{2}\eta\right\} = u_0^* + B\left(0, \frac{1}{2}\eta\right) \subset C_0^*. \tag{2.6}$$

Since $\{x_n\}_{n=1}^{\infty} \subset C_0$, by the formulas (2.5), we have the following inequalities

$$\lim_{n\to\infty} x_n(y_0^*) = 1 \geq \limsup_{n\to\infty}\left(\sup\{x_n(x^*) : x^* \in C_0^*\}\right).$$

Therefore, from the above inequalities and formula (2.6), we obtain the following inequalities

$$\begin{aligned}\lim_{n\to\infty} x_n(y_0^*) &\geq \limsup_{n\to\infty} \left(\sup\{x_n(x^*) : x^* \in C_0^*\}\right)\\ &\geq \limsup_{n\to\infty}\left(\sup\left\{x_n(x^*) : x^* \in u_0^* + B\left(0, \frac{1}{2}\eta\right)\right\}\right)\\ &= \lim_{n\to\infty} x_n(u_0^*) + \limsup_{n\to\infty}\left(\sup\left\{x_n(x^*) : x^* \in B\left(0, \frac{1}{2}\eta\right)\right\}\right).\end{aligned}$$

Therefore, by the above inequalities, we have the following inequalities

$$\lim_{n\to\infty} \langle x_n, y_0^* - u_0^*\rangle \geq \limsup_{n\to\infty}\left(\sup\left\{x_n(x^*) : x^* \in B\left(0, \frac{1}{2}\eta\right)\right\}\right).$$

Therefore, by the formulas $y_0^* = u_0^* + v_0^*$ and $v_0^* \in B(0, \eta/2)$, we obtain that

$$\lim_{n\to\infty} \langle x_n, y_0^* - u_0^*\rangle = \limsup_{n\to\infty}\left(\sup\left\{x_n(x^*) : x^* \in B\left(0, \frac{1}{2}\eta\right)\right\}\right). \tag{2.7}$$

Since the sequence $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence, we may assume without loss of generality that $\{\|x_n\|\}_{n=1}^{\infty}$ is a Cauchy sequence. This implies that

$$\lim_{n\to\infty} \langle x_n, y_0^* - u_0^*\rangle = \limsup_{n\to\infty}\left(\sup\left\{x_n(x^*) : x^* \in B\left(0, \frac{\eta}{2}\right)\right\}\right) = \lim_{n\to\infty} \frac{\eta}{2}\|x_n\|_1 .$$

Moreover, there exists a sequence $\{k_n\}_{n=1}^{\infty} \subset R^+$ such that $\|k_n x_n\| = \|x_0\|$ for all $n \in N$. Hence we have the following equations

$$\lim_{n\to\infty} \langle k_n x_n, v_0^*\rangle = \lim_{n\to\infty} \langle k_n x_n, y_0^* - u_0^*\rangle = \lim_{n\to\infty} \frac{1}{2}\eta\,\|k_n x_n\|_1 = \frac{1}{2}\eta\,\|x_0\|_1 . \tag{2.8}$$

Therefore, by the formula $\left\langle x_0, v_0^*\right\rangle = (\eta\,\|x_0\|_1)/2$ and the formula (2.8), we obtain the following equations

$$\lim_{n\to\infty} \langle k_n x_n, v_0^*\rangle + \langle x_0, v_0^*\rangle = \frac{1}{2}\eta\,\|x_0\|_1 + \frac{1}{2}\eta\,\|x_0\|_1 = \eta\,\|x_0\|_1 .$$

Therefore, by the above equations, we have the following equation

$$\lim_{n\to\infty}\left\langle \frac{1}{\|x_0\|_1} k_n x_n + \frac{1}{\|x_0\|_1} x_0, \frac{2}{\eta} v_0^*\right\rangle = 2.$$

Moreover, since $\|k_n x_n\|_1 = \|x_0\|_1$ and $\|v_0^*\|_1 = \eta/2$, by the above equation, we get the following equations

$$2 \geq \liminf_{n\to\infty}\left\|\frac{1}{\|x_0\|_1} k_n x_n + \frac{1}{\|x_0\|_1} x_0\right\|_1 = \lim_{n\to\infty}\left\langle \frac{1}{\|x_0\|_1} k_n x_n + \frac{1}{\|x_0\|_1} x_0, \frac{2}{\eta} v_0^*\right\rangle = 2.$$

Since the space $(X, \|\cdot\|_1)$ is a locally uniformly convex space, we get that $k_n x_n \to x_0$ as $n \to \infty$. Moreover, by the formula $x_n(y_0^*) \to x_0(y_0^*)$, we obtain that $k_n \to 1$ as $n \to \infty$. This implies that $\|x_n - x_0\| \to 0$ as $n \to \infty$. Hence the point $d_F\|y_0^*\|_0 \in X$ is a strongly exposed point of $B(X, \|\cdot\|_0)$. Therefore, by Lemma

1.12, we get that $d_F\|y_0^*\|_0 \in X$ is a w^*-strongly exposed point of $B(X^{**}, \|\cdot\|_0)$. Then, by the definition of G, we get that the norm $\|\cdot\|_0$ is Frechet differentiable at every point of G and the point $d_F\|x^*\|_0 \in X$ is a w^*-strongly exposed point of $B(X^{**}, \|\cdot\|_0)$ whenever $x^* \in G$.

(c) Let $p_0(x^{***}) = \|x^{***}\|_0$ for each $x^{***} \in X^{***}$. Then, by the proof of (b), we know that if x^* is norm attainable on $S(X, \|\cdot\|_0)$, then x^* is a Frechet differentiable point of p_0 and $d_F p_0(x^*) \in X$. Then we get that

$$S(X, \|\cdot\|_0) = \{d_F p_0(x^*) \in X : \text{there is a point } x \in C_0 \text{ so that } x^*(x) = p_0(x^*)\}.$$

Since the space X is separable, we get that every subset of X is separable. Let

$$A_0 = \{x \in B(X, \|\cdot\|_0) : x \text{ is a strongly exposed point of } B(X, \|\cdot\|_0)\}.$$

Then there exists a sequence $\{x_n\}_{n=1}^\infty \subset A_0$ such that $\overline{\{x_n\}_{n=1}^\infty} = A_0$. Therefore, by the formula $\overline{\{x_n\}_{n=1}^\infty} = A_0$, we have the following equations

$$\inf_{x^* \in S(X^*, \|\cdot\|_0)} \sup_{n \in N} \langle x^*, x_n \rangle = \inf_{x^* \in S(X^*, \|\cdot\|_0)} \sup\{\langle x^*, x \rangle : x \in B(X, \|\cdot\|_0)\} = 1. \tag{2.9}$$

Pick a point $x_0^* \in X^*$ such that $\|x_0^*\| = 1$. Then, from the proof of (a), we obtain that $\|x_0^*\|_0 \le \|x_0^*\| \le (1+\varepsilon/2)\|x_0^*\|_0$. Then we get that $(1+\varepsilon/2)^{-1} \le \|x_0^*\|_0$. This implies that $\|(1+\varepsilon/2)x_0^*\|_0 \ge 1$. Define the set

$$W_0 = \bigcup_{x^* \in S(X^*, \|\cdot\|)} \left\{\lambda x^* + (1-\lambda)\left(1+\frac{\varepsilon}{2}\right)x^* \in X^* : \lambda \in [0,1]\right\}. \tag{2.10}$$

Then we get that $S(X^*, \|\cdot\|) \subset W_0$ and $S(X^*, \|\cdot\|_0) \subset W_0$. Pick a point $z^* \in X^*$ such that $\|z^*\| = 1$. Then $\|z^*\|_0 \ge (1+\varepsilon/2)^{-1}$. Therefore, by the formulas (2.9) and (2.10), we have the following inequalities

$$\begin{aligned}
\sup_{n\in N}\left\langle \lambda z^* + (1-\lambda)\left(1+\frac{\varepsilon}{2}\right)z^*, x_n\right\rangle &= \sup_{n\in N}\left[\lambda + (1-\lambda)\left(1+\frac{\varepsilon}{2}\right)\right]\cdot\langle z^*, x_n\rangle \\
&\ge \left[\lambda + (1-\lambda)\left(1+\frac{\varepsilon}{2}\right)\right]\left(1+\frac{\varepsilon}{2}\right)^{-1} \\
&\ge \lambda\left(1+\frac{\varepsilon}{2}\right)^{-1} + (1-\lambda) \ge \left(1+\frac{\varepsilon}{2}\right)^{-1}.
\end{aligned}$$

Therefore, by the above inequalities and the definition of W_0, we obtain that

$$\inf_{x^* \in W_0}\left(\sup_{n \in N}\langle x^*, x_n\rangle\right) \ge \left(1+\frac{\varepsilon}{2}\right)^{-1}. \tag{2.11}$$

Therefore, from the proof of (b) and the definition of A_0, there exists $\{x_n^*\}_{n=1}^\infty \subset S(X^*)$ so that $\|x_n^*\|_0 = 1$ and $x_n = d_F p_0(x_n^*)$ for every $n \in N$. Hence we have $\{x_n^*\}_{n=1}^\infty \subset G$. Moreover, we can assume without loss of generality that $(1+\varepsilon) < 2$. Define the closed ball sequences

$$B_{i,m}^0 = B\left(\left(m + \frac{1}{1+\varepsilon}\right)x_i^*, -\frac{1}{m} + m\right), \quad m = 3, 4, 5, \ldots \qquad i = 1, 2, 3, \ldots$$

in $(X^*, \|\cdot\|_0)$. Then, by the formula $\|x_i^*\|_0 = 1$, we get that if $y^* \in B_{i,m}^0$, then

$$\|y^*\|_0 \ \ge \ \left\|\left(m+\frac{1}{1+\varepsilon}\right)x_i^*\right\|_0 - \left\|\left(m+\frac{1}{1+\varepsilon}\right)x_i^* - y^*\right\|_0$$

$$\begin{aligned} &\geq \left\| \left(m + \frac{1}{1+\varepsilon}\right) x_i^* \right\|_0 - m \left\|x_i^*\right\|_0 + \frac{1}{m} \\ &= m \left\|x_i^*\right\|_0 + \frac{1}{1+\varepsilon} \left\|x_i^*\right\|_0 - m \left\|x_i^*\right\|_0 + \frac{1}{m} \geq \frac{1}{1+\varepsilon}. \end{aligned}$$

Hence we get that $B_{i,m}^0$ has a positive distance $1/(1+\varepsilon)$ from the origin. Suppose that there exists a point $y^* \in W_0$ such that for every $i \in N$ and $m \in N$, we have $y^* \notin B_{i,m}^0$. Moreover, by the formula (2.11), there exists a natural number $n_0 \in N$ such that $\langle y^*, x_{n_0} \rangle = \eta > 1/(1+\varepsilon)$. Define the hyperplane

$$H_{n_0} = \{x^* \in X^* : \langle x^*, x_{n_0} \rangle = 0\}$$

in X^*. Hence there exists a point $h_{n_0}^* \in H_{n_0}$ such that $y^* = \eta x_{n_0}^* + h_{n_0}^*$. Therefore, by the formulas $y^* \notin B_{i,m}^0$ and $y^* = \eta x_{n_0}^* + h_{n_0}^*$, we get that

$$m - \frac{1}{m} \leq \left\| \left(m + \frac{1}{1+\varepsilon}\right) x_{n_0}^* - y^* \right\|_0 = \left\| \left(m + \frac{1}{1+\varepsilon} - \eta\right) x_{n_0}^* - h_{n_0}^* \right\|_0 .$$

Therefore, by the above inequalities, we have the following inequalities

$$\begin{aligned} -\frac{1}{m} &\leq \left\| \left(m + \frac{1}{1+\varepsilon} - \eta\right) x_{n_0}^* - h_{n_0}^* \right\|_0 - m \left\|x_{n_0}^*\right\|_0 \\ &\leq \left\|(m-\eta) x_{n_0}^* - h_{n_0}^*\right\|_0 - m \left\|x_{n_0}^*\right\|_0 + \frac{1}{1+\varepsilon} \left\|x_{n_0}^*\right\|_0 \\ &\leq (m-\eta)\left[\left\|x_{n_0}^* - \frac{1}{m-\eta} h_{n_0}^*\right\|_0 - \left\|x_{n_0}^*\right\|_0\right] - \eta + \frac{1}{1+\varepsilon} \\ &\leq \frac{1}{t}\left[\left\|x_{n_0}^* - t h_{n_0}^*\right\|_0 - \left\|x_{n_0}^*\right\|_0\right] - \eta + \frac{1}{1+\varepsilon}, \end{aligned}$$

where $t = 1/(m-\eta)$. Moreover, since the point $x_{n_0}^*$ is a Gâteaux differentiable point of norm $\|\cdot\|_0$ in X^*, form the above inequalities and the formula $h_{n_0}^* \in H_{n_0}$, we have the following inequalities

$$\begin{aligned} 0 &\leq \lim_{t \to 0}\left[\frac{1}{t}\left[\left\|x_{n_0}^* - t h_{n_0}^*\right\|_0 - \left\|x_{n_0}^*\right\|_0\right] - \eta + \frac{1}{1+\varepsilon}\right] \\ &= \left\langle h_{n_0}^*, x_{n_0} \right\rangle - \eta + \frac{1}{1+\varepsilon} = -\eta + \frac{1}{1+\varepsilon} < 0, \end{aligned}$$

this is a contradiction. This implies that

$$W_0 \subset \bigcup \left\{B_{i,m}^0 : m = 3, 4, 5, ... \qquad i = 1, 2, 3, ...\right\}.$$

Hence, there exists a sequence $\{x_i^*\}_{i=1}^{\infty} \subset G$ such that ball-covering $\{B(x_i^*, r_i)\}_{i=1}^{\infty}$ of $(X^*, \|\cdot\|_0)$ is $(1+\varepsilon)^{-1}$-off the origin and $S(X^*, \|\cdot\|) \subset \cup_{i=1}^{\infty} B(x_i^*, r_i)$.

Sufficiency. Since the space $(X^*, \|\cdot\|_0)$ has the ball-covering property, we get that the space X is separable, which finishes the proof.

Next, we will study what conditions can guarantee that the dual space of a separable space has the ball-covering property.

Theorem 2.2. *Suppose that the space X is weakly locally uniform convex. Then X is a separable space if and only if for every $\alpha \in (0, 1)$, there exists a sequence $\{x_i^*\}_{i=1}^{\infty}$ such that*

(1) the ball-covering $\{B(x_i^*, r_i)\}_{i=1}^{\infty}$ of X^* is α-off the origin;

(2) the norm of X^* is Gâteaux differentiable at every point of $\{x_i^*\}_{i=1}^{\infty}$;

(3) the point $d_G\|x_i^*\|$ belongs to X for each $i \in N$.

Proof. Necessity. (a) We first will prove that the norm of X^* is Gâteaux differentiable on a dense subset of $S(X^*)$. In fact, by the Bishop-Phelps Theorem, we get that $A_0^* \subset S(X^*)$ and $\overline{A_0^*} = S(X^*)$, where

$$A_0^* = \{x^* \in S(X^*) : \text{there is a point } x \in S(X) \text{ so that } x^*(x) = 1 = \|x^*\|\}.$$

We claim that every point of A_0^* is a Gâteaux differentiable point of X^*. In fact, pick a point $x_0^* \in A_0^*$. Since the space X is weakly locally uniform convex, there exists an unique point $x_0 \in S(X)$ such that $x_0^*(x_0) = 1$. Suppose that there exists a functional $x_0^{**} \in S(X^{**})$ such that $x_0^{**}(x_0^*) = 1$ and $x_0^{**} \neq x_0$. Then there exists a weak* neighbourhood V of origin in X^{**}. such that

$$(x_0^{**} + V) \cap (x_0 + V) = \emptyset. \tag{2.12}$$

Moreover, for every natural number $n \in N$, we define the weak* neighbourhood

$$V_n = \left\{ z^{**} \in X^{**} : \left| z^{**}(x_0^*) - x_0^{**}(x_0^*) \right| < \frac{1}{n} \right\} \tag{2.13}$$

of x_0^{**} in X^{**}. Therefore, from the Goldstine Theorem, there is a point $x_n \in B(X)$ such that $x_n \in \left(x_0^{**} + V\right) \cap V_n$ for all $n \in N$. Hence we have $x_0^*(x_n) \to 1$ as $n \to \infty$. Therefore, by the formula $x_0^*(x_0) = 1$, we have

$$2 \geq \limsup_{n\to\infty} \|x_n + x_0\| \geq \limsup_{n\to\infty} \left|\langle x_0^*, x_n + x_0\rangle\right| = \lim_{n\to\infty} \langle x_0^*, x_n + x_0\rangle = 2.$$

Since the space X is weakly locally uniform convex, we obtain that $x_n \xrightarrow{w} x_0$ as $n \to \infty$. Since V is a weak* neighbourhood of origin in X^{**}, we can assume that $x_n \in x_0 + V$. However $x_n \in \left(x_0^{**} + V\right) \cap V_n$ for each $n \in N$, which contradicts the formula (2.12). This implies that every point of A_0^* is Gâteaux differentiable point of X^*.

(b) Let $\alpha \in (0, 1)$. Pick $\varepsilon \in (0, 1)$. Moreover, since the space X is a separable space, there exists a sequence $\{x_n\}_{n=1}^{\infty} \subset S(X)$ such that $\overline{\{x_n\}_{n=1}^{\infty}} = S(X)$. Then we have the following equations

$$\inf_{x^* \in S(X^*)} \sup_{n \in N} \langle x^*, x_n\rangle = \inf_{x^* \in S(X^*)} \sup \{\langle x^*, x\rangle : x \in B(X)\} = 1. \tag{2.14}$$

Therefore, from the previous proof, we define the following the set

$$G = \{x^* \in X^* : \text{there is a point } x \in \{x_n\}_{n=1}^{\infty} \text{ so that } x^*(x) = \|x^*\|\}.$$

Therefore, from the previous proof, we obtain that the norm of X^* is Gâteaux differentiable at every point of G and every point of G is norm attainable on set $S(X)$. Hence, for every natural number $n \in N$, there exists a point $x_n^* \in G$ with $\|x_n^*\| = 1$ such that $x_n^*(x_n) = 1$. Hence we define a sequence $\{x_n^*\}_{n=1}^{\infty} \subset S(X^*)$.

Pick a point $y^* \in S(X^*)$. Then, by the formula (2.14), it is easy to see that there exists a natural number $n_0 \in N$ such that $1 \geq \langle y^*, x_{n_0} \rangle > 1/(1+2\varepsilon)$. We define the following hyperplane

$$H_{n_0} = \{x^* \in X^* : \langle x^*, x_{n_0} \rangle = 0\}$$

in X^*. Then, by formula (2.13), there exists a point $h^*_{n_0} \in H_{n_0}$ and a real number $\eta \in (0, +\infty)$ such that $y^* = \eta x^*_{n_0} + h^*_{n_0}$. Then, by the inequalities $1 \geq \langle y^*, x_{n_0} \rangle > 1/(1+2\varepsilon)$ and $y^* = \eta x^*_{n_0} + h^*_{n_0}$, we have the following inequalities

$$1 \geq \langle y^*, x_{n_0} \rangle = \eta \left\langle x^*_{n_0}, x_{n_0} \right\rangle + \left\langle h^*_{n_0}, x_{n_0} \right\rangle = \eta \left\langle x^*_{n_0}, x_{n_0} \right\rangle > \frac{1}{1+2\varepsilon}. \tag{2.15}$$

Moreover, since $\langle x^*_{n_0}, x_{n_0} \rangle = 1$, by formula (2.15) and the formula $y^* = \eta x^*_{n_0} + h^*_{n_0}$, we have the following inequalities

$$\frac{1}{1+2\varepsilon} < \frac{1}{1+2\varepsilon} \cdot \frac{1}{\left\langle x^*_{n_0}, x_{n_0} \right\rangle} < \eta < \frac{1}{\left\langle x^*_{n_0}, x_{n_0} \right\rangle} < 1+2\varepsilon. \tag{2.16}$$

Hence, for each natural numbers i and m, we define the closed ball sequences

$$B_{i,m} = B\left(\left(m + \frac{1}{1+2\varepsilon}\right) x^*_i, \ -\frac{1}{m} + m \left\| x^*_i \right\|\right), \ \ m = 3, 4, 5, ... \ \ i = 1, 2, 3, ...$$

in X^*. Suppose that $y^* \notin B_{i,m}$ for all $i \in N$ and $m \in N$. Then for every natural numbers $m \in N$, we obtain that

$$m \left\| x^*_{n_0} \right\| - \frac{1}{m} \leq \left\| \left(m + \frac{1}{1+2\varepsilon}\right) x^*_{n_0} - y^* \right\| = \left\| \left(m + \frac{1}{1+2\varepsilon} - \eta\right) x^*_{n_0} - h^*_{n_0} \right\|.$$

Therefore, by the above inequalities, we have the following inequalities

$$\begin{aligned}
-\frac{1}{m} &\leq \left\| \left(m + \frac{1}{1+2\varepsilon} - \eta\right) x^*_{n_0} + h^*_{n_0} \right\| - m \left\| x^*_{n_0} \right\| \\
&\leq (m-\eta) \left[\left\| x^*_{n_0} + \frac{1}{m-\eta} h^*_{n_0} \right\| - \left\| x^*_{n_0} \right\| \right] - \eta + \frac{1}{1+2\varepsilon} \\
&\leq \frac{1}{t} \left[\left\| x^*_{n_0} + t h^*_{n_0} \right\| - \left\| x^*_{n_0} \right\| \right] - \eta + \frac{1}{1+2\varepsilon},
\end{aligned}$$

where $t = 1/(m-\eta)$. Moreover, since the point $x^*_{n_0}$ is a Gâteaux differentiable point of norm of X^* and $\langle x^*_{n_0}, x_{n_0} \rangle = 1$, we have $d_G \|x^*_{n_0}\| = x_{n_0} \in X$. Therefore, by the formula (2.16), we have the following inequalities

$$\begin{aligned}
0 &\leq \lim_{t \to 0} \left[\frac{1}{t} \left[\left\| x^*_{n_0} + t h^*_{n_0} \right\| - \left\| x^*_{n_0} \right\| \right] - \eta + \frac{1}{1+2\varepsilon} \right], \\
&= \left\langle x_{n_0}, h^*_{n_0} \right\rangle - \eta + \frac{1}{1+2\varepsilon} = -\eta + \frac{1}{1+2\varepsilon} < 0,
\end{aligned}$$

this is a contradiction. Hence we have the following formula

$$S(X^*) \subset \bigcup \{B_{i,m} : m = 3, 4, 5, ... \qquad i = 1, 2, 3, ...\}.$$

Since $\varepsilon \in (0, 1)$, we can assume without loss of generality that $(1 + 2\varepsilon)^{-1} - \alpha > 0$. Pick a point $y^* \in B_{i,m}$. Since $(1 + 2\varepsilon)^{-1} > \alpha$, by the formula $\|x_i^*\| = 1$ and the triangle inequality, we have the following inequalities

$$\begin{aligned} \|y^*\| &\geq \left\|\left(m + \frac{1}{1+2\varepsilon}\right)x_i^*\right\| - \left\|\left(m + \frac{1}{1+2\varepsilon}\right)x_i^* - y^*\right\| \\ &\geq \left\|\left(m + \frac{1}{1+2\varepsilon}\right)x_i^*\right\| - m\left\|x_i^*\right\| + \frac{1}{m} = \frac{1}{1+2\varepsilon} + \frac{1}{m} > \alpha. \end{aligned}$$

Therefore, by the formula $x_i^* \in G$, we obtain that for every $0 < \alpha < 1$, there is a sequence $\{x_i^*\}_{i=1}^{\infty}$ of norm Gâteaux differentiable points such that the ball-covering $\{B(x_i^*, r_i)\}_{i=1}^{\infty}$ of X^* is α-off the origin. Hence, we get that the conditions (1) and (2) are true. Moreover, from the previous proof, it is that $d_G\|x_i^*\| = x_i \in X$ for every $i \in N$. The condition (3) is true.

Sufficiency. Since the space X^* has the ball-covering property, we get that the space X is separable, which finishes the proof.

Corollary 2.3. *Suppose that the space X is locally uniform convex. Then X is a separable space iff for every $\alpha \in (0, 1)$, there exists a sequence $\{x_i^*\}_{i=1}^{\infty}$ such that*

(1) the ball-covering $\{B(x_i^*, r_i)\}_{i=1}^{\infty}$ of X^* is α-off the origin;

(2) the norm of X^* is Frechet differentiable at every point of $\{x_i^*\}_{i=1}^{\infty}$;

(3) the point $d_F\|x_i^*\|$ belongs to X for each $i \in N$.

Proof. By the proof of Theorem 2.1 and Theorem 2.2, we get that Corollary 2.3 is true, which finishes the proof.

3. Application to Orlicz sequence spaces

In this section, we use the results of the ball-covering of Banach space to study the ball-covering theory of Orlicz sequence space. On the other hand, since the Orlicz sequence space is a kind of specific Banach space, we can get a more perfect conclusion on the Orlicz sequence space than the general Banach space.

Definition 3.1. (see [14]) A function $M : R \to R$ is said to be an Orlicz function if it has the following properties:

(1) M is even, continuous, convex and $M(0) = 0$;

(2) $M(u) > 0$ for all $u > 0$;

(3) $\lim_{u\to 0} M(u)/u = 0$ and $\lim_{u\to\infty} M(u)/u = \infty$.

Definition 3.2. (see [14]) Let M be an Orlicz function, p be the right derivative of M, and $q(s) = \sup\{t : p(t) \leq s\}$. Then, we call that

$$N(v) = \int_0^{|v|} q(s)\mathrm{d}s$$

is the complementary function of M.

By [14], we know that if the function M is an Orlicz function, then the complementary function N of M is an Orlicz function. Moreover, by [14], we know that the complementary function of N is M. Hence we say that M and N are complementary to each other (see [14]).

Definition 3.3. (see [14]) An Orlicz function M is said to be satisfies condition Δ_2 if there exist $K > 2$ and $u_0 \geq 0$ such that

$$M(2u) \leq KM(u) \qquad \text{whenever} \qquad u \geq u_0.$$

In this case, we write $M \in \Delta_2$ or $N \in \nabla_2$, where N is the complementary function of M.

For any sequence $x = (x(1), x(2), ...)$, we define its modular by

$$\rho_M(x) = \sum_{i=1}^{\infty} M\left(x(i)\right).$$

Then the Orlicz sequence space l_M and its subspace h_M are defined as follows:

$$l_M = \{x : \rho_M(\lambda x) < +\infty \quad \text{for} \quad \text{some} \quad \lambda > 0\},$$

$$h_M = \{x : \rho_M(\lambda x) < +\infty \quad \text{for} \quad \text{all} \quad \lambda > 0\}.$$

For each $x \in l_M$, we define the Luxemburg norm

$$\|x\| = \inf\left\{\lambda > 0 : \rho_M\left(\frac{x}{\lambda}\right) \leq 1\right\}$$

or the Orlicz norm

$$\|x\|^0 = \inf_{k>0} \frac{1}{k}\left[1 + \rho_M(kx)\right]$$

in l_M (see [14]). It is well known that l_M and l_M^0 are two Banach spaces (see [14]). Moreover, we know that h_M is a closed subspace of l_M and h_M^0 is a closed subspace of l_M^0 (see [14]). It is well known that h_M and h_M^0 are separable spaces (see [14]). Moreover, it is well known that $(h_N^0)^* = l_M$ and $(h_N)^* = l_M^0$(see [14]). It is well known that $l_M(l_M^0)$ is separable if and only if $M \in \Delta_2$ (see [14]).

Theorem 3.4. *Suppose that $M \in \Delta_2$ or $M \in \nabla_2$. Then for any $\alpha \in (0, 1)$, there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset l_M$ of norm Gâteaux differentiable points such that*

(1) the ball-covering $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of l_M is α-off the origin;

(2) the norm of l_M is Gâteaux differentiable at every point of $\{x_i\}_{i=1}^{\infty}$.

Proof. Suppose that $M \in \Delta_2$. Then the space l_M is a separable space. Hence l_M is a weak Asplund space. It is easy to see that for any $\alpha \in (0, 1)$, there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset l_M$ of norm Gâteaux differentiable points such that (1) the ball-covering $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of l_M is α-off the origin; (2) the norm of l_M is Gâteaux differentiable at every point of $\{x_i\}_{i=1}^{\infty}$.

Suppose that $M \in \nabla_2$. Then we get that $N \in \Delta_2$. This implies that h_N^0 has the Radon-Nikodym property. Therefore, by $(h_N^0)^* = l_M$, we get that the norm of l_M are Gâteaux differentiable on a dense subset of X^*. Hence, the norm of l_M are Gâteaux differentiable on a dense subset of $S(X^*)$. Therefore, from the proof of Theorem 2.2, we get that for every $\alpha \in (0, 1)$, there exists a sequence $\{x_i^*\}_{i=1}^{\infty}$ such that (1) the ball-covering $\{B(x_i^*, r_i)\}_{i=1}^{\infty}$ of X^* is α-off the origin; (2) the norm of X^* is Gâteaux differentiable at every point of $\{x_i^*\}_{i=1}^{\infty}$, which finishes the proof.

Theorem 3.5. *Suppose that $M \in \Delta_2$ or $M \in \nabla_2$. Then for any $\alpha \in (0,1)$, there exists a sequence $\{x_i\}_{i=1}^{\infty} \subset l_M^0$ of norm Gâteaux differentiable points such that*

(1) the ball-covering $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of l_M^0 is α-off the origin;

(2) the norm of l_M^0 is Gâteaux differentiable at every point of $\{x_i\}_{i=1}^{\infty}$.

Proof. Similar to the proof of Theorem 3.4, we obtain that Theorem 3.5 is true, which finishes the proof.

Theorem 3.6. *Let l_M be an Orlicz sequence space. Then l_M has a ball-covering $\{B(x_i, r_i)\}_{i=1}^{\infty}$ such that* $\sup_{i\in N} r_i < +\infty$.

Proof. Let Q denote rational number set. Then, for every natural number $n \in N$, we define the set

$$Q_n = \{(r_1, ..., r_n, 0, 0, ...) : r_i \in Q\}.$$

Define the set $Q_0 = \cup_{n\in N} Q_n$. Then the set Q_0 is a countable set. Let $A = \{x \in Q_0 : \|x\| \in [2,4]\}$. Since the set A is a countable set, we can order it as a sequence $A = \{x_i\}_{i=1}^{\infty}$. Then we define the closed ball sequences

$$B_{i,m} = B\left(x_i, \|x_i\| - \frac{1}{m}\right) = \left\{x \in l_M : \|x - x_i\| \le \|x_i\| - \frac{1}{m}\right\}, \qquad i = 1, 2, ...$$

Pick a point $x \in S(l_M)$. Then we obtain that $\rho_M(x) \le 1$. Let $x = (x(1), x(2), ...)$. Then there exists a natural number $i_0 \in N$ such that

$$0 \le \sum_{i=i_0+1}^{\infty} M\left(\frac{1}{3}x(i)\right) < \frac{1}{4}\rho_M\left(\frac{1}{3}x\right) \le \frac{1}{4}.$$

Therefore, by the above inequalities, we have the following inequalities

$$\begin{aligned}\sum_{i=1}^{i_0} M\left(\frac{1}{3}x(i)\right) &= \sum_{i=1}^{\infty} M\left(\frac{1}{3}x(i)\right) - \sum_{i=i_0+1}^{\infty} M\left(\frac{1}{3}x(i)\right) \\ &\ge \rho_M\left(\frac{1}{3}x\right) - \frac{1}{4}\rho_M\left(\frac{1}{3}x\right) \\ &= \frac{3}{4}\rho_M\left(\frac{1}{3}x\right).\end{aligned}$$

Since $\rho_M(x/3) > 0$, by the above inequalities, we obtain that

$$\sum_{i=1}^{i_0} M\left(\frac{1}{3}x(i)\right) - \sum_{i=i_0+1}^{\infty} M\left(\frac{1}{3}x(i)\right) \ge \frac{3}{4}\rho_M\left(\frac{1}{3}x\right) - \frac{1}{4}\rho_M\left(\frac{1}{3}x\right) > 0. \tag{3.1}$$

Moreover, we can assume without loss of generality that $x(i_0 + 1) \in [0, +\infty)$. We pick a real number $u_0 \in (0, +\infty)$ such that $\|x_0\| = 3$, where

$$x_0 = (x(1), x(2), ..., x(i_0), u_0, 0, 0, ...) \in l_M.$$

Therefore, by the formula (3.1) and $\rho_M(x) \leq 1$, we obtain that $u_0 - x(i_0+1) > 0$. Therefore, by the formulas $x(i_0+1) \in [0, +\infty)$ and $u_0 \in (0, +\infty)$, we obtain the following inequality

$$M\left(u_0 - x(i_0+1)\right) \leq M\left(u_0\right). \tag{3.2}$$

Since $\rho_M(x) \leq 1$, by the formula $\|x_0\| = 3$ and the definition of x_0, we get that $\rho_M\left(x_0/3\right) = 1$. Therefore, by the formulas (3.1)-(3.2) and the definition of x_0, we have the following inequalities

$$\begin{aligned}
\rho_M\left(\frac{x_0 - x}{3}\right) &= M\left(\frac{1}{3}(u_0 - x(i_0+1))\right) + \sum_{i=i_0+1}^{\infty} M\left(\frac{1}{3}x(i)\right) \\
&\leq M\left(\frac{1}{3}u_0\right) + \sum_{i=i_0+1}^{\infty} M\left(\frac{1}{3}x(i)\right) \\
&< M\left(\frac{1}{3}u_0\right) + \sum_{i=1}^{i_0} M\left(\frac{1}{3}x(i)\right) \\
&= \rho_M\left(\frac{1}{3}x_0\right) = 1.
\end{aligned}$$

Since $\rho_M(x) \leq 1$ and $\rho_M\left(x_0 - x\right) < +\infty$, there is a real number $\lambda_0 \in (0,3)$ such that $\rho_M\left((x_0 - x)/\lambda_0\right) = 1$. Hence we obtain that $\|x_0 - x\| = \lambda_0$. Moreover, by the definition of $\{x_i\}_{i=1}^{\infty}$ and the formula $\|x_0\| = 3$, there exists a point $x_j \in \{x_i\}_{i=1}^{\infty}$ such that $\|x_j\| \geq 3$ and $\|x_j - x_0\| < (3 - \lambda_0)/2$. Then, by $\lambda_0 \in (0,3)$, we get that

$$\left\|x - x_j\right\| \leq \left\|x_j - x_0\right\| + \|x - x_0\| < \frac{1}{2}(3 - \lambda_0) + \lambda_0 < 3 \leq \left\|x_j\right\|.$$

Therefore, by the above inequalities, there is a natural number $m_0 \in N$ so that

$$x \in B_{j,m_0} = B\left(x_j, \left\|x_j\right\| - \frac{1}{m_0}\right).$$

This implies that l_M has a ball-covering $\{B_{i,m}\}_{i=1,\ m=1}^{\infty}$ such that $\sup_{i\in N} r_i < 4 < +\infty$, which finishes the proof.

Corollary 3.7. *Orlicz sequence space l_M has the ball-covering property.*

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Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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