



Research article

Nontrivial solutions for the Laplace equation with a nonlinear Goldstein-Wentzell boundary condition

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Abstract: The paper deals with the existence and multiplicity of nontrivial solutions for the doubly elliptic problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ -\Delta_{\Gamma} u + \partial_{\nu} u = |u|^{p-2} u & \text{on } \Gamma_1, \end{cases}$$

where Ω is a bounded open subset of \mathbb{R}^N ($N \geq 2$) with C^1 boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, Γ_1 being nonempty and relatively open on Γ , $\mathcal{H}^{N-1}(\Gamma_0) > 0$ and $p > 2$ being subcritical with respect to Sobolev embedding on $\partial\Omega$.

We prove that the problem admits nontrivial solutions at the potential-well depth energy level, which is the minimal energy level for nontrivial solutions. We also prove that the problem has infinitely many solutions at higher energy levels.

Keywords: Laplace equation; Laplace-Beltrami operator; existence and multiplicity for nontrivial solutions; Wentzell boundary conditions; Vencel boundary conditions; Mountain Pass Theorem

Mathematics Subject Classification: 35D30, 35J05, 35J20, 25J25, 35J61, 35J67

1. Introduction and main results

We deal with the doubly elliptic problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ -\Delta_{\Gamma} u + \partial_{\nu} u = |u|^{p-2} u & \text{on } \Gamma_1, \end{cases} \quad (1.1)$$

where Ω is a bounded open subset of \mathbb{R}^N ($N \geq 2$) with C^1 boundary (see [1]). We denote $\Gamma = \partial\Omega$ and we assume $\Gamma = \Gamma_0 \cup \Gamma_1$, $\Gamma_0 \cap \Gamma_1 = \emptyset$, Γ_1 is nonempty and relatively open on Γ (or equivalently, $\overline{\Gamma_0} = \Gamma_0$).

Denoting by \mathcal{H}^{N-1} the Hausdorff measure, we assume that $\mathcal{H}^{N-1}(\bar{\Gamma}_0 \cap \bar{\Gamma}_1) = 0$ and $\mathcal{H}^{N-1}(\Gamma_0) > 0$. These properties of Ω , Γ_0 and Γ_1 will be assumed, without further comments, throughout the paper. Moreover, in (1.1), we consider $p > 2$ and we respectively denote by Δ and Δ_Γ the Laplace and the Laplace-Beltrami operators, while ν stands for the outward normal to Ω .

Elliptic equations with nonlinear Neumann boundary conditions, such as problem (1.1) without the Laplace-Beltrami term, have a wide literature. Without any aim of completeness, here we refer to [2–7].

Boundary conditions like the one in (1.1), but without the nonlinear source $|u|^{p-2}u$, are known in the literature as generalized Wentzell (sometimes spelled as Ventcel) or Goldstein-Wentzell boundary condition, since they have been the subject of several papers in the framework of linear evolution problems. See for example [8–13] and [14], to which we refer for the physical motivations of this kind of problems.

The same boundary conditions also appear in the context of bulk-surface elliptic problems. See for example [15–19] for linear eigenvalue problems related to the Wentzell boundary condition. We also would like to refer to [20] (also giving a physical derivation of the boundary condition) and the references therein, for a related doubly parabolic problem. See also [21] for another doubly parabolic related problem.

On the other hand, to the author's knowledge, a Goldstein-Wentzell boundary condition with a nonlinear source like $|u|^{p-2}u$ in connection with the Laplace equation has never been considered in the literature. The motivation for studying it comes from a series of papers by the author concerning the wave equation with hyperbolic dynamical boundary conditions with boundary damping and source terms. The prototype of this kind of problem is the evolutionary boundary value problem

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } (0, \infty) \times \Omega, \\ u = 0 & \text{on } (0, \infty) \times \Gamma_0, \\ u_{tt} + \partial_\nu u - \Delta_\Gamma u + |u_t|^{m-2}u_t = |u|^{p-2}u & \text{on } (0, \infty) \times \Gamma_1, \end{cases} \quad (1.2)$$

where $u = u(t, x)$, $t \geq 0$, $x \in \Omega$ and $\Delta = \Delta_x$ denotes the Laplacian operator with respect to the space variable. Its associated initial-value problem was introduced in [22] and then studied, as a particular case, in [23–25]. We refer to [23, 26] for the physical derivation of the problem, describing the vibrations of a membrane with a part of the boundary carrying a linear density of kinetic energy.

In order to give clear-cut criteria on the initial data to discriminate between global existence and blow-up for solutions of (1.2) it is useful to know if it possesses nontrivial stationary solutions, which turns out to be solutions of (1.1), at some specific energy level.

In particular in the present paper we shall consider the case when the nonlinearity $|u|^{p-2}u$ is subcritical with respect to the Sobolev Embedding $H^1(\Gamma) \hookrightarrow L^p(\Gamma)$, that is we shall assume that

$$2 < p < r, \quad \text{where } r = \begin{cases} \frac{2(N-1)}{N-3} & \text{if } N \geq 4, \\ \infty & \text{if } N = 2, 3. \end{cases} \quad (1.3)$$

Moreover, when dealing with problem (1.2), we shall also assume that

$$m > 1, \quad p \leq 1 + r/\bar{m}', \quad \text{where } \bar{m} := \max\{2, m\}, \quad (1.4)$$

the last assumption being related to well-posedness issues (see the papers quoted above). * We also

* Assumption (1.4) may be skipped when dealing with stationary solutions, but we prefer to keep it to avoid re-discussing problem (1.2) here

remark that, although the case $p \geq r$ (when $N \geq 4$) was also considered there, only the case $p < r$ is of interest when dealing with the dichotomy between global existence and blow-up, see [25, Remark 1, p. 6].

To state our main results we first introduce some basic notation. Subsequently, we shall identify $L^p(\Gamma_1)$, for $1 \leq p \leq \infty$, with its isometric image in $L^p(\Gamma)$, that is

$$L^p(\Gamma_1) = \{u \in L^p(\Gamma) : u = 0 \text{ a.e. on } \Gamma_0\}. \quad (1.5)$$

Moreover we shall denote by Tr the trace operator from $H^1(\Omega)$ onto $H^{1/2}(\Gamma)$ and, for simplicity of notation, $\text{Tr } u = u|_\Gamma$.

We introduce the Hilbert spaces $H^0 = L^2(\Omega) \times L^2(\Gamma_1)$ and

$$H^1 = \{(u, v) \in H^1(\Omega) \times H^1(\Gamma) : v = u|_\Gamma, v = 0 \text{ on } \Gamma_0\}, \quad (1.6)$$

with the topologies inherited from the products. For the sake of simplicity we shall identify, when useful, H^1 with its isomorphic counterpart

$$H_{\Gamma_0}^1(\Omega, \Gamma) = \{u \in H^1(\Omega) : u|_\Gamma \in H^1(\Gamma) \cap L^2(\Gamma_1)\}, \quad (1.7)$$

studied for example in [27], through the identification $(u, u|_\Gamma) \mapsto u$. So we shall write, without further mention, $u \in H^1$ for functions defined on Ω . Moreover we shall drop the notation $u|_\Gamma$, when useful, so we shall write $\|u\|_{L^2(\Gamma)}$ and so on, for $u \in H^1$, referring to the restriction of the Hausdorff measure \mathcal{H}^{N-1} to measurable subsets of Γ . We shall also drop the notation $d\mathcal{H}^{N-1}$ in boundary integrals, so writing $\int_\Gamma u = \int_\Gamma u d\mathcal{H}^{N-1}$.

By assumption (1.3) we can introduce in H^1 the nonlinear functional $I \in C^1(H^1) = C^1(H^1; \mathbb{R})$ defined by [†]

$$I(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{2} \int_{\Gamma_1} |\nabla_\Gamma u|_\Gamma^2 - \frac{1}{p} \int_{\Gamma_1} |u|^p, \quad (1.8)$$

which represents the potential energy associated to problem (1.2). For this reason we shall call it the *energy functional* when dealing with (1.1).

We also introduce the potential-well depth d given by

$$d = \inf_{u \in H^1, u|_\Gamma \neq 0} \sup_{\lambda > 0} I(\lambda u) = \inf_{u \in H^1 \setminus \{0\}} \sup_{\lambda > 0} I(\lambda u), \quad (1.9)$$

noticing that the identity between the two infima in (1.9) is essentially trivial and that we shall prove that $d > 0$.

Our first main result shows that problem (1.1) admits nontrivial weak solutions, see Definition 3 below, coinciding with critical points of the functional I , at the positive energy level d . We shall also recognize that they are stationary weak solutions of (1.2) provided this class of solutions is well-defined, see Definition 2 below.

Theorem 1. *When (1.3) holds, problem (1.1) has at least a couple $(u, -u)$ of antipodal weak solutions in H^1 such that $I(u) = I(-u) = d > 0$. When (1.4) holds they are also stationary weak solutions of problem (1.2).*

[†]Here ∇_Γ denotes the Riemannian gradient on Γ and $|\cdot|_\Gamma$, the norm associated to the Riemannian scalar product on the tangent bundle of Γ . See Section 2.

Moreover d coincides with the Mountain Pass level of the functional I , that is

$$d = \inf_{\sigma \in \Sigma} \max_{t \in [0,1]} I(\sigma(t)), \quad \text{where } \Sigma = \{\sigma \in C([0, 1]; H^1) : \sigma(0) = 0, I(\sigma(1)) < 0\}.$$

Theorem 1 will be proved by applying a variant, maybe less well-known than other ones, of the Mountain Pass Theorem, explicitly given in § 2.

To show the relevance of the potential-well depth d , beside its interest in evolution problems, we give some relevant properties of weak solutions of (1.1) having energy d . At first we introduce the norm B of the bounded linear trace operator from H^1 to $L^p(\Gamma_1)$, that is

$$B = \sup_{u \in H^1 \setminus \{0\}} \frac{\|u\|_{L^p(\Gamma_1)}}{\left(\|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_{\Gamma} u\|_{L^2(\Gamma_1)}^2\right)^{1/2}}, \quad (1.10)$$

noticing that we shall prove that $B < \infty$. We can then state our second main result.

Theorem 2. *Let (1.3) hold and set*

$$\lambda_1 = B^{-p/(p-2)}, \quad \text{and} \quad \lambda_2 = B^{-2/(p-2)}. \quad (1.11)$$

Then we have

$$d = \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2 = \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_2^p. \quad (1.12)$$

Moreover, if u is a weak solution of (1.1) with $I(u) = d$ we also have

$$\|\nabla u\|_{L^2(\Omega)}^2 + \|\nabla_{\Gamma} u\|_{L^2(\Gamma_1)}^2 = \lambda_1^2 \quad \text{and} \quad \|u\|_{L^p(\Gamma_1)} = \lambda_2. \quad (1.13)$$

Finally weak solutions at the energy level d are the lowest energy non-trivial solutions of (1.1), that is for any non-trivial weak solutions u of (1.1) one has $I(u) \geq d$.

The proof of the minimality of the energy of solutions at level d , stated in Theorem 2, is of elementary nature. Hence it is easier than most proofs in the literature for internal sources, see for example [28–30]. By the way the homogeneity of the source $|u|^{p-2}u$ allows for this simple approach.

Finally, to show that the minimality asserted in Theorem 2 is of some use, since there are solutions at an higher level, we give our last main result, which can also be of independent interest.

Theorem 3. *When (1.3) holds there is a sequence $(u_n)_n$ of nontrivial weak solutions of (1.1) such that $I(u_n) \rightarrow \infty$ as $n \rightarrow \infty$.*

The proof of Theorem 3 relies on applying the \mathbb{Z}_2 -version of the Mountain Pass Theorem in a different variational setting, which turns out to be equivalent to the one illustrated in this Section. See § 4 for details.

We would like to mention that, although in the paper we give Theorems 1-3 for the prototype nonlinearity $f(x, u) = |u|^{p-2}u$, they can be easily extended to the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ -\Delta_{\Gamma} u + \partial_{\nu} u = f(x, u) & \text{on } \Gamma_1, \end{cases}$$

under suitable assumptions on f that have been widely used in the literature. Here, for the sake of simplicity, we preferred to concentrate on the prototype problem.

Moreover, clearly Theorems 1-3 also show the existence of stationary solutions for other evolution problems not considered in the present paper, which can be of future interest.

The paper is organized as follows: in Section 2 we shall give all preliminaries needed in the paper. Section 3 will be devoted to prove Theorems 1 and 2, while in Section 4 we shall prove Theorem 3.

2. Preliminaries

2.1. Notation

We shall adopt the standard notation for (real) Lebesgue and Sobolev spaces in Ω , referring to [31]. For simplicity we shall denote by $\|\cdot\|_\tau$, for $1 \leq \tau \leq \infty$, the norms in $L^\tau(\Omega)$ and in $L^\tau(\Omega; \mathbb{R}^N)$.

Given a Banach space X we shall denote by X' its dual and by $\langle \cdot, \cdot \rangle_X$ the duality product between them. Moreover we shall use the standard notation for X -valued Lebesgue and Sobolev spaces in a real interval. When another Banach space Y is given we shall denote by $\mathcal{L}(X, Y)$ the space of bounded linear operators between X and Y and by $\|\cdot\|_{\mathcal{L}(X, Y)}$ the standard norm on it.

2.2. Function spaces and Riemannian operators on Γ .

Lebesgue spaces on Γ and Γ_1 will be intended with respect to (the restriction to measurable subset of them of) the Hausdorff measure \mathcal{H}^{N-1} , and for simplicity we shall denote, for $1 \leq \tau \leq \infty$, $\|\cdot\|_{\tau, \Gamma} = \|\cdot\|_{L^\tau(\Gamma)}$ and $\|\cdot\|_{\tau, \Gamma_1} = \|\cdot\|_{L^\tau(\Gamma_1)}$.

Sobolev spaces on Γ and on its relatively open subsets are classical objects, and we shall use the standard notation for them. We refer to [1] for their definition in the present case in which Γ is merely C^1 .

Since Γ is C^1 , it inherits from \mathbb{R}^N the structure of a Riemannian C^1 manifold (see [32]), so in the sequel we shall use some notation of geometric nature, which is quite common when Γ is smooth (see [33–36]), and which can be easily extended to the C^1 case, see for example [37]. Moreover, since Γ_1 is relatively open on Γ , this notation will apply (by restriction) to it, without further mention.

We shall denote by $T(\Gamma)$ and $T^*(\Gamma)$ the tangent and cotangent bundles, and by $(\cdot, \cdot)_\Gamma$ the Riemannian metric inherited from \mathbb{R}^N , given in local coordinates by $(u, v)_\Gamma = g_{ij}u^i v^j$ for all $u, v \in T(\Gamma)$ (here and in the sequel the summation convention being in use). The metric induces the fiber-wise defined musical isomorphisms $\flat : T(\Gamma) \rightarrow T^*(\Gamma)$ and $\sharp = \flat^{-1} : T^*(\Gamma) \rightarrow T(\Gamma)$ defined by $\langle \flat u, v \rangle_{T(\Gamma)} = (v, u)_\Gamma$ for $u, v \in T(\Gamma)$, where $\langle \cdot, \cdot \rangle_{T(\Gamma)}$ denotes the fiber-wise defined duality pairing. The induced bundle metric on $T^*(\Gamma)$, still denoted by $(\cdot, \cdot)_\Gamma$, is then defined by the formula $(\alpha, \beta)_\Gamma = \langle \alpha, \sharp \beta \rangle_{T(\Gamma)}$ for all $\alpha, \beta \in T^*(\Gamma)$, so that

$$(\alpha, \beta)_\Gamma = (\sharp \beta, \sharp \alpha)_\Gamma, \quad \text{for all } \alpha, \beta \in T^*(\Gamma). \quad (2.1)$$

By $|\cdot|_\Gamma^2 = (\cdot, \cdot)_\Gamma$ we shall denote the associated bundle norms on $T(\Gamma)$ and $T^*(\Gamma)$.

Denoting by d_Γ the standard differential on Γ , the Riemannian gradient operator ∇_Γ is defined by setting, for $u \in C^1(\Gamma)$ and thus by density for $u \in H^1(\Gamma)$, $\nabla_\Gamma u = \sharp d_\Gamma u$, so $\nabla_\Gamma u = g^{ij} \partial_j u \partial_i$ in local coordinates, where $(g^{ij}) = (g_{ij})^{-1}$. By (2.1), one trivially gets that $(\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma = (d_\Gamma u, d_\Gamma v)_\Gamma$ for all $u, v \in H^1(\Gamma)$, so in the sequel the use of vectors or forms is optional.

It is well known, see for example [37, Chapter 3], that $H^1(\Gamma)$ can be equipped with the equivalent

norm $\|\cdot\|_{H^1(\Gamma)}$ given by

$$\|u\|_{H^1(\Gamma)}^2 = \|u\|_{2,\Gamma}^2 + \|\nabla_\Gamma u\|_{2,\Gamma}^2, \quad \text{where} \quad \|\nabla_\Gamma u\|_{2,\Gamma}^2 := \int_\Gamma |\nabla_\Gamma u|_\Gamma^2.$$

In the sequel we shall also deal with the closed subspace of $H^1(\Gamma)$

$$H_{\Gamma_0}^1(\Gamma) = \{u \in H^1(\Gamma) : u = 0 \quad \text{a.e. on } \Gamma_0\}, \quad (2.2)$$

endowed with the norm $\|\cdot\|_{H^1(\Gamma)}$, which is then a Hilbert space. Since for all $u \in H_{\Gamma_0}^1(\Gamma)$ one has $\nabla_\Gamma u = 0$ a.e. on $\Gamma \setminus \overline{\Gamma_1}$ and $\mathcal{H}^{N-1}(\overline{\Gamma_0} \cap \overline{\Gamma_1}) = 0$, we have

$$\|u\|_{H^1(\Gamma)}^2 = \|u\|_{2,\Gamma_1}^2 + \|\nabla_\Gamma u\|_{2,\Gamma_1}^2 \quad \text{for all } u \in H_{\Gamma_0}^1(\Gamma), \quad (2.3)$$

where $\|\nabla_\Gamma u\|_{2,\Gamma_1}^2 := \int_{\Gamma_1} |\nabla_\Gamma u|_\Gamma^2$.

Remark 1. Although the definition of the space $H_{\Gamma_0}^1(\Gamma)$ given above is adequate for our purpose, we would like to point out two characterizations of it in two different geometrical situations:

- i) When $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, both Γ_0 and Γ_1 are relatively open, and by identifying the elements of $H^1(\Gamma_i)$, $i = 0, 1$, with their trivial extensions to Γ , one easily gets the splitting $H^1(\Gamma) = H^1(\Gamma_0) \oplus H^1(\Gamma_1)$, and consequently $H_{\Gamma_0}^1(\Gamma)$ is isometrically isomorphic to $H^1(\Gamma_1)$.
- ii) When $\overline{\Gamma_0} \cap \overline{\Gamma_1} \neq \emptyset$, such a characterization is false, since one easily sees that the characteristic function χ_{Γ_1} of Γ_1 does not belong to $H_{\Gamma_0}^1(\Gamma)$, while its restriction to Γ_1 trivially belongs to $H^1(\Gamma_1)$. Indeed in this case the elements of $H_{\Gamma_0}^1(\Gamma)$ “vanish” at the relative boundary $\partial\Gamma_1 = \overline{\Gamma_0} \cap \overline{\Gamma_1}$ of Γ_1 on Γ , although such a notion can be made more precise only when $\partial\Gamma_1$ is regular enough. For example, when Γ is smooth and $\overline{\Gamma_1}$ is a manifold with boundary $\partial\Gamma_1$, see [36, §5.1], $H_{\Gamma_0}^1(\Gamma)$ is isometrically isomorphic to the space

$$H_0^1(\Gamma_1) := \overline{C_c^\infty(\Gamma_1)}^{\|\cdot\|_{H^1(\Gamma_1)}}.$$

The Laplace-Beltrami operator Δ_Γ can be defined in a geometrically elegant way by using ∇_Γ and the Riemannian divergence operator, as in [37, § 2.3], at least when Γ is C^2 . To avoid the need of introducing Sobolev spaces of tensor fields we shall adopt here a less elegant approach. Indeed we set, when Γ is C^2 and $u \in C^2(\Gamma')$, $\Gamma' \subset \Gamma$ relatively open,

$$\Delta_\Gamma u = g^{-1/2} \partial_i (g^{1/2} g^{ij} \partial_j u), \quad \text{where } g = \det(g_{ij}), \quad (2.4)$$

in local coordinates. Since g, g^{ij} are continuous and Γ is compact, formula (2.4) extends by density to $u \in H^2(\Gamma)$, so defining an operator $-\Delta_\Gamma \in \mathcal{L}(H^2(\Gamma); L^2(\Gamma))$, which restricts to $-\Delta_\Gamma \in \mathcal{L}(H^2(\Gamma'); L^2(\Gamma'))$ for relatively open subsets Γ' of Γ . Since Γ is compact, by (2.4), integrating by parts and using a C^2 partition of the unity one gets that

$$-\int_\Gamma \Delta_\Gamma u v = \int_\Gamma (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma \quad \text{for all } u \in H^2(\Gamma) \text{ and } v \in H^1(\Gamma). \quad (2.5)$$

Formula (2.5) motivates the definition of the operator $-\Delta_\Gamma \in \mathcal{L}(H^1(\Gamma); H^{-1}(\Gamma))$, also when Γ is merely C^1 , given by

$$\langle -\Delta_\Gamma u, v \rangle_{H^1(\Gamma)} = \int_\Gamma (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma \quad \text{for all } u, v \in H^1(\Gamma). \quad (2.6)$$

By density, when Γ is C^2 , the so-defined operator is the unique extension of $-\Delta_\Gamma \in \mathcal{L}(H^2(\Gamma); L^2(\Gamma))$.

In § 4 we shall deal with the realization of $-\Delta_\Gamma$ between the space $H_{\Gamma_0}^1(\Gamma)$ and its dual. The different nature of the space $H_{\Gamma_0}^1(\Gamma)$ in the two cases i) and ii) has been pointed out in Remark 1. To explain the definition of the realization we shall give we recall that, when $\overline{\Gamma_0} \cap \overline{\Gamma_1} = \emptyset$, so Γ_1 is compact, formula (2.5) holds, when Γ is C^2 , also when replacing Γ with Γ_1 , so making natural to set $-\Delta_{\Gamma_1} \in \mathcal{L}(H^1(\Gamma_1); H^{-1}(\Gamma_1))$ by

$$\langle -\Delta_{\Gamma_1} u, v \rangle_{H^1(\Gamma_1)} = \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma \quad \text{for all } u, v \in H^1(\Gamma_1). \quad (2.7)$$

When $\overline{\Gamma_0} \cap \overline{\Gamma_1} \neq \emptyset$, Γ is smooth and $\overline{\Gamma_1}$ is a manifold with boundary $\partial\Gamma_1$, formula (2.5) fails to hold on Γ_1 , since a boundary integral on $\partial\Gamma_1$ appears. On the other hand, taking into account the homogeneous Dirichlet boundary condition in the space $H_0^1(\Gamma_1)$, it is natural to set $-\Delta_{\Gamma_{1D}} \in \mathcal{L}(H_0^1(\Gamma_1); H^{-1}(\Gamma_1))$ by

$$\langle -\Delta_{\Gamma_{1D}} u, v \rangle_{H_0^1(\Gamma_1)} = \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma \quad \text{for all } u, v \in H_0^1(\Gamma_1). \quad (2.8)$$

Hence, taking into account the characterizations given in Remark 1, to simultaneously deal with the two cases i) and ii), in the sequel we shall deal with the operator $-\Delta_{\Gamma_{1(D)}} \in \mathcal{L}(H_{\Gamma_0}^1(\Gamma); [H_{\Gamma_0}^1(\Gamma)]')$ defined by

$$\langle -\Delta_{\Gamma_{1(D)}} u, v \rangle_{H_{\Gamma_0}^1(\Gamma)} = \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma \quad \text{for all } u, v \in H_{\Gamma_0}^1(\Gamma), \quad (2.9)$$

noticing that, by (2.6), $-\Delta_{\Gamma_{1(D)}} u = -\Delta_\Gamma u|_{H_{\Gamma_0}^1(\Gamma)}$ for all $u \in H_{\Gamma_0}^1(\Gamma)$.

2.3. The space H^1 .

We recall [13, Lemma 1, p. 2147], which trivially extends to Γ of class C^1 , that the space

$$H^1(\Omega; \Gamma) = \{(u, v) \in H^1(\Omega) \times H^1(\Gamma) : v = u|_\Gamma\},$$

with the topology inherited from the product, can be identified with the space $\{u \in H^1(\Omega) : u|_\Gamma \in H^1(\Gamma)\}$ and equivalently equipped with the norm $\|\cdot\|_{H^1(\Omega, \Gamma)}$ given by

$$\|u\|_{H^1(\Omega, \Gamma)}^2 = \|\nabla u\|_2^2 + \|\nabla_\Gamma u\|_{2, \Gamma}^2 + \|u\|_{2, \Gamma}^2.$$

The identification made in § 1 between the spaces H^1 and $H_{\Gamma_0}^1(\Omega, \Gamma)$, respectively defined by (1.6) and (1.7), is a simple consequence of the identification above, and, by (2.3), H^1 can be equivalently equipped with the norm $\|\cdot\|_{H^1}$ given by

$$\|u\|_{H^1}^2 = \|\nabla u\|_2^2 + \|\nabla_\Gamma u\|_{2, \Gamma_1}^2 + \|u\|_{2, \Gamma_1}^2. \quad (2.10)$$

On the other hand, to get the advantage of the assumption $\mathcal{H}^{N-1}(\Gamma_0) > 0$, made in the present paper, we point out the following well-known result, the proof of which is given only for the reader's convenience.

Lemma 1. *Let $\mathcal{H}^{N-1}(\Gamma_0) > 0$. Then, setting, for $u, v \in H^1$,*

$$(u, v)_{H^1} = \int_\Omega \nabla u \nabla v + \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma v)_\Gamma \quad \text{and} \quad \|\cdot\|_{H^1} = (\cdot, \cdot)_{H^1}^{1/2}, \quad (2.11)$$

$\|\cdot\|_{H^1}$ defines on H^1 a norm equivalent to $\|\cdot\|_{H^1}$.

Proof. By combining [38, Chapter 2, Theorem 2.6.16, p. 75] and [38, Chapter 4, Corollary 4.5.2, p. 195] one gets the following Poincaré-type inequality: there is a positive constant $c_1 = c_1(\Omega, \Gamma_0)$ such that

$$\|u\|_2 \leq c_1 \|\nabla u\|_2 \quad \text{for all } u \in H^1. \quad (2.12)$$

By the Trace Theorem there is a positive constant $c_2 = c_2(\Omega, \Gamma_0)$ such that

$$\|u\|_{2,\Gamma_1} \leq c_2 \|u\|_{H^1(\Omega)} \quad \text{for all } u \in H^1(\Omega), \quad (2.13)$$

where $\|\cdot\|_{H^1(\Omega)}$ is the standard norm of $H^1(\Omega)$. Since $H^1 \subset H^1(\Omega)$, by combining (2.10), (2.12) and (2.13) we get

$$\|u\|_{H^1}^2 \leq \|\nabla u\|_2^2 + \|\nabla_{\Gamma} u\|_{2,\Gamma_1}^2 + c_2(1 + c_1)\|\nabla u\|_2^2 \leq c_3 \|u\|_{H^1}^2$$

for all $u \in H^1$, where $c_3 = 1 + c_2(1 + c_1)$, from which the statement trivially follows.

2.4. Some results from Critical Point Theory

We now recall some well-known notions of Critical Point Theory for a functional $\mathcal{I} \in C^1(X) = C^1(X; \mathbb{R})$ on any Banach space X with norm $\|\cdot\|_X$. By $\mathcal{I}' \in C(X; X')$ we shall denote the Fréchet differential of \mathcal{I} and (PS) will stand, in short, for Palais-Smale. See [39].

Definition 1. Let $\mathcal{I} \in C^1(X)$. We say that a sequence $(u_n)_n$ in X is a (PS) sequence if $(\mathcal{I}(u_n))_n$ is bounded and $\mathcal{I}'(u_n) \rightarrow 0$ in X' . We also say that $\mathcal{I} \in C^1(X)$ satisfies the (PS) condition if any (PS) sequence has a (strongly) convergent subsequence.

The following result is a well-known version of the celebrated Mountain Pass Theorem, see [39, Chapter 1, p. 4].

Theorem 4. Let $\mathcal{I} \in C^1(X)$ satisfy the (PS) condition and

- i) $\mathcal{I}(0) = 0$;
- ii) there are $\rho, \alpha > 0$ such that $\mathcal{I}(u) \geq \alpha$ for all $u \in X$ such that $\|u\|_X = \rho$;
- iii) there is $l \in X$ such that $\|l\|_X > \rho$ and $\mathcal{I}(l) \leq 0$.

Then \mathcal{I} possesses a critical value $c_l \geq \alpha$ given by

$$c_l = \inf_{\sigma \in \Sigma_l} \max_{t \in [0,1]} \mathcal{I}(\sigma(t)), \quad \text{where } \Sigma_l = \{\sigma \in C([0,1]; X) : \sigma(0) = 0, \sigma(1) = l\}. \quad (2.14)$$

It is rarely pointed out in textbooks that the critical level c_l above may depend on l , as the following trivial example shows:

Example 1. Let $X = \mathbb{R}$ and

$$\mathcal{I}(x) = \begin{cases} x^2 - x^4 & \text{if } x \geq 0, \\ x^2 - 2x^4 & \text{if } x < 0. \end{cases}$$

Trivially $\mathcal{I} \in C^1(\mathbb{R})$ satisfies the (PS) condition as well as assumptions i)-iii). Moreover its critical points are exactly $x = 0, -1/2, \sqrt{2}/2$ from which one easily sees that

$$c_l = \begin{cases} \mathcal{I}(\sqrt{2}/2) = 1/4 & \text{if } l > 0, \\ \mathcal{I}(-1/2) = 3/16 & \text{if } l < 0. \end{cases}$$

Since in the present paper we are interested in characterizing our critical level as the potential-well depth of the functional I , we now state a less known variant of the Mountain Pass Theorem under slightly more restrictive assumptions on the functional that look similar (although not identical) to the assumptions of [40, Theorem 2.1, p. 354], this one being the first version of this celebrated result.

Theorem 5. *Let $I \in C^1(X)$ satisfy the (PS) condition, assumptions i)-iii) in Theorem 4 and*

iv) $I(u) > 0$ for all $u \in X \setminus \{0\}$ such that $\|u\|_X \leq \rho$.

Then I possesses a critical value $c \geq \alpha$ given by

$$c = \inf_{\sigma \in \Sigma} \max_{t \in [0,1]} I(\sigma(t)), \text{ where } \Sigma = \{\sigma \in C([0,1]; X) : \sigma(0) = 0, I(\sigma(1)) < 0\}. \quad (2.15)$$

Proof. One can repeat the proof of [40, Theorem 2.1, p. 354] verbatim, since by assumption iv) for any $\sigma \in \Sigma$ one has $\|\sigma(1)\|_X > \rho$. Alternatively one can also deduce the statement from Theorem 4 by the following simple argument. By assumption iv) one has $\Sigma = \bigcup_{l \in A} \Sigma_l$, where $A = \{l \in X : \|l\|_X > \rho \text{ and } I(l) < 0\}$. Hence, by (2.14) and (2.15) we have $c = \inf_{l \in A} c_l$. Hence, since by assumption ii) one has $\alpha \leq c < \infty$, there is a sequence $(l_n)_n$ in A such that $c_{l_n} \rightarrow c$. By Theorem 4 there is a corresponding sequence $(u_n)_n$ in X such that $I(u_n) = c_{l_n}$ and $I'(u_n) = 0$, which is then a (PS) sequence and consequently, up to a subsequence, $u_n \rightarrow u$. Then $I'(u) = 0$ and $I(u) = c$, concluding the proof.

In the sequel we shall also use the following well-known \mathbb{Z}_2 - version [‡] of the Mountain Pass Theorem, see [39, Chapter 9, Theorem 9.12, p. 55 and Proposition 9.33, p. 58].

Theorem 6. *Let X be an infinite dimensional space and $I \in C^1(X)$ be even, satisfying the (PS) condition, assumptions i)-ii) of Theorem 4 and*

v) for each finite dimensional subspace Y of X there is $R_Y > 0$ such that $I(u) \leq 0$ for all $u \in Y$ such that $\|u\|_X > R_Y$.

Then I possesses a sequence $(u_n)_n$ of critical points such that $I(u_n) \rightarrow \infty$.

3. Existence of solutions of (1.1) at level d and their minimality

The aim of this section is to prove Theorems 1 and 2. We start by recalling what we mean by a weak solution of (1.2), referring to [24, §2.2 and Definition 3.1, p. 4896]. We shall also make precise the use of the term "stationary" when referring to them.

Definition 2. Let (1.3) and (1.4) hold. A *weak solution* of (1.2) is

$$u \in L_{\text{loc}}^\infty([0, \infty); H^1) \cap W_{\text{loc}}^{1,\infty}([0, \infty); H^0), \quad (u|_{\Gamma_1})_t \in L_{\text{loc}}^m(0, \infty); L^m(\Gamma_1)), \quad (3.1)$$

satisfying the distribution identity

$$\int_0^\infty \left[- \int_\Omega u_t \psi_t - \int_{\Gamma_1} (u|_{\Gamma_1})_t (\psi|_{\Gamma_1})_t + \int_\Omega \nabla u \nabla \psi + \int_{\Gamma_1} (\nabla_\Gamma u, \nabla_\Gamma \psi)_\Gamma - \int_{\Gamma_1} |u|^{p-2} u \psi \right] = 0, \quad (3.2)$$

for all $\psi \in C_c((0, \infty); H^1) \cap C_c^1((0, \infty); H^0)$ such that $(\psi|_{\Gamma_1})_t \in L_{\text{loc}}^m((0, \infty); L^m(\Gamma_1))$. We say that u is *stationary* if $u(t) \equiv u_0 \in H^1$ in $(0, \infty)$.

[‡]The name is justified by the fact that I is supposed to be invariant with respect to the group $\{-\text{Id}, \text{Id}\}$, which is isomorphic to the unique cyclic group of order two, that is \mathbb{Z}_2 .

We also make precise what we mean by weak solutions of (1.1).

Definition 3. Let $2 < p < r$. A weak solution of (1.1) is $u \in H^1$ such that

$$\int_{\Omega} \nabla u \nabla \phi + \int_{\Gamma_1} (\nabla_{\Gamma} u, \nabla_{\Gamma} \phi)_{\Gamma} - \int_{\Gamma_1} |u|^{p-2} u \phi = 0 \quad \text{for all } \phi \in H^1. \quad (3.3)$$

Actually weak solutions of (1.1) and stationary weak solutions of (1.2) coincide when they are both defined, as the following result shows.

Lemma 2. Let (1.3), (1.4) hold and $u_0 \in H^1$. Then $u \equiv u_0$ is a stationary weak solution of (1.2) if and only if u_0 is a weak solution of (1.1).

Proof. If u_0 is a weak solution of (1.1) by (3.3), one immediately gets that $u \equiv u_0$ satisfies (3.2). To prove the converse we recall that, by [24, Lemma 3.3, p. 4896], any weak solution u of (1.2) satisfies the alternative form of the distribution identity (3.2)

$$\left[\int_{\Omega} u_t \psi + \int_{\Gamma_1} (u_{|\Gamma_1})_t \psi \right]_0^T + \int_0^T \left[- \int_{\Omega} u_t \psi_t - \int_{\Gamma_1} (u_{|\Gamma_1})_t (\psi_{|\Gamma_1})_t + \int_{\Omega} \nabla u \nabla \psi + \int_{\Gamma_1} (\nabla_{\Gamma} u, \nabla_{\Gamma} \psi)_{\Gamma} - \int_{\Gamma_1} |u|^{p-2} u \psi \right] = 0, \quad (3.4)$$

for all $T > 0$ and $\psi \in C([0, T]; H^1) \cap C^1([0, T]; H^0)$, $(\psi_{|\Gamma_1})_t \in L^m((0, T) \times \Gamma_1)$. Hence, when $u \equiv u_0 \in H^1$ is a stationary weak solution of (1.2), taking in (3.4) test functions $\psi \equiv \phi \in H^1$ for an arbitrary $T > 0$ we get (3.3), concluding the proof.

Equation (3.3) has a variational nature, as the next result highlights.

Lemma 3. If (1.3) holds, the functional I defined in (1.8) belongs to $C^1(H^1)$ and its critical points coincide with the weak solutions of (1.1).

Proof. By classical arguments, see [41, Chapter 1, Theorem 2.9, p. 22], the potential operator $F : H^1 \rightarrow \mathbb{R}$, defined by $F(u) = \frac{1}{p} \|u\|_{p, \Gamma_1}^p$, is Fréchet differentiable and, for all $u, \phi \in H^1$ one has

$$\langle F'(u), \phi \rangle_{H^1} = \int_{\Gamma_1} |u|^{p-2} u \phi. \quad (3.5)$$

Consequently, since

$$I(u) = \frac{1}{2} \|u\|_{H^1}^2 - F(u) \quad \text{for all } u \in H^1, \quad (3.6)$$

trivially $I \in C^1(H^1)$ and

$$\langle I'(u), \phi \rangle_{H^1} = (u, \phi)_{H^1} - \langle F'(u), \phi \rangle_{H^1} \quad \text{for all } u, \phi \in H^1. \quad (3.7)$$

By (2.11), (3.5) and (3.7) one immediately gets that (3.3) is rewritten as $I'(u) = 0$.

We now establish some geometrical properties of the functional I .

Lemma 4. If (1.3) holds, the functional I satisfies the assumptions i)-iii) of Theorem 4 and iv) of Theorem 5.

Proof. By (1.8), trivially $I(0) = 0$, proving i). To prove ii) and iv) we remark that, by Sobolev Embedding Theorem there is $c_4 = c_4(p, \Omega) > 0$ such that

$$\|u\|_{p,\Gamma} \leq c_4 \|u\|_{H^1(\Gamma)} \quad \text{for all } u \in H^1(\Gamma).$$

Consequently, by Lemma 1, there is $c_5 = c_5(p, \Omega, \Gamma_0) > 0$ such that

$$\|u\|_{p,\Gamma_1} \leq c_5 \|u\|_{H^1} \quad \text{for all } u \in H^1. \quad (3.8)$$

Hence, by (1.8), for all $u \in H^1$ we have

$$I(u) = \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{p} \|u\|_{p,\Gamma_1}^p \geq \left(\frac{1}{2} - \frac{c_5^p}{p} \|u\|_{H^1}^{p-2} \right) \|u\|_{H^1}^2,$$

from which ii) and iv) follow, by taking for example

$$\rho = \left(\frac{p}{4c_5^p} \right)^{\frac{1}{p-2}}, \quad \text{and} \quad \alpha = \frac{1}{4} \left(\frac{p}{4c_5^p} \right)^{\frac{2}{p-2}}.$$

To prove iii) we remark that, for any $u \in H^1$ such that $u|_{\Gamma_1} \neq 0$ and $s > 0$ we have

$$I(su) = \frac{1}{2} \|u\|_{H^1}^2 s^2 - \frac{1}{p} \|u\|_{p,\Gamma_1}^p s^p \rightarrow -\infty \quad \text{as } s \rightarrow \infty.$$

The last relevant property of I is given by the following result:

Lemma 5. *If (1.3) holds, the functional I satisfies the (PS) condition.*

Proof. Let $(u_n)_n$ be a (PS) sequence in H^1 . Then there are $c_6, c_7 \geq 0$, depending on $(u_n)_n$, such that

$$I(u_n) \leq c_6, \quad \text{and} \quad |\langle I'(u_n), u_n \rangle_{H^1}| \leq c_7 \|u_n\|_{H^1} \quad \text{for all } n \in \mathbb{N}. \quad (3.9)$$

Since, by (1.8) and (3.7),

$$pI(u_n) - \langle I'(u_n), u_n \rangle_{H^1} = \left(\frac{p}{2} - 1 \right) \|u_n\|_{H^1}^2,$$

by (3.9) we get

$$\left(\frac{p}{2} - 1 \right) \|u_n\|_{H^1}^2 \leq pc_6 + c_7 \|u_n\|_{H^1},$$

from which one immediately yields that $(u_n)_n$ is bounded in H^1 . Consequently, up to a subsequence, $u_n \rightharpoonup u$ in H^1 . To prove that the convergence is actually strong we remark that, by (3.7), for all $\phi \in H^1$ we have

$$\langle u_n - u, \phi \rangle_{H^1} = \langle I'(u_n) - I'(u), \phi \rangle_{H^1} + \int_{\Gamma_1} (|u_n|^{p-2} u_n - |u|^{p-2} u) \phi,$$

so taking $\phi = u_n - u$ we get

$$\|u_n - u\|_{H^1}^2 = \langle I'(u_n), u_n - u \rangle_{H^1} - \langle I'(u), u_n - u \rangle_{H^1} + \int_{\Gamma_1} (|u_n|^{p-2} u_n - |u|^{p-2} u)(u_n - u). \quad (3.10)$$

The first two terms on the right hand side of (3.10) converge to 0 since $I'(u_n) \rightarrow 0$ in $(H^1)'$ and $u_n - u \rightarrow 0$, hence it is norm bounded. As to the third term in it, since the embedding $H^1(\Gamma) \hookrightarrow L^p(\Gamma)$ is compact and the operator $u \mapsto u|_{\Gamma}$ from H^1 to $H^1(\Gamma)$ is bounded, up to a subsequence we have $u_n|_{\Gamma} \rightarrow u|_{\Gamma}$ strongly in $L^p(\Gamma)$. By standard properties of the Nemitskii operators, see [41, Chapter 1, Theorem 2.2, p. 16], we also have $|u_n|^{p-2} u_n|_{\Gamma} \rightarrow |u|^{p-2} u|_{\Gamma}$ strongly in $L^{p'}(\Gamma)$. Consequently, the third term in the right hand side of (3.10) converges to 0 by the Hölder inequality.

We can then give the proof of our first main result:

Proof of Theorem 1. By simply combining Lemmas 2-5, Theorem 5, with $I = \mathcal{I}$, and the fact that I is even, we get the statement, but for one fact. The only exception is the value of the critical level, since by Theorem 5 we have that $I(u) = I(-u) = c$, where c is given by (2.15). To complete the proof we then have to recognize that $c = d_1 = d_2$, where

$$d_1 = \inf_{u \in H^1, u_{\Gamma} \neq 0} \sup_{\lambda > 0} I(\lambda u), \quad \text{and} \quad d_2 = \inf_{u \in H^1 \setminus \{0\}} \sup_{\lambda > 0} I(\lambda u).$$

Since, for any $\lambda > 0$,

$$I(\lambda u) = \frac{1}{2} \|u\|_{H^1}^2 \lambda^2 - \frac{1}{p} \|u\|_{p, \Gamma_1}^p \lambda^p, \quad (3.11)$$

trivially $\sup_{\lambda > 0} I(\lambda u) < \infty$ when $u_{\Gamma} \neq 0$, while $\sup_{\lambda > 0} I(\lambda u) = \infty$ when $u \in H^1 \setminus \{0\}$ and $u_{\Gamma} \equiv 0$. Hence $d_1 = d_2 = d$, and consequently it remains to be proved that $d = c$.

This fact is well-known for similar problems, see for example [42, Chapter 8, p. 117] and [28, 29, 43–45]. Here we shall essentially conveniently adapt the argument in [45]. By (3.11), for any $u \in H^1$ with $u_{\Gamma} \neq 0$, the function $\lambda \mapsto I(\lambda u)$ has a unique critical point $\lambda_u > 0$, $\max_{\lambda > 0} I(\lambda u) = I(\lambda_u u)$ and $I(\lambda u) \rightarrow -\infty$ as $\lambda \rightarrow \infty$, for any such u , defining $\sigma_u \in C([0, 1]; H^1)$ by $\sigma_u(s) = Rsu$ for $s \in [0, 1]$, with $R > 0$ so large that $I(Ru) < 0$, we have $\sigma_u \in \Sigma$ and consequently $I(\lambda_u u) = \max_{t \in [0, 1]} I(\sigma_u(t)) \geq c$. Hence $d \geq c$. On the other hand, if u is a critical point of I with $I(u) = c$, already found above, by (3.7) we have $\|u\|_{H^1}^2 = \|u\|_{p, \Gamma_1}^p$. Then, since $u \neq 0$, we also get that $u_{\Gamma} \neq 0$. Consequently, since $\frac{d}{d\lambda} I(\lambda u) = \langle I'(u), u \rangle_{H^1}$, we have $\lambda_u = 1$ and consequently $c = I(u) = \max_{\lambda > 0} I(\lambda u) \geq d$, completing the proof.

We now turn to proving Theorem 2. We remark at first that the number B defined in (1.10) is finite because of the estimate (3.8). We now introduce the auxiliary functional $K \in C^1(H^1)$ given by

$$K(u) = \langle I'(u), u \rangle_{H^1} = \|u\|_{H^1}^2 - \|u\|_{p, \Gamma_1}^p. \quad (3.12)$$

The key point in the proof of Theorem 2 is the following result, of possible independent interest:

Lemma 6. *Let (1.3) hold and λ_1, λ_2 be given by (1.11). Then (1.12) holds. Moreover, for any $u \in H^1$ such that $u_{\Gamma} \neq 0$ and $I(u) \leq d$, the following implications hold:*

$$\begin{aligned} K(u) \geq 0 &\iff \|u\|_{H^1} \leq \lambda_1 \iff \|u\|_{p, \Gamma_1} \leq \lambda_2, \\ K(u) \leq 0 &\iff \|u\|_{H^1} \geq \lambda_1 \iff \|u\|_{p, \Gamma_1} \geq \lambda_2. \end{aligned} \quad (3.13)$$

Proof. To prove (1.12) we remark that, for any $u \in H^1$ with $u_{\Gamma} \neq 0$ an easy calculation shows that

$$\max_{\lambda > 0} I(\lambda u) = I(\lambda_u u) \quad \text{where} \quad \lambda_u = \|u\|_{H^1}^{2/(p-2)} \|u\|_{p, \Gamma_1}^{-p/(p-2)}$$

so that

$$\max_{\lambda > 0} I(\lambda u) = \left(\frac{1}{2} - \frac{1}{p} \right) \left(\frac{\|u\|_{H^1}}{\|u\|_{p, \Gamma_1}} \right)^{\frac{2p}{p-2}}$$

and consequently

$$d = \left(\frac{1}{2} - \frac{1}{p} \right) \left(\sup_{u \in H^1, u_{\Gamma} \neq 0} \frac{\|u\|_{H^1}}{\|u\|_{p, \Gamma_1}} \right)^{\frac{-2p}{p-2}}.$$

By using (1.10) and (1.11) in the last formula we get (1.12).

Now let $u \in H^1$ such that $u|_{\Gamma} \neq 0$ and $I(u) \leq d$. To prove (3.13) we shall first prove the implications

$$K(u) \geq 0 \implies \|u\|_{H^1} \leq \lambda_1 \implies \|u\|_{p,\Gamma_1} \leq \lambda_2 \implies K(u) \geq 0. \quad (3.14)$$

If $K(u) \geq 0$, supposing by contradiction that $\|u\|_{H^1} > \lambda_1$, by (3.6) we get

$$I(u) \geq \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_{H^1}^2 > \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_2^2 = d,$$

a contradiction. If $\|u\|_{H^1} \leq \lambda_1$, since by (1.10) we have

$$\|u\|_{p,\Gamma_1} \leq B\|u\|_{H^1} \quad \text{for all } u \in H^1, \quad (3.15)$$

and $\lambda_2 = B\lambda_1$, we immediately get $\|u\|_{p,\Gamma_1} \leq \lambda_2$. If $\|u\|_{p,\Gamma_1} \leq \lambda_2$, by (1.11) and (3.15) we have

$$\|u\|_{p,\Gamma_1}^p \leq \lambda_2^{p-2} \|u\|_{p,\Gamma_1}^2 = B^{-2} \|u\|_{p,\Gamma_1}^2 \leq \|u\|_{H^1}^2,$$

so $K(u) \geq 0$, concluding the proof of (3.16).

To complete the proof of (3.13) we are then going to prove the further implications

$$K(u) \leq 0 \implies \|u\|_{p,\Gamma_1} \geq \lambda_2 \implies \|u\|_{H^1} \geq \lambda_1 \implies K(u) \leq 0. \quad (3.16)$$

If $K(u) \leq 0$, by (3.15), we have

$$B^{-2} \|u\|_{p,\Gamma_1}^2 \leq \|u\|_{H^1}^2 \leq \|u\|_{p,\Gamma_1}^p$$

which, since $u|_{\Gamma} \neq 0$, gives $\|u\|_{p,\Gamma_1} \geq B^{-2/(p-2)} = \lambda_2$. If $\|u\|_{p,\Gamma_1} \geq \lambda_2$ by (3.15) we have $B\|u\|_{H^1} \geq \lambda_1$, i.e., $\|u\|_{H^1} \geq B^{-1}\lambda_2 = \lambda_1$. If $\|u\|_{H^1} \geq \lambda_1$, then assuming by contradiction that $K(u) > 0$, i.e., $\|u\|_{p,\Gamma_1}^p < \|u\|_{H^1}^2$, by (3.6) we get

$$I(u) = \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{p} \|u\|_{p,\Gamma_1}^p > \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_{H^1}^2 \geq \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2 = d,$$

a contradiction, so $K(u) \leq 0$ and the proof is complete.

We can now prove our second main result.

Proof of Theorem 2. Formula (1.12) has been already proved in Lemma 6. Moreover, if u is a weak solution of (1.1) with $I(u) = d$, by Lemma 3 and (3.12) we have $K(u) = 0$, and since $d > 0$ necessarily $u \neq 0$. Hence $\|u\|_{p,\Gamma_1}^p = \|u\|_{H^1}^2 > 0$, so by Lemma 6 we immediately get (1.13).

Now let u be a non-trivial weak solution of (1.1). By Lemma 3 and (3.12) we have $K(u) = 0$. Then, by (3.6) and (3.15) we get

$$I(u) \geq \frac{1}{2} \|u\|_{H^1}^2 - \frac{B^p}{p} \|u\|_{H^1}^p, \quad (3.17)$$

and since $K(u) = 0$, also

$$I(u) = \frac{1}{2} \|u\|_{H^1}^2 - \frac{1}{p} \|u\|_{H^1}^2 = \left(\frac{1}{2} - \frac{1}{p}\right) \|u\|_{H^1}^2. \quad (3.18)$$

By combining (3.17) and (3.18) we get $B^p \|u\|_{H^1}^p \geq \|u\|_{H^1}^2$, so since $u \neq 0$ we have $\|u\|_{H^1} \geq B^{-p/(p-2)} = \lambda_1$. Then, using (3.18) again we obtain $I(u) \geq \left(\frac{1}{2} - \frac{1}{p}\right) \lambda_1^2 = d$, concluding the proof.

4. Multiplicity

The aim of this section is to prove Theorem 3 and the strategy of the proof consists in applying Theorem 6. Unfortunately, the simple variational setting used in the previous section is not adequate for this purpose. Indeed the functional I defined in (1.8) does not satisfy assumption (v) in Theorem 6, as it is evident by considering its restriction to the space $H_0^1(\Omega)$.

To introduce a convenient setting we first recall that by standard elliptic theory, see for example [46, Chapter I], for any $v \in H^{1/2}(\Gamma)$ the nonhomogeneous Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ u = v & \text{on } \Gamma, \end{cases} \quad (4.1)$$

has a unique solution $u \in H^1(\Omega)$, continuously depending on v in the topologies of the respective spaces, where the Laplace equation $\Delta u = 0$ is taken in the sense of distributions, or equivalently in the space $H^{-1}(\Omega)$, that is

$$\int_{\Omega} \nabla u \nabla \phi = 0 \quad \text{for all } \phi \in H_0^1(\Omega), \quad (4.2)$$

while the boundary condition is taken in the trace sense. Trivially $u \in H^1$ when $v \in H_{\Gamma_0}^1(\Gamma)$, see (2.2), so we obtain the bounded linear Dirichlet operator $v \mapsto u$ from $H_{\Gamma_0}^1(\Gamma)$ into H^1 . Trivially its range is the closed subspace \mathbb{A}^1 of H^1 defined by

$$\mathbb{A}^1 = \{u \in H^1 : (4.2) \text{ holds}\}. \quad (4.3)$$

Hence, denoting $u = \mathbb{D}v$, we get the bijective isomorphism

$$\mathbb{D} \in \mathcal{L}(H_{\Gamma_0}^1(\Gamma); \mathbb{A}^1), \quad \text{with } \mathbb{D}^{-1} = \text{Tr}_{\Gamma|\mathbb{A}^1}. \quad (4.4)$$

The starting point of the analysis is that the space \mathbb{A}^1 is a natural constraint for problem (1.1), since equations (1.1)₁ and (1.1)₂ hold in it. To write equation (1.1)₃ in a weak form we shall use the realization $-\Delta_{\Gamma_1(\mathbb{D})}$ of the Laplace-Beltrami operator introduced in (2.2) as well as the Dirichlet-to-Neumann operator $\mathcal{A} \in \mathcal{L}(H_{\Gamma_0}^1(\Gamma); [H_{\Gamma_0}^1(\Gamma)]')$ given by

$$\langle \mathcal{A}v, w \rangle_{H_{\Gamma_0}^1(\Gamma)} = \int_{\Omega} \nabla(\mathbb{D}v) \nabla(\mathbb{D}w) \quad \text{for all } v, w \in H_{\Gamma_0}^1(\Gamma). \quad (4.5)$$

By (4.3) and (4.4), since problem (4.1) has a unique solution, we also get that

$$\langle \mathcal{A}v, w \rangle_{H_{\Gamma_0}^1(\Gamma)} = \int_{\Omega} \nabla(\mathbb{D}v) \nabla \phi, \quad (4.6)$$

for all $v, w \in H_{\Gamma_0}^1(\Gamma)$ and $\phi \in H^1(\Omega)$ such that $\phi|_{\Gamma} = w$. One can then abstractly write equation (1.1)₃ in the space $[H_{\Gamma_0}^1(\Gamma)]'$ as follows:

Definition 4. Let (1.3) hold. We say that $v \in H_{\Gamma_0}^1(\Gamma)$ is a solution of

$$-\Delta_{\Gamma_1(\mathbb{D})}v + \mathcal{A}v = |v|^{p-2}v \quad \text{in } [H_{\Gamma_0}^1(\Gamma)]', \quad (4.7)$$

if

$$\int_{\Omega} \nabla(\mathbb{D}v) \nabla(\mathbb{D}\psi) + \int_{\Gamma_1} (\nabla_{\Gamma}v, \nabla_{\Gamma}\psi)_{\Gamma} - \int_{\Gamma_1} |v|^{p-2}v\psi = 0, \quad (4.8)$$

for all $\psi \in H_{\Gamma_0}^1(\Gamma)$.

We immediately get the following result:

Lemma 7. *Let $u \in H^1$. Then u is a weak solution of (1.1) if and only if $u \in \mathbb{A}^1$ and $v = u|_\Gamma$ is a solution of (4.7). Moreover, in this case $u = \mathbb{D}v$.*

Proof. If $u \in H^1$ is a weak solution of (1.1), by (3.3) one immediately gets that $u \in \mathbb{A}^1$, so $v = u|_\Gamma$ satisfies (4.8) and $u = \mathbb{D}v$. Conversely, if $v \in H_{\Gamma_0}^1(\Gamma)$ satisfies (4.8) then $u := \mathbb{D}v \in \mathbb{A}^1$ and, by combining (4.5), (4.6) and (4.8), we get (3.3).

Trivially, equation (4.8) has a variational structure. To write it down we introduce the functional $J : H_{\Gamma_0}^1(\Gamma) \rightarrow \mathbb{R}$ given by

$$J(v) = \frac{1}{2} \int_{\Omega} |\nabla(\mathbb{D}v)|^2 + \frac{1}{2} \int_{\Gamma_1} |\nabla_{\Gamma} v|_{\Gamma}^2 - \frac{1}{p} \int_{\Gamma_1} |v|^p. \quad (4.9)$$

The following result points out some trivial properties of J .

Lemma 8. *If (1.3) holds we have $J \in C^1(H_{\Gamma_0}^1(\Gamma))$, its Fréchet derivative being given for all $v, \psi \in H_{\Gamma_0}^1(\Gamma)$ by*

$$\langle J'(v), \psi \rangle_{H_{\Gamma_0}^1(\Gamma)} = \int_{\Omega} \nabla(\mathbb{D}v) \nabla(\mathbb{D}\psi) + \int_{\Gamma_1} (\nabla_{\Gamma} v, \nabla_{\Gamma} \psi)_{\Gamma} - \int_{\Gamma_1} |v|^{p-2} v \psi. \quad (4.10)$$

Consequently, solutions of (4.7) are exactly critical points of J . Moreover $J = I \cdot \mathbb{D}$, J is even and $J(0) = 0$.

Proof. Trivially, J is even, $J(0) = 0$ and $J = I \cdot \mathbb{D}$. Hence, since $I \in C^1(H^1)$, we also have $J \in C^1(H_{\Gamma_0}^1(\Gamma))$ and (4.10) holds true. The correspondence between solutions of (4.7) and critical points of J immediately follows by (4.8) and (4.10).

The next result shows that J satisfies the remaining geometrical assumptions of Theorem 6.

Lemma 9. *If (1.3) holds, the functional J satisfies assumption ii) in Theorem 4 and assumption v) in Theorem 6.*

Proof. Since $J = I \cdot \mathbb{D}$, by (3.8) there is $c_8 = c_8(p, \Omega, \Gamma_0) > 0$ such that

$$\|v\|_{p, \Gamma_1} \leq c_8 \|\mathbb{D}v\|_{H^1} \quad \text{for all } v \in H_{\Gamma_0}^1(\Gamma).$$

Consequently, by (2.13) and (4.9), we get the estimate

$$J(v) \geq \left(\frac{1}{2} - \frac{c_8^p}{p} \|\mathbb{D}v\|_{H^1}^{p-2} \right) \|\mathbb{D}v\|_{H^1}^2 \quad \text{for all } v \in H_{\Gamma_0}^1(\Gamma),$$

and then

$$J(v) \geq \frac{1}{4} \|\mathbb{D}v\|_{H^1}^2 \quad \text{for all } v \in H_{\Gamma_0}^1(\Gamma) \text{ such that } \|\mathbb{D}v\|_{H^1} \leq \left(\frac{p}{4c_8^p} \right)^{\frac{1}{p-2}}. \quad (4.11)$$

Since, by (4.4), $\|\mathbb{D}(\cdot)\|_{H^1}$ is on $H_{\Gamma_0}^1(\Gamma)$ a norm equivalent to the one inherited from $H^1(\Gamma)$, assumption ii) in Theorem 4, for suitable $\rho, \alpha > 0$, trivially follows from (4.11).

To prove that also assumption v) in Theorem 6 holds true, let Y be any finite dimensional subspace of $H_{\Gamma_0}^1(\Gamma)$. Since on Y all norms are equivalent, there are $c_9 = c_9(p, Y) > 0$ and $c_{10} = c_{10}(p, Y) > 0$ such that

$$c_9 \|v\|_{p, \Gamma_1} \leq \|v\|_{H^1(\Gamma)} \leq c_{10} \|v\|_{p, \Gamma_1} \quad \text{for all } v \in Y. \quad (4.12)$$

Consequently, using again (4.4), there is $c_{11} = c_{11}(p, Y, \Omega, \Gamma_0) > 0$ such that

$$\|\mathbb{D}v\|_{H^1} \leq c_{11} \|v\|_{p, \Gamma_1} \quad \text{for all } v \in Y.$$

Using it in (4.9) we then get

$$J(v) = \frac{1}{2} \|\mathbb{D}v\|_{H^1}^2 - \frac{1}{p} \|v\|_{p, \Gamma_1}^p \leq \frac{1}{2c_{11}^2} \|v\|_{p, \Gamma_1}^2 - \frac{1}{p} \|v\|_{p, \Gamma_1}^p,$$

from which, since $p > 2$, one gets that $J(v) \leq 0$ for $\|v\|_{p, \Gamma_1}$ sufficiently large. By (4.12) then assumption v) in Theorem 6 holds true.

To complete checking of the assumptions of Theorem 6 we give the following result:

Lemma 10. *If (1.3) holds, the functional J satisfies the (PS) condition.*

Proof. We shall prove the statement by using Lemma 5. With this aim we make some preliminary remarks concerning the space H^1 and the functional I . At first H^1 admits the orthogonal (with respect to $(\cdot, \cdot)_{H^1}$ given in (2.11)) splitting

$$H^1 = \mathbb{A}^1 \oplus H_0^1(\Omega), \quad (4.13)$$

the respective orthogonal projectors $\Pi_{\mathbb{A}^1} \in \mathcal{L}(H^1, \mathbb{A}^1)$ and $\Pi_{H_0^1(\Omega)} \in \mathcal{L}(H^1, H_0^1(\Omega))$ being given by

$$\Pi_{\mathbb{A}^1} u = \mathbb{D}u|_{\Gamma} \quad \text{and} \quad \Pi_{H_0^1(\Omega)} u = u - \mathbb{D}u|_{\Gamma} \quad \text{for all } u \in H^1.$$

Using this splitting we can rewrite (3.7), for any $u \in \mathbb{A}^1$, as

$$\begin{aligned} \langle I'(u), \phi + \psi \rangle_{H^1} &= (u, \phi + \psi)_{H^1} + \int_{\Gamma_1} |u|^{p-2} u (\phi + \psi) \\ &= (u, \phi)_{H^1} + \int_{\Gamma_1} |u|^{p-2} u \phi \\ &= \langle I'_{\mathbb{A}^1}(u), \phi \rangle_{\mathbb{A}^1} \quad \text{for all } \phi \in \mathbb{A}^1 \text{ and } \psi \in H_0^1(\Omega), \end{aligned} \quad (4.14)$$

where $I_{\mathbb{A}^1} = I|_{\mathbb{A}^1}$, that is $I_{\mathbb{A}^1}$ is the restriction of I to \mathbb{A}^1 .[§]

To prove the statement let now $(v_n)_n$ be a (PS) sequence for J . Since $J = I \cdot \mathbb{D} = I_{\mathbb{A}^1} \cdot \mathbb{D}$, by (4.4) we get that $(\mathbb{D}v_n)_n$ is a (PS) sequence for the functional $I_{\mathbb{A}^1}$ in \mathbb{A}^1 . Hence $(I(\mathbb{D}v_n))_n$ is bounded in H^1 and, using (4.14), $I'(\mathbb{D}v_n) \rightarrow 0$ in $[H^1]'$, so $(\mathbb{D}v_n)_n$ is a (PS) sequence for I .

By Lemma 5 then, up to a subsequence, $(\mathbb{D}v_n)_n$ converges in H^1 which, using (4.4) again, concludes the proof.

We can finally give the proof of our last main result.

Proof of Theorem 3. By Lemmas 8-10 the functional J satisfies the assumptions of Theorem 6, so it possesses a sequence $(v_n)_n$ of critical points such that $J(v_n) \rightarrow \infty$. By Lemma 7 the sequence $(u_n)_n$ given by $u_n = \mathbb{D}v_n$ for all $n \in \mathbb{N}$ is thus a sequence of critical points of I with $I(u_n) = J(v_n) \rightarrow \infty$. By Lemma 4 the proof is thus concluded.

[§]Once one recognizes that critical points of I belong to \mathbb{A}^1 , formula (4.14) gives a different proof of Lemma 7.

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Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflict of interest.

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